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On The Continuity of the Spectrum of Fields of Operators

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Special thanks to my advisor Erik Christopher Bedos, and his amazingly sharp eye for details.

To my father: Happy birthday, dad.

Abstract

Based on a paper by Beckus and Bellisard, we study the continuity of fields of operators, mainly by studying their associated spectra. We discuss and extend Beckus and Bellisard's results about self-adjoint operators, including unbounded, self-adjoint operators, and we also look briefly at bounded, normal operators.

Introduction

This is Christian Aarset's Master's thesis for the course MAT5960 at the University of Oslo.

In [3], Beckus and Bellissard discuss the following problem: *Given a family of bounded self-adjoint operators $(A_t)_{t\in T}$ indexed by a parameter t in some topological space T, what are the different interesting forms of continuity of the spectrum of these operators, and how are the different types of continuity connected?* They conclude that for bounded, self-adjoint operators, there is an equivalence between continuity of the gap edges, $p2$-continuity of the field and Fell continuity of the spectrum; furthermore, for metric space settings, they give results relating $p2$-α-Hölder continuity of $(A_t)_{t\in T}$ with α-Hölder continuity of the gap edges and $\alpha/2$-Hölder continuity of the width of gaps.

In this article, we will extend several of Beckus and Bellissard's ideas, as well as giving detailed and precise proofs of all their claims. This is particularly true in the case of unbounded, self-adjoint operators, which are only treated very briefly by Beckus and Bellisard.

The first chapter serves as a refresher and an introduction to several important concepts we will make extensive use of. The second chapter deals with the core results for fields of bounded, self-adjoint operators. The third chapter deals with Hölder continuity of fields of self-adjoint, bounded operators, proving and in some cases improving Beckus and Bellisard's estimates. The fourth chapter extends some of the earlier results to the case of unbounded, self-adjoint operators. The fifth chapter explores some possibilities for giving results for fields of normal operators, and discusses the difficulties involved in this approach.

Contents

1 Preliminaries

1.1 Basic definitions

The main concept is that of a *field of operators*:

Definition 1.1. For any given topological space T, a family $H = (H_t)_{t\in T}$ of Hilbert spaces is called a *field of Hilbert spaces*. If we for each $t \in T$ define a linear operator A_t in H_t, then the family $A = (A_t)_{t\in T}$ is called a *field of operators*.

We will often assume that the H_t are already given, and refer to fields of operators without explicitly mentioning any H_t. We will also just use the word "operator" to refer to *linear* operators. Whenever we are given a point $t_0 \in T$, we will often use the convention $A_0 := A_{t_0}$ to simplify notation.

Given some field $A = (A_t)_{t\in T}$ and a point $t_0 \in T$, it is natural to ask whether A is, in some way, continuous at t_0. However, many of the common ways of assessing continuity of families of operators fail with fields. For example, norm continuity, strong continuity and weak continuity are all meaningless concepts when the underlying Hilbert spaces of the different operators are allowed to be different, that is, if for every neighbourhood $U \subset T$ of t_0 there exists a $t \in U$ such that $H_t \neq H_{t_0}$. Indeed, it is easy to construct cases where the underlying Hilbert spaces are pairwise different, that is, $H_t \neq H_s$ for all pairs $t, s \in T$ with $t \neq s$.

It follows that in order to have a meaningful definition of continuity, we need some sort of property that allows for comparison. Fortunately, any linear operator, no matter how exotic, can be related to a subset of $\mathbb{C}$ or $\mathbb{R}$ through the notion of the *spectrum*.

Definition 1.2. For any Hilbert space H over $\mathbb{C}$, we denote the space of all bounded linear operators in H by $\mathbf{B}(H)$ and the space of all invertible bounded linear operators by $GL(H)$, that is,

$$GL(H) := \{B \in \mathbf{B}(H) \mid \text{There exists some } C \in \mathbf{B}(H) \text{ such that } BC = CB = I\}$$

Definition 1.3. For any Hilbert space H over $\mathbb{C}$ and any bounded operator $B : H \to H$, the *spectrum* of B is defined as the subset $\sigma(B) \subset \mathbb{C}$ where

$$\sigma(B) := \{\lambda \in \mathbb{C} \mid \lambda I - B \notin GL(H)\}$$

It is well known that the spectrum of a bounded operator B is a non-empty, closed and bounded (i.e., compact) subset of $\mathbb{C}$ such that

$$r(B) := \sup |\sigma(B)| \leq \|B\|$$

with equality if B is normal, that is, if it commutes with its adjoint. Similarly, it is known that B is self-adjoint if and only if the spectrum is contained in $\mathbb{R}$. Furthermore, for any bounded, normal operator B and any complex function f whose restriction to $\sigma(B)$ is continuous, we have that

$$f(\sigma(B)) = \sigma(f(B))$$

and the same is more generally true for any bounded operator B and any polynomial p. These facts will be used throughout the article.

The complement of the spectrum, the *resolvent set*, is denoted by $\rho(B)$. From the notes above, it follows that for a bounded operator, the resolvent set is an open, non-empty subset of $\mathbb{C}$.

For any $\lambda \in \rho(B)$, the operator

$$R(\lambda, B) := (\lambda I - B)^{-1}$$

(alternatively denoted by $R_\lambda(B)$) is called *the resolvent* (of B at λ).

As spectra are generally closed, we will be working extensively with closed sets, so we introduce for every topological space X the space $\mathscr{C}(X)$ of all closed subsets of X. Initially we will focus on the case where $X = \mathbb{R}$, but as we progress we will consider cases where X is a subset of $\mathbb{R}$ or even $\mathbb{C}$.

One important word of caution: Many authors choose to let $\mathscr{C}(X)$ (or $\mathscr{K}(X)$) denote the space of *compact, non-empty* subsets of X. However, *we do not do this*. Per our definition, we have $F \in \mathscr{C}(X)$ for *any* closed set $F \subset X$, including empty and non-compact F. Although this choice will result in some extra work, it will pay off later, when we start considering unbounded operators.

Another important construction is the *compactification*. More accurately, we have the following definition from [9], page 237:

Definition 1.4. A compactification of a space X is a compact Hausdorff space Y containing (a copy of) X as a subspace such that $\overline{X} = Y$. Two compactifications Y_1 and Y_2 of X are said to be equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for every $x \in X$.

A fairly common compactification of $\mathbb{R}$ is the so-called "one-point compactification" $\mathbb{R} \cup \{\infty\}$, which is homeomorphic to S^1. However, this compactification will not be very useful to us. Its main issue is that it "does not differentiate between sequences that diverge towards the left or towards the right" - for example, note that in $\mathbb{R} \cup \{\infty\}$, we get that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty = \lim_{x\to 0} -\frac{1}{x^2}$$

The way to address this is to work with the *two-point compactification* of $\mathbb{R}$, denoted by $\mathbb{R} \cup \{\pm\infty\}$ or $[-\infty, \infty]$, where ∞ and $-\infty$ are the classic plus and minus infinity symbols. The basis for the topology on $[-\infty, \infty]$ consists of all intervals on the form (a, b), $[-\infty, a)$ and $(b, \infty]$ where $a, b \in \mathbb{R}$ and $a < b$; see [1], page 57.

Throughout this paper, we will often look at sets that are on the form $X \cup \{\infty\}$, $X \cup \{-\infty\}$ or $X \cup \{\pm\infty\}$, where X is some subset of $\mathbb{R}$. These will always be treated as subsets of $[-\infty, \infty]$, and, where necessary, will be be endowed with the subspace topology inherited from $[-\infty, \infty]$. For example, the space $[0, \infty]$ (occasionally written as $\mathbb{R}^+ \cup \{\infty\}$), which is the one-point compactification of $\mathbb{R}^+$, will be treated as a subset of $[-\infty, \infty]$.

1.2 Gap edge continuity

One possible method of determining "continuity of the spectrum" is by looking not at the spectra themselves, but at their so-called *gaps*, that is, the connected components of their complements. If these gaps "vary continuously" in some manner, then it would be reasonable to say that the spectrum, too, is continuous in that manner. More accurately, we have the following definition:

Definition 1.5. Take $F \in \mathscr{C}(\mathbb{R})$. A *gap* of F is a connected component of its complement.

It is well known that any non-empty open subset of $\mathbb{R}$ can be written as a countable union of disjoint open intervals. As such, any $F \in \mathscr{C}(\mathbb{R})$ with $F \neq \mathbb{R}$ has at most countably many gaps. In other words, a gap of F is an interval $(a, b) \subset \mathbb{R}\backslash F$ with $a \in F \cup \{-\infty\}$ and $b \in F \cup \{\infty\}$. The points a and b are sometimes referred to as *gap edges* of F. The gap edges $\inf F$ and $\sup F$ (if finite) are sometimes referred to as the *outer gap edges*, while other gap edges are sometimes called the *inner gap edges*.

For each $F \in \mathscr{C}(\mathbb{R})$, there is at most one gap of F where $a = -\infty$ and at most one gap where $b = \infty$, and if F is unbounded there may be no such gaps. If $F = \emptyset$, its only gap is $(-\infty, \infty) = \mathbb{R}$, while if $F = \mathbb{R}$ it has no gaps (unless we consider $\emptyset$ as a gap, which we generally will not do).

In particular, since the spectra of self-adjoint operators are closed subsets of $\mathbb{R}$, the concepts of gaps and gap edges are well defined for them.

A useful fact to note is that if $F \in \mathscr{C}(\mathbb{R})$ and $K \subset F^c$ is a compact subset of $\mathbb{R}$, then K is contained in the union of only finitely many gaps of F. For since the gaps of F are open sets, they form an open covering of disjoint subsets of $\mathbb{R}$ covering K, and by the compactness of K there exists a finite subcover. Said differently, every connected component of K is contained in exactly one gap of F.

We will need a formal definition of what it means for "the gap edges to be a continuous function of t":

Definition 1.6. Consider a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$. For any $t_0 \in T$, we say that the map $t \mapsto F_t$ is *gap edge continuous at* t_0 if the following conditions are both satisfied:

- For every gap (a, b) of F_{t_0}, there exists an open neighbourhood $U \subset T$ of t_0 such that for every $t \in U$, there exists a gap (a_t, b_t) of F_t so that $-\infty \leq a_t < b_t \leq \infty$, with

$$\lim_{t \to t_0} a_t = a \text{ and } \lim_{t \to t_0} b_t = b$$

- Assume $\{t_\iota\}_{\iota \in I}$ is a net of points in T converging to t_0, and $\{a_\iota\}_{\iota \in I}$, $\{b_\iota\}_{\iota \in I}$ are two convergent nets of points in $[-\infty, \infty]$ such that

$$a := \lim_{\iota} a_\iota \, , \, b := \lim_{\iota} b_\iota$$

  and (a_ι, b_ι) is a gap of F_{t_ι} for every $\iota \in I$. Then either (a, b) is a gap of F_{t_0}, or $a = b$.

We will say that the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is *gap edge continuous* if it is gap edge continuous at all points $t \in T$.

For a field of self-adjoint operators $A = (A_t)_{t \in T}$ and any $t_0 \in T$, we say that *A is gap edge continuous at t_0* if the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at t_0 as defined above. We say that *A is gap edge continuous* or that *the gap edges of A are continuous* if A is gap edge continuous at every $t \in T$.

Intuitively speaking, the above definition says that gap edge continuity at t_0 means that gaps cannot "appear" or "disappear" at t_0.

Note that in the second condition of gap edge continuity, we clearly have that a, $b \in F_{t_0} \cup \{\pm\infty\}$ if (a, b) is a gap of F_{t_0}, by the definition of gaps. However, we did not say anything about the case where $a = b$. Fortunately, we have that a, $b \in F_{t_0} \cup \{\pm\infty\}$ in this case as well.

Proposition 1.7. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at some point $t_0 \in T$. Assume also that we are given a net $\{t_\iota\}_{\iota \in I}$ of points in T converging to t_0, as well as two convergent nets $\{a_\iota\}_{\iota \in I}$, $\{b_\iota\}_{\iota \in I}$ of points in $[-\infty, \infty]$ such that each (a_ι, b_ι) is a gap of F_{t_ι}. If*

$$a := \lim_\iota a_\iota = \lim_\iota b_\iota =: b$$

then $a = b \in F_{t_0} \cup \{\pm\infty\}$.

Proof. Set $F_0 := F_{t_0}$, and assume that $a = b \notin F_0 \cup \{\pm\infty\}$. We will show that this causes a contradiction.

Assume that $F_0 \neq \mathbb{R}$, and thus has at least one gap, as otherwise the proof is trivial. Since $a = b \notin F_0 \cup \{\pm\infty\}$, it must lie in a gap of F_0; by the comments following Definition 1.5, it follows that this gap is an interval (c, d) with c, $d \in F_0 \cup \{\pm\infty\}$ and $c < a = b < d$.

Assume first that $-\infty < c < d < \infty$. By the first condition of gap edge continuity, there exists a neighbourhood $U \subset T$ of t_0 such that for each $t \in U$, there exists a gap (c_t, d_t) of F_t where

$$|c_t - c| < \frac{a - c}{2} \text{ and } |d_t - d| < \frac{d - a}{2}$$

However, this is impossible. For by assumption, each (a_ι, b_ι) is a gap of F_{t_ι}, and eventually we have

$$\max\{|a_\iota - a|, |b_\iota - a|\} < \min\left\{\frac{a - c}{2}, \frac{d - a}{2}\right\}$$

But this implies that eventually we have $c_{t_\iota} < a_\iota < b_\iota < d_{t_\iota}$, so eventually $(a_\iota, b_\iota) \subsetneq (c_{t_\iota}, d_{t_\iota})$, while at the same time both $(c_{t_\iota}, d_{t_\iota})$ and (a_ι, b_ι) are gaps of F_{t_ι}, which is impossible as any two different gaps are necessarily disjoint. This proves that $a = b \in F_0 \cup \{\pm\infty\}$ in the case where $-\infty < c < d < \infty$.

Next, assume that $-\infty < c < d = \infty$. By the first condition of gap edge continuity, we have that for any $N > 0$ there exists a neighbourhood $U \subset T$ of t_0 such that for each $t \in U$, there exists a gap (c_t, d_t) of F_t with

$$|c_t - c| < \frac{a-c}{2} \text{ and } d_t > N$$

Again, this is impossible. For, since N was arbitrary, we can choose N so $N > a+1$. But since each (a_ι, b_ι) is a gap of F_{t_ι}, and since $a_\iota \to a$ and $b_\iota \to a$, we eventually have that

$$\max\{|a_\iota - a|, |b_\iota - a|\} < \min\left\{\frac{a-c}{2}, 1\right\}$$

Since t_ι is eventually in U, we must eventually have that $c_{t_\iota} < a_\iota < b_\iota < d_{t_\iota}$, so eventually $(a_\iota, b_\iota) \subsetneq (c_{t_\iota}, d_{t_\iota})$, while at the same time both $(c_{t_\iota}, d_{t_\iota})$ and (a_ι, b_ι) are gaps of F_{t_ι}. But this is impossible, as any two different gaps are necessarily disjoint. This proves that $a = b \in F_0 \cup \{\pm\infty\}$ in the case where $-\infty < c < d = \infty$.

The case where $-\infty = c < d < \infty$ is completely analoguous to the previous case.

Finally, if $-\infty = c < d = \infty$, it follows that $F_0 = \emptyset$. By the first condition of gap edge continuity, we know that for any $N > 0$ there exists a neighbourhood $U \subset T$ of t_0 such that for every $t \in U$, there is a gap (c_t, d_t) of F_t such that

$$c_t < -N < N < d_t$$

As N was arbitrary, we can choose it so that $N > a + 1$. However, as $a_\iota \to a$ and $b_\iota \to a$, we must eventually have

$$\min\{|a_\iota - a|, |b_\iota - a|\} < 1$$

Since t_ι is eventually in U, we must eventually have $(a_\iota, b_\iota) \subsetneq (c_{t_\iota}, d_{t_\iota})$ while both (a_ι, b_ι) and $(c_{t_\iota}, d_{t_\iota})$ are gaps of F_{t_ι}. Again, this is impossible as any two different gaps are necessarily disjoint. Thus $a = b \in F_0 \cup \{\pm\infty\}$ also in this case; as this exhausts all possible cases, we are done.

□

The following examples serve to illustrate gap edge continuity.

Example 1.8. Set $T = [0, 1]$ with the standard topology it inherits as a subset of $\mathbb{R}$, and consider the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ given by

$$F_t = (-\infty, -t] \cup [t, \infty)$$

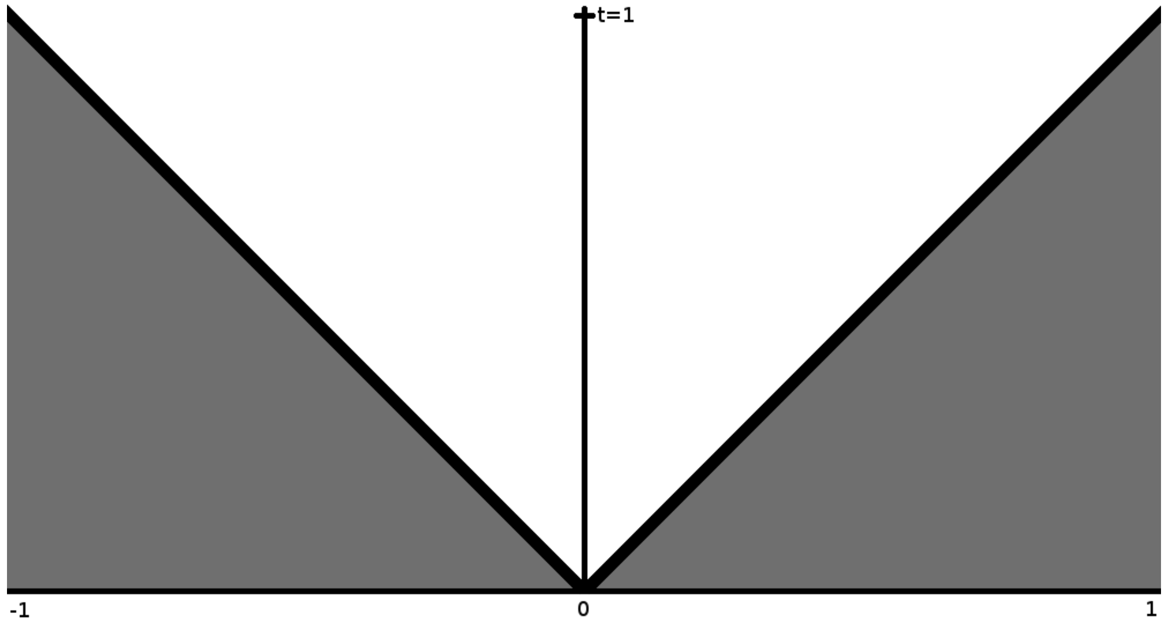

Figure 1: A part of the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ pictured as a subset of $T \times [-1, 1] = [0, 1] \times [-1, 1]$.

We claim that the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at all points $t \in T$.

For all $t \neq 0$, we see that $(-t, t)$ is a gap of F_t, and is indeed the only gap of F_t. Thus for any $t_0 \neq 0$, it follows that if we set $a = -t_0$, $b = t_0$, $U = (0, 1]$ and for every $t \in U$ set $a_t = -t$ and $b_t = t$, then (a, b) is a gap of F_{t_0} and (a_t, b_t) is a gap of F_t for $t \in U$. Clearly,

$$\lim_{t \to t_0} a_t = a \text{ and } \lim_{t \to t_0} b_t = b$$

Thus, the first condition of gap edge continuity is satisfied at $t_0 \neq 0$.

For the second condition, we have that if $\{t_\iota\}_{\iota \in I}$ is a net in T converging to t_0 such that for each $\iota \in I$ we know that $(a_{t_\iota}, b_{t_\iota})$ is a gap of F_{t_ι}, then again clearly we must have $a_{t_\iota} = -t_\iota$ and $b_{t_\iota} = t_\iota$, so it follows that

$$\lim_\iota a_{t_\iota} = -t_0 \text{ and } \lim_\iota b_{t_\iota} = t_0$$

and so the second condition is satisfied as well at $t_0 \neq 0$.

Next, note that for $t_0 = 0$, $F_0 = \mathbb{R}$ is a connected set and has no gaps, so the first condition is trivally satisfied at $t_0 = 0$. Furthermore, if $\{t_\iota\}_{\iota \in I}$ is a net in T converging to 0 such that for each $\iota \in I$ we know that $(a_{t_\iota}, b_{t_\iota})$ is a gap of F_{t_ι}, then clearly we must have $a_{t_\iota} = -t_\iota$ and $b_{t_\iota} = t_\iota$, so it follows that

$$\lim_{t_\iota \to 0} a_{t_\iota} = 0 = \lim_{t_\iota \to 0} b_{t_\iota} \in F_0$$

and so the second condition is satisfied as well at $t_0 = 0$.

Example 1.9. Consider any bounded, self-adjoint operator A_0, and define the field $A := (A_t)_{t\in T} := (f(t)A_0)_{t\in T}$ for any function $f : T \to \mathbb{R}$ that is continuous at some point $t_0 \in T$. Since $\sigma(A_t) = \sigma(f(t)A_0) = f(t)\sigma(A_0)$ for all $t \in T$, it follows that A is gap edge continuous at t_0, as can easily be verified.

Example 1.10. We claim that the standard procedure to construct the Cantor set is gap edge continuous. More explicitly, define $T := \mathbb{N}_0 \cup \{\infty\}$, and equip it with the subspace topology inherited from $[-\infty, \infty]$. We make the following recursive definition:

$$C_0 := [0, 1]$$
$$C_n := \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \text{ for } n \in \mathbb{N}$$

and set

$$C_\infty := \bigcap_{n\in\mathbb{N}_0} C_n$$

Figure 2: The steps from $n = 0$ to $n = 6$ of the Cantor set generation process. Image courtesy of [13].

Since C_0 is closed, an inductive argument shows that C_n is a closed set for every $n \in \mathbb{N}_0$. Since C_∞ is a countable intersection of closed sets, it is itself closed. It follows that we have that $C_n \in \mathscr{C}(\mathbb{R})$ for every $n \in T$.

As can be seen in [5], C_∞ is the standard Cantor set. Our claim that "the standard procedure to construct the Cantor set is gap edge continuous" now becomes equivalent to the statement that "the map $n \in T \mapsto C_n \in \mathscr{C}(\mathbb{R})$ is gap edge continuous".

The first few sets can be expressed as

$$C_1 = C_0 \backslash \left(\frac{1}{3}, \frac{2}{3}\right)$$
$$C_2 = C_1 \backslash \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right)$$
$$C_3 = C_2 \backslash \left(\left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right)\right)$$

An inductive argument shows that if we for each $n \in \mathbb{N}$ set

$$\mathscr{A}_n := \{0,2\}^n$$

and

$$\mathscr{B}_0 = \{0\}$$
$$\mathscr{B}_n = \left\{ \left(\sum_{i=1}^{n} \frac{\alpha_i}{3^i} \right) \in [0,1] \mid \boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathscr{A}_n \right\}$$

then in general,

$$\begin{aligned} C_n &= C_{n-1} \setminus \bigcup_{\beta \in \mathscr{B}_{n-1}} \left(\beta + \frac{1}{3^n}, \beta + \frac{2}{3^n} \right) \\ &= [0,1] \setminus \bigcup_{k=1}^{n} \bigcup_{\beta \in \mathscr{B}_{k-1}} \left(\beta + \frac{1}{3^k}, \beta + \frac{2}{3^k} \right) \end{aligned} \tag{1}$$

for $1 \leq n < \infty$, where all the sets in the union are disjoint, and the union is finite. From this it follows that

$$C_\infty = [0,1] \setminus \left[\bigcup_{k \in \mathbb{N}} \bigcup_{\beta \in \mathscr{B}_{k-1}} \left(\beta + \frac{1}{3^k}, \beta + \frac{2}{3^k} \right) \right] \tag{2}$$

As $\{n\}$ is an open set for all $n \in T$ with $n \neq \infty$, the map is trivially gap edge continuous at all points $n_0 \neq \infty$.

Next, we prove gap edge continuity at $n_0 = \infty$. Observe that from (2) we see that any gap G of C_∞ is either $(-\infty, 0)$, $(1, \infty)$ or on the form $\left(\beta + \frac{1}{3^m}, \beta + \frac{2}{3^m}\right)$ for some $m \in \mathbb{N}$ and some $\beta \in \mathscr{B}_{m-1}$.

In the first two cases, the first condition of gap edge continuity is trivially satisfied, as $(-\infty, 0)$ and $(1, \infty)$ is a gap of C_n for all $n \in T$.

So assume $G \neq (-\infty, 0)$ and $G \neq (1, \infty)$. From (1) we see that G is a gap of C_m, and in fact it is a gap for all C_n with $n \geq m$. Since $\{n \in \mathbb{N} \mid n \geq m\} \cup \{\infty\}$ is an open neighbourhood of $n_0 = \infty$, the first condition clearly holds.

For the second condition, consider a net $\{n_\iota\}_{\iota \in I}$ of points in T converging to $n_0 = \infty$, as well as two nets $\{a_\iota\}_{\iota \in I}$ and $\{b_\iota\}_{\iota \in I}$ of points in $[-\infty, \infty]$. Assume that we are given that for every $\iota \in I$, we have that (a_ι, b_ι) is a gap of C_{n_ι}, and that $a_\iota \to a$, $b_\iota \to b$ for some $a, b \in [-\infty, \infty]$. We must show that either $a = b$ or that (a, b) is a gap of C_∞.

Assume first that $a = -\infty$. As $C_n \cap (-\infty, 0) = \emptyset$ for all $n \in T$, it follows that we in order to have $a_\iota \to -\infty$, we must eventually have $a_\iota = -\infty$. Thus, we must also eventually have $b_\iota = 0$, so the second condition is clearly satisfied in this case. The case where $b = \infty$ is similar.

Assume next that $a \neq -\infty$ and $b \neq \infty$. As each a_ι and b_ι is a gap edge of C_{n_ι}, it follows by the preceding discussion that $a_\iota, b_\iota \in C_\infty \cup \{\pm\infty\}$ for all $\iota \in I$. As $C_\infty \cup \{\pm\infty\}$

is closed, we get that a, $b \in C_\infty \cup \{\pm\infty\}$, and by assumption $a \neq -\infty$ and $b \neq \infty$, so we must have $a, b \in C_\infty$.

Assume that $a \neq b$; otherwise we are done. Begin by noting that by (1), it follows that for every $\iota \in I$ we have that

$$a_\iota = \beta_\iota + \frac{1}{3^{m_\iota}} \text{ and } b_\iota = \beta_\iota + \frac{2}{3^{m_\iota}}$$

for some $m_\iota \in \mathbb{N}$ and some $\beta_\iota \in \mathscr{B}_{m_\iota - 1}$. It follows that

$$\beta_\iota = 2a_\iota - b_\iota \text{ and } m_\iota = -\frac{\log(b_\iota - a_\iota)}{\log 3}$$

Thus we see that the nets $\{\beta_\iota\}_{\iota\in I}$ and $\{m_\iota\}_{\iota\in I}$ converge to

$$\beta = 2a - b \text{ and } m = -\frac{\log(b - a)}{\log 3}$$

respectively. We have $m_\iota \in \mathbb{N} \cup \{\infty\}$ for every $\iota \in I$, and $\mathbb{N} \cup \{\infty\}$ is a closed subset of $[-\infty, \infty]$, so $m \in \mathbb{N} \cup \{\infty\}$. Since $b \neq a$, it follows that $m \neq \infty$. But this implies that eventually, $m_\iota = m \in \mathbb{N}$, and so eventually

$$a_\iota = \beta_\iota + \frac{1}{3^m} \text{ and } b_\iota = \beta_\iota + \frac{2}{3^m}$$

with $\beta_\iota \in \mathscr{B}_{m-1}$. As $\mathscr{B}_{m-1}$ is just a finite collection of singletons, it is a closed set. Since $\{\beta_\iota\}_{\iota\in I}$ converges to β and eventually $\beta_\iota \in \mathscr{B}_{m-1}$, we must have $\beta \in \mathscr{B}_{m-1}$.

Since we have

$$a = \beta + \frac{1}{3^m} \text{ and } b = \beta + \frac{2}{3^m}$$

with $m \in \mathbb{N}$ and $\beta \in \mathscr{B}_{m-1}$, it follows from (2) that (a, b) is a gap of C_∞. This proves that the second condition of gap edge continuity holds at $n_0 = \infty$, so we are done.

A very nice consequence of gap edge continuity is that if $t \mapsto F_t$ is gap edge continuous at a point t_0, then we can always find a neighbourhood U of t_0 such that each F_t with $t \in U$ has "roughly as many" gaps as F_{t_0}. More formally, we have the following definition and result.

Definition 1.11. Define the map $\#_G : \mathscr{C}(\mathbb{R}) \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\#_G(F) := \begin{cases} N & \text{if } F \text{ has exactly } N \text{ gaps for some } N \in \mathbb{N}_0 \\ \infty & \text{if } F \text{ has countably infinitely many gaps} \end{cases}$$

Note that, by the discussion following Definition 1.5, the map $\#_G$ is well defined, and clearly $\#_G$ is a surjection from $\mathscr{C}(\mathbb{R})$ onto $\mathbb{N}_0 \cup \{\infty\}$.

Proposition 1.12. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at some point $t_0 \in T$. Then the following statements both hold:*

- *If $\#_G(F_{t_0}) = N$ for some $N \in \mathbb{N}_0$, then there exists a neighbourhood $U \subset T$ of t_0 such that $\#_G(F_t) \geq N$ for all $t \in U$.*

- *If $\#_G(F_{t_0}) = \infty$, then for every $N \in \mathbb{N}_0$ there exists a neighbourhood $U_N \subset T$ of t_0 such that $\#_G(F_t) \geq N$ for all $t \in U_N$.*

Proof. Write $F_0 := F_{t_0}$. If $F_0 = \mathbb{R}$, then $\#_G(F_0) = 0$, and there is nothing to prove. So assume $F_0 \neq \mathbb{R}$ and thus $\#_G(F_0) \neq 0$.

Start by taking any $N \in \mathbb{N}$ with $N \leq \#_G(F_0)$. As each gap of F_0 is on the form (a, b) for some $a, b \in [-\infty, \infty]$ with $a < b$, and as $\mathbb{R}$ with the standard order is a totally ordered set, it follows that we can find a family $\{(a_n, b_n)\}_{n=1}^N$ of intervals in $\mathbb{R}$ such that for each n with $1 \leq n \leq N$, (a_n, b_n) is a gap of F_0 and such that

$$a_n < b_n \leq a_{n+1} < b_{n+1}$$

for all n with $1 \leq n \leq N - 1$. As $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for all $n \neq m$, it follows the family $\{(a_n, b_n)\}_{n=1}^N$ consists of exactly N disjoint gaps of F_0.

Next, define

$$\begin{aligned}
\epsilon_1 &:= \frac{a_2 - a_1}{2} \\
\epsilon_n &:= \min\left\{\frac{a_{n+1} - a_n}{2}, \frac{a_n - a_{n-1}}{2}\right\} \text{ for } 2 \leq n \leq N - 1 \\
\epsilon_N &:= \frac{a_N - a_{N-1}}{2}
\end{aligned}$$

By the first condition of gap edge continuity, we can for each n find a neighbourhood $U_n \subset T$ of t_0 such that for each $t \in U_n$, there exists a gap $(a_{n,t}, b_{n,t})$ of F_t with

$$|a_{n,t} - a_n| < \epsilon_n$$

Now set $U := \bigcap_{n=1}^N U_n$; as it is a finite intersection of neighbourhoods of t_0, it is itself a neighbourhood of t_0. For every $t \in U$ and every n, we now have a gap $(a_{n,t}, b_{n,t})$ of F_t, and from the definition of the ϵ_n's we see that

$$\begin{aligned}
a_{1,t} &< \frac{a_2 + a_1}{2} \\
\frac{a_n - a_{n-1}}{2} < a_{n,t} &< \frac{a_{n+1} + a_n}{2} \\
\frac{a_N - a_{N-1}}{2} &< a_{N,t}
\end{aligned}$$

for every $t \in U$ and for every n with $2 \leq n \leq N - 1$. Note that this implies that $a_{n,t} < \frac{a_{n+1}+a_n}{2} < a_{n+1,t}$ for every $t \in U$ and every n with $1 \leq N \leq N - 1$. But this implies that we must have

$$a_{n,t} < b_{n,t} \leq a_{n+1,t} < b_{n+1,t}$$

for every $t \in U$ and for every n with $1 \leq n \leq N-1$. For if not, then we must have $a_{n+1,t} < b_{n,t}$ for at least one n and at least one $t \in U$. Therefore, we must have

$$(a_{n,t}, b_{n,t}) \cap (a_{n+1,t}, b_{n+1,t}) = (a_{n+1,t}, \min\{b_{n,t}, b_{n+1,t}\}) \neq \emptyset$$

while at the same time $(a_{n,t}, b_{n,t}) \neq (a_{n+1,t}, b_{n+1,t})$ as $a_{n,t} < a_{n+1,t}$. But this is impossible, as both $(a_{n,t}, b_{n,t})$ and $(a_{n+1,t}, b_{n+1,t})$ are gaps of F_t, and any two different gaps are necessarily disjoint.

Thus for every $t \in U$, the family $\{(a_{n,t}, b_{n,t})\}_{n=1}^{N}$ of intervals in $\mathbb{R}$ consists of exactly N disjoint gaps of F_t, implying that F_t has at least N gaps, so it follows that we must have $\#_G(F_t) \geq N$ for every $t \in U$.

As N was arbitrary (under the restrictions $N \leq \#_G(F_0)$ and $N \in \mathbb{N}$), the proof of the second statement follows immediately, and the proof of the first statement follows by taking $N := \#_G(F_0)$.

□

Note that neither of the statements in the above Proposition can be improved, as these two examples show.

Example 1.13. Set $T = [0,1]$ with the standard topology inherited from $\mathbb{R}$, and consider the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ given by

$$F_t := \begin{cases} \mathbb{R} \setminus \bigcup_{k \in \mathbb{Z}} (2k - t, 2k + t) & \text{if } t \neq 0 \\ \mathbb{R} & \text{if } t = 0 \end{cases}$$

It is not hard to verify that $t \mapsto F_t$ is gap edge continuous at all points $t \in T$. We see that $\#_G(F_0) = 0$ while $\#_G(F_t) = \infty$ for all $t \neq 0$.

Example 1.14. Consider $n \in T \mapsto C_n \in \mathscr{C}(\mathbb{R})$ as defined in Example 1.10. It is not hard to see that $\#_G(C_\infty) = \infty$, while $\#_G(C_n) < \infty$ for all $n \in \mathbb{N}_0$.

We will not go much further down the path of counting and labeling gaps, although we will reference the above Proposition later to explain the difficulties involved in working in $\mathbb{C}$.

1.3 Topologies on $\mathscr{C}(X)$

In addition to the previous notion of gap edge continuity, we would like to have more topological interpretations of the concept of continuity of families of closed sets. Recall that for any topological space X, we let $\mathscr{C}(X)$ be defined as the set of all closed subsets of X.

When X is a metric space - for example, any subset of $\mathbb{C}$ - we often work with the *Hausdorff distance* on the space of closed subsets of X. Many authors work with Hausdorff distance only on the space of *compact, non-empty subsets of* X. However, we want to give a slightly more general formulation:

Definition 1.15. Given a metric space (X, d), we define *the Hausdorff distance* between points in X and elements in $\mathscr{C}(X)$ by the function dist : $X \times \mathscr{C}(X) \to \mathbb{R}^+ \cup \{\infty\}$, given by

$$\text{dist}(x, Y) := \inf_{y \in Y} d(x, y)$$

Similarly, we define *the Hausdorff distance* between elements in $\mathscr{C}(X)$ by the identically named function dist : $\mathscr{C}(X) \times \mathscr{C}(X) \to \mathbb{R}^+ \cup \{\infty\}$, defined as

$$\begin{aligned}\text{dist}(Y, Z) &:= \max\left\{\sup_{y \in Y} \text{dist}(y, Z), \sup_{z \in Z} \text{dist}(z, Y)\right\} \\ &= \max\left\{\sup_{y \in Y} \inf_{z \in Z} d(y, z), \sup_{z \in Z} \inf_{y \in Y} d(z, y)\right\}\end{aligned}$$

Note that as by definition $\inf \emptyset = \infty$ (∞ is the biggest number that is smaller than all numbers in the empty set), we have $\text{dist}(x, \emptyset) = \infty$ for all $x \in X$, and so

$$\text{dist}(\emptyset, Y) = \text{dist}(Y, \emptyset) = \infty$$

for all $Y \in \mathscr{C}(X)$ with $Y \neq \emptyset$.

It is somewhat more difficult to define $\text{dist}(\emptyset, \emptyset)$. One reasonable way to define it would be to use the fact that $\sup \emptyset = -\infty$ ($-\infty$ is the smallest number that is greater than all numbers in the empty set), but a more practical convention, which was suggested in for example [1] and which we will use, is to simply set $\text{dist}(\emptyset, \emptyset) = 0$.

One thing to be careful about with Hausdorff distance is that, in general, for any given $x \in X$ and $F \in \mathscr{C}(X)$ we do *not* have that $\text{dist}(x, F) = \text{dist}(\{x\}, F)$. For example, when $X = \mathbb{R}$ with the standard metric we have $\text{dist}(0, [0, 1]) = 0$, but $\text{dist}(\{0\}, [0, 1]) = 1$.

As long as all the quantities involved are finite, we have, for example from [1], page 110, that

$$\begin{aligned}&\text{dist}(Y, Z) \geq 0 \\ &\text{dist}(Y, Z) = 0 \Leftrightarrow Y = Z \\ &\text{dist}(Y, Z) = \text{dist}(Z, Y) \\ &\text{dist}(Y, Z) \leq \text{dist}(Y, W) + \text{dist}(W, Z)\end{aligned}$$

for all $Y, Z, W \in \mathscr{C}(X)$, and indeed, if we restrict ourselves to non-empty, compact sets, then we see, also from [1], that the Hausdorff distance is a metric. However, in general the Hausdorff distance on $\mathscr{C}(X)$ is *not* a metric, as infinite distances can occur; for example, if $X = \mathbb{R}$ with the standard metric, then

$$\text{dist}\Big((-\infty, 0], [0, \infty)\Big) = \infty$$

There is, however, still a natural definition of *Hausdorff continuity* of maps of the form $t \in T \mapsto F_t \in \mathscr{C}(X)$:

Definition 1.16. Given a metric space (X, d), a map $t \in T \mapsto F_t \in \mathscr{C}(X)$ and any $t_0 \in T$, we say that *the map $t \mapsto F_t$ is Hausdorff continuous at t_0* if the map $t \in T \mapsto \text{dist}(F_{t_0}, F_t) \in \mathbb{R}^+ \cup \{\infty\}$ is continuous at t_0, that is, if

$$\lim_{t \to t_0} \text{dist}(F_{t_0}, F_t) = 0$$

We can use this to define *Hausdorff continuity of the spectra of self-adjoint operators*:

Definition 1.17. Given a field A of self-adjoint operators and any $t_0 \in T$, we say that *the spectrum function of A is Hausdorff continuous at t_0* if the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is Hausdorff continuous at t_0; that is, if the map

$$t \in T \mapsto \text{dist}\Big(\sigma(A_{t_0}), \sigma(A_t)\Big) \in \mathbb{R}^+ \cup \{\infty\}$$

is continuous at t_0.

If the spectrum function of the field is Hausdorff continuous at every point $t \in T$, we will say that *the spectrum function of the field A of self-adjoint operators is Hausdorff continuous*.

To shorthand notation a bit, we will usually not mention the "spectrum function" $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$, and will often just say that "the spectrum of A is Hausdorff continuous" or even just "A is Hausdorff continuous " (alternatively "Hausdorff continuous at t_0").

It is well known - see for example Theorem 6.2.1(v) in [2] - that if H is a Hilbert space, $t \in T \mapsto A_t \in \mathbf{B}(H)$ is a map that is norm continuous at t_0 and A_t is a normal operator for every $t \in T$, then the map

$$t \in T \mapsto \text{dist}\Big(\sigma(A_{t_0}), \sigma(A_t)\Big) \in \mathbb{R}^+ \cup \{\infty\}$$

is continuous at t_0, that is, $t \mapsto A_t$ is Hausdorff continuous at t_0. However, this result is not very useful to us, as it both requires norm continuity, which is a very strict condition, and that the underlying Hilbert space does not change with t.

As noted above, the Hausdorff distance is a metric on the space of non-empty, compact subsets. Since self-adjoint, bounded operators always have non-empty, compact spectra, it is reasonable to consider the above definition of Hausdorff continuity of the spectra of operators as a proper topological concept.

A more explicit topological framework on $\mathscr{C}(X)$, which makes sense even when X is not a metric space, turns out to be the *Fell topology*, first showcased by Fell himself in [6]. We will now introduce the Fell topology as it is defined in [3].

Definition 1.18. Let X be a topological space. For any $K \subset X$ compact and for any finite family $\mathscr{F}$ of open sets in X, define

$$\mathscr{U}(K, \mathscr{F}) := \{C \in \mathscr{C}(X) \mid C \cap K = \emptyset,\ C \cap O \neq \emptyset \text{ for all } O \in \mathscr{F}\}$$

The collection of all $\mathscr{U}(K, \mathscr{F})$ is the basis for the Fell topology on $\mathscr{C}(X)$.

This clearly gives us a definition of *Fell continuity* of maps to $\mathscr{C}(X)$, that is, continuity with respect to the Fell topology. In simple terms, a map to $\mathscr{C}(X)$ is Fell continuous if it never "jumps" into a compact set or out of a family of open sets. The following Lemma, taken from [12], page 454, characterizes convergence in the Fell topology:

Lemma 1.19. *Let X be a locally compact topological space. Let $\{F_\iota\}_{\iota \in I}$ be a net in $\mathscr{C}(X)$, and take $F \in \mathscr{C}(X)$. Then $F_\iota \to F$ in the Fell topology on $\mathscr{C}(X)$ if and only if*

1. *given $x_\iota \in F_\iota$ such that $x_\iota \to x$ for some $x \in X$, then $x \in F$ and*
2. *given $x \in F$, then there is a subnet $\{F_{\iota_\kappa}\}$ and $x_{\iota_\kappa} \in F_{\iota_\kappa}$ such that $x_{\iota_\kappa} \to x$.*

Proof. We start by proving the "only if" parts. To prove 1., we assume that we are given $F_\iota \to F$, $x_\iota \in F_\iota$ and $x_\iota \to x$. Assume $x \notin F$. Since F is closed, there exists a compact neighbourhood $K \subset X$ of x with $F \cap K = \emptyset$; since $x_\iota \to x$, we must eventually have that $x_\iota \in K$, which implies that we eventually have $F_\iota \notin \mathscr{U}(K, \emptyset)$.

However, since $F \cap K = \emptyset$ we have $F \in \mathscr{U}(K, \emptyset)$, which is now impossible since we eventually have that $F_\iota \notin \mathscr{U}(K, \emptyset)$ while we simultaneously know that $F_\iota \to F$ in the Fell topology. Thus we must have $x \in F$, proving 1.

To prove 2., let $\prec$ be the ordering on I. Given any $x \in F$, define

$$\Gamma := \{(\iota, U) \mid \iota \in I, U \text{ is an open neighbourhood of } x, F_\iota \cap U \neq \emptyset\}$$

Γ is not empty as $(\iota, X) \in \Gamma$ for all $\iota \in I$. We will show that Γ is a directed set under the ordering $\preceq$, where $(\iota, U) \preceq (\iota', U')$ if $\iota \prec \iota'$ and $U \supseteq U'$. It is clearly reflexive; $\iota \prec \iota$ and $U \supseteq U$ so $(\iota, U) \preceq (\iota, U)$, and it is also transitive, for if $(\iota, U) \preceq (\iota', U')$ and $(\iota', U') \preceq (\iota'', U'')$ then we have $\iota \prec \iota'$, $\iota' \prec \iota''$, $U \supseteq U'$ and $U' \supseteq U''$; by the transitivity of $\prec$ and $\supseteq$ it follows that $\iota \prec \iota''$ and $U \supseteq U''$ so $(\iota, U) \preceq (\iota'', U'')$.

Finally, it also has the upper bound property. For take any two (ι, U), $(\iota', U') \in \Gamma$. Then if we define $U'' := U \cap U'$, we have $U \supseteq U''$ and $U' \supseteq U''$, and U'' is an open neighbourhood of x. Since x is also in F by assumption, $F \in \mathscr{U}(\emptyset, \{U''\})$ (where $\{U''\}$ is the finite family of open subsets of X consisting only of U''), and since $F_\iota \to F$ it follows that there exists some ι'' such that $\iota \prec \iota''$, $\iota' \prec \iota''$ and $F_{\iota''} \in \mathscr{U}(\emptyset, \{U''\})$, so $(\iota'', U'') \in \Gamma$ and we have both $(\iota, U) \preceq (\iota'', U'')$ and $(\iota', U') \preceq (\iota'', U'')$. Thus Γ is directed under the ordering $\preceq$.

Now for each $(\kappa, U) \in \Gamma$, choose some $y_{(\kappa,U)}$ in $F_\kappa \cap U$ (by the definition of Γ, these exist); $\{y_\gamma\}_{\gamma \in \Gamma}$ is now a net converging to x. If we for each $(\kappa, U) \in \Gamma$ set $F_{\iota_{(\kappa,U)}} := F_\kappa$ and $x_{\iota_{(\kappa,U)}} := y_{(\kappa,U)}$, it follows that $\left\{F_{\iota_\gamma}\right\}_{\gamma \in \Gamma}$ is a subnet of $\{F_\iota\}_{\iota \in I}$ and $\left\{x_{\iota_\gamma}\right\}_{\gamma \in \Gamma}$ is the required subnet converging to x with $x_{\iota_\gamma} \in F_{\iota_\gamma}$ for every ι_γ. This proves 2., and so the "only if" part is done.

To prove the "if" part, assume that both 1. and 2. hold. Suppose $F \in \mathscr{U}(K, \mathscr{F})$ for some compact $K \subset X$ and some finite family $\mathscr{F}$ of open subsets of X.

If we don't eventually have $F_\iota \cap K = \emptyset$, then there exists some subnet $\{F_{\iota_\kappa}\}$ such that $F_{\iota_\kappa} \cap K \neq \emptyset$ for all ι_κ, and thus we can find a net $\{x_{\iota_\kappa}\}$ of points in X such that $x_{\iota_\kappa} \in F_{\iota_\kappa} \cap K$ for every ι_κ. As K is compact and $x_{\iota_\kappa} \in K$ for all ι_κ, it follows that there exists some subnet $\left\{x_{\iota_{\kappa_\gamma}}\right\}$ that converges to some $x \in K$. By 1., it also follows that $x \in F$, so $x \in F \cap K$, contradicting the fact that $F \cap K = \emptyset$, and so we must eventually have $F_\iota \cap K = \emptyset$.

Next, take any $U \in \mathscr{F}$, and assume that we don't eventually have $F_\iota \cap U \neq \emptyset$. Then there exists some subnet $\{F_{\iota_\kappa}\}$ such that $F_{\iota_\kappa} \cap U = \emptyset$ for all ι_κ. Since $F \cap U \neq \emptyset$, there exists at least one $x \in F \cap U$. By 2., this implies that there exists a sub-subnet $F_{\iota_{\kappa_\gamma}}$ such that we can find a net $\left\{x_{\iota_{\kappa_\gamma}}\right\}$ with $x_{\iota_{\kappa_\gamma}} \in F_{\iota_{\kappa_\gamma}}$ with $x_{\iota_{\kappa_\gamma}} \to x$. Since $x \in U$ we must eventually have $x_{\iota_{\kappa_\gamma}} \in F_{\iota_{\kappa_\gamma}} \cap U$, contradicting our earlier assumption that $F_{\iota_\kappa} \cap U = \emptyset$ for all ι_κ, and so we must eventually have $F_\iota \cap U \neq \emptyset$.

As there are only finitely many U, it follows that eventually we must have that $F_\iota \cap K = \emptyset$ and that $F_\iota \cap U \neq \emptyset$ for all $U \in \mathscr{F}$, which implies that we eventually have $F_\iota \in \mathscr{U}(K, \mathscr{F})$. As K and $\mathscr{F}$ were arbitrary, this means that $F_\iota \to F$ in the Fell topology, and we are done.

□

Since the spectra of operators are always closed, we can extend the concept of Fell continuity to also cover fields of operators:

Definition 1.20. Given a field A of self-adjoint operators and any $t_0 \in T$, we say that *the spectrum function of A is Fell continuous at t_0* if the map

$$t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$$

is Fell continuous at t_0; that is, if for every basis element $\mathscr{U}(K, \mathscr{F})$ of the Fell topology on $\mathscr{C}(\mathbb{R})$ such that $\sigma(A_{t_0}) \in \mathscr{U}(K, \mathscr{F})$, there exists a neighbourhood $U \subset T$ of t_0 such that $\sigma(A_t) \in \mathscr{U}(K, \mathscr{F})$ for all $t \in U$.

If the spectrum function of the field is Fell continuous at every point $t \in T$, we will say that *the spectrum function of the field A of self-adjoint operators is Fell continuous*.

Once again we will shorthand notation a bit; we will usually not mention the "spectrum function" $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$, and will often just say that "the spectrum of A is Fell continuous" or even just "A is Fell continuous " (alternatively "Fell continuous at t_0).

Although we will mainly be working with the Hausdorff and Fell topologies, it is convenient for some purposes to introduce a third topology on $\mathscr{C}(X)$ when X is a metric space - the Wijsman topology, as detailed in for example [4], page 34. It turns out to be equivalent to the Fell topology whenever both are defined, and so it will mostly serve as an aid in certain proofs, or as a starting point for readers more familiar with this sort of topology than they are with the Fell topology.

Definition 1.21. Let (X, d) be a metric space. For every $x \in X$ and every $\alpha > 0$, define the two sets

$$\{F \in \mathscr{C}(X) \mid \text{dist}(x, F) < \alpha\}$$

and

$$\{F \in \mathscr{C}(X) \mid \text{dist}(x, F) > \alpha\}$$

The collection of all such sets is a subbase for the *Wijsman topology* on $\mathscr{C}(X)$.

Alternatively, one can say that a net $\{F_\iota\}_{\iota \in I}$ of elements in $\mathscr{C}(X)$ converges to some $F \in \mathscr{C}(X)$ in the Wijsman topology on $\mathscr{C}(X)$ if and only if

$$\lim_\iota \text{dist}(x, F_\iota) = \text{dist}(x, F)$$

for every $x \in X$.

Lemma 1.22. *Assume that we are given a metric space (X, d), some map $t \in T \mapsto F_t \in \mathscr{C}(X)$ and some point $t_0 \in T$. Assume $t \in T \mapsto F_t \in \mathscr{C}(X)$ is Fell continuous at t_0. Then it is Wijsman continuous at t_0.*

Proof. To show that $t \mapsto F_t$ is Wijsman continuous at t_0, we will show that the map

$$t \in T \mapsto \text{dist}(x, F_t) \in \mathbb{R}^+$$

is continuous at t_0 for every $x \in X$.

Choose any $x \in X$. Write $F_0 := F_{t_0}$. Assume first that that $x \notin F_0$.

Set $r := \text{dist}(x, F_0) > 0$. It follows that for any $0 < \epsilon < r$, we have

$$F_0 \in \mathscr{U}\left(\overline{B}_{r-\epsilon}(x), \{B_{r+\epsilon}(x)\}\right)$$

Note that $\{B_{r+\epsilon}(x)\}$ is the finite family of open sets consisting of the single element $B_{r+\epsilon}(x)$. Thus by the Fell-continuity of $t \mapsto F_t$ at t_0, there exists a neighbourhood $U \subset T$ of t_0 such that

$$F_t \in \mathscr{U}\left(\overline{B}_{r-\epsilon}(x), \{B_{r+\epsilon}(x)\}\right)$$

for every $t \in U$. This implies that for every $t \in U$, the point in F_t closest to λ is contained in $B_{r+\epsilon}(x) \backslash \overline{B}_{r-\epsilon}(x)$, that is, $r - \epsilon < \text{dist}(x, F_t) < r + \epsilon$, so

$$|\text{dist}(x, F_0) - \text{dist}(x, F_t)| < \epsilon$$

for all $t \in U$. Since ϵ was arbitrary, we have proven the continuity of the map $t \mapsto \text{dist}(x, F_t)$ for all $x \notin F_0$.

Assume next that $x \in F_0$. The proof in this case is exactly as in the previous case, just by using the fact that

$$F_0 \in \mathscr{U}\left(\emptyset, \{B_\epsilon(x)\}\right)$$

and that by the Fell-continuity of $t \mapsto F_t$ at t_0, there exists a neighbourhood $U \subset T$ of t_0 such that

$$F_t \in \mathscr{U}\left(\emptyset, \{B_\epsilon(x)\}\right)$$

for all $t \in U$. This proves Wijsman continuity at t_0.

□

Lemma 1.23. *Assume that we are given a metric space (X, d), some map $t \in T \mapsto F_t \in \mathscr{C}(X)$ and some point $t_0 \in T$. Assume $t \in T \mapsto F_t \in \mathscr{C}(X)$ is Wijsman continuous at t_0. Then it is Fell continuous at t_0.*

Proof. To show that $t \mapsto F_t$ is Fell continuous at t_0, we will show that for every compact $K \subset X$ and every finite family $\mathscr{F}$ of open sets $O \subset X$ such that

$$F_0 \in \mathscr{U}\left(K, \mathscr{F}\right)$$

there exists some neighbourhood $W \subset T$ of t_0 such that

$$F_t \in \mathscr{U}\left(K, \mathscr{F}\right)$$

for every $t \in W$.

1) For every $x \in K$, consider $r_x = \text{dist}(x, F) > 0$. Clearly we have

$$K \subset \bigcup_{x \in K} B_{r_x/2}(x)$$

and by the compactness of K there exists a finite subcollection $x_k \in K$, $1 \leq k \leq l$ such that if we write $r_k := r_{x_k}$ then

$$K \subset \bigcup_{k=1}^{l} B_{r_k/2}(x_k)$$

Since for each k the map $t \mapsto \text{dist}(x_k, F_t)$ is continuous at t_0, we can for each k find an open neighbourhood $U_k \subset T$ of t_0 such that

$$|\text{dist}(x_k, F_0) - \text{dist}(x_k, F_t)| < r_k/2$$

for every $t \in U_k$, which implies that

$$\text{dist}(x_k, F_t) > r_k/2$$

for every $t \in U_k$.

It follows that for each k with $1 \leq k \leq l$, we have $B_{r_k/2}(x_k) \cap F_t = \emptyset$ for all $t \in U_k$. Thus if we define $U := \bigcap_{k=1}^{l} U_k$, then for all $t \in U$ we have

$$K \cap F_t \subset \left(\bigcup_{k=1}^{l} B_{r_k/2}(x_k) \right) \cap F_t = \emptyset$$

As U is a finite intersection of open neighbourhoods of t_0, it is itself an open neighbourhood of t_0, and the first part of the proof is complete.

2) Next, fix $O \in \mathscr{F}$. We know by the fact that $O \cap F_0 \neq \emptyset$ that there exists a point $x \in O \cap F_0 \subset O$. Furthermore, since O is open, there exists a real number $r > 0$ such that $B_r(x) \subset O$.

Clearly dist$(x, F_0) = 0$. By the continuity of $t \mapsto$ dist(x, F_0) at t_0, there exists an open neighbourhood $V_O \subset T$ of t_0 such that dist$(x, F_t) < r$ for all $t \in V_O$. But this means that

$$B_r(x) \cap F_t \neq \emptyset$$

for all $t \in V_O$. Since $B_r(x) \subset O$, it follows that

$$O \cap F_t \supset B_r(x) \cap F_t \neq \emptyset$$

for all $t \in V_O$.

Now set $V := \bigcap_{O \in \mathscr{F}} V_O$; it is a finite - remember, $\mathscr{F}$ is a finite family! - intersection of open neighbourhoods of t_0, so it is itself an open neighbourhood of t_0, and $O \cap F_t \neq \emptyset$ for all $t \in V$ and all $O \in \mathscr{F}$.

3) It follows that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all $t \in W := U \cap V$. This completes the proof.

□

As such, we will generally not mention Wijsman continuity as anything else than a tool for some proofs.

We would like Hausdorff continuity and Fell continuity to be equivalent to each other. Unfortunately, as the next example shows, this is not the case.

Example 1.24. Set $T = [0,1]$ with the standard subspace topology it inherits from $\mathbb{R}$, and consider a field A of bounded, self-adjoint operators with spectra given by

$$\sigma(A_t) := \begin{cases} \{0\} & \text{if } t = 0 \\ \{0\} \cup \{1/t\} & \text{if } t > 0 \end{cases}$$

We claim that this field is Fell continuous at $t_0 = 0$, but not Hausdorff continuous there.

1) First, we check Fell continuity at $t_0 = 0$. Take any $\mathscr{U}(K, \mathscr{F})$ such that we have $\sigma(A_0) \in \mathscr{U}(K, \mathscr{F})$. To prove continuity, we must find an open neighbourhood $W \subset T$ of 0 such that $\sigma(A_t) \in \mathscr{U}(K, \mathscr{F})$ for all $t \in W$.

Assume $k := \max K < 0$. Then it follows that $\sigma(A_t) \cap K \subseteq [0, \infty) \cap (-\infty, 0) = \emptyset$ for all $t \in T$.

We cannot have $k := \max K = 0$ since $\{0\} \cap K = \emptyset$. So assume next that $k := \max K > 0$. Now note that $K \subset (-\infty, 0) \cup (0, k]$. But if we set $U := [0, 1/k)$, then it follows that

$$\sigma(A_t) \subset \{0\} \cup (k, \infty)$$

for all $t \in U$. Then clearly $\sigma(A_t) \cap K = \emptyset$ for all $t \in U$.

Next, take any $O \in \mathscr{F}$. Since $\sigma(A_0) \cap O \neq \emptyset$, it follows that $0 \in O$, so $\sigma(A_t) \cap O \neq \emptyset$ for all $t \in T$. This is true for every $O \in \mathscr{F}$, so we can set $W := U \cap T = U$, which is the desired neighbourhood.

This proves Fell continuity at t_0.

2) To see that the spectrum of A is not Hausdorff continuous at t_0, we note that for all $t \neq 0$,

$$\text{dist}(\sigma(A_0), \sigma(A_t)) = \begin{cases} 0 & \text{if } t = 0 \\ \text{dist}\left(\{0\}, \{0\} \cup \{1/t\}\right) = 1/t & \text{if } t > 0 \end{cases}$$

so the map $t \mapsto \text{dist}(\sigma(A_0), \sigma(A_t))$ is clearly discontinuous at 0.

We see that Fell continuity and Hausdorff continuity are not equivalent. However, as is easily verified, the field *is* Fell continuous and Hausdorff continuous at each $t \neq 0$. Indeed, it turns out that the only thing that "can go wrong" is the behaviour presented in this example - specifically, the fact that even though the field consists of bounded operators, there exists no neighbourhood $U \subset T$ of 0 such that $\sup_{t \in U} \|A_t\| < \infty$.

Definition 1.25. Let $(X, \|\cdot\|)$ be a normed space equipped with the topology induced by the norm, and consider a map $t \in T \mapsto F_t \in \mathscr{C}(X)$. We say that the map is *locally uniformly bounded at* t_0 if there exists some neighbourhood $U \subset T$ of t_0 such that

$$\sup_{t \in U} \sup_{x \in F_t} \|x\| < \infty$$

Definition 1.26. Let A be a field of operators. We say that A is *locally uniformly bounded at* t_0 if the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{C})$ is locally uniformly bounded at t_0. If A is a field of bounded, normal operators, this is equivalent to demanding that there exists some neighbourhood $U \subset T$ of t_0 such that

$$\sup_{t \in U} \|A_t\| < \infty$$

In many situations, we shall demand that a field be *Fell continuous and locally uniformly bounded at* t_0; to be clear, this just means that we require it to both be Fell continuous at t_0 *and* locally uniformly bounded at t_0. As the following Lemmas show, the condition of local uniform boundedness is sufficient to prove equivalence between Fell continuity and Hausdorff continuity.

Lemma 1.27. *Let X be a subset of $\mathbb{C}$ with the subspace metric, and assume that some map $t \in T \mapsto F_t \in \mathscr{C}(X)$ is Fell continuous and uniformly locally bounded at some point $t_0 \in T$. Then $t \in T \mapsto F_t \in \mathscr{C}(X)$ is Hausdorff continuous at t_0.*

Proof. Fix any $\epsilon > 0$, and write $F_0 := F_{t_0}$. We must show that there exists a neighbourhood of t_0 such that $\text{dist}(F_0, F_t) < \epsilon$ for all $t \in U$.

As $t \mapsto F_t$ is uniformly locally bounded at t_0, there exists an $m > 0$ and a neighbourhood $W \subset T$ of t_0 such that $\sup_{t \in W} \sup |F_t| < m$; in particular, $F_t = F_t \cap \overline{B}_m(0)$ for all $t \in W$. As such, if we define

$$K := \overline{B}_m(0) \cap \{x \in X \mid \text{dist}(x, F_0) \geq \epsilon\}$$

then K is a bounded subset of X with $F_0 \cap K = \emptyset$, and it is clearly also closed, so K is compact.

Note that since F_0 is closed and bounded, we have that F_0 is a compact set. It follows that since the collection $\left\{B_{\epsilon/2}(x)\right\}_{x \in F_0}$ of open balls covers F_0, there exists a finite subcollection $\left\{B_{\epsilon/2}(x_k)\right\}_{k=1}^{l}$ which also covers F_0. So set $\mathscr{F} := \left\{B_{\epsilon/2}(x_k)\right\}_{k=1}^{l}$.

We clearly have $F_0 \in \mathscr{U}(K, \mathscr{F})$. By the Fell continuity of $t \mapsto F_t$ at t_0, it follows that we can find a neighbourhood $U \subset T$ of t_0 such that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all $t \in U$. Since $F_t \cap K = \emptyset$ for all $t \in U$, it follows that for all $t \in U \cap W$ we have

$$\begin{aligned} F_t \cap \{x \in \mathbb{R} \mid d(F_0, x) \geq \epsilon\} &= F_t \cap [-m, m] \cap \{x \in \mathbb{R} \mid d(F_0, x) \geq \epsilon\} \\ &= F_t \cap K \\ &= \emptyset \end{aligned}$$

and so in particular $\text{dist}(x_t, F_0) < \epsilon$ for all $x_t \in F_t$. Since F_t is closed, we actually have $\sup_{x_t \in F_t} \text{dist}(x_t, F_0) < \epsilon$ for all $t \in U \cap W$ and all $x_t \in F_t$.

Fix any $x_0 \in F_0$. As $\mathscr{F}$ covers F_0, there exists a k_0 with $1 \leq k_0 \leq l$ such that $x_0 \in B_{\epsilon/2}(x_{k_0})$. Since we have $F_t \cap B_\epsilon(x_{k_0}) \neq \emptyset$ for every $t \in U$, it follows that for every $t \in U$ there exists a point $x_t \in F_t \cap B_{\epsilon/2}(x_{k_0})$, so

$$\begin{aligned} |x_0 - x_t| &\leq |x_0 - x_{k_0}| + |x_{k_0} - x_t| \\ &< \epsilon/2 + \epsilon/2 < \epsilon \end{aligned}$$

for every $t \in U$.

Hence $\text{dist}(x_0, F_t) < \epsilon$ for every $t \in U$. x_0 was arbitrary, so we must have that $\text{dist}(x_0, F_t) < \epsilon$ for every $x_0 \in F_0$ and every $t \in U$. Since F_0 is closed, we actually have $\sup_{x_0 \in F_0} \text{dist}(x_0, F_t) < \epsilon$ for every $t \in U$.

As such we now have

$$\begin{aligned}\text{dist}(F_0, F_t) &= \max\left\{\sup_{x_0 \in F_0} \text{dist}(x_0, F_t), \sup_{x_t \in F_t} \text{dist}(x_t, F_0)\right\} \\ &< \max\{\epsilon, \epsilon\} \\ &= \epsilon\end{aligned}$$

for all $t \in U \cap W$. As ϵ was arbitrary, this proves that $t \mapsto F_t$ is Hausdorff continuous at t_0.

□

Lemma 1.28. *Let X be a subset of $\mathbb{C}$ with the subspace metric, and assume that some map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Hausdorff continuous at some point $t_0 \in T$ and that F_{t_0} is bounded. Then $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous and locally uniformly bounded at t_0.*

Proof. Take any compact $K \subset X$ and any finite family $\mathscr{F}$ of open subsets of X such that $F_0 := F_{t_0} \in \mathscr{U}(K, \mathscr{F})$, that is, such that $F_0 \cap K = \emptyset$ and $F_0 \cap O \neq \emptyset$ for every $O \in \mathscr{F}$. We must find a neighbourhood of t_0 such that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all t in this neighbourhood.

1) We will first find a neighbourhood $U \subset T$ of t_0 such that $F_t \cap K = \emptyset$ for all $t \in U$. Since $F_0 \cap K = \emptyset$, there exists for every $x \in K$ a real number $r_x > 0$ such that

$$F_0 \cap B_{r_x}(x) = \emptyset$$

Clearly the collection of smaller balls $\left\{B_{r_x/2}(x)\right\}_{x \in K}$ is an open covering of K, and since K by definition is compact, there exists some finite collection $\{x_k\}_{k=1}^{l}$ of points of K such that if we set $r_k := r(x_k)$, then $\left\{B_{r_k/2}(x_k)\right\}_{k=1}^{l}$ is also an open cover of K.

By the Hausdorff continuity of $t \mapsto F_t$ at t_0, it follows that for each k we can find an open neighbourhood $U_k \subset T$ of t_0 such that $\text{dist}(F_0, F_t) < r_k/2$ for all $t \in U_k$. We claim that $F_t \cap B_{r_k/2}(x_k) = \emptyset$ for all $t \in U_k$. For if not, there exists a point $y_k \in F_t \cap B_{r_k/2}(x_k)$, which implies that

$$|y_k - x_k| < r_k/2$$

and

$$\text{dist}(y_k, F_0) \leq \text{dist}(F_t, F_0) < r_k/2$$

But then

$$\begin{aligned}\text{dist}(x_k, F_0) &\leq \text{dist}(y_k, F_0) + |y_k - x_k| \\ &< r_k/2 + r_k/2 \\ &= r_k\end{aligned}$$

contradicting the fact that $B_{r_k}(x_k) \cap F_0 = \emptyset$. Thus $B_{r_k/2}(x_k) \cap F_t = \emptyset$ for all $t \in U_k$.

If we set $U := \bigcap_{k=1}^{l} U_k$, then $U \subset T$ is an open neighbourhood of t_0 and

$$F_t \cap K \subset F_t \cap \left(\bigcup_{k=1}^{l} B_{r_k/2}(x_k) \right) = \emptyset$$

for all $t \in U$, so the first part is done.

2) For the second part, we must find a neighbourhood $V \subset T$ of t_0 such that we have $F_t \cap O \neq \emptyset$ for all $t \in V$ and all $O \in \mathscr{F}$. Start by fixing $O \in \mathscr{F}$.

Choose any $x \in F_0 \cap O$; as O is open, we can choose some $r > 0$ such that $B_r(x) \subset O$. Since $t \mapsto F_t$ is Hausdorff continuous at t_0, we can find a subset $V_O \subset T$ such that

$$\text{dist}(F_0, F_t) < r$$

for all $t \in V_O$. Now note that for every $t \in V_O$, there must exist at least one point $x_t \in F_t$ that is also in $B_r(x)$; for otherwise, we would have

$$\text{dist}(F_0, F_t) \geq \text{dist}(x, F_t) \geq r$$

for at least one $t \in V_O$, which is impossible.

This implies that the required x_t exists, so we have

$$x_t \in F_t \cap B_r(x) \subset F_t \cap O$$

for all $t \in V_O$, giving us $F_t \cap O \neq \emptyset$ for all $t \in V_O$.

Since $\mathscr{F}$ is finite, we see that $V := \cap_{O \in \mathscr{F}} V_O$ is an open neighbourhood of t_0, and $F_t \cap O \neq \emptyset$ for every $O \in \mathscr{F}$ and every $t \in V$. This proves the second claim.

3) We now see that for all $t \in U \cap V$, we have $F_t \cap K = \emptyset$ and $F_t \cap O \neq \emptyset$ for all $O \in \mathscr{F}$, so $F_t \in \mathscr{U}(K, \mathscr{F})$ for all $t \in U \cap V$.

This proves Fell continuity of $t \mapsto F_t$ at t_0, as K and $\mathscr{F}$ were arbitrary.

4) To prove local uniform boundedness, we must find a neighbourhood $W \subset T$ of t_0 such that $\sup_{t \in W} \sup |F_t| < \infty$. Recall that, by assumption, we have $\sup |F_0| = m < \infty$.

By Hausdorff continuity, we can find a neighbourhood $W \subset T$ of t_0 such that $d(F_0, F_t) < 1$ for all $t \in W$. Since F_t is a closed set for every $t \in T$, we see that for any $t \in W$ we cannot have $\sup F_t \geq \sup F_0 + 1$ or $\inf F_t \leq \inf F_0 - 1$, as otherwise we would have

$$d(F_0, F_t) \geq \max \{ |\sup F_0 - \sup F_t| , |\inf F_0 - \inf F_t| \} \geq 1$$

which is impossible by assumption. But then for all $t \in W$,

$$\sup |F_t| = \max \{ |\sup F_t| , |\inf F_t| \} < m + 1 < \infty$$

which proves local uniform boundedness, and the proof is complete.

□

The next chapter will be devoted to proving that as long as we have this property of local uniform boundedness, then *all* the types of continuity we have discussed so far are indeed equivalent.

2 Bounded, self-adjoint operators

2.1 Forms of continuity for fields

Now that we have laid the groundwork, we can start considering how to define continuity of a field A. The different notions of continuity for maps $t \in T \mapsto F_t \in \mathscr{C}(X)$ are certainly interesting, but we would like to have notions of continuity that relate more directly to the nature of the operators involved.

For *families of bounded operators* $\{A_t\}_{t\in T}$ such that each A_t is an operator on the *same* Hilbert space H, we have several well-known forms of continuity. Among the more famous ones are *norm continuity, strong continuity and weak continuity*. In the following discussion, we will use the term *family of bounded operators* to refer specifically to such families where each operator acts on the same Hilbert space.

Definition 2.1. Let $\{A_t\}_{t\in T}$ be a family of bounded operators on some Hilbert space H, and take any $t_0 \in T$. We say that $\{A_t\}_{t\in T}$ is norm continuous at t_0 if

$$\lim_{t\to t_0} \|A_{t_0} - A_t\| := \lim_{t\to t_0} \sup_{\|x\|\le 1} \|(A_{t_0} - A_t)\,x\| = 0$$

Definition 2.2. Let $\{A_t\}_{t\in T}$ be a family of bounded operators on some Hilbert space H, and take any $t_0 \in T$. We say that $\{A_t\}_{t\in T}$ is strongly continuous at t_0 if

$$\lim_{t\to t_0} \|(A_{t_0} - A_t)\,x\| = 0$$

for every $x \in H$.

Definition 2.3. Let $\{A_t\}_{t\in T}$ be a family of bounded operators on some Hilbert space H, and take any $t_0 \in T$. We say that $\{A_t\}_{t\in T}$ is weakly continuous at t_0 if

$$\lim_{t\to t_0} \Big\langle (A_{t_0} - A_t)\,x, y \Big\rangle = 0$$

for every pair $x, y \in H$.

However, these definitions do not necessarily make sense in the case of a field of operators, as the H_t are allowed to be different. The term $(A_{t_0} - A_t)\,x$ is only well defined when both A_{t_0} and A_t are both actually able to operate on x, and it appears in all three definitions. We will need to find some common area that we can use to evaluate continuity.

Our first idea is a variation of norm continuity. While norm continuity as above is not useful to us, we could try to see whether

$$\lim_{t\to t_0} \Big| \|A_{t_0}\| - \|A_t\| \Big| = 0$$

as this is always well defined. However, this would be far too weak a test of continuity. For just consider the case where for all $t \in T$ we have $H_t = H$ for some Hilbert space H and we define $A = \{A_t\}_{t\in T}$ by

$$A_t := \begin{cases} \|B\| I & \text{if } t = t_0 \\ B & \text{otherwise} \end{cases}$$

for some point $t_0 \in T$ and some bounded operator B. Then by the above notion of continuity, we would have

$$\lim_{t \to t_0} \Big| \|A_{t_0}\| - \|A_t\| \Big| = \lim_{t \to t_0} \Big| \big\| \|B\| I \big\| - \|B\| \Big| = 0$$

suggesting that A is continuous at t_0, which seems quite counterintuitive, as B was any arbitrary operator in $\mathbf{B}(H)$, and may be very different from the identity operator.

In [3], Beckus and Bellissard suggested that a better approach would be to use the following method:

Definition 2.4. We say that a field $A = (A_t)_{t \in T}$ of bounded operators is *$p2$-continuous at a point $t_0 \in T$* if for every polynomial p of degree two or less with real coefficients, the map

$$t \in T \mapsto \|p(A_t)\| \in \mathbb{R}^+$$

is continuous at t_0, that is, if

$$\lim_{t \to t_0} \Big| \|p(A_{t_0})\| - \|p(A_t)\| \Big| = 0$$

for every p as above.

We say that *A is $p2$-continuous* if A is $p2$-continuous at every point $t \in T$.

We can show that this definition is weaker than or equal to norm continuity in situations where both definitions make sense, and under the assumption of local uniform boundedness.

Proposition 2.5. *Let $A = (A_t)_{t \in T}$ be a field of self-adjoint, bounded operators, take $t_0 \in T$ and set $H := H_{t_0}$. Assume that there exists a neighbourhood $U \subset T$ of t_0 such that*

- *$H_t = H$ for every $t \in U$,*
- *$m := \sup_{t \in U} \|A_t\| < \infty$,*
- *$t \in U \mapsto A_t \in \boldsymbol{B}(H)$ is norm continuous at t_0*

Then A is $p2$-continuous at t_0.

Proof. If $\{t_\kappa\}_{\kappa \in J}$ is a net of points in T converging to t_0, then eventually $t_\kappa \in U$, so there exists $\kappa' \in J$ such that $t_\kappa \in U$ for every $\kappa \succ \kappa'$. It follows that eventually, $\|A_{t_0} - A_{t_\kappa}\|$ is well defined and

$$\lim_{\kappa} \|A_{t_0} - A_{t_\kappa}\| = 0$$

But for any polynomial of degree two or less with real coefficients, say $p(x) = p_2 x^2 + p_1 x + p_0$, we have

$$\begin{aligned} \Big| \|p(A_{t_0})\| - \|p(A_{t_\kappa})\| \Big| &\leq \|p(A_{t_0}) - p(A_{t_\kappa})\| \\ &= \|p_2 A_{t_0}^2 + p_1 A_{t_0} + p_0 - p_2 A_{t_\kappa}^2 - p_1 A_{t_\kappa} - p_0\| \\ &\leq |p_2| \, \|A_{t_0} - A_{t_\kappa}\| \, \|A_{t_0} + A_{t_\kappa}\| + |p_1| \, \|A_{t_0} - A_{t_\kappa}\| \\ &\leq (2|p_2| m + |p_1|) \, \|A_{t_0} - A_{t_\kappa}\| \end{aligned}$$

for every $\kappa \succ \kappa'$. It follows that

$$\lim_{\kappa} \Big| \|p(A_{t_0})\| - \|p(A_{t_\kappa})\| \Big| = 0$$

and we are done.

□

Example 2.6. Set $T = \mathbb{R}$ with the standard topology, set for each $t \in T$

$$H_t := \begin{cases} L^2([0,1]) & \text{if } t \in \mathbb{Q} \\ \mathbb{C}^2 & \text{if } t \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

with their standard Hilbert space structure, and set $A_t := tI$ for every $t \in T$.

Intuitively, the field $A = (A_t)_{t\in T}$ "should" be continuous due to its simple nature, but as H_t changes as we move along T, we cannot apply the definitions of norm, strong or weak continuity. However, A is $p2$-continuous at every $t_0 \in T$. For take any polynomial $p(x) = p_2x^2 + p_1x + p_0$ with real coefficients of degree two or less, and note that

$$\begin{aligned} \Big| \|p(A_{t_0})\| - \|p(A_t)\| \Big| &= \Big| \|p(t_0)I\| - \|p(t)I\| \Big| \\ &= \Big| |p(t_0)| - |p(t)| \Big| \\ &\leq \Big| p(t_0) - p(t) \Big| \end{aligned}$$

and that $\lim_{t\to t_0} \Big| p(t_0) - p(t) \Big| = 0$.

Example 2.7. Set T and H_t as in the previous example, but assume now that the operators $A_t \in \mathbf{B}(H_t)$ are self-adjoint and satisfy

$$\sigma(A_t) := \begin{cases} [0,1] & \text{if } t \in \mathbb{Q} \\ \{0\} \cup \{1\} & \text{if } t \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

We claim that in this case, A is *not* $p2$-continuous. For any polynomial p, it follows from the functional calculus that

$$\|p(A_t)\| = \sup |p(\sigma(A_t)|$$

Set $p(x) = 1 - (x - 1/2)^2 = x^2 - x - 3/4$, and take any $t_0 \in \mathbb{Q}$. Now note that for any $t \in \mathbb{R}\backslash\mathbb{Q}$, we have

$$\begin{aligned} \Big| \|p(A_{t_0})\| - \|p(A_t)\| \Big| &= \Big| \Big(\sup |p([0,1])| \Big) - \Big(\sup |p(\{0\} \cup \{1\})| \Big) \Big| \\ &= |1 - 3/4| = 1/4 \end{aligned}$$

Since any neighbourhood of t_0 in T will contain at least one point in $\mathbb{R}\backslash\mathbb{Q}$ (in fact, uncountably many), we have

$$\lim_{t\to t_0} \Big| \|p(A_{t_0})\| - \|p(A_t)\| \Big| \neq 0$$

and since p is a polynomial of degree two with real coefficients, we are done in this case. By reversing the argument, we see that this also holds for $t_0 \in \mathbb{R}\backslash\mathbb{Q}$, so A is nowhere $p2$-continuous.

As an aside, note that $\|A_t\| = \sup|\sigma(A_t)| = 1$ for every $t \in T$, so by our naïve first attempt at defining continuity of fields, A would indeed be everywhere continuous.

As the last example illustrates, the truly interesting properties of an operator (with regards to evaluating continuity) seem to lie not in its norm, but in the shape and size of its spectrum, justifying our work so far.

We shall make a short detour, and consider what would happen if we replaced our polynomials in the definition of $p2$-continuity with other sorts of functions. Although not intuitively obvious, the choice falls on *proper* functions, as these will arise naturally in our considerations later.

Definition 2.8. A continuous map $f : X \to Y$ between two topological spaces X and Y is said to be *proper* if the inverse image of any compact subset is compact; that is, if we have that $f^{-1}(K)$ is compact in X for every compact subset $K \subset Y$.

Definition 2.9. A field $A = (A_t)_{t\in T}$ of self-adjoint, bounded operators is *proper-continuous at a point* $t_0 \in T$ if for every proper, continuous function $f : \mathbb{R} \to \mathbb{R}$, the map

$$t \in T \mapsto \|f(A_t)\| \in \mathbb{R}^+$$

is continuous at t_0, that is, if for every f as above we have

$$\lim_{t\to t_0}\Big|\|f(A_{t_0})\| - \|f(A_t)\|\Big| = 0$$

We say that A *is proper-continuous* if A is proper-continuous at every point $t \in T$.

Proposition 2.10. *All polynomials of degree two or less with real coefficients, that is, all polynomials on the form* $p : x \in \mathbb{R} \mapsto ax^2 + bx + c \in \mathbb{R}$, *are either proper (in the standard topology on* $\mathbb{R}$*) or constant functions.*

Proof. For $p(x) = ax^2 + bx + c$, choose any compact subset $K \in \mathbb{R}$; we must either show that

$$p^{-1}(K) = \Big\{\lambda \in \mathbb{R} \mid a\lambda^2 + b\lambda + c \in K\Big\}$$

is a compact subset of $\mathbb{R}$, or that p is a constant function.

Assume that $a \neq 0$. Remember that the equation $ax^2 + bx + c = y$ has the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a}$$

whenever $b^2 - 4a(c - y) \geq 0$, and no (real) solutions otherwise.

Assume $a > 0$. It follows that if we set

$$L := \left\{ y \in \mathbb{R} | y \geq c - \frac{b^2}{4a} \right\}$$

then $p^{-1}(\{y\}) = \emptyset$ for all $y \notin L$, hence $p^{-1}(U) = p^{-1}(U \cap L)$ for all subsets $U \subset \mathbb{R}$. L is clearly closed.

It follows that if we define the two functions $q_1, q_2 : L \to R$ by

$$q_1 : y \in L \mapsto \frac{-b + \sqrt{b^2 - 4a(c-y)}}{2a}$$

and

$$q_2 : y \in L \mapsto \frac{-b - \sqrt{b^2 - 4a(c-y)}}{2a}$$

then $p^{-1}(K) = p^{-1}(K \cap L) = q_1(K \cap L) \cup q_2(K \cap L)$. Both q_1 and q_2 are clearly continuous functions, and $K \cap L$ is compact in L by the definition of the subspace topology. It follows that $q_1(K \cap L)$ and $q_2(K \cap L)$ are compact in $\mathbb{R}$, so $p^{-1}(K)$ is compact, and we are done with this case.

The case $a < 0$ is similar. If $a = 0$, then we have to consider the two cases $b = 0$ and $b \neq 0$.

If $b \neq 0$ then $p : x \in \mathbb{R} \mapsto bx + c \in \mathbb{R}$. If we define the (clearly continuous) function $q : y \in \mathbb{R} \mapsto \frac{y-c}{b} \in \mathbb{R}$ then clearly $p^{-1}(U) = q(U)$ for any subset $U \subset \mathbb{R}$; in particular, if K is compact then $p^{-1}(K) = q(K)$, which is compact by the continuity of q, so in this case we are done.

If $b = 0$ then p is just the constant function $p : x \in \mathbb{R} \mapsto c \in \mathbb{R}$, and we are done.

□

Corollary 2.11. *Let $A = (A_t)_{t \in T}$ be a field of self-adjoint, bounded operators, and choose any $t_0 \in T$. If A is proper-continuous at t_0, then A is $p2$-continuous at t_0.*

Proof. The continuity of $t \in T \mapsto \|aA_t^2 + bA_t + c\| \in \mathbb{R}^+$ in the case where at least one of a and b are non-zero follows directly from Proposition 2.10. When $a = b = 0$, we have

$$\left\| aA_t^2 + bA_t + c \right\| = \sup |\{c\}| = |c|$$

and the map $t \in T \mapsto |c| \in \mathbb{R}^+$ is obviously continuous, which covers the case where $p : x \in \mathbb{R} \mapsto c \in \mathbb{R}$ is a constant function; this clarification is necessary, as constant functions from non-compact spaces are not proper.

□

It follows that the definition of proper-continuity is stronger than or equal to that of $p2$-continuity. It turns out, maybe surprisingly, that in fact these two definitions are equivalent, and we will prove this in the next section.

So far we have discussed various different forms of continuity. We will repeat Example 1.24 to illustrate that once again, we will need local uniform boundedness to prove that they are all equivalent.

Example 2.12. Set $T = [0, 1]$ with the standard subspace topology it inherits from $\mathbb{R}$, and consider a field A of bounded, self-adjoint operators with spectra given by

$$\sigma(A_t) := \begin{cases} \{0\} & \text{if } t = 0 \\ \{0\} \cup \{1/t\} & \text{if } t > 0 \end{cases}$$

We showed in Example 1.24 that this field is Fell continuous at $t_0 = 0$, but not Hausdorff continuous there. We will now show that the field is gap edge continuous at t_0, but neither $p2$-continuous nor proper-continuous there.

1) First, we show that $t \mapsto \sigma(A_t)$ is gap edge continuous at t_0.

For the first condition of gap edge continuity, note that $\sigma(A_0)$ has two gaps, $(-\infty, 0)$ and $(0, \infty)$. $(-\infty, 0)$ is also a gap of $\sigma(A_t)$ for all $t \in T$, so the first condition trivially holds for this gap. For the gap $(0, \infty)$, we note that for any $N > 0$, the set

$$U_N := \{t \in T \mid 1/t > N\}$$

is an open neighbourhood of $t_0 = 0$, and for every $t \in U_N \setminus \{0\}$, we see that $\sigma(A_t)$ has a gap on the form $(0, b_t)$ where $b_t > N$. But this implies that $\lim_{t \to 0} b_t = \infty$, and the first condition of gap edge continuity holds here as well.

For the second condition of gap edge continuity, note that for any $t \in T \setminus \{0\}$, we know that $\sigma(A_t)$ has three gaps: $(-\infty, 0)$, $(0, 1/t)$ and $(1/t, \infty)$. Let $\{t_\iota\}_{\iota \in I}$ be some net of points in T converging to $t_0 = 0$, and let $\{a_\iota\}_{\iota \in I}$ and $\{b_\iota\}_{\iota \in I}$ be two convergent nets of points in $\mathbb{R} \cup \{\pm\infty\}$. Assume that we are given that for every $\iota \in I$, (a_ι, b_ι) is a gap of $\sigma(A_{t_\iota})$. We must show that if $a_\iota \to a$ and $b_\iota \to b$, then we have that $a = b$ or that (a, b) is a gap of $\sigma(A_0)$.

If $a < 0$, it follows we eventually have $a_\iota < 0$. But since a_ι is a gap edge of $\sigma(A_{t_\iota})$, it follows that eventually $a_\iota = -\infty$. Since (a_ι, b_ι) is a gap of $\sigma(A_{t_\iota})$, it follows that eventually, we have $b_\iota = 0$. But then clearly $a = -\infty$ and $b = 0$, and $(a, b) = (-\infty, 0)$ is a gap of $\sigma(A_0)$.

If $a = 0$, then eventually $|a_\iota| < 1$. But, again since a_ι is a gap edge of $\sigma(A_{t_\iota})$, it follows that eventually we have $a_\iota = 0$, and so $a = 0$. From this it follows that eventually $b_\iota = 1/t_\iota$, and so

$$b = \lim_\iota b_\iota = \lim_\iota 1/t_\iota = \infty$$

and $(a, b) = (0, \infty)$ is a gap of $\sigma(A_0)$.

If $a > 0$, then eventually we have that $a_\iota > 0$. Again, we can use this and the fact that a_ι is a gap edge of $\sigma(A_{t_\iota})$ to conclude that eventually, $a_\iota = 1/t_\iota$, which in turn leads us to conclude that eventually $b_\iota = \infty$. But this implies that

$$a = \lim_\iota a_\iota = \lim_\iota 1/t_\iota = \infty$$

and $b = \infty$, so $a = b$, and we are done.

As this exhausts all possible cases, the second condition of gap edge continuity holds. Since both conditions of gap edge continuity hold, it follows that $t \mapsto \sigma(A_t)$ is gap edge continuous at $t_0 = 0$.

2) To prove that the field is neither $p2$-continuous nor proper-continuous at $t_0 = 0$, notice that if we set $p(z) = z$ then $p(z)$ is a polynomial of degree one, and is also proper, being the identity function, and

$$\|p(A_t)\| = \|A_t\| = \begin{cases} 0 & \text{if } t = 0 \\ 1/t & \text{if } t > 0 \end{cases}$$

so $t \mapsto \|p(A_t)\|$ is clearly discontinuous at $t_0 = 0$.

Combining the two examples, we have now demonstrated that Fell continuity and gap edge continuity are not equivalent to the other forms of continuity. However, as is again easily verified, the field *is* Fell continuous, Hausdorff continuous, gap edge continuous, $p2$-continuous and proper-continuous at each $t \neq 0$, and we claim that once again, the only "problem" is the lack of local uniform boundedness at $t_0 = 0$ - a claim we will prove in the next section.

2.2 Equivalence of different forms of continuity

So far we've considered several main forms of continuity of spectra: Gap edge continuity, $p2$-continuity, proper-continuity, Hausdorff continuity and Fell continuity. The first major result of [3] shows for locally uniformly bounded fields of self-adjoint, bounded operators, gap edge continuity, $p2$-continuity and Fell continuity are equivalent. Here, we will prove an extended version of the theorem, stating that in this case, all five forms of continuity are equivalent.

Theorem 2.13. *Let $A = (A_t)_{t \in T}$ be a field of self-adjoint, bounded operators, and take any $t_0 \in T$. Consider the following statements:*

1. *A is gap edge continuous at t_0.*
2. *A is Fell continuous at t_0*
3. *A is $p2$-continuous at t_0.*
4. *A is proper-continuous at t_0.*
5. *A is Hausdorff continuous at t_0.*

Then 1. and 2. are equivalent. If A is locally uniformly bounded at t_0, then all the above statements are equivalent.

It should be noted that in [3], this theorem is formulated rather differently, ignoring the condition of local uniform boundedness. However, Beckus and Bellisard do acknowledge the necessity of local uniform boundedness in the proofs they give later on in the article, although they never directly name the property. The concepts of proper-continuity and Hausdorff continuity are not explicitly introduced in [3].

As such, our first goal is to prove the various equivalences in Theorem 2.13, while clarifying the notation from [3] as much as possible. First, a few technical results.

Proposition 2.14. *Let $-\infty \leq a < b < \infty$, and set $X = (a, b]$. Then the map*

$$F \in \mathscr{C}(X) \mapsto \sup F \in \mathbb{R}$$

is continuous with regards to the Fell topology on $\mathscr{C}(X)$ and the standard topology on $\mathbb{R}$.

Proof. Fix $F_0 \in \mathscr{C}(X)$ with $\sup F_0 = \lambda$; we have $\lambda \in F_0$ since F_0 is closed and $\lambda < \infty$ since $F_0 \in \mathscr{C}(X)$. To show that the map defined above is continuous, we must show that for any real number $\epsilon > 0$, there exists an open (in the Fell topology) neighbourhood $\mathscr{U}(K_\epsilon, \mathscr{F}_\epsilon)$ around F_0 such that

$$|\sup F_0 - \sup F| < \epsilon$$

for all $F \in \mathscr{U}(K_\epsilon, \mathscr{F}_\epsilon)$.

We will begin by assuming that $-\infty < a$.

First assume $\lambda < b$. For any given ϵ, let us define

$$\tilde{\epsilon} := \min\left\{\epsilon, \frac{\lambda - a}{2}, \frac{b-\lambda}{2}\right\}$$

Now set $K_\epsilon := [\lambda + \tilde{\epsilon}, b]$ and $O_\epsilon := (\lambda - \tilde{\epsilon}, b]$; clearly K_ϵ is compact in X and O_ϵ is open in X. Furthermore $F_0 \cap K_\epsilon = \emptyset$ and $F_0 \cap O_\epsilon \neq \emptyset$, so $F_0 \in \mathscr{U}(K_\epsilon, \{O_\epsilon\})$.

Take any other $F \in \mathscr{U}(K_\epsilon, \{O_\epsilon\})$. It follows that $F \cap K_\epsilon = \emptyset$, which implies that $\sup\{F\} < \lambda + \tilde{\epsilon}$, since F is closed. Furthermore, $F \cap O_\epsilon \neq \emptyset$, which similarly implies that $\lambda - \tilde{\epsilon} < \sup F$.

From the above, it is clear that

$$|\sup F_0 - \sup F| < \tilde{\epsilon} \leq \epsilon$$

for all $F \in \mathscr{U}(K_\epsilon, \{O_\epsilon\})$, which proves continuity for $\lambda < b$. But for $\lambda = b$, the exact same proof with $K_\epsilon = \emptyset$ and $\tilde{\epsilon} := \min\left\{\epsilon, \frac{\lambda - a}{2}\right\}$ proves the result in the case $-\infty < a$.

In the case $a = -\infty$, the method above works in its entirety, although one could choose to remove the mention of $\frac{\lambda-a}{2}$ in the definitions of $\tilde{\epsilon}$. This completes the proof. □

Proposition 2.15. *Let X be a topological space, Y be a locally compact Hausdorff space, and $f : X \to Y$ be a proper, continuous function. Then the map $\hat{f} : F \in \mathscr{C}(X) \mapsto f(F) \in \mathscr{C}(Y)$ is Fell continuous.*

Proof. A version of The Closed Map Lemma says that any proper, continuous function from a topological space to a locally compact Hausdorff space is closed, that is, it sends closed sets to closed sets; for a proof of this version, see [8].

Thus we see that the map $\hat{f}$ is well defined. It remains to demonstrate Fell-continuity; that is, we need to show that for every basis element $\mathscr{U}(K, \mathscr{F})$ in the Fell topology on Y, the set $f^{-1}(\mathscr{U}(K, \mathscr{F}))$ is open in the Fell topology on X. In order to do this, we will show that there exists an open set $W \subset \mathscr{C}(X)$ in the Fell topology on X such that for every $F \in \mathscr{U}(K, \mathscr{F}) \subset \mathscr{C}(Y)$, we have

$$\tilde{F} := f^{-1}(F) \in W \subset f^{-1}(\mathscr{U}(K, \mathscr{F})) \subset \mathscr{C}(X)$$

which will prove continuity.

Given $\mathscr{U}(K, \mathscr{F}) \subset \mathscr{C}(Y)$, we know that $K \subset Y$ is compact and that $\mathscr{F}$ is a finite family of open subsets $O \subset Y$. Since f is continuous and proper, we know that $f^{-1}(O) \subset X$ is open for every $O \in \mathscr{F}$ and that $f^{-1}(K) \subset X$ is compact. It follows that the set $\mathscr{U}(f^{-1}(K), f^{-1}(\mathscr{F}))$ is open in the Fell topology on $\mathscr{C}(X)$; we set

$$W := \mathscr{U}\left(f^{-1}(K), f^{-1}(\mathscr{F})\right)$$

Now fix an $F \in \mathscr{U}(K, \mathscr{F})$. By definition, $F \cap K = \emptyset$ and $F \cap O \neq \emptyset$ for every $O \in \mathscr{F}$. We have that

$$\tilde{F} \cap f^{-1}(K) = \emptyset$$

For, if there existed an $x \in X$ with $x \in \tilde{F} \cap f^{-1}(K)$, then

$$f(x) \in f\left(f^{-1}(F)\right) = F \text{ and } f(x) \in f\left(f^{-1}(K)\right) = K$$

which is impossible. We obviously have $\tilde{F} \cap O \neq \emptyset$ for every $O \in \mathscr{F}$; thus

$$\tilde{F} \in \mathscr{U}\left(f^{-1}(K), f^{-1}(\mathscr{F})\right)$$

Next consider any $G \in \mathscr{U}\left(f^{-1}(K), f^{-1}(\mathscr{F})\right)$; to show the inclusion

$$\mathscr{U}\left(f^{-1}(K), f^{-1}(\mathscr{F})\right) \subset f^{-1}\left(\mathscr{U}(K, \mathscr{F})\right)$$

we need to show that $G \in f^{-1}\left(\mathscr{U}(K, \mathscr{F})\right)$, i.e. that $f(G) \in \mathscr{U}(K, \mathscr{F})$. Similar to above, we must have $f(G) \cap K \neq \emptyset$; for otherwise there would exist an $x \in G$ such that $f(x) \in K$, which implies $x \in f^{-1}(K)$, which cannot be true. Similarly it is clear that $f(G) \cap f(O) \neq \emptyset$ for all $O \in \mathscr{F}$.

□

Proposition 2.16. *Let B be a bounded, self-adjoint linear operator. For any given $m > \|B\|$, define the polynomial $p(z) := m^2 - z^2$. Then for any $r < m$, we have*

$$\|p(B)\| \leq m^2 - r^2$$

if and only if $B_r(0) \cap \sigma(B) = \emptyset$, where $B_r(0) = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$.

Equivalently,

$$\|p(B)\| > m^2 - r^2$$

if and only if $B_r(0) \cap \sigma(B) \neq \emptyset$.

Proof. Since B is self-adjoint, $\sigma(B) \subset \mathbb{R}$, and for any polynomial P, $\sigma(P(B)) = P(\sigma(B))$. Since $|\lambda| \leq \|B\|$ for all $\lambda \in \sigma(B)$, it follows that

$$p(\lambda) = m^2 - \lambda^2 \geq m^2 - \|B\|^2 > 0$$

for all $\lambda \in \sigma(B)$; in particular $|m^2 - \lambda^2| = m^2 - \lambda^2$ for all $\lambda \in \sigma(B)$.

Note also that $B_r(0) \cap \sigma(B) = \emptyset$ implies that $|\lambda| \geq r$ for all $\lambda \in \sigma(B)$. Thus

$$\begin{aligned}
\|p(B)\| &= \sup |\sigma(p(B))| \\
&= \sup |p(\sigma(B))| \\
&= \sup_{\lambda \in \sigma(B)} |p(\lambda)| \\
&= \sup_{\lambda \in \sigma(B)} \left\{\left|m^2 - \lambda^2\right|\right\} \\
&= \sup_{\lambda \in \sigma(B)} \left\{m^2 - \lambda^2\right\} \\
&\leq m^2 - r^2
\end{aligned}$$

To prove the reverse implication, note that everything we did except the last line is still valid, so $\|p(B)\| = \sup_{\lambda\in\sigma(B)} \{m^2 - \lambda^2\}$. If we assume that $\|p(B)\| \leq m^2 - r^2$, then we have

$$\sup_{\lambda\in\sigma(B)} \left\{m^2 - \lambda^2\right\} \leq m^2 - r^2$$
$$\Downarrow$$
$$\inf_{\lambda\in\sigma(B)} \lambda^2 \geq r^2$$
$$\Downarrow$$
$$\inf_{\lambda\in\sigma(B)} |\lambda| \geq r$$

which clearly implies $|\lambda| \geq r$ for all $\lambda \in \sigma(B)$, which is equivalent to saying that $B_r(0) \cap \sigma(B) = \emptyset$. This completes the proof of the first statement.

The second statement is logically equivalent to the first, so we are done.

□

We are now ready to prove the results that in turn will prove Theorem 2.13.

Lemma 2.17. *Let $A = (A_t)_{t\in T}$ be a field of self-adjoint, bounded operators, let t_0 be any point of T, and assume that A is $p2$-continuous at t_0. Then A is Fell continuous and locally uniformly bounded at t_0.*

Proof. Fix $t_0 \in T$, and remember our convention of writing $A_0 := A_{t_0}$; we must show both that for every basis element $\mathscr{U}(K, \mathscr{F})$ of the Fell topology on $\mathscr{C}(\mathbb{R})$ such that $\sigma(A_0) \in \mathscr{U}(K, \mathscr{F})$, there exists a neighbourhood $S \subset T$ of t_0 such that $\sigma(A_t) \in \mathscr{U}(K, \mathscr{F})$ for all $t \in S$, and that there exists some neighbourhood W of t_0 such that $\sup_{t\in W} \|A_t\| < \infty$.

1) Since $K \cap \sigma(A_0) = \emptyset$, there exists for every $x \in K$ a real number $r(x) > 0$ such that $B_{r(x)}(x) \cap \sigma(A_0) = \emptyset$. Clearly the collection of (smaller) balls $\left\{B_{r(x)/2}(x)\right\}_{x\in K}$ is an open covering of K, and since K by definition is compact, there exists some finite collection $\{x_k\}_{k=1}^{l}$ of points of K such that if we set $r_k := r(x_k)$ then $\left\{B_{r_k/2}(x_k)\right\}_{k=1}^{l}$ is also an open cover of K. Clearly we still have $B_{r_k}(x_k) \cap \sigma(A_0) = \emptyset$ for all $1 \leq k \leq l$.

By the $p2$-continuity of A, there exists a neighbourhood U_0 of t_0 such that for all $t \in U_0$,

$$\Big| \|A_t\| - \|A_0\| \Big| < 1$$

This implies that $\|A_t\| < 1 + \|A_0\|$, and, in particular, $\sup_{t\in U_0} \|A_t\| < \infty$. Since by compactness $\sup_{x\in K} |x| < \infty$, it follows that we can fix a real number

$$m > 2 \sup_{t\in U_0} \|A_t\| + \sup_{x\in K} |x|$$

Furthermore, since $B_{r_k}(x_k) \cap \sigma(A_0) = \emptyset$, we can use Proposition 2.16 to conclude that

$$\left\|m^2 - (A_0 - x_k)^2\right\| < m^2 - r_k^2$$

By $p2$-continuity of A, it follows that for each $k \in \{1, ..., l\}$ there must exist an open neighbourhood $U_k \subset U_0 \subset T$ of t_0 such that

$$\left|\left\|m^2 - (A_t - x_k)^2\right\| - \left\|m^2 - (A_0 - x_k)^2\right\|\right| < 3r_k^2/4$$

for all $t \in U_k$ for each k. This gives us

$$\begin{aligned}\left\|m^2 - (A_t - x_k)^2\right\| &< \left\|m^2 - (A_0 - x_k)^2\right\| + 3r_k^2/4 \\ &< m^2 - r_k^2/4\end{aligned}$$

for all $t \in U_k$ for each k. It now follows that if we set $U := \bigcap_{k=0}^{l} U_k$, we can again use Proposition 2.16 to conclude that $B_{r_k/2}(x_k) \cap \sigma(A_t) = \emptyset$ for every $k \in \{1, ..., l\}$ and all $t \in U$. Since $K \subset \bigcup_{k=0}^{l} B_{r_k/2}(x_k)$, it follows that $K \cap \sigma(A_t) = \emptyset$ for all $t \in U$.

2) Take any $O \in \mathscr{F}$. Since we have $O \cap \sigma(A_0) \neq \emptyset$, it follows that for any given $x \in O \cap \sigma(A_0)$ there exists a real number $r(x) > 0$ such that $B_{r(x)}(x) \subset O$, by the openness of O. Since $x \in \sigma(A_0)$, we have $|x| \leq \|A_0\|$, and accordingly $\|A_0 - x\| \leq 2\|A_0\| < m$ with m as earlier. Since $B_{r(x)/2}(x) \cap \sigma(A_0) \neq \emptyset$ (as $x \in \sigma(A_0)$), we get from Proposition 2.16 that $\|m^2 - (A_0 - x)^2\| > m^2 - r(x)^2/4$.

Now by $p2$-continuity, there exists a neighbourhood $V_O \subset T$ of t_0 such that for all $t \in V_O$,

$$\left|\|m^2 - (A_0 - x)^2\| - \|m^2 - (A_t - x)^2\|\right| < 3r(x)^2/4$$

which gives us

$$\|m^2 - (A_0 - x)^2\| - \|m^2 - (A_t - x)^2\| < 3r(x)^2/4$$

for all $t \in V_O$, implying

$$m^2 - r(x)^2/4 - 3r(x)^2/4 < \|m^2 - (A_t - x)^2\|$$

for all $t \in V_O$, so we can finally conclude

$$\|m^2 - (A_t - x)^2\| > m^2 - r(x)^2$$

for all $t \in V_O$. Again by Proposition 2.16, we thus have $B_{r(x)}(x) \cap \sigma(A_t) \neq \emptyset$ and so $O \cap \sigma(A_t) \neq \emptyset$ for all $t \in V_O$. Since $\mathscr{F}$ is a finite family, $V := \bigcap_{O \in \mathscr{F}} V_O$ is an open, non-empty neighbourhood of t_0 and $O \cap \sigma(A_t) \neq \emptyset$ for all $O \in \mathscr{F}$ and all $t \in V$.

3) We now have that $S := U \cap V$ is the desired neighbourhood of t_0 such that $\sigma(A_t) \in \mathscr{U}(K, \mathscr{F})$ for all $t \in S$. Since K and $\mathscr{F}$ were arbitrary, the proof of the first statement now follows.

4) Local uniform boundedness at t_0 was "accidentally" proved in part 1); we get the desired W by setting $W := U_0$. This concludes the proof.

□

We will need a refresher on proper functions before our next result.

Proposition 2.18. *The function $|\cdot| : x \in \mathbb{R} \mapsto |x| \in \mathbb{R}$ is proper (in the standard topology on $\mathbb{R}$).*

Proof. Choose any compact subset $K \in \mathbb{R}$; we must show that

$$|\cdot|^{-1}(K) = \{\lambda \in \mathbb{C} \mid |\lambda| \in K\}$$

is a compact subset of $\mathbb{R}$.

As K is compact, it is bounded, so there exists a finite $r > 0$ such that $\sup |K| = r$. It follows that

$$\begin{aligned} \sup \left| |\cdot|^{-1}(K) \right| &= \sup \left| \{\lambda \in \mathbb{R} \mid \ |\lambda| \in K\} \right| \\ &= \sup |K| = r \end{aligned}$$

so $|\cdot|^{-1}(K)$ is also bounded. By the continuity of $|\cdot|$, it is also closed, which completes the proof since closed, bounded subsets of $\mathbb{R}$ are compact.

□

Proposition 2.19. *Let X and Y be two topological spaces, and let $f : X \to Y$ be a continuous proper function. If $X_R \subset X$ is a closed subset of X and $Y_R \subset Y$ is a closed subset of Y with $f(X_R) \subset Y_R$, then the restriction $f_R : X_R \to Y_R$ of f is a continuous proper function when X_R and Y_R are equipped with their respective subspace topologies.*

Proof. Proving continuity is trivial, so we will focus on proving that f_R is proper.

Take any compact subset $K_R \subset Y_R$. As Y_R is closed, K_R is a closed subset of Y.

Let $\{V_\iota\}_{\iota \in I}$ be a covering of K_R by sets open in Y. Then

$$\{V_\iota \cap Y_R\}_{\iota \in I}$$

is a covering of K_R by sets open in Y_R, and as such there exists some finite subcovering $\{V_{\iota_n} \cap Y_R\}_{n=1}^k$. This implies that $\{V_{\iota_n}\}_{n=1}^k$ is a finite subcollection of $\{V_\iota\}_{\iota \in I}$ that covers K_R, so K_R is compact in Y.

As f is proper, $f^{-1}(K_R)$ is compact in X. Since

$$f_R^{-1}(K_R) = f^{-1}(K_R) \cap X_R$$

it is by definition closed as a subset of X_R.

Let $\{U_{R,\iota}\}_{\iota\in I}$ be a covering of $f_R^{-1}(K_R)$ by open subsets of X_R. Consider the family

$$\{U_{R,\iota} \cup (X \backslash X_R)\}_{\iota\in I}$$

We claim that this is a family of open subsets of X.

For any given $\iota \in I$, consider any net $\{x_\kappa\}_{\kappa\in J}$ of points in $(U_{R,\iota} \cup (X\backslash X_R))^c$. As $x_\kappa \in X_R$ for every $\kappa \in J$ and X_R is closed in X, it follows that if $\{x_\kappa\}_{\kappa\in J}$ converges towards some $x \in X$ (in the topology on X), then $x \in X_R$.

Furthermore, if $x \in U_{R,\iota}$ then it follows that $x_\kappa \to x$ also in the subspace topology on X_R. But this contradicts the fact that $U_{R,\iota}$ is open in X_R. It follows that if x exists, we have

$$x \in (U_{R,\iota} \cup (X\backslash X_R))^c$$

so $(U_{R,\iota} \cup (X\backslash X_R))^c$ is a closed set in X and $U_{R,\iota} \cup (X\backslash X_R)$ is open in X.

Now $\{U_{R,\iota} \cup (X\backslash X_R)\}_{\iota\in I}$ is a family of open subsets of X covering $f_R^{-1}(K_R)$, which is compact in X. Thus we have a finite subcollection

$$\{U_{R,\iota_n} \cup (X\backslash X_R)\}_{n=1}^k$$

that also covers $f_R^{-1}(K_R)$.

Finally,

$$\{(U_{R,\iota_n} \cup (X\backslash X_R)) \cap X_R\}_{n=1}^k = \{U_{R,\iota_n}\}_{n=1}^k$$

is now an open subfamily of $\{U_{R,\iota}\}_{\iota\in I}$ covering $f_R^{-1}(K_R)$. Thus $f_R^{-1}(K_R)$ is compact as a subset of X_R, and the proof is complete.

□

Lemma 2.20. *Let $A = (A_t)_{t\in T}$ be a field of bounded, self-adjoint operators, and choose any $t_0 \in T$. If A is Fell continuous and locally uniformly bounded at t_0, then A is proper-continuous at t_0.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, proper function. Recall that since each A_t is self-adjoint and bounded, $\sigma(A_t)$ is compact in $\mathbb{R}$ and

$$f(\sigma(A_t)) = \sigma(f(A_t)) \subset \mathbb{R}$$

Furthermore, recall from Proposition 2.15 that for any continuous, proper function $h : X \to Y$ between two locally compact Hausdorff spaces, $\hat{h}$ is the (Fell-continuous) map

$$F \in \mathscr{C}(X) \to h(F) \in \mathscr{C}(Y)$$

We start by claiming that the field $\{f(A_t)\}_{t\in T}$ is also locally uniformly bounded at t_0. To prove this, note first that for each $t \in T$, we have

$$\|f(A_t)\| = \sup |\sigma(f(A_t))|$$

Since A is locally uniformly bounded at t_0, there exists a neighbourhood $U \subset T$ of t_0 and a $R > 0$ such that $\sigma(A_t) \subset [-R, R]$ for all $t \in U$. Since f is continuous, it follows that there exists some $\tilde{R} > 0$ such that

$$\sigma(f(A_t)) = f(\sigma(A_t)) \subset f([-R, R]) \subset \left[-\tilde{R}, \tilde{R}\right]$$

for all $t \in U$. Since $[R, R]$ is compact and f is continuous, $\left[-\tilde{R}, \tilde{R}\right]$ is also compact.

It now follows that for all $t \in U$, we have $\sigma(f(A_t)) \in \mathscr{C}\left(\left[-\tilde{R}, \tilde{R}\right]\right)$ and $|\sigma(f(A_t))| \in \mathscr{C}\left(\left(-\infty, \tilde{R}\right]\right)$. If we now look at the restrictions

$$f_R : [-R, R] \to \left[-\tilde{R}, \tilde{R}\right]$$

and

$$|\cdot|_R : \left[-\tilde{R}, \tilde{R}\right] \to \left(-\infty, \tilde{R}\right]$$

of f and $|\cdot|$, respectively, then we see that we can write the map $t \in U \mapsto \|f(A_t)\| \in \mathbb{R}$ as the composition

$$\begin{aligned} t \in U \mapsto \quad & \sigma(A_t) \in \mathscr{C}([-R, R]) \quad \overset{\widehat{f_R}}{\mapsto} \quad \sigma(f(A_t)) \in \mathscr{C}(\left[-\tilde{R}, \tilde{R}\right]) \\ & \overset{\widehat{|\cdot|_R}}{\mapsto} |\sigma(f(A_t))| \in \mathscr{C}\left(\left(-\infty, \tilde{R}\,\right]\right) \mapsto \sup |\sigma(f(A_t))| \in \mathbb{R} \end{aligned}$$

The first map is continuous at t_0 by assumption. The second and third maps are continuous by Proposition 2.15, since $|\cdot|$ is proper by Proposition 2.18 and the two restricted maps f_R and $|\cdot|_R$ are continuous and proper by Proposition 2.19. The last map is continuous by Lemma 2.14. This proves continuity at t_0.

The comment at the end of the lemma immediately follows from the fact that all polynomials of degree equal to or less than two with real coefficients are either proper functions or constant functions, as shown in Proposition 2.10, and the map is obviously continuous in the constant case.

□

Lemma 2.21. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous at some point $t_0 \in T$. Then $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at t_0.*

Proof. 1) We will first show that the first condition of gap edge continuity holds. Let (a, b) be any gap of $F_0 := F_{t_0}$ - if F_0 has no gaps, then $F_0 = \mathbb{R}$, and the first condition of gap edge continuity is trivially satisfied.

Assume first that $-\infty < a < b < \infty$. We will show that for any $\epsilon > 0$ such that $\epsilon < \frac{b-a}{4}$, there exists a neighbourhood $U \subset T$ of t_0 such that for each $t \in U$, there exists a gap (a_t, b_t) of F_t such that

$$|a_t - a| < \epsilon \text{ and } |b_t - b| < \epsilon$$

Note that if we set

$$K := [a + \epsilon, b - \epsilon]\ , O_a := (a - \epsilon, a + \epsilon) \text{ and } O_b := (b - \epsilon, b + \epsilon)$$

then K is a compact set and $\mathscr{F} := \{O_a, O_b\}$ is a finite family of open sets, and we have

$$F_0 \cap K = \emptyset \text{ and } F_0 \cap O \neq \emptyset$$

for all $O \in \mathscr{F}$. It follows that $F_0 \in \mathscr{U}(K, \mathscr{F})$. By the Fell continuity of $t \mapsto F_t$, there exists a neighbourhood $U \subset T$ of t_0 such that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all $t \in U$. This implies that for all $t \in U$, we have

$$F_t \cap [a + \epsilon, b - \epsilon] = \emptyset\ , F_t \cap (a - \epsilon, a + \epsilon) \neq \emptyset \text{ and } F_t \cap (b - \epsilon, b + \epsilon) \neq \emptyset$$

But this tells us that for each $t \in U$, there exists some gap (a_t, b_t) of F_t containing $[a + \epsilon, b - \epsilon]$ such that its gap edges lie in $(a - \epsilon, a + \epsilon)$ and $(b - \epsilon, b + \epsilon)$, respectively. Thus we must have that

$$|a_t - a| < \epsilon \text{ and } |b_t - b| < \epsilon$$

for all $t \in U$, and we are done in this case.

Next, assume that $-\infty < a < b = \infty$, so that (a, ∞) is a gap of F_0. Choose any $\epsilon > 0$ and any $N > 0$ such that $N > a + \epsilon$. We will show that there exists a neighbourhood $U \subset T$ of t_0 such that for each $t \in U$, there is a gap (a_t, b_t) of F_t with

$$|a_t - a| < \epsilon \text{ and } b_t > N$$

Note that if we define $K := [a + \epsilon, N]$ and $O := (a - \epsilon, a + \epsilon)$, then K is compact, O is open, and we have

$$F_0 \cap K = 0 \text{ and } F_0 \cap O \neq \emptyset$$

Thus we have $F_0 \in \mathscr{U}(K, \{O\}$. It follows from the Fell continuity of the map $t \mapsto F_t$ that there exists a neighbourhood U of t_0 in T such that we have $F_t \in \mathscr{U}(K, \{O\})$ for all $t \in U$. This implies that for all $t \in U$, we have

$$F_t \cap [a + \epsilon, N] = \emptyset \text{ and } F_t \cap (a - \epsilon, a + \epsilon)$$

This implies that for each $t \in U$, there must exist a gap (a_t, b_t) of F_t such that it contains $[a+\epsilon, N]$ and such that its gap edges lie in $(a-\epsilon, a+\epsilon)$ and (N, ∞), respectively. But this means that

$$|a_t - a| < \epsilon \text{ and } b_t > N$$

for all $t \in U$, and we are done in this case as well.

Proving that the first condition of gap edge continuity holds in the case where $-\infty = a < b < \infty$ is completely analogous.

Finally, if $-\infty = a < b = \infty$, then $F_0 = \emptyset$, and given any $N > 0$ we will show that there exists some neighbourhood $U \subset T$ of t_0 such that for every $t \in U$, there exists a gap (a_t, b_t) of F_t such that

$$a_t < -N \text{ and } b_t > N$$

We see that if we set $K := [-N, N]$, then clearly $F_0 \in \mathscr{U}(K, \emptyset)$. By the Fell continuity of $t \mapsto F_t$, there exists a neighbourhood $U \subset T$ of t_0 such that $F_t \in \mathscr{U}(K, \emptyset)$ for all $t \in U$. Thus for all $t \in U$, we have

$$F_t \cap [-N, N] = \emptyset$$

This implies that for each $t \in U$, there exists a gap (a_t, b_t) of F_t such that it contains $[-N, N]$ and such that its gap edges lie in $(-\infty, -N)$ and (N, ∞), respectively. But this means that

$$a_t < -N \text{ and } b_t > N$$

for all $t \in U$, and we are done.

This covers all possibilities, and proves that the first condition of gap edge continuity holds at t_0.

2) Assume that we are given a net $\{t_\iota\}_{\iota \in I}$ of points in T converging to t_0, as well as two nets $\{a_{t_\iota}\}_{\iota \in I}$, $\{b_{t_\iota}\}_{\iota \in I}$ of points in $[-\infty, \infty]$ such that $(a_{t_\iota}, b_{t_\iota})$ is a gap of F_{t_ι} for every $\iota \in I$, and such that $a_{t_\iota} \to a$ and $b_{t_\iota} \to b$ for some $a, b \in [-\infty, \infty]$.

In order to prove that the second condition of gap edge continuity holds, we must show that either $a = b$ or that (a, b) is a gap of F_0. So assume to the contrary that $a \neq b$ and that (a, b) is not a gap of F_0. We will show that this causes a contradiction.

Since $a \neq b$ and $a_{t_\iota} < b_{t_\iota}$ for all $\iota \in I$, it follows that $a < b$. Since (a, b) is not a gap of F_0, at least one of the following must hold:

- $a \notin F_0 \cup \{-\infty\}$,
- $b \notin F_0 \cup \{\infty\}$ or
- there exists a $c \in F_0$ with $a < c < b$.

Assume that $a \notin F_0 \cup \{-\infty\}$. Then $a \in (F_0 \cup \{-\infty\})^c$, which is an open set. Since a lies in an open set and $a \neq \pm\infty$ (if we had $a = \infty$ then we could not have $a < b$), there exists an $\epsilon > 0$ such that $F_0 \cap (a - \epsilon, a + \epsilon) = \emptyset$. If we set $K := \left[a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right]$ then we see that $F_0 \cap K = \emptyset$, so $F_0 \in \mathscr{U}(K, \emptyset)$.

By the Fell continuity of $t \mapsto F_t$ at t_0, there exists a neighbourhood $U \subset T$ of t_0 such that $F_t \in \mathscr{U}(K, \emptyset)$ for all $t \in U$. Since $t_\iota \to t_0$, we eventually have that

$$F_{t_\iota} \cap K = F_{t_\iota} \cap \left[a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right] = \emptyset$$

Since $a_{t_\iota} \to a$, we eventually have that $|a_{t_\iota} - a| < \frac{\epsilon}{2}$. But this implies that we eventually have $a_{t_\iota} \in \left[a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right] = K$, which is impossible, since by assumption $a_{t_\iota} \in F_{t_\iota}$ for all $\iota \in I$ and eventually we have $F_{t_\iota} \cap K = \emptyset$. Thus we have obtained a contradiction in the case where $a \notin F_0 \cup \{-\infty\}$.

The proof in the case where we assume that $b \notin F_0 \cup \{\infty\}$ is analogous to the case where we assumed that $a \notin F_0 \cup \{-\infty\}$.

Finally, assume that there exists a $c \in F_0$ with $a < c < b$. If we set

$$O := \left(\frac{a+c}{2}, \frac{b+c}{2}\right)$$

then we have $c \in O$. Thus $F_0 \in \mathscr{U}(\emptyset, \{O\})$ (remember that $\{O\}$ is the family of open sets consisting only of O). By Fell continuity, we can choose a neighbourhood $U \subset T$ of t_0 such that $F_t \in \mathscr{U}(\emptyset, \{O\})$ for all $t \in U$, that is, such that $F_t \cap O \neq \emptyset$ for all $t \in U$.

Since $t_\iota \to t_0$, it follows that we eventually have $t_\iota \in U$. Furthermore, since $a_{t_\iota} \to a$ and $b_{t_\iota} \to b$ we eventually have that $|a_{t_\iota} - a| < \frac{a+c}{2}$ and $|b_{t_\iota} - b| < \frac{c+b}{2}$. This implies that we eventually have that

$$a_{t_\iota} < \frac{a+c}{2} < \frac{c+b}{2} < b_{t_\iota}$$

and so eventually $O \subset (a_{t_\iota}, b_{t_\iota})$. But this is impossible - for since, by Fell continuity, eventually we have $F_{t_\iota} \cap O \neq \emptyset$, it follows that eventually we have $F_t \cap (a_{t_\iota}, b_{t_\iota}) \neq \emptyset$, and so eventually $(a_{t_\iota}, b_{t_\iota})$ is not a gap of F_{t_ι}. This contradicts our starting assumptions, and completes the proof.

□

Lemma 2.22. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at some point $t_0 \in T$. Then the map $t \mapsto F_t$ is Fell continuous at t_0.*

Proof. Choose any compact subset $K \subset \mathbb{R}$ and any finite family $\mathscr{F}$ of open subsets of $\mathbb{R}$ such that $F_0 := F_{t_0} \in \mathscr{U}(K, \mathscr{F})$. We must find a neighbourhood of t_0 such that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all t in this neighbourhood.

1) We will first find a neighbourhood $U \subset T$ of t_0 such that $F_t \cap K = \emptyset$ for all $t \in U$.

Choose any $x \in K$. As $F_0 \cap K = \emptyset$, we have $x \in F_0^c$. This implies that $x \in (a_x, b_x)$ where (a_x, b_x) is a gap of F_0.

Assume first that $-\infty < a_x < b_x < \infty$. By the gap edge continuity of $t \mapsto F_t$ at t_0, there must exist some neighbourhood $U_x \subset T$ of t_0 such that for each $t \in U_x$, there exists a gap $(a_{x,t}, b_{x,t})$ of F_t with

$$\max\{|a_{x,t} - a_x|, |b_{x,t} - b_x|\} < \min\left\{\frac{x - a_x}{2}, \frac{b_x - x}{2}\right\}$$

This implies that if we set $c_x := \frac{x-a_x}{2}$ and $d_x := \frac{b_x - x}{2}$, it follows that we have $x \in (c_x, d_x)$ and $F_t \cap (c_x, d_x) = \emptyset$ for all $t \in U_x$.

Next, assume that $-\infty < a_x < b_x = \infty$. By the first condition of gap edge continuity, we can find some neighbourhood $U_x \subset T$ of t_0 such that for each $t \in U_x$, there exists a gap $(a_{x,t}, b_{x,t})$ of F_t such that

$$|a_{x,t} - a_x| < \frac{x - a_x}{2} \text{ and } b_{x,t} > |x| + 1$$

Thus if we set $c_x := \frac{x-a_x}{2}$ and $d_x := |x| + 1$, it follows that $x \in (c_x, d_x)$ and $F_t \cap (c_x, d_x) = \emptyset$ for all $t \in U_x$.

The case where $-\infty = a_x < b_x < \infty$ is completely analoguous to the previous case.

If $-\infty = a_x < b_x = \infty$, we can, again by the first condition of gap edge continuity, find some neighbourhood $U_x \subset T$ of t_0 such that for each $t \in U_x$, there exists a gap $(a_{x,t}, b_{x,t})$ such that

$$a_{x,t} < -|x| - 1 \text{ and } b_{x,t} > |x| + 1$$

It follows that if we set $c_x = -|x| - 1$ and $d_x = |x| + 1$, then $x \in (c_x, d_x)$ and $F_t \cap (c_x, d_x) = \emptyset$ for all $t \in U_x$.

Now the collection of all (c_x, d_x) for $x \in K$ form an open covering of K, so by the compactness of K there exists a finite subcollection $\{x_k\}_{k=1}^{l}$ so that the collection of all (c_{x_k}, d_{x_k}) for $1 \leq k \leq l$ also covers K.

Define $U = \bigcap_{k=1}^{l} U_{x_k}$; it follows that U is an open neighbourhood of t_0 such that for all $t \in U$,

$$F_t \cap K \subset F_t \cap \left(\bigcup_{k=1}^{l} (c_{x_k}, d_{x_k}) \right) = \emptyset$$

so we have found the required U.

2) For the second part, we want to find a neighbourhood $V \subset T$ of t_0 such that $F_t \cap O \neq \emptyset$ for all $t \in V$ and all $O \in \mathscr{F}$. Start by fixing $O \in \mathscr{F}$.

Choose any $x \in F_0 \cap O$, and choose any $r > 0$ such that $B_r(x) \subset O$. Now there must exist an open neighbourhood $V_O \subset T$ of t_0 such that $F_t \cap B_r(x) \neq \emptyset$ for all $t \in V_O$. For if not, then for every open neighbourhood $V \subset T$ of t_0, there would have to exist a $t_V \in V$ such that $F_{t_V} \cap B_r(x) = \emptyset$ - that is, there must exist a gap (a_V, b_V) of F_{t_V} with

$$a_V \in [-\infty, x - r] \text{ and } b_V \in [x + r, \infty]$$

The net $\{t_V\}$ (where the V's are open neighbourhoods of t_0, ordered under reverse inclusion) converges to t_0. Furthermore, as all elements of the net $\{a_V\}$ lie in the compact set $[-\infty, x - r]$, it follows that there exists some convergent subnet $\{a_{V_\iota}\}_{\iota \in I}$ such that $a_{V_\iota} \to a$ for some $a \in [-\infty, x - r]$. Now all elements of the net $\{b_{V_\iota}\}_{\iota \in I}$ lie in the compact set $[x + r, \infty]$, so there exists a convergent subnet $\left\{ b_{V_{\iota_\kappa}} \right\}_{\kappa \in J}$ such that $b_{V_{\iota_\kappa}} \to b$ for some $b \in [x + r, \infty]$.

Accordingly, we have that

$$t_{V_{\iota_\kappa}} \to t_0 \text{ , } a_{V_{\iota_\kappa}} \to a \text{ and } b_{V_{\iota_\kappa}} \to b$$

and that $(a_{V_{\iota_\kappa}}, b_{V_{\iota_\kappa}})$ is a gap of $F_{V_{\iota_\kappa}}$ for each $\kappa \in J$. We also have

$$a \in [-\infty, x - r] \text{ and } b \in [x + r, \infty]$$

Thus we can use the second condition of gap edge continuity of $t \mapsto F_t$ at t_0 to conclude that (a, b) must be a gap of F_0.

Now we also have $x \in (a, b)$, that is, x lies in a gap of F_0, so we have $x \notin F_0$, which is impossible. Therefore the required V_O such $F_t \cap O \neq \emptyset$ for all $t \in V_O$ must exist. Since $\mathscr{F}$ is finite, it follows that $V := \bigcap_{O \in \mathscr{F}} V_O$ is the required neighbourhood of t_0 such that

$$F_t \cap O \neq \emptyset$$

for all $t \in V$ and all $O \in \mathscr{F}$.

3) It now follows that $F_t \in \mathscr{U}(K, \mathscr{F})$ for every $t \in U \cap V$, with $U \cap V$ being an open neighbourhood of t_0, so we are done.

□

Proof of Theorem 2.13:
Recall that we are considering a field $A = (A_t)_{t \in T}$ of self-adjoint, bounded operators; in particular, $\sigma(A_{t_0})$ is a compact subset of $\mathbb{R}$. Now:

- Fell continuity at t_0 implies gap edge continuity at t_0 by Lemma 2.21.
- Gap edge continuity at t_0 implies Fell continuity at t_0 by Lemma 2.22.
- Proper-continuity at t_0 implies $p2$-continuity at t_0 by Corollary 2.11.
- $p2$-continuity at t_0 implies Fell continuity and local uniform boundedness at t_0 by Lemma 2.17.
- Fell continuity and local uniform boundedness at t_0 implies proper-continuity at t_0 by Lemma 2.20.
- Fell continuity and local uniform boundedness at t_0 implies Hausdorff continuity at t_0 by Lemma 1.27.
- Hausdorff continuity at t_0 implies Fell continuity and local uniform boundedness at t_0 by Lemma 1.28.

□

2.3 R-continuity

Another reasonable way of considering continuity of spectra is to instead look at the norm of the resolvent, that is,

$$\left\|(\lambda I - B)^{-1}\right\|$$

for $\lambda \in \rho(B)$. For self-adjoint operators, this is comparatively straightforward; you "only" need to calculate the norm of the resolvent for λ in $\mathbb{C}\backslash\mathbb{R}$, as opposed to λ in all of $\mathbb{C}$ or arbitrary subsets thereof. The set $\mathbb{C}\backslash\mathbb{R}$ will always be included in the resolvent since the spectrum is a subset of $\mathbb{R}$; furthermore, as the resolvent set is open and the resolvent is continuous in λ, the norm of the resolvent at any point in the intersection of the resolvent and $\mathbb{R}$ can be approximated arbitrarily well by approaching it from above or below, thus justifying that we do not need to verify continuity for $\lambda \in \mathbb{R}\backslash\sigma(B)$.

The second result from [3] states that R-continuity is also equivalent with gap edge continuity, thus allowing us to apply Theorem 2.13. In [3], the proof for this theorem was extremely brief, so the majority of this section has been written from the ground up. In particular, we have chosen to demonstrate the equivalence between R-continuity and *Fell-continuity*, rather than gap edge continuity, as it both provides a cleaner proof and generalizes more easily.

We begin with a relevant definition.

Definition 2.23. A field $A = (A_t)_{t\in T}$ of self-adjoint operators is called R-continuous if the map $t \in T \mapsto \|(\lambda I - A_t)^{-1}\| \in \mathbb{R}^+$ is continuous for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

We usually just write λ for the operator λI in cases where there can be no confusion about this.

As promised, the main result of this section is that this definition of continuity is the same as those we have encountered before:

Theorem 2.24. *Let $A = (A_t)_{t\in T}$ be a field of bounded, self-adjoint operators. Then A is R-continuous at a point $t_0 \in T$ if and only if the spectrum of A is Fell-continuous at t_0.*

The main idea behind proving this will be to use the connection between the norm of the resolvent at λ and the *distance between λ and the spectrum*, allowing for a much simpler proof based on our earlier topological considerations.

Lemma 2.25. *Let B be a bounded, normal operator, and take any $\lambda \in \rho(B)$. Then*

$$\left\|(\lambda - B)^{-1}\right\| = \frac{1}{dist(\lambda, \sigma(B))}$$

Proof. By the continuous functional calculus for normal bounded operators, we have

$$\begin{aligned}\left\|(\lambda - B)^{-1}\right\| &= \sup\left|\frac{1}{\sigma(\lambda - B)}\right| \\ &= \frac{1}{\inf_{x\in\sigma(B)}|\lambda - x|} \\ &= \frac{1}{\operatorname{dist}(\lambda, \sigma(B))}\end{aligned}$$

□

Proof of Theorem 2.24:

As each A_t is self-adjoint, $\sigma(A_t) \subset \mathbb{R}$ for every $t \in T$. Lemma 2.25 thus implies that for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$, the map $t \in T \mapsto \|(\lambda - A_t)^{-1}\| \in \mathbb{R}^+$ is continuous at t_0 if and only if the map $t \in T \mapsto \text{dist}(\lambda, \sigma(A_t))$ is.

For any $\lambda \in \mathbb{R}$, we have for every $t \in T$ that since $\sigma(A_t) \subset \mathbb{R}$, we can use the Pythagorean theorem to see that

$$\begin{aligned}\text{dist}(\lambda, \sigma(A_t))^2 &= \inf_{x \in \sigma(A_t)} |\lambda - x|^2 \\ &= \inf_{x \in \sigma(A_t)} |\lambda + ir - x|^2 - r^2 \\ &= \text{dist}(\lambda + ir, \sigma(A_t))^2 - r^2\end{aligned}$$

for any $r \in \mathbb{R}$, and so it follows that the map $t \in T \mapsto \text{dist}(\lambda, \sigma(A_t))$ is continuous at t_0 if and only if the map $t \in T \mapsto \text{dist}(\lambda + ir, \sigma(A_t))$ is continuous at t_0 for every $r \in \mathbb{R}$.

Thus, we see that the map $t \in T \mapsto \|(\lambda - A_t)^{-1}\| \in \mathbb{R}^+$ is continuous at t_0 if and only if the map $t \in T \mapsto \text{dist}(\lambda, \sigma(A_t))$ is continuous at t_0 for *every* $\lambda \in \mathbb{C}$.

But this is exactly the same as demanding that $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ be Wijsman continuous at t_0, and Lemmas 1.22 and 1.23 imply that Wijsman continuity at t_0 is equivalent to Fell continuity at t_0, so the result follows.

□

3 Hölder Estimates

3.1 ∞-metrics

In this chapter, we will study what happens when we are given not just a topological structure on T, but a metric structure. In this section, we will mostly just be dealing with technicalities that are needed for the rest of the chapter, and the reader can safely skip this section.

As a refresher, a *metric space* is a space X together with a map

$$d : X \times X \to \mathbb{R}^+$$

- the so-called *metric* - such that for any $x, y, z \in X$, the following conditions all hold:

- $d(x, y) = 0 \Leftrightarrow x = y$ *(identity of indiscernibles)*
- $d(x, y) = d(y, x)$ *(symmetry)*
- $d(x, y) \leq d(x, z) + d(z, y)$ *(subadditivity)*

Any metric space is also a topological space, with its basis the collection of open balls $B_r(x) := \{y \in X \mid d(x, y) < r\}$ for all $r > 0$ and all $x \in X$. However, not all topologies arise from a metric structure. For example, let X be any non-Hausdorff topological space. Then there exists a pair $x, y \in X$ such that $x \neq y$, and such that for every open set $U \subset X$ we have $x \in U$ if and only if $y \in U$. If the topology on X was induced by a metric d, then we would need to have $y \in B_r(x)$ for all $r > 0$, implying that $d(x, y) = 0$ even though $x \neq y$, which is impossible.

$\mathbb{R}$ with the Euclidean metric is a classic example of a metric space, and the standard topology on $\mathbb{R}$ is induced by the Euclidean metric. The topology $\mathscr{T}$ on the two-point compactification $[-\infty, \infty]$ of $\mathbb{R}$, as we defined it in Section 1.1, is also induced by a metric, but it is not simply an extension of the Euclidean metric. For define the map $d_\infty : [-\infty, \infty] \times [-\infty, \infty] \to \mathbb{R}^+ \cup \{\infty\}$ by

$$d_\infty(x, y) = \begin{cases} |x - y| & \text{if } x, y \in (-\infty, \infty) \\ 0 & \text{if } x = y \in \{-\infty, \infty\} \\ \infty & \text{otherwise} \end{cases} \tag{*}$$

Although d_∞ is an intuitive extension of the Euclidean metric, it is, in fact, not a metric itself, and even if we did define a topology on $[-\infty, \infty]$ by letting its basis consist of all balls $B_r(x)$, it would not induce $\mathscr{T}$. As d_∞ takes values in $\mathbb{R}^+ \cup \{\infty\}$, it does not match our definition of metrics - although it should be noted that it *is* what we will call an ∞-*metric* - but the issue of not inducing $\mathscr{T}$ is the more pressing one. To see this, simply note that $B_r(\infty) = \{\infty\}$ for all $0 < r < \infty$, even though $\{\infty\} \notin \mathscr{T}$.

As we introduced the term ∞-metric, we would do well to define it properly.

Definition 3.1. Let X be a space. A map $d_\infty : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ is called an ∞-*metric on* X if, for every $x, y, z \in X$, the following conditions all hold:

- $d_\infty(x,y) = 0 \Leftrightarrow x = y$
- $d_\infty(x,y) = d_\infty(y,x)$
- $d_\infty(x,y) \leq d_\infty(x,z) + d_\infty(z,y)$, with the convention that $\infty \leq \infty + a \leq \infty + \infty$ for all $a \geq 0$.

The space X together with the ∞-metric d_∞ is called an ∞-*metric space*.

One reason why ∞-metrics are not always considered is that it is quite straightforward to transform an ∞-metric into a (regular) metric. For, given an ∞-metric d_∞ on a space X, define either $d_1(x,y) := \min\{1, d_\infty(x,y)\}$ or

$$d_2(x,y) := \begin{cases} \frac{d_\infty(x,y)}{1+d_\infty(x,y)} & \text{if } d_\infty(x,y) < \infty \\ 1 & \text{otherwise} \end{cases}$$

It is not difficult to confirm that both d_1 and d_2 will be metrics on X. However, they both have a large flaw in our case - they do, again, not induce the standard topology $\mathscr{T}$ on $[-\infty, \infty]$, as

$$\{x \in [-\infty,\infty] \mid d_1(x,\infty) < r\} = \{x \in [-\infty,\infty] \mid d_2(x,\infty) < r\} = \{\infty\} \notin \mathscr{T}$$

for all $r \leq 1$.

There is, however, a way to construct the "correct" metric from d_∞.

Lemma 3.2. *Define $d_\infty : [-\infty,\infty] \times [-\infty,\infty] \to \mathbb{R}^+ \cup \{\infty\}$ as in (*), and define the map sign : $[-\infty,\infty] \to \{-1,1\}$ by*

$$\mathit{sign}(x) := \begin{cases} -1 & \text{if } x \in [-\infty, 0) \\ 1 & \text{otherwise} \end{cases}$$

Then the map $d : [-\infty,\infty] \times [-\infty,\infty] \to \mathbb{R}^+$ defined by

$$d(x,y) := \left| \mathit{sign}(x) e^{-1/d_\infty(x,0)} - \mathit{sign}(y) e^{-1/d_\infty(y,0)} \right|$$

is a metric on $[-\infty,\infty]$, and it induces the standard topology $\mathscr{T}$ on $[-\infty,\infty]$.

Proof. It is easy to see that $d(x,y) = 0$ if and only if both $d_\infty(x,0) = d_\infty(y,0)$ and $\text{sign}(x) = \text{sign}(y)$. Referring to the definition of d_∞, we see that this is the case if and only if $x = y$, so d satisfies identity of indiscernibles.

The symmetry of d is trivial.

To see that d is subadditive, note that for any $x, y, z \in [-\infty,\infty]$ we have

$$\begin{aligned} d(x,y) &= \left| \text{sign}(x) e^{-1/d_\infty(x,0)} - \text{sign}(y) e^{-1/d_\infty(y,0)} \right| \\ &= \Big| \text{sign}(x) e^{-1/d_\infty(x,0)} - \text{sign}(z) e^{-1/d_\infty(z,0)} \\ &\quad + \text{sign}(z) e^{-1/d_\infty(z,0)} - \text{sign}(y) e^{-1/d_\infty(y,0)} \Big| \\ &\leq d(x,z) + d(z,y) \end{aligned}$$

To see that the topology generated by the basis consisting of all sets of the type $B_r^d(x) := \{y \in [-\infty, \infty] \mid d(x, y) < r\}$ for $x \in [-\infty, \infty]$ and $r > 0$ is equivalent to $\mathscr{T}$, it is sufficient to show that for every $x \in [-\infty, \infty]$ and for every $U \in \mathscr{T}$ with $x \in U$, there exists some $r > 0$ and some $x' \in [-\infty, \infty]$ such that $B_r^d(x') \subset U$, c.f. [9].

Note that by definition,

$$\begin{aligned} \text{sign}(x)e^{-1/d_\infty(x,0)} &= \begin{cases} 1 & \text{if } x = \infty \\ \text{sign}(x)e^{-1/|x|} & \text{if } x \in \mathbb{R}\setminus\{0\} \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x = -\infty \end{cases} \\ &= \begin{cases} 1 & \text{if } x = \infty \\ e^{-1/x} & \text{if } x \in (0, \infty) \\ 0 & \text{if } x = 0 \\ -e^{1/x} & \text{if } x \in (-\infty, 0) \\ -1 & \text{if } x = -\infty \end{cases} \end{aligned}$$

Assume first that $x = \infty$, and fix any $U \in \mathscr{T}$ with $x \in U$. By the definition of $\mathscr{T}$, there exists some $a \in \mathbb{R}$ such that $(a, \infty] \subset U$. If $a \leq 0$, then it is easily verified that $B_1^d(\infty) = \{y \in [-\infty, \infty] \mid d(\infty, y) < 1\} \subset (0, \infty] \subset (a, \infty) \subset U$.

Assume $a > 0$. If $y \in B_r^d(\infty)$ for some r with $0 < r \leq 1$, then we see that we must have $y > 0$ and

$$r > \left|e^{-1/\infty} - e^{-1/y}\right| = 1 - e^{-1/y}$$

which implies that

$$e^{-1/y} > 1 - r$$

and so $y > -\frac{1}{\ln(1-r)}$. It follows that if $r < 1 - e^{-1/a}$, then $y > a$ for all $y \in B_r^d(\infty)$, so $B_r^d(\infty) \subset (a, \infty) \subset U$.

If $x \in (0, \infty)$, then from the definition of $\mathscr{T}$ there exists some $r' > 0$ and some interval $(x - r', x + r') \subset U$. As the map $x \in (0, \infty) \mapsto e^{-1/x} \in \mathbb{R}$ is bicontinuous at all points in $(0, \infty)$, we can always find some $r > 0$ such that for all $y \in (0, \infty)$ with $\left|e^{-1/x} - e^{-1/y}\right| < r$, we have $|x - y| < r'$, so

$$B_r^d(x) \subset (x - r', x + r') \subset U$$

If $x = 0$, then, again by the definition of $\mathscr{T}$ there exists some $r' > 0$ such that $(-r', r') \subset U$. Thus we need to find some $r > 0$ such that $|x| < r'$ whenever $\left|\text{sign}(x)e^{-1/|x|}\right| < r$. Clearly $|\text{sign}(x)| = 1$ and $e^{-1/|x|} \geq 0$ for all $x \in [-\infty, \infty]$. Thus if $\left|\text{sign}(x)e^{-1/|x|}\right| < r$ then $e^{-1/|x|} < r$, so by similar manipulations as we did earlier we get that $|x| < r'$ whenever $e^{-1/|x|} < e^{-1/r'}$, implying that

$$B_{e^{-1/r'}}^d(0) \subset (-r', r) \subset U$$

If $x \in (-\infty, 0)$ or $x = -\infty$, then we can use methods completely analogous to those above to find $B_r^d(x) \subset U$ for any $U \in \mathscr{T}$ with $x \in U$. This proves that d induces $\mathscr{T}$, and we are done.

□

Note in particular that the work above means that as long as we are working only bounded subsets (in the Euclidean metric) of $(-\infty, \infty)$, then we can use the Euclidean metric and the "new" metric d quite interchangeably, and we will do so several times without explicitly remarking this. This is hardly news - indeed, by how we initially defined the topology on $[-\infty, \infty]$, clearly a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in $[-\infty, \infty]$ converges to a point $x \in (-\infty, \infty)$ if and only if for every $\epsilon > 0$, there exists some n' such that $|x_n - x| < \epsilon$ for all $n \geq n'$. Similarly, $x_n \to \infty$ if and only if for every $N \in \mathbb{N}$ there exists some n' such that $x_n > N$ for all $n \geq n'$, and $x_n \to -\infty$ if and only if for every $N \in \mathbb{N}$ there exists some n' such that $x_n < -N$ for all $n \geq n'$. In conclusion, the metric d on $[-\infty, \infty]$ will not find many practical uses, if any, and we will always use the Euclidean metric whenever we can get away with it.

After all this work, one might expect that we will be actively avoiding ∞-metrics; however, this is not the case. In fact, we will do quite the opposite: For the remainder of this text, we will refer to *both metrics and ∞-metrics* whenever we speak of "metrics", unless we explicitly say otherwise. The main motivation for this is the fact - that we stated without proof earlier - that the Hausdorff distance that we defined in Section 1.3 is an ∞-metric.

Lemma 3.3. *Let (X, d) be any (non-∞) metric space, and let dist : $\mathscr{C}(X) \times \mathscr{C}(X) \to \mathbb{R}^+ \cup \{\infty\}$ be the Hausdorff distance defined by*

$$\textit{dist}(Y, Z) := \max\left\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\right\}$$

if Y, $Z \neq \emptyset$, with dist$(\emptyset, \emptyset) = 0$ and dist$(\emptyset, Y) =$ dist$(Y, \emptyset) = \infty$ for all non-empty $Y \in \mathscr{C}(X)$. Then dist is an ∞-metric on $\mathscr{C}(X)$.

Proof. It is not hard to see that

$$\text{dist}(Y, Y) = \sup_{y\in Y} d(y,y) = 0$$

for every non-empty $Y \in \mathscr{C}(X)$. Conversely, if dist$(Y, Z) = 0$ for any Y, $Z \in \mathscr{C}(X)$, then

$$0 = \max\left\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\right\}$$

implying that $\inf_{z\in Z} d(y,z) = 0$ for every $y \in Y$ and $\inf_{y\in Y} d(y,z) = 0$ for every $z \in Z$. As Y and Z are closed sets, this implies that for every $y \in Y$ we can find some $z \in Z$ such that $d(y,z) = 0$, and vice versa. As d is a metric, this is equivalent to saying that for every $y \in Y$ we can find a $z \in Z$ such that $y = z$, and vice versa; this implies that

$Y \subset Z$ and $Z \subset Y$, so we must have $Y = Z$. This proves that dist satisfies identity of indiscernibles.

As d is a metric, proving symmetry is completely trivial.

To see that dist satisfies subadditivity, start by noting that as $\text{dist}(\emptyset, \emptyset) = 0$ and $\text{dist}(\emptyset, Y) = \infty$ for all $Y \in \mathscr{C}(X) \setminus \{\emptyset\}$, we trivially have

$$\text{dist}(Y, Z) \leq \text{dist}(Y, W) + \text{dist}(W, Z)$$

if any of Y, Z and $W \in \mathscr{C}(X)$ are the empty set. So for the rest of the proof, assume that we work only with non-empty Y, Z and $W \in \mathscr{C}(X)$.

Note that as d is a metric, we have $d(y, z) \leq d(y, w) + d(w, z)$ for all y, z, $w \in X$, and recall that we defined the map $\text{dist} : X \times \mathscr{C}(X) \to \mathbb{R}^+$ by $\text{dist}(y, Z) = \inf_{z \in Z} d(y, z)$. We see that for any Y, Z and $W \in \mathscr{C}(X)$, any $y \in Y$, $z \in Z$ and $w \in W$ we have

$$\begin{aligned}
\text{dist}(y, Z) &= \inf_{z \in Z} d(y, z) \\
&\leq \inf_{z \in Z} \left(d(y, w) + d(w, z)\right) \\
&= d(y, w) + \text{dist}(w, Z) \\
&\leq d(y, w) + \left(\sup_{w \in W} \text{dist}(w, Z)\right) \\
&\leq d(y, w) + \text{dist}(W, Z)
\end{aligned}$$

As this holds for all $w \in W$, we also have

$$\begin{aligned}
\text{dist}(y, Z) &\leq \inf_{w \in W} \left(d(y, w) + \text{dist}(W, Z)\right) \\
&= \text{dist}(y, W) + \text{dist}(W, Z) \\
&\leq \left(\sup_{y \in Y} \text{dist}(y, W)\right) + \text{dist}(W, Z) \\
&\leq \text{dist}(Y, W) + \text{dist}(W, Z)
\end{aligned}$$

Taking supremum over Y on both sides, it follows that

$$\sup_{y \in Y} \text{dist}(y, Z) \leq \text{dist}(Y, W) + \text{dist}(W, Z)$$

By interchanging the roles of Y and Z, we see that we also have

$$\sup_{z \in Z} \text{dist}(z, Y) \leq \text{dist}(Y, W) + \text{dist}(W, Z)$$

Thus,

$$\begin{aligned}
\text{dist}(Y, Z) &= \max\left\{\sup_{y \in Y} \inf_{z \in Z} d(y, z), \sup_{z \in Z} \inf_{y \in Y} d(y, z)\right\} \\
&= \max\left\{\sup_{y \in Y} \text{dist}(y, Z), \sup_{z \in Z} \text{dist}(z, Y)\right\} \\
&\leq \text{dist}(Y, W) + \text{dist}(W, Z)
\end{aligned}$$

□

Even so, we will attempt to be careful, and will continue to refer to dist as *the Hausdorff distance*, rather than *the Hausdorff metric*, even though we shall treat it as an (∞-)metric for all upcoming purposes. This is mostly to avoid confusing readers who are familiar with the Hausdorff metric defined on the space of non-empty compact subsets, which *is* a (traditional) metric, but which we shall not make use of in this paper.

3.2 Hölder continuity

In the case where T is a metric space, we can obtain additional information about the continuity of the spectrum by considering Hölder continuity. More precisely, we find a relation, albeit imperfect, between Hölder continuity of "second-order polynomials of the fields" and Hölder continuity of the spectrums themselves with respect to Hausdorff distance.

Our first order of business should be to define the different types of Hölder continuity.

Definition 3.4. Given a real number $\alpha > 0$ and two metric spaces (X, d_X) and (Y, d_Y), as well as a function $f : X \to Y$, we define *the α-Hölder constant of f* as

$$\mathrm{Hol}^\alpha(f) := \sup_{x \neq x'} \frac{d_Y\left(f(x), f(x')\right)}{d_X\left(x, x'\right)^\alpha}$$

We say that f is α-Hölder continuous if $\mathrm{Hol}^\alpha(f) < \infty$.

Given a family $\mathscr{F}$ of functions $f : X_f \to Y_f$ together with two families $\left\{(X_f, d_{X_f})\right\}_{f \in \mathscr{F}}$ and $\left\{(Y_f, d_{Y_f})\right\}_{f \in \mathscr{F}}$ of metric spaces, we define the *α-Hölder constant of $\mathscr{F}$* as

$$\mathrm{Hol}^\alpha(\mathscr{F}) := \sup_{f \in \mathscr{F}} \mathrm{Hol}^\alpha(f)$$

We say that $\mathscr{F}$ is α-Hölder continuous if $\mathrm{Hol}^\alpha(\mathscr{F}) < \infty$.

Note that, as discussed in the previous section, both (X, d_X) and (Y, d_Y) are allowed to be ∞-metric spaces.

Example 3.5. The function $x \in [0,1] \mapsto x^{1/2} \in \mathbb{R}^+$ is a typical example of a Hölder continuous function; indeed, it is α-Hölder continuous for every α with $0 < \alpha \leq 1/2$.

To see that it is $1/2$-Hölder continuous, begin by noting that for any x, $y \in [0,1]$, we have by the triangle inequality that

$$\left|x^{1/2} - y^{1/2}\right| \leq \left|x^{1/2}\right| + \left|y^{1/2}\right| = \left|x^{1/2} + y^{1/2}\right|$$

By multiplying each side with $\left|x^{1/2} - y^{1/2}\right|$, it follows that

$$\left|x^{1/2} - y^{1/2}\right|^2 \leq \left|x^{1/2} - y^{1/2}\right| \left|x^{1/2} + y^{1/2}\right| = |x - y|$$

so $\left|x^{1/2} - y^{1/2}\right| \leq |x - y|^{1/2}$ for all x, $y \in [0,1]$ and we are done. To see that this also holds for all α with $0 < \alpha \leq 1/2$, just note that as $|x - y| \leq 1$, we have $|x - y|^{1/2} \leq |x - y|^\alpha$, and so

$$\left|x^{1/2} - y^{1/2}\right| \leq |x - y|^{1/2} \leq |x - y|^\alpha$$

This implies that $\text{Hol}^\alpha(x^{1/2}) \leq 1$ for all α with $0 < \alpha \leq 1/2$, and since

$$\frac{\left|1^{1/2} - 0^{1/2}\right|}{|1-0|^\alpha} = 1$$

it follows that $\text{Hol}^\alpha(x^{1/2}) = 1$ for every such α, that is, the α-Hölder constant of $x^{1/2}$ on the interval $[0,1]$ is 1 for every α with $0 < \alpha \leq 1/2$.

Note that if $\alpha = 1$, then α-Hölder continuity is just the same as the more widely known Lipschitz continuity.

Definition 3.6. Given any $M > 0$, we define $\mathscr{P}_2(M)$ as the set of polynomials $p : x \in \mathbb{R} \mapsto p_0 + p_1 x + p_2 x^2 \in \mathbb{R}$ with real coefficients such that

$$\|p\|_1 := |p_0| + |p_1| + |p_2| \leq M$$

Given any field $A = (A_t)_{t\in T}$ of bounded operators where (T,d) is a metric space, we define for every polynomial p the function $\Phi_p : T \to \mathbb{R}^+$ given by

$$\Phi_p(t) = \|p(A_t)\|$$

Considering $\mathbb{R}$ with its standard metric, and for any α, $M > 0$, we further define

$$C_{\alpha,M} := \text{Hol}^\alpha\left(\{\Phi_p \mid p \in \mathscr{P}_2(M)\}\right)$$

Keep in mind that the $C_{\alpha,M}$ is also dependent on the underlying field A. We will typically only be working with one field at a time, so generally there should be no confusion as to "which" $C_{\alpha,M}$ we are referring to.

Definition 3.7. Let (T,d) be a metric space, and take any $\alpha > 0$. A field $A = (A_t)_{t\in T}$ of bounded operators is said to be *$p2$-α-Hölder continuous* if, for all $M > 0$, the family $\{\Phi_p\}_{p\in\mathscr{P}_2(M)}$ of maps $\Phi_p : t \in T \mapsto \|p(A_t)\| \in \mathbb{R}^+$ is uniformly α-Hölder, that is, if for all $M > 0$ we have

$$C_{\alpha,M} < \infty$$

Proposition 3.8. *Let (T,d) be a metric space, and let $A = (A_t)_{t\in T}$ be a field of bounded operators. For any α, M, $N > 0$ we have that $C_{\alpha,M} < \infty$ if and only if $C_{\alpha,N} < \infty$, and in this case $C_{\alpha,N} = \frac{N}{M} C_{\alpha,M}$.*

Proof. Assume that $M \neq N$ (otherwise we are done), and assume that we are given $C_{\alpha,M} < \infty$. Note first that if $p(x) = p_2 x^2 + p_1 x + p_0 \in \mathscr{P}_2(N)$ and we define $q(x) := \frac{M}{N} p(x)$, then

$$\|q\|_1 = \frac{M}{N}\|p\|_1 \leq M$$

and so $q \in \mathscr{P}_2(M)$. Since the map $x \in \mathbb{R}^+ \mapsto \frac{M}{N}x \in \mathbb{R}^+$ is a bijection for any $M, N > 0$, it follows that

$$\mathscr{P}_2(M) = \frac{M}{N}\mathscr{P}_2(N)$$

Thus we see that

$$\begin{aligned}
C_{\alpha,N} &= \sup_{p\in\mathscr{P}_2(N)} \sup_{t\neq s} \frac{\big|\|p(A_t)\| - \|p(A_s)\|\big|}{d(t,s)^\alpha} \\
&= \frac{N}{M} \sup_{p\in\mathscr{P}_2(N)} \sup_{t\neq s} \frac{\big|\|\frac{M}{N}p(A_t)\| - \|\frac{M}{N}p(A_s)\|\big|}{d(t,s)^\alpha} \\
&= \frac{N}{M} \sup_{q\in\mathscr{P}_2(M)} \sup_{t\neq s} \frac{\big|\|q(A_t)\| - \|q(A_s)\|\big|}{d(t,s)^\alpha} \\
&= \frac{N}{M} C_{\alpha,M} < \infty
\end{aligned}$$

The reverse implication follows immediately from switching the roles of N and M in the argument above, so we are done.

□

As a consequence of Proposition 3.8, proving that a given field A of bounded operators over a metric space (T, d) is $p2$-α-Hölder continuous is just the same as proving that

$$C_{\alpha,1} := \sup_{p\in\mathscr{P}_2(1)} \sup_{t\neq s} \frac{\big|\|p(A_t)\| - \|p(A_s)\|\big|}{d(t,s)^\alpha} < \infty$$

and, indeed, since we just proved that $C_{\alpha,M} = MC_{\alpha,1}$, we will generally just write $C_\alpha := C_{\alpha,1}$, which reduces the need for computations significantly.

Another convenient consequence of $p2$-α-Hölder continuity is that it implies local uniform boundedness at every $t_0 \in T$.

Proposition 3.9. *Let (T, d) be a metric space, take any $\alpha > 0$, and let $A = (A_t)_{t\in T}$ be a field of bounded operators. If A is $p2$-α-Hölder continuous, then A is locally uniformly bounded at every $t_0 \in T$.*

Proof. Since A is $p2$-α-Hölder continuous, we have that

$$\sup_{t\neq s} \frac{\big|\|p(A_t)\| - \|p(A_s)\|\big|}{d(t,s)^\alpha} \leq MC_\alpha$$

for any $p \in \mathscr{P}_2(M)$. In particular, with $p(x) = x$ we have

$$\sup_{t\neq s} \frac{\big|\|A_t\| - \|A_s\|\big|}{d(t,s)^\alpha} \leq C_\alpha$$

Fix $t_0 \in T$. If we set $U := B_1(t_0) \subset T$, then it follows that

$$\begin{aligned}\sup_{t \in U \setminus \{t_0\}} \Big| \|A_t\| - \|A_{t_0}\| \Big| &\leq \sup_{t \in U \setminus \{t_0\}} \frac{\Big| \|A_t\| - \|A_{t_0}\| \Big|}{d(t,t_0)^\alpha} \\ &\leq \sup_{t \neq t_0} \frac{\Big| \|A_t\| - \|A_{t_0}\| \Big|}{d(t,t_0)^\alpha} \\ &\leq C_\alpha < \infty\end{aligned}$$

and, as $\Big| \|A_{t_0}\| - \|A_{t_0}\| \Big| = 0$, it follows that

$$\sup_{t \in U} \Big| \|A_t\| - \|A_{t_0}\| \Big| < \infty$$

and hence $\sup_{t \in U} \|A_t\| \leq \sup_{t \in U} \Big| \|A_t\| - \|A_{t_0}\| \Big| + \|A_{t_0}\| < \infty$.

As t_0 was arbitrary, this completes the proof.

□

Note that this does *not* mean that $p2$-α-Hölder continuity of A implies that $\sup_{t \in T} \|A_t\| < \infty$ - it *is* true if T is a bounded metric space, but not necessarily otherwise.

Definition 3.10. Let (T, d) be a metric space, take any $\alpha > 0$, and let $A = (A_t)_{t \in T}$ be a field of self-adjoint, bounded operators. We say that *the spectrum of A is α-Hölder continuous* if the function $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is α-Hölder continuous with respect to Hausdorff distance, that is, if the α-Hölder constant

$$\sup_{t \neq s} \frac{\text{dist}(\sigma(A_t), \sigma(A_s))}{d(t,s)^\alpha}$$

is finite.

We are now ready to formulate the main result of the section:

Theorem 3.11. *Let (T, d) be a metric space, take any $\alpha > 0$, and let $A = (A_t)_{t \in T}$ be a field of self-adjoint, bounded operators such that $\sup_{t \in T} \|A_t\| < \infty$. Then the following both hold:*

1. *If A is $p2$-α-Hölder continuous, then the spectrum of A is $\alpha/2$-Hölder continuous.*
2. *If the spectrum of A is α-Hölder continuous, then A is $p2$-α-Hölder continuous.*

We will need a few technical results before proving this.

Proposition 3.12. *Let (X, d_X) and (Y, d_Y) be metric spaces, take any $\alpha > 0$, and take any α-Hölder continuous proper function $f : X \to Y$. Then, the map*

$$\hat{f} : F \in \mathscr{C}(X) \mapsto f(F) \in \mathscr{C}(Y)$$

is α-Hölder continuous, that is, there exists a constant $C > 0$ such that for every pair K, $L \in \mathscr{C}(X)$, we have

$$\mathit{dist}(f(K), f(L)) \leq C \mathit{dist}(K, L)^\alpha$$

Proof. As in Proposition 2.15, $\hat{f}$ is well-defined since we demand that f be continuous and proper. We can set $C := \mathrm{Hol}^{\alpha}(f)$ and use the α-Hölder continuity of f to conclude that

$$\begin{aligned}\text{dist}(f(x), f(L)) &= \inf\{d_Y(f(x), f(l)) \mid l \in L\} \\ &\leq \inf\{C\, d_Y(x,l)^{\alpha} \mid l \in L\} \\ &= C\,\text{dist}(x, L)^{\alpha}\end{aligned}$$

Accordingly,

$$\begin{aligned}\text{dist}(f(K), f(L)) &= \max\left\{\sup_{k\in K}\text{dist}(f(k), f(L)), \sup_{l\in L}\text{dist}(f(K), f(l))\right\} \\ &\leq \max\left\{\sup_{k\in K} C\,\text{dist}(k, L)^{\alpha}, \sup_{l\in L} C\,\text{dist}(K, l)^{\alpha}\right\} \\ &= C\,\text{dist}(K, L)^{\alpha}\end{aligned}$$

which completes the proof.

□

Lemma 3.13. *Let B be a bounded, self-adjoint operator, choose any $m \geq \|B\|$, and define $p := 4m^2 - (z-c)^2$ for any $c \in \mathbb{R}$ with $|c| \leq m$. Then $p \in \mathscr{P}_2(4m^2+2)$, and*

$$\|p(B)\| = 4m^2 - \inf_{t\in\sigma(B)} |t-c|^2$$

which can be alternatively written as

$$\left\|4m^2 - (B-c)^2\right\| = 4m^2 - \mathit{dist}(c, \sigma(B))^2$$

Proof. For the first part, we know that $m \geq |c|$ so $|4m^2 - c^2| = 4m^2 - c^2$. It follows that

$$\begin{aligned}\|p\|_1 &= \left\|4m^2 - c^2 + 2cz - z^2\right\|_1 \\ &= \left|4m^2 - c^2\right| + 2c + 1 \\ &= 4m^2 - c^2 + 2c + 1 \\ &= 4m^2 + 2 - c^2 + 2c - 1 \\ &= 4m^2 + 2 - (c-1)^2 \\ &\leq 4m^2 + 2\end{aligned}$$

For the second part, we first note that

$$\begin{aligned}\left\|(B-c)^2\right\| &\leq \|B-c\|^2 \\ &\leq (\|B\| + |c|)^2 \\ &\leq 4m^2\end{aligned}$$

It follows that $4m^2 \geq (t-c)^2$ for all $t \in \sigma(B)$, so we have

$$\begin{aligned}\|p(B)\| &= \left\|4m^2 - (B-c)^2\right\| \\ &= \sup_{t\in\sigma(B)} \left|4m^2 - (t-c)^2\right| \\ &= \sup_{t\in\sigma(B)} \left(4m^2 - (t-c)^2\right) \\ &= 4m^2 - \inf_{t\in\sigma(B)} (t-c)^2\end{aligned}$$

Finally, from Definition 1.15 we see that

$$\begin{aligned}\inf_{t\in\sigma(B)} (t-c)^2 &= \inf_{t\in\sigma(B)} |t-c|^2 \\ &= \text{dist}(c, \sigma(B))^2\end{aligned}$$

so the proof is finished. □

Lemma 3.14. *Let $A = (A_t)_{t\in T}$ be a p2-α-Hölder continuous field of self-adjoint, bounded operators such that $m := \sup_{t\in T} \|A_t\| < \infty$. Then the spectrum of A is $\alpha/2$-Hölder continuous with $\alpha/2$-Hölder constant less than or equal to $\sqrt{(4m^2+2)C_\alpha}$.*

Proof. Consider $s, t \in T$. Choose any $\lambda \in \sigma(A_t)$; we will show that

$$\text{dist}(\lambda, \sigma(A_s)) \leq \sqrt{(4m^2+2)C_\alpha}\, d(s,t)^{\alpha/2}$$

If $\lambda \in \sigma(A_t) \cap \sigma(A_s)$, then clearly $\text{dist}(\lambda, \sigma(A_s)) = 0$ and we are done. So assume that $\lambda \in \sigma(A_t)\backslash\sigma(A_s)$. Since $\lambda \in \sigma(A_t)$, it follows by Lemma 3.13 that

$$\left\|4m^2 - (A_t - \lambda)^2\right\| = 4m^2$$

Similarly we have

$$\left\|4m^2 - (A_s - \lambda)^2\right\| = 4m^2 - \text{dist}\left(\lambda, \sigma(A_s)\right)^2$$

If we define $p(z) := 4m^2 - (z-\lambda)^2$, then, again by Lemma 3.13, we have that $p \in \mathscr{P}_2(4m^2+2)$. It follows that

$$\begin{aligned}\text{dist}\left(\lambda, \sigma(A_s)\right)^2 &= 4m^2 - \left(4m^2 - \text{dist}\left(\lambda, \sigma(A_s)\right)^2\right) \\ &= \left|\|p(A_t)\| - \|p(A_s)\|\right| \\ &\leq C_{\alpha,4m^2+2}\, d(t,s)^\alpha \\ &= (4m^2+2)C_\alpha\, d(t,s)^\alpha\end{aligned}$$

Since this is true for every $\lambda \in \sigma(A_t)$, and since we can interchange the roles of s and t in the discussion above, it follows that

$$\text{dist}\left(\sigma(A_s), \sigma(A_t)\right) \leq \sqrt{(4m^2+2)C_\alpha}\, d(s,t)^{\alpha/2}$$

which completes the proof.

□

Proposition 3.15. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces, and assume that we are given functions $f : X \to Y$ and $g : Y \to Z$ such that f is α-Hölder continuous with α-Hölder constant C and g is β-Hölder continuous with β-Hölder constant D for some α, β, C, $D > 0$. Then the composition $g \circ f : X \to Z$ is $\alpha\beta$-Hölder continuous with $\alpha\beta$-Hölder constant less than or equal to $C^\beta D$.*

Proof. Since f is α-Hölder continuous , we have

$$d_Y\left(f(x), f(x')\right) \leq C\, d_X(x, x')^\alpha$$

for any x, $x' \in X$. Similarly

$$d_Z\left(g(y), g(y')\right) \leq D\, d_Y(y, y')^\beta$$

for any y, $y' \in Y$. It follows that

$$\begin{aligned} d_Z\Big(g(f(x)), g(f(x'))\Big) &\leq D\, d_Y\Big(f(x), f(x')\Big)^\beta \\ &\leq DC^\beta \Big(d_X(x, x')^\alpha\Big)^\beta \end{aligned}$$

for any x, $x' \in X$, and we are done. □

Lemma 3.16. *Let $A = (A_t)_{t\in T}$ be a field of self-adjoint, bounded operators such that $m := \sup_{t\in T} \|A_t\| < \infty$. If the spectrum $\sigma(A_t)$ is α-Hölder continuous with Hölder constant C, then A is $p2$-α-Hölder continuous with $C_\alpha \leq 2nC$, where $n := \max\{2m, 1\}$.*

Proof. As in Lemma 2.20, the map $t \in T \mapsto \|p(A_t)\| \in \mathbb{R}^+$ can be seen as a composition of the maps

$$t \mapsto \sigma(A_t) \overset{\hat{p}}{\mapsto} \sigma(p(A_t)) \overset{\hat{|\cdot|}}{\mapsto} |\sigma(p(A_t))| \overset{\sup}{\mapsto} \|p(A_t)\|$$

The map $t \mapsto \sigma(A_t)$ is α-Hölder continuous with Hölder constant C by assumption. Next, we have that if $p = p_2x^2 + p_1x + p_0$ is any polynomial of degree two or less with real coefficients, then its restriction to the compact set $[-m, m]$ is 1-Hölder continuous with Hölder constant less than or equal to $2m\,|p_2| + |p_1|$, as

$$\begin{aligned} \Big|p_2x^2 + p_1x + p_0 - p_2x'^2 - p_1x' - p_0\Big| &\leq \Big|p_2(x^2 - x'^2) + p_1(x - x')\Big| \\ &\leq (|p_2|\,|x + x'| + |p_1|)\,|x - x'| \\ &\leq (2m\,|p_2| + |p_1|)\,|x - x'| \end{aligned}$$

for all x, $x' \in [-m, m]$. Furthermore, there exists some $\tilde{m} > 0$ such that $p([-m, m]) \subset [-\tilde{m}, \tilde{m}]$ for any $p \in \mathscr{P}_2(1)$. For set $\tilde{m} := \max\{1, m, m^2\}$, and take any $p \in \mathscr{P}_2(1)$. We see that for any $x \in [-m, m]$ we have

$$\begin{aligned} \Big|p_2x^2 + p_1x + p_0\Big| &\leq |p_2|\,x^2 + |p_1|\,|x| + |p_0| \\ &\leq (|p_2| + |p_1| + |p_0|)\,\tilde{m} \\ &\leq \tilde{m} \end{aligned}$$

and so $p([-m,m]) \subset [-\tilde{m},\tilde{m}]$ for all $p \in \mathscr{P}_2(1)$.

Trivially, the map $|\cdot| : [-\tilde{m},\tilde{m}] \to [0,\tilde{m}]$ is also 1-Hölder continuous with Hölder constant 1.

Finally, the map $\sup : \mathscr{C}([0,\tilde{m}]) \to \mathbb{R}$ is also 1-Hölder continuous with Hölder constant less than or equal to 1. For take any F, $F' \in \mathscr{C}([0,\tilde{m}])$, and note that $\sup F \in F$ and $\sup F' \in F'$ since F and F' are compact sets. Assume that $\sup F' \leq \sup F$. Now we must have

$$\inf_{x' \in F'} |\sup F - x'| = |\sup F - \sup F'|$$

and so it follows that

$$|\sup F - \sup F'| \leq \sup_{x \in F} \inf_{x' \in F'} |x - x'|$$

If conversely we assume that $\sup F' \leq \sup F$, then we see by the same argument as above that

$$|\sup F - \sup F'| \leq \sup_{x' \in F'} \inf_{x \in F} |x - x'|$$

But then in either case we must have

$$\begin{aligned} |\sup F - \sup F'| &\leq \max\left\{ \sup_{x \in F} \inf_{x' \in F'} |x - x'|\,, \sup_{x' \in F'} \inf_{x \in F} |x - x'| \right\} \\ &= \operatorname{dist}(F, F') \end{aligned}$$

As F, $F' \in \mathscr{C}([0,\tilde{m}])$ were arbitrary, it follows that the map

$$\sup : \mathscr{C}([0,\tilde{m}]) \to \mathbb{R}$$

is 1-Hölder with Hölder constant less than or equal to 1.

Now by combining Proposition 3.12 and Proposition 3.15, it follows that for each $p \in \mathscr{P}_2(1)$, the map $\sigma(A_t) \mapsto \|p(A_t)\|$, that is, the composition

$$\sup \circ \widehat{|\cdot|} \circ \hat{p}$$

is α-Hölder continuous with Hölder constant less than or equal to $2m\,|p_2| + |p_1|$. Thus again by Proposition 3.15, we see that for each $p \in \mathscr{P}_2(1)$ we have

$$\begin{aligned} \Big| \|p(A_s)\| - \|p(A_t)\| \Big| &\leq (2m\,|p_2| + |p_1|) C\, d(s,t)^\alpha \\ &\leq (|p_2| + |p_1|)\, 2nC\, d(s,t)^\alpha \\ &\leq 2nC\, d(s,t)^\alpha \end{aligned}$$

As $n < \infty$, it follows that

$$\sup_{t \neq s} \sup_{p \in \mathscr{P}_2(1)} \frac{\Big| \|p(A_s)\| - \|p(A_t)\| \Big|}{d(s,t)^\alpha} \leq 2nC < \infty$$

By the comments following Proposition 3.8, it follows that A is $p2$-α-Hölder continuous, and we are done. □

3.3 Hölder continuity of gap edges

In a sense, the result in the previous section can be seen as an analogy, if imperfect, to the proof that $p2$-continuity and Hausdorff continuity are equivalent, in a Hölder continuity setting. It turns out that we are also able to provide a similar analogy to the equivalence of $p2$-continuity and gap edge continuity.

There are a few problems involved in trying to get Hölder estimates for the position of gap edges. For one, "continuous families of gaps" only exist locally, and their exact area of definition is often unclear at best. The more serious issue is that even if a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at a point t_0, then the nature of the gaps may be very chaotic except quite close to t_0, which can make the Hölder constant blow up disproportionally or even become ∞.

We can, however, make use of a "local" generalization of the concept of Hölder continuity - the property known as the *dilation*:

Definition 3.17. Given two metric spaces (X, d_x), (Y, d_Y), a point $x \in X$, a function $f : X \to Y$ and some $r > 0$, we define for each $\alpha > 0$

$$\mathrm{Hol}_r^\alpha(f) := \sup_{\{x' \mid 0 < d_X(x,x') < r\}} \frac{d_Y(f(x), f(x'))}{d_X(x, x')^\alpha}$$

We say that f *is locally α-Hölder continuous at x* if $\mathrm{Hol}_r^\alpha(f) < \infty$ for some $r > 0$, in which case we set

$$\mathrm{dil}^\alpha(f)(x) := \lim_{r \downarrow 0} \mathrm{Hol}_r^\alpha(f)$$

As clearly $0 \leq \mathrm{Hol}_r^\alpha(f) \leq \mathrm{Hol}_{r'}^\alpha(f)$ for any $0 < r \leq r'$, the limit is well defined and finite, and we call $\mathrm{dil}^\alpha(f)(x)$ the *α-dilation of f at x.*

Example 3.18. Set $X := \bigcup_{n \in \mathbb{Z}} (n, n+1) \subset \mathbb{R}$ with the standard metric it inherits from $\mathbb{R}$. We claim that the function $f : X \to \mathbb{R}$ given by

$$f(x) := n^2 \text{ if } n < x < n+1 \text{ for } n \in \mathbb{Z}$$

is not 1-Hölder continuous, but is locally α-Hölder continuous with α-dilation equal to zero for all $\alpha \geq 0$ at every $x \in X$.

For note that for any $x, x' \in X$, we have that if

$$n < x < n+1 \text{ and } n' < x' < n'+1$$

for some $n, n' \in \mathbb{Z}$ then $|x - x'| \leq |n - n'| + 1$. It follows that

$$\begin{aligned} \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|} &\geq \sup_{n \neq n'} \frac{|n^2 - n'^2|}{|n - n'| + 1} \\ &\geq \sup_{n \neq 0} \frac{|n^2|}{|n| + 1} = \infty \end{aligned}$$

To show that the α-dilation is 0 at every $x \in X$, start by fixing x. As $|x - x'| \to 0$ we must eventually have that $x, x' \in (n_x, n_x + 1)$ for some $n_x \in \mathbb{Z}$, so

$$\begin{aligned}\lim_{r\downarrow 0} \sup_{0<|x-x'|<r} \frac{|f(x) - f(x')|}{|x - x'|^{\alpha}} &= \lim_{r\downarrow 0} \sup_{0<|x-x'|<r} \frac{|n_x^2 - n_x^2|}{|x - x'|^{\alpha}} \\ &= \lim_{r\downarrow 0} \sup_{0<|x-x'|<r} \frac{0}{|x - x'|^{\alpha}} = 0\end{aligned}$$

As the example shows, a function $f : X \to Y$ can be locally α-Hölder continuous at every $x \in X$ without being α-Hölder continuous. On the other hand, it is not difficult to see from the definitions that if f *is* α-Hölder continuous, then it is also locally α-Hölder continuous at every $x \in X$.

With the notion of local α-Hölder continuity developed, we want to be able to speak of α-Hölder continuity of gaps. Before that, we will make a small remark: It is well known that whenever we work in a topological space that is also a metric space, we can always consider *sequences* instead of nets, and we will always do so. As such, any definition or result that considers metric spaces will always only concern itself with *sequences*, while any definition or result that considers topological spaces will concern itself with *nets*.

Definition 3.19. For a metric space (T, d), let $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ be a map, and choose any $t_0 \in T$. We say that the map $t \mapsto F_t$ is *α-Hölder gap edge continuous at t_0* if the following both hold:

- For every gap (a, b) of F_{t_0}, there exists an $R > 0$ and two maps

  $$t \in B_R(t_0) \mapsto a_t \in F_t \cup \{\pm\infty\} \text{ and } t \in B_R(t_0) \mapsto b_t \in F_t \cup \{\pm\infty\}$$

  such that for each $t \in B_R(t_0)$ we have that (a_t, b_t) is a gap of F_t with $-\infty \leq a_t < b_t \leq \infty$, and such that both these maps are locally α-Hölder continuous at t_0 - that is, such that

  $$\mathrm{dil}^{\alpha}(t \mapsto a_t)(t_0) := \lim_{r\downarrow 0} \sup_{\{t|0<d(t_0,t)<r\leq R\}} \frac{|a - a_t|}{d(t_0, t)^{\alpha}} < \infty$$

  and

  $$\mathrm{dil}^{\alpha}(t \mapsto b_t)(t_0) := \lim_{r\downarrow 0} \sup_{\{t|0<d(t_0,t)<r\leq R\}} \frac{|b - b_t|}{d(t_0, t)^{\alpha}} < \infty$$

- Assume $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of points in T converging to t_0, and $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are two convergent nets of points in $[-\infty, \infty]$ such that

  $$a := \lim_n a_n \text{ and } b := \lim_n b_n$$

and (a_n, b_n) is a gap of F_{t_n} for every $n \in \mathbb{N}$. Then either (a, b) is a gap of F_{t_0} or $a = b$, and in either case

$$\lim_{r\downarrow 0} \sup_{\{n|0<d(t_0,t_n)<r\}} \frac{|a - a_n|}{d(t_0, t_n)^\alpha} < \infty$$

and

$$\lim_{r\downarrow 0} \sup_{\{n|0<d(t_0,t_n)<r\}} \frac{|b - b_n|}{d(t_0, t_n)^\alpha} < \infty$$

On occasion, we shall say that only some given family of gaps of $t \mapsto F_t$ are α-Hölder gap edge continuous at t_0; this just means that only these specific gaps satisfy the equations above.

If $t \mapsto F_t$ is α-Hölder gap edge continuous at t_0, and given some family $\{(a_\kappa, b_\kappa)\}_{\kappa\in J}$ of gaps of F_{t_0} - for example, the collection of all inner gaps of F_{t_0} - we call the quantity

$$\sup_{\kappa\in J} \max \{\mathrm{dil}^\alpha(t \mapsto (a_\kappa)_t), \mathrm{dil}^\alpha(t \mapsto (b_\kappa)_t)\}$$

the *α-dilation of the family*, where the maps are as they were defined above.

Given a field $A = (A_t)_{t\in T}$ of self-adjoint, bounded operators, we say that *the gaps of A are α-Hölder continuous at t_0* if the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is α-Hölder gap edge continuous at t_0.

Theorem 3.20. *For a metric space (T, d), let $A = (A_t)_{t\in T}$ be a $p2$-α-Hölder continuous field of bounded, self-adjoint operators. Then A is α-Hölder gap edge continuous at every $t \in T$.*

We will break down the proof into two lemmas: One where we show gap edge continuity for outer gaps, and one where we show gap edge continuity for inner gaps.

Lemma 3.21. *For a metric space (T, d), let $A = (A_t)_{t\in T}$ be a $p2$-α-Hölder continuous field of self-adjoint, bounded operators. Then the outer gap edges of A are α-Hölder gap edge continuous at every $t \in T$, and for each $t \in T$ there exists an m_t with $0 < m_t < \infty$ such that the α-dilation of the outer gaps of A at t is less than or equal to $(1 + m_t)C_\alpha$.*

Proof. Begin by fixing some $t_0 \in T$. We will begin by showing that the first condition holds.

For each $t \in T$, let $l_t = \inf(\sigma(A_t))$ be the lower outer edge of $\sigma(A_t)$ and $u_t = \sup(\sigma(A_t))$ be the upper outer edge of $\sigma(A_t)$. By Proposition 3.9, there exists some neighbourhood $W \subset T$ of t_0 and some $m_t = m > 0$ such that $\|A_t\| \leq m$ for all $t \in W$.

We have $\|A_t\| = \sup |\sigma(A_t)| = \max \{|l_t| , |u_t|\}$. As $0 \leq l_t + m \leq u_t + m$ for all $t \in W$, it follows that

$$\begin{aligned}
\|A_t + m\| &= \sup |\sigma(A_t + m)| \\
&= \sup |\sigma(A_t) + m| \\
&= \max \{|l_t + m| , |u_t + m|\} \\
&= u_t + m
\end{aligned}$$

for all $t \in W$.

Similarly, we have $l_t - m \leq u_t - m \leq 0$ for all $t \in W$, so

$$\begin{aligned}\|A_t + m\| &= \sup |\sigma(A_t - m)| \\ &= \sup |\sigma(A_t) - m| \\ &= \max \{|l_t - m|, |u_t - m|\} \\ &= m - l_t\end{aligned}$$

for all $t \in W$.

Now by the $p2$-α-Hölder continuity of A, we know that for all $M > 0$ there exists a real number $C_{\alpha,M} > 0$ such that for any $p \in \mathscr{P}_2(M)$,

$$\begin{aligned}\Big| \|p(A_s)\| - \|p(A_t)\| \Big| &\leq C_{\alpha,M}\, d(s,t)^\alpha \\ &= MC_\alpha\, d(s,t)^\alpha\end{aligned}$$

With $p_1 := z - m \in \mathscr{P}_2(1+m)$ and $p_2 := z + m \in \mathscr{P}_2(1+m)$, we see that for all $t \in W$ we have

$$\begin{aligned}\|l_{t_0} - l_t\| &= \|(m - l_{t_0}) - (m - l_t)\| \\ &= \Big| \|A_{t_0} - m\| - \|A_t - m\| \Big| \\ &= \Big| \|p_1(A_{t_0})\| - \|p_1(A_t)\| \Big| \\ &\leq (1+m)C_\alpha\, d(t_0,t)^\alpha\end{aligned}$$

and similarly,

$$\begin{aligned}|u_{t_0} - u_t| &= |(u_{t_0} + m) - (u_t + m)| \\ &= \Big| \|A_{t_0} + m\| - \|A_t + m\| \Big| \\ &= \Big| \|p_2(A_{t_0})\| - \|p_2(A_t)\| \Big| \\ &\leq (1+m)C_\alpha\, d(t_0,t)^\alpha\end{aligned}$$

It follows that we have

$$\begin{aligned}\mathrm{dil}^\alpha(t \in W \mapsto l_t \in \mathbb{R})(t_0) &= \lim_{r \downarrow 0} \sup_{\{t \in W | 0 < d(t_0,t) < r\}} \frac{|l_{t_0} - l_t|}{d(t_0,t)^\alpha} \\ &\leq \sup_{t \in W \setminus \{t_0\}} \frac{|l_{t_0} - l_t|}{d(t_0,t)^\alpha} \\ &\leq (1+m)C_\alpha < \infty\end{aligned}$$

and

$$\begin{aligned}\mathrm{dil}^\alpha(t \in W \mapsto u_t \in \mathbb{R})(t_0) &= \lim_{r \downarrow 0} \sup_{\{t | 0 < d(t_0,t) < r\}} \frac{|u_{t_0} - u_t|}{d(t_0,t)^\alpha} \\ &\leq \sup_{t \in W \setminus \{t_0\}} \frac{|u_{t_0} - u_t|}{d(t_0,t)^\alpha} \\ &\leq (1+m)C_\alpha < \infty\end{aligned}$$

As t_0 was arbitrary, the proof follows.

□

Corollary 3.22. *For a metric space (T, d), let $A = (A_t)_{t\in T}$ be a $p2$-α-Hölder continuous field of self-adjoint, bounded operators, and assume that $m := \sup_{t\in T} \|A_t\| < \infty$. Then the outer gap edges of A are α-Hölder gap edge continuous with α-dilation less than or equal to $C_{\alpha,1+m}$ at every $t \in T$.*

Lemma 3.23. *Let $A = (A_t)_{t\in T}$ be a $p2$-α-Hölder continuous field of self-adjoint, bounded operators. Then the inner gaps of A are α-Hölder continuous at every $t \in T$, and for each $t \in T$ there exists a $m > 0$ such that each gap (a, b) of A_t has α-dilation less than or equal to $(8m^2 + 4)C_\alpha/(b - a)$.*

Proof. Fix $t_0 \in T$ and some inner gap (a, b) of A_{t_0}, and divide (a, b) into six equal intervals

$$(a + (n - 1)r, a + nr)$$

for $n \in \{1, 2, 3, 4, 5, 6\}$ and $r := (b - a)/6$. Set also $c := a + 4r = b - 2r$, i.e. the point 2/3rds of the way between a and b. Clearly, the $p2$-α-Hölder continuity of A implies that A is $p2$-continuous at t_0, so, by Theorem 2.13, A is gap edge continuous at t_0. So there exists some open neighbourhood $U \subset T$ of t_0 such that for each $t \in U$, $\sigma(A_t)$ has a gap (a_t, b_t) with $|a_t - a| < r$ and $|b_t - b| < r$. It follows that for each $t \in U$, we have

$$b_t - a_t = (b_t - b) + (b - a) + (a - a_t) > -r + 6r + -r > 4r > 0$$

and

$$b_t - c = (b_t - b) + (b - c) > -r + 2r = r$$

as well as

$$b_t - c = (b_t - b) + (b - c) < r + 2r = 3r$$

which yields $r < b_t - c < 3r$. Finally,

$$c - a_t = (c - a) + (a - a_t) > 4r - r = 3r$$

It follows that $0 < b_t - c < 3r < c - a_t$. In particular, this implies that $a_t < c < b_t$, so c lies in the gap (a_t, b_t) of F_t, and as $|b_t - c| < |a_t - c|$ we have

$$\text{dist}(\sigma(A_t), c) = b_t - c$$

By Proposition 3.9, we see that the $p2$-α-Hölder continuity of A implies there exists some neighbourhood $V \subset T$ of t_0 and some $m > 0$ such that

$$\sup_{t\in V} \|A_t\| \le m < \infty$$

It follows that for all $t \in U \cap V$, $|c| \leq \|A_t\| \leq m$. Lemma 3.13 gives us

$$4m^2 - (b_t - c)^2 = 4m^2 - \text{dist}(c, \sigma(A_t))$$

for all $t \in U \cap V$. Set now $p(z) := m^2 - (z-c)^2$. Again by Lemma 3.13,

$$\begin{aligned}\Big| \|p(A_t)\| - \|p(A_{t_0})\| \Big| &= \Big| 4m^2 - (b_t - c)^2 - 4m^2 + (b-c)^2 \Big| \\ &= |b_t - b|\, |b_t + b - 2c| \\ &= |b_t - b|\, |b_t - c + 2r|\end{aligned}$$

as $b - c = 2r$ by definition.

By our earlier inequalities, we have, for all $t \in U \cap V$, that $b_t - c > r > 0$, and clearly $2r > 0$. It follows that

$$|b_t - c + 2r| = b_t - c + 2r > 3r$$

Since $\|p\|_1 \leq 4m^2 + 2$ (cf. Lemma 3.13), we have by the $p2$-α-Hölder continuity of A that

$$\begin{aligned}\Big| \|p(A_t)\| - \|p(A_s)\| \Big| &\leq C_{\alpha,4m^2+2}\, d(t,s)^\alpha \\ &= (4m^2+2) C_\alpha\, d(t,s)^\alpha\end{aligned}$$

Thus for $t \in U \cap V$, we get

$$\begin{aligned}|b_t - b| &= \frac{\Big| \|p(A_t)\| - \|p(A_{t_0})\| \Big|}{|b_t - c + 2r|} \\ &\leq \frac{(4m^2+2)C_\alpha}{3r} d(t,t_0)^\alpha \\ &= \frac{2(4m^2+2)C_\alpha}{6r} d(t,t_0)^\alpha \\ &= \frac{(8m^2+4)C_\alpha}{b-a} d(t,t_0)^\alpha\end{aligned}$$

The same statement holds with a_t and a_s instead of b_t and b_s by instead choosing $c := a + 2r = b - 4r$, i.e. closer to a than to b. As t_0 was arbitrary, this completes the proof.

□

A note: The original result in [3] showed that the Hölder constant was equal to or less than $\frac{(12m^2+6)C_{\alpha,}}{b-a}$; however, we were able to improve upon this result.

Proof of Theorem 3.20:

Proof. This is just a combination of the two previous lemmas.

□

3.4 Hölder continuity of gap widths

The final result inspired by [3] concerns itself with *gap widths* and the behaviour around so-called *closed gaps*, and linking the α-Hölder continuity of the gap widths to the $p2$-α-Hölder continuity of the field itself. Since we are working with bounded, self-adjoint operators, the gap widths are easily defined - the width of a bounded gap $G = (a, b)$ is simply $b - a$. The "width" of the two unbounded gaps is not considered.

We will first need to introduce one new concept - we want to give a name to what happens when a gap "closes", that is, that "a gap disappears" at certain values of t:

Definition 3.24. For a topological space T and for some $t_0 \in T$, let $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ be a map that is gap edge continuous at t_0. We say that a point $c \in F_{t_0} \cup \{\pm\infty\}$ is a *closed gap* of F_{t_0} if there exists some net $\{t_\iota\}_{\iota \in I}$ of points in $T \setminus \{t_0\}$ converging to t_0, and two convergent nets $\{a_\iota\}_{\iota \in I}$ and $\{b_\iota\}_{\iota \in I}$ of points in $[-\infty, \infty]$ with

$$\lim_\iota a_\iota = c = \lim_\iota b_\iota$$

and such that (a_ι, b_ι) is a gap of F_{t_ι} for every $\iota \in I$.

Remembering Definition 1.6, we see that a gap is closed if it satisfies the second option of the second condition, that is, if $a = b$.

It may seem strange that we allow $c = \pm\infty$ to be a closed gap. However, we have already seen an example where it is appropriate to say that $c = \infty$ is a closed gap - consider the map $t \in [0, 1] \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ as we defined it in Example 1.24, that is,

$$\sigma(A_t) := \begin{cases} \{0\} & \text{if } t = 0 \\ \{0\} \cup \{1/t\} & \text{if } t > 0 \end{cases}$$

Now $(1/t, \infty)$ is a gap of $\sigma(A_t)$ for all $t \neq 0$, and as $\lim_{t \downarrow 0} 1/t = \infty$ it is quite natural to consider ∞ as a closed gap of $\sigma(A_0)$.

Example 3.25. Recall the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ from Example 1.8 defined by

$$F_t = (-\infty, -t] \cup [t, \infty)$$

We claim that the point 0 is a closed gap of F_{t_0} for $t_0 = 0$. For recall that any gap of F_t for $t \neq 0$ is on the form $(-t, t)$, so if $t_n \to t_0$ and (a_n, b_n) is a gap of F_{t_n} for every $n \in \mathbb{N}$, then $a_n = -t_n$, $b_n = t_n$ and

$$\lim_n a_n = 0 = \lim_n b_n$$

and we are done.

Example 3.26. Consider the standard procedure to construct the Cantor set and its associated map $n \in T \mapsto C_n \in \mathscr{C}(\mathbb{R})$, as introduced in Example 1.10. It is well known that in addition to containing points on the form $\beta + \frac{1}{3^n}$ or $\beta + \frac{2}{3^n}$, that is, points that are end points of removed intervals, the Cantor set C_∞ also contains uncountably many other points. To shed some light on the nature of these points, we claim that *every point of the Cantor set is a closed gap*.

So fix any point $c \in C_\infty$. It suffices to show that there for each $n \in \mathbb{N}$ exists some gap (a_n, b_n) of C_n such that

$$\max\left\{|c - a_n|, |c - b_n|\right\} \leq \frac{2}{3^n}$$

This will prove that c is a closed gap, as

$$\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$$

A simple inductive argument shows that each C_n is composed of intervals of length $\frac{1}{3^n}$, and to either the left or right of each such interval there is a gap (a_n, b_n) of length $\frac{1}{3^n}$. Since c necessarily lies in one of these intervals, we must have

$$\max\{|c - a_n|, |c - b_n|\} \leq \frac{1}{3^n} + \frac{1}{3^n} = \frac{2}{3^n}$$

which proves the claim.

When T is a metric space, we will often want to consider *every* pair of sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ satisfying the conditions in Definition 3.24. First, we define for any space X the *space of sequences on* X:

Definition 3.27. For any space X, we define the space $X^{\mathbb{N}}$ by

$$X^{\mathbb{N}} := \left\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X \text{ for all } n \in \mathbb{N}\right\}$$

Definition 3.28. For any metric space (T, d), any $t_0 \in T$ and any map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$, define for each closed gap c of F_{t_0} with $c \neq \pm\infty$ the subset $\mathscr{G}^c_{t_0} \subset T^{\mathbb{N}} \times (\mathbb{R}^2)^{\mathbb{N}}$ by

$$\mathscr{G}^c_{t_0} := \Bigg\{\Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}}\Big) \;\Big|\; \lim t_n = t_0,\ \lim a_n = \lim b_n = c,$$
$$t_n \neq t_0 \text{ and } (a_n, b_n) \text{ is a gap of } F_{t_n} \text{ for each } n \in \mathbb{N}\Bigg\}$$

If we are given a field $A = (A_t)_{t\in T}$, then we define $\mathscr{G}^c_{t_0}(A)$ as $\mathscr{G}^c_{t_0}$ for the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$.

By the definition of closed gaps, $\mathscr{G}^c_{t_0}$ is always non-empty. Note in particular that we do not define $\mathscr{G}^c_{t_0}$ for $c = \pm\infty$, and that we do not accept elements $\Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}}\Big)$ such that $a_n = -\infty$ or $b_n = \infty$ for any $n \in \mathbb{N}$. The reason for this is, as we discussed earlier, that we do not want to work with the width of the outer gaps.

By the terminology of Beckus and Bellisard, $\mathscr{G}^c_{t_0}$ would be "the space of all families of gaps closing on c at t_0" (although $\mathscr{G}^c_{t_0}$ was never formally introduced or used in [3]). We will usually use the symbol $\mathfrak{g}$ to denote elements in a space $\mathscr{G}^c_{t_0}$, which means that any $\mathfrak{g}$ will be on the form

$$\Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}}\Big) \in T^{\mathbb{N}} \times \left(\mathbb{R}^2\right)^{\mathbb{N}}$$

Example 3.29. $\mathscr{G}_{t_0}^c$ can be quite large. For example, set $T = [0,1]$ with the standard metric it inherits from $\mathbb{R}$, and define the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ by

$$F_t := \bigcup_{k \in \mathbb{Z}} \{tk\}$$

It is not hard to see that

$$\mathscr{G}_0^0 = \Big\{ \Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}} \Big) \;\Big|\; \lim t_n = 0,$$

$$t_n \neq t_0 \text{ and } a_n = t_n k_n,\, b_n = t_n(k_n + 1) \text{ for some } k_n \in \mathbb{Z} \text{ for each } n \in \mathbb{N} \Big\}$$

As, for every sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim t_n = 0$ and $t_n \neq t_0$ for all $n \in \mathbb{N}$, there are countably infinitely many choices of k_n for every $n \in \mathbb{N}$, it follows that $\mathscr{G}_0^0$ is homeomorphic to $T^{\mathbb{N}} \times \mathbb{N}^2$.

Definition 3.30. For a metric space (T, d) and for some $t_0 \in T$, let $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ be a map that is gap edge continuous at t_0. If c is a closed gap of F_{t_0}, then we say that *the width of the gaps of $t \mapsto F_t$ closing on c at t_0 is α-Hölder continuous* if

$$\sup_{\mathfrak{g} \in \mathscr{G}_{t_0}^c} \lim_{r \downarrow 0} \sup_{\{n | 0 < d(t_0, t_n) < r\}} \frac{b_n - a_n}{d(t_0, t_n)^\alpha} < \infty$$

If $A = (A_t)_{t\in T}$ is a field of self-adjoint operators, we say that *the width of the gaps of A closing on c at t_0 is α-Hölder continuous* if

$$\sup_{\mathfrak{g} \in \mathscr{G}_{t_0}^c(A)} \lim_{r \downarrow 0} \sup_{\{n | 0 < d(t_0, t_n) < r\}} \frac{b_n - a_n}{d(t_0, t_n)^\alpha} < \infty$$

and we refer to this quantity as *the α-dilation of the width of the gaps of A closing on c at t_0*.

Theorem 3.31. *For a metric space (T, d), let $A = (A_t)_{t\in T}$ be a $p2$-α-Hölder continuous field of bounded, self-adjoint operators, and take any $t_0 \in T$. Assume that c is a closed gap of $\sigma(A_{t_0})$ such that c is an interior point of $\sigma(A_{t_0})$. Then the width of the gaps of A closing on c at t_0 is $\alpha/2$-Hölder continuous at t_0, with $\alpha/2$-dilation less than or equal to $2\sqrt{(4m^2+2)C_\alpha}$.*

Proof. For simplicity, we write $F_t := \sigma(A_t)$ and $F_0 := \sigma(A_{t_0})$.

Let us consider some

$$\mathfrak{g} = \Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}} \Big) \in \mathscr{G}_{t_0}^c(A)$$

Define for all $n \in \mathbb{N}$ the number $\lambda_n := (a_n + b_n)/2$. Since c is an interior point of F_0, it follows that there exists some $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset F_0$. As $\lim b_n = \lim a_n = c$, it follows that $\lim \lambda_n = c$, so there exists some $n' \in \mathbb{N}$ such that $|\lambda_n - c| < \epsilon$ for all $n \geq n'$, that is, such that $\lambda_n \in F_0$ for all $n \geq n'$.

By Proposition 3.9, there exists some $m > 0$ and some neighbourhood $W \subset T$ of t_0 such that $\sup_{t \in W} |F_t| < m$. As $t_n \to t_0$, there exists some $n'' \in \mathbb{N}$ with $n'' \geq n'$ such that $t_n \in W$ for all $n \geq n''$. Since c is an interior point of F_0, we have $-\infty < c < \infty$, and so

$$-\infty < -m < a_n < \lambda_n < b_n < m < \infty$$

for all $n \geq n''$. Since $\max\{\|A_{t_n}\|, |\lambda_n|\} < m$, we get that

$$4m^2 = (2m)^2 > \left(\sup_{x \in F_{t_n}} |\lambda_n - x|\right)^2$$

for every $n \geq n''$. It follows from Lemma 3.13 that

$$\begin{aligned}\left\|4m^2 - (A_{t_0} - \lambda_n)^2\right\| &= 4m^2 - \text{dist}(\lambda_n, F_0) \\ &= 4m^2\end{aligned}$$

and

$$\begin{aligned}\left\|4m^2 - (A_{t_n} - \lambda_n)^2\right\| &= 4m^2 - \text{dist}(\lambda_n, F_{t_n}) \\ &= 4m^2 - (b_n - a_n)^2/4\end{aligned}$$

for all $n \geq n''$, as $\text{dist}(\lambda_n, F_0) = 0$ for $n \geq n'' \geq n'$.

Finally, we observe that if we for each $n \geq n''$ define the map $p_n : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}p_n(x) &:= 4m^2 - \lambda_n^2 + 2\lambda_n x - x^2 \\ &= 4m^2 - (x - \lambda_n)^2\end{aligned}$$

then by Lemma 3.13, $\|p_n\|_1 \leq 4m^2 + 2$ for all $n \geq n''$, which means that we have $p_n \in \mathscr{P}_2(4m^2 + 2)$ for all $n \geq n''$.

Using the $p2$-α-Hölder continuity of A at t_0, we see that

$$\begin{aligned}\frac{1}{4}(b_n - a_n)^2 &= \left(4m^2\right) - \left(4m^2 - (b_n - a_n)^2/4\right) \\ &= \left|\|4m^2 - (A_{t_0} - \lambda_n)^2\| - \|4m^2 - (A_t - \lambda_n)^2\|\right| \\ &= \left|\|p_n(A_{t_0})\| - \|p_n(A_{t_n})\|\right| \\ &\leq C_{\alpha, 4m^2+2}\, d(t_0, t_n)^\alpha \\ &= (4m^2 + 2) C_\alpha\, d(t_0, t_n)^\alpha\end{aligned}$$

for each $n \geq n''$, again by Lemma 3.13. It follows that

$$\frac{1}{4}(b_n - a_n)^2 \leq (4m^2 + 2) C_\alpha\, d(t_0, t_n)^\alpha$$

for all $n \geq n''$, alternatively

$$b_n - a_n \leq 2\sqrt{(4m^2 + 2) C_\alpha}\, d(t_0, t_n)^{\alpha/2}$$

Finally, as $t_n \to t_0$ and $t_n \neq t_0$ for all $n \in \mathbb{N}$, there exists some $r' > 0$ such that $d(t_0, t_n) < r'$ if and only if $n \geq n''$, and so

$$\begin{aligned}\lim_{r\downarrow 0} \sup_{\{n|0<d(t_0,t_n)<r\}} \frac{b_n - a_n}{d(t_0, t_n)^{\alpha/2}} &= \lim_{r\downarrow 0} \sup_{\{n|0<d(t_0,t_n)<r\leq r'\}} \frac{b_n - a_n}{d(t_0, t_n)^{\alpha/2}} \\ &\leq 2\sqrt{(4m^2+2)C_\alpha}\end{aligned}$$

As $\mathfrak{g} = \Big(\{t_n\}_{n\in\mathbb{N}}, \{(a_n, b_n)\}_{n\in\mathbb{N}}\Big) \in \mathscr{G}^c_{t_0}(A)$ was arbitrary, it follows that

$$\sup_{\mathfrak{g}\in\mathscr{G}^c_{t_0}(A)} \lim_{r\downarrow 0} \sup_{\{n|0<d(t_0,t_n)<r\}} \frac{b_n - a_n}{d(t_0, t_n)^{\alpha}} \leq 2\sqrt{(4m^2+2)C_\alpha}$$

□

4 Unbounded Operators

4.1 Unbounded, self-adjoint operators

So far we have worked chiefly with bounded operators. However, many interesting operators are not bounded. Unbounded operators are, in many ways, much less well behaved than bounded operators.

One notable problem is that any unbounded operator B on a Hilbert space H is generally *not defined on all of* H, but rather only on some linear subspace $D(B) \subset H$, the *domain of* B. Typically, we will demand that $D(B)$ is dense as a subset of H, that is, that B is *densely defined*. We therefore use the phrase "operator on H" to refer to any operator, bounded or unbounded, that is defined on some dense linear subspace of H.

The following definitions and results are all taken from [10]:

Definition 4.1. Let B be an operator on some Hilbert space H. We say that B is *closed* if

$$\Gamma(B) := \left\{ \langle x, Bx \rangle \,\middle|\, x \in D(B) \right\}$$

is a closed subset of $H \times H$.

Definition 4.2. Let B and B' be operators on some Hilbert space H. If $\Gamma(B') \supset \Gamma(B)$, we say that B' is an *extension* of B, and we write $B' \supset B$. Equivalently, $B' \supset B$ if and only if $D(B') \supset D(B)$ and $B'x = Bx$ for all $x \in D(B)$.

Definition 4.3. Let B be a densely defined linear operator on a Hilbert space H. Let $D(B^*)$ be the set of $y \in H$ for which there is a $z \in H$ with

$$\langle Bx, y \rangle = \langle x, z \rangle$$

for all $x \in D(B)$. For each such $y \in D(B^*)$, we define

$$B^* y := z$$

B^* is called the *adjoint* of B.

Definition 4.4. A densely defined operator B on a Hilbert space H is called *symmetric* if $B \subset B^*$, that is, if $D(B) \subset D(B^*)$ and $Bx = B^*x$ for all $x \in D(B)$. Equivalently, B is symmetric if and only if

$$\langle Bx, y \rangle = \langle x, By \rangle$$

for all $x, y \in D(B)$.

Definition 4.5. A densely defined operator B on a Hilbert space H is called *self-adjoint* if $B = B^*$, that is, if and only if B is symmetric and $D(B) = D(B^*)$.

Theorem 4.6. *Let B be a densely defined operator on a Hilbert space H. Then B^* is closed.*

As an immediate consequence, any self-adjoint operator is necessarily closed, and so whenever we speak of a *self-adjoint operator* B, we will understand B to be densely defined, symmetric and closed, with $B = B^*$ and $D(B) = D(B^*)$.

In order to work with the inverses of unbounded operators, we need to carefully redefine the *resolvent*:

Definition 4.7. Let B be a closed operator on a Hilbert space. A complex number λ is in the *resolvent set*, $\rho(B)$, if $\lambda I - B$ is a bijection of $D(B)$ onto H with a bounded inverse. If $\lambda \in \rho(B)$,

$$R_\lambda(B) := (\lambda I - B)^{-1}$$

is called the *resolvent* of B at λ.

To be clear, the inverse of $\lambda - B$ is the unique bounded operator $R_\lambda(B) \in \mathbf{B}(H)$ such that $R_\lambda(B)(H) = D(B)$ and such that

$$R_\lambda(B)(\lambda I_{D(B)} - B) = I_{D(B)}$$

and

$$(\lambda I_{D(B)} - B)R_\lambda(B) = I_H$$

where $I_{D(B)}$ is the identity operator on $D(B)$ and I_H is the identity operator on H.

We shall usually not bother with the subscripts and just write I instead of $I_{D(B)}$ or I_H. Furthermore, as before, we shall usually write λ instead of λI whenever this can not cause any confusion. We will also interchangably use the notation $R_\lambda(B)$ and $(\lambda - B)^{-1}$ for the inverse of $\lambda - B$, depending on which is more convenient.

Now that we have defined the resolvent $\rho(B)$ of B, we can naturally define $\sigma(B)$, the *spectrum* of B, as the complement of $\rho(B)$ in $\mathbb{C}$.

Like in the bounded case, we would like to have some control over the properties of the spectrum of B. Fortunately, as long as B is self-adjoint, $\sigma(B)$ will always be a closed subset of $\mathbb{R}$.

Theorem 4.8. *Let B be a closed densely defined linear operator. Then the resolvent set of B is an open subset of the complex plane on which the resolvent is an analytic operator-valued function. Furthermore,*

$$\{R_\lambda(B) \mid \lambda \in \rho(B)\}$$

is a commuting family of bounded operators satisfying

$$R_\lambda(B) - R_\mu(B) = (\mu - \lambda)R_\mu(B)R_\lambda(B)$$

Theorem 4.9. *Let B be a closed symmetric operator on a Hilbert space H. Then the spectrum of B is* one *of the following:*

1. *The closed upper half-plane.*
2. *The closed lower half-plane.*
3. *The entire plane.*
4. *A subset of the real axis.*

B is self-adjoint if and only if case 4 holds and the spectrum is non-empty.

However, unlike in the bounded case, the spectrum on an unbounded self-adjoint operator *can be empty*, and if not empty it is in general *an unbounded closed subset of* $\mathbb{R}$, as the two following examples, both also taken from [10] (pages 291 and 254, respectively), illustrate.

Example 4.10. Set $H = L^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$, and let the operator B on H be multiplication by x, that is, $Bf(x) = xf(x)$ for every $x \in \mathbb{R}$ and every $f \in H$.

B is obviously a linear operator. We claim that B is unbounded. For note that for any $r \in \mathbb{R}$, the function $\chi_{[r,r+1]}$, that is, the indicator function for the interval $[r, r+1]$, has L^2-norm of 1, as

$$\begin{aligned}\|\chi_{[r,r+1]}\|_{L^2}^2 &= \int_{\mathbb{R}} |\chi_{[r,r+1]}(x)|^2 \mathrm{d}x \\ &= \int_{[r,r+1]} 1 \mathrm{d}x = 1\end{aligned}$$

However, $\|B\chi_{[r,r+1]}\|_{L^2}$ grows to be arbitrarily large as r does, since

$$\begin{aligned}\|B\chi_{[r,r+1]}\|_{L^2}^2 &= \int_{\mathbb{R}} |x\chi_{[r,r+1]}(x)|^2 \mathrm{d}x \\ &= \int_{[r,r+1]} x^2 \mathrm{d}x = r^2 + r + \frac{1}{3}\end{aligned}$$

which tends to infinity as $|r|$ does. Thus B is unbounded.

Next, we claim that $\sigma(B) = \mathbb{R}$. It is not hard to see that B is self-adjoint, and so we must have $\sigma(B) \subset \mathbb{R}$. Take any $\lambda \in \mathbb{R}$. Then the operator $\lambda - B$ is given by

$$(\lambda - B)f(x) = (\lambda - x)f(x)$$

for every $x \in \mathbb{R}$ and every $f \in L^2(\mathbb{R})$. It follows that *there cannot exist any linear operator S_λ on H such that $S_\lambda(\lambda - B) = (\lambda - B)S_\lambda = I$*, as

$$(\lambda - B)f(\lambda) = (\lambda - \lambda)f(\lambda) = 0$$

for every $f \in L^2(\mathbb{R})$, and if S_λ was a linear operator satisfying $S_\lambda(\lambda - B) = I$ we would necessarily have

$$f(\lambda) = If(\lambda) = S_\lambda(\lambda - B)f(\lambda) = S_\lambda 0 = 0$$

for any $f \in L^2(\mathbb{R})$. As there clearly exist $f \in L^2(\mathbb{R})$ with $f(\lambda) \neq 0$, we have proven that there exists no inverse S_λ of $(\lambda - B)$. As $\lambda \in \mathbb{R}$ was arbitrary, it follows that $\sigma(B) = \mathbb{R}$.

As a side note, the above example demonstrates that there are indeed unbounded operators whose domain of definition is all of H; however, generally speaking, unbounded operators are not defined on all of H. In the below example, we will see an unbounded operator that is defined only on a dense, proper linear subspace of the Hilbert space it acts on.

Example 4.11. Set $H = L^2([0,1])$, the space of square integrable functions on $[0,1]$, and denote by $AC([0,1])$ the dense linear subspace of all absolutely continuous functions on $[0,1]$ whose derivatives are in $L^2([0,1])$; the norm on H is the L^2-norm. Define the unbounded linear operator $B = i\frac{d}{dx}$ with domain

$$D(B) = \{f \in AC([0,1]) \mid f(0) = 0\}$$

Again, it is simple to confirm that B is a linear operator, $D(B)$ is clearly a dense linear subspace of $AC([0,1])$ and thus of H, and unboundedness of the derivation operator is also rather trivial. Fix $\lambda \in \mathbb{R}$. We claim that the map S_λ defined by

$$(S_\lambda g)(x) = i\int_0^x e^{-i\lambda(x-y)} g(y)\,\mathrm{d}y$$

for $g \in H$ is a bounded linear operator on H with $S_\lambda(H) = D(B)$, and satisfies the conditions $(\lambda - B)S_\lambda = I_H$ and $S_\lambda(\lambda - B) = I_{D(B)}$.

Linearity is obvious. To see that S_λ is a bounded operator taking values in $D(B)$, note that for any $g \in H$ we have that

$$\begin{aligned}
\|S_\lambda g\|_{L^2}^2 &= \int_0^1 |(S_\lambda g)(x)|^2\,\mathrm{d}x \\
&\leq \left(\sup_{x\in[0,1]} |(S_\lambda g)(x)|^2\right) \\
&= \left(\sup_{x\in[0,1]} \left|i\int_0^x e^{-i\lambda(x-y)} g(y)\,\mathrm{d}y\right|^2\right) \\
&\leq \left(\sup_{x\in[0,1]} \int_0^x \left|e^{-i\lambda(x-y)}\right|^2 \mathrm{d}y\right)\left(\sup_{x\in[0,1]} \int_0^x |g(y)|^2\,\mathrm{d}y\right) \\
&= \left(\sup_{x\in[0,1]} \int_0^x \left|e^{-i\lambda(x-y)}\right|^2 \mathrm{d}y\right)\|g\|_{L^2}^2
\end{aligned}$$

where the quantity $\left(\sup_{x\in[0,1]} \int_0^x \left|e^{-i\lambda(x-y)}\right|^2 \mathrm{d}y\right)$ is finite and depends only on λ, and clearly

$$(S_\lambda g)(0) = i\int_0^0 e^{-i\lambda(x-y)} g(y)\,\mathrm{d}y = 0$$

so $S_\lambda(H) \subset D(B)$. To see that $S_\lambda(H) = D(B)$, note that if we for any given $f \in D(B)$ set $g(x) := -ie^{-i\lambda x}\frac{d}{dx}\left[f(x)e^{i\lambda x}\right]$, then $g \in H$ as $f \in AC([0,1])$, and we have by the fundamental theorem of calculus that

$$\begin{aligned}(S_\lambda g)(x) &= i\int_0^x e^{-i\lambda(x-y)}\,(-i)\,e^{-i\lambda y}\frac{d}{dy}\left[f(y)e^{i\lambda y}\right]\mathrm{d}y\\ &= e^{-i\lambda x}\int_0^x \frac{d}{dy}\left[f(y)e^{i\lambda y}\right]\mathrm{d}y\\ &= e^{-i\lambda x}\left(f(x)e^{i\lambda x} - f(0)e^0\right) = f(x)\end{aligned}$$

as $f(0) = 0$ since $f \in D(B)$.

To see that S_λ is the inverse of $\lambda - B$, note that for any $g \in H$, we have, again by the fundamental theorem of calculus, that

$$\begin{aligned}\left((\lambda - B)S_\lambda g\right)(x) &= (\lambda - B)\left(i\int_0^x e^{-i\lambda(x-y)}g(y)\,\mathrm{d}y\right)\\ &= \lambda\left(i\int_0^x e^{-i\lambda(x-y)}g(y)\,\mathrm{d}y\right) - i\frac{d}{dx}\left[\left(i\int_0^x e^{-i\lambda(x-y)}g(y)\,\mathrm{d}y\right)\right]\\ &= i\lambda\int_0^x e^{-i\lambda(x-y)}g(y)\,\mathrm{d}y + \frac{d}{dx}\left[e^{-i\lambda x}\right]\int_0^x e^{i\lambda y}g(y)\,\mathrm{d}y + e^{-i\lambda x}\frac{d}{dx}\left[\int_0^x e^{i\lambda y}g(y)\,\mathrm{d}y\right]\\ &= i\lambda\int_0^x e^{-i\lambda(x-y)}g(y)\,\mathrm{d}y - i\lambda e^{-i\lambda x}\int_0^x e^{i\lambda y}g(y)\,\mathrm{d}y + e^{-i\lambda x}e^{i\lambda x}g(x)\\ &= g(x)\end{aligned}$$

while for any $f \in D(B)$ we can use integration by parts to see that

$$\begin{aligned}S_\lambda\left((\lambda - B)f\right)(x) &= i\int_0^x e^{-i\lambda(x-y)}\Big(\lambda f(y) - if'(y)\Big)\,\mathrm{d}y\\ &= i\lambda\int_0^x e^{-i\lambda(x-y)}f(y)\,\mathrm{d}y + \int_0^x e^{-i\lambda(x-y)}f'(y)\,\mathrm{d}y\\ &= i\lambda\int_0^x e^{-i\lambda(x-y)}f(y)\,\mathrm{d}y + \left[e^{-i\lambda(x-y)}f(y)\right]_{y=0}^{y=x} - \int_0^x \frac{d}{dy}\left[e^{-i\lambda(x-y)}\right]f(y)\,\mathrm{d}y\\ &= i\lambda\int_0^x e^{-i\lambda(x-y)}f(y)\,\mathrm{d}y + e^{-i\lambda 0}f(x) - e^{-i\lambda x}f(0) - i\lambda\int_0^x e^{-i\lambda(x-y)}f(y)\,\mathrm{d}y\\ &= f(x)\end{aligned}$$

as we have $f(0) = 0$ by assumption.

As $\lambda \in \mathbb{R}$ was arbitrary, this shows that $\sigma(B) \cap \mathbb{R} = \emptyset$. As Theorem 4.9 shows that the spectrum of B must be either all of $\mathbb{C}$, the upper or lower (closed) half-plane or a subset of $\mathbb{R}$, this implies that the only possibility left is $\sigma(B) = \emptyset$. This completes the proof.

As another remark, the above example can be manipulated to show just how important our choice of $D(B)$ is. For had we set $D(B) = AC([0,1])$, then B would still be a well-defined unbounded operator, but its spectrum would be *all of* $\mathbb{C}$, as we have $e^{-i\lambda x} \in AC([0,1])$ and

$$\begin{aligned}(\lambda - B)e^{-i\lambda x} &= \lambda e^{-i\lambda x} - ie^{-i\lambda x}\frac{d}{dx}\\ &= \lambda e^{-i\lambda x} + i^2\lambda e^{-i\lambda x}\\ &= 0\end{aligned}$$

for all $\lambda \in \mathbb{C}$, and a similar argument as in Example 4.10 shows that there thus cannot exist a bounded inverse for any $\lambda \in \mathbb{C}$. With our old definition of $D(B)$, however, we would have $e^{-i\lambda x} \notin D(B)$ for all $\lambda \in \mathbb{C}$. As such, one must always understand an unbounded operator as being defined not only by its "form", but also its domain of definition, as even operators who agree on a fellow domain of definition may behave dramatically differently.

4.2 Continuity for fields of unbounded operators

It is reasonable to ask whether the ideas and results from Chapter 2 can be extended to fields of self-adjoint, not necessarily bounded operators. The answer is, at least in part, yes, due to our insistence on working with $\mathscr{C}(\mathbb{R})$ and $[-\infty, \infty]$ in Chapters 1 and 2, as opposed to just working with $\mathscr{K}(\mathbb{R})$ (the space of non-empty compact subsets of $\mathbb{R}$) and $\mathbb{R}$.

The concept of $p2$-continuity can, unfortunately, not be applied to general fields $A = (A_t)_{t\in T}$ of unbounded operators. One problem already appears when we try to define $p(A_t)$. [10] does assert that there exists a functional calculus for unbounded, self-adjoint operators, such that $f(B)$ is well defined for every bounded continuous (in fact, Borel) function f on $\mathbb{R}$, and that we have

$$\|f(B)\| \leq \sup_{x\in\sigma(B)} |f(x)|$$

However, non-constant polynomials, while continuous, are most certainly not bounded, as $\lim_{x\to\infty} |p(x)| = \infty$ for all non-constant polynomials p. Moreover, the fact that we only have an inequality, as opposed to an equality, makes any implementation of concepts similar to $p2$-continuity tedious at best.

The definition of gap edge continuity, however, can be applied to unbounded operators *directly from Definition 1.6*. In fact, note that in this definition, we never assumed that the sets or operators involved were bounded, and Example 1.8 is an example of gap edge continuity for a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ where every F_t is unbounded. Similarly, both the definition of Hausdorff continuity in Definition 1.17 and Fell continuity in Definition 1.20 were defined for any field of self-adjoint operators, bounded or unbounded. In fact, Lemma 2.21 proved that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is gap edge continuous at a point t_0 if and only if it is Fell continuous at t_0, without requiring the F_t to be bounded, and so we have already shown that a field $A = (A_t)_{t\in T}$ of self-adjoint, not necessarily bounded operators is gap edge continuous at a point t_0 if and only if it is Fell continuous at t_0.

However, as we demonstrated in Example 1.24, Fell continuity and Hausdorff continuity are in general not equivalent for maps $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$. The problem with unbounded sets is that there can easily be cases where $t \mapsto F_t$ is Fell continuous at some $t_0 \in T$, but $\text{dist}(F_{t_0}, F_t) = \infty$ for all $t \neq t_0$; for example, set $T = [0, 1]$ and

$$F_t := \begin{cases} \mathbb{R} & \text{if } t = 0 \\ [-1/t, \infty] & \text{otherwise} \end{cases}$$

In Chapter 1, we overcame this difficulty by demanding that $t \mapsto F_t$ be locally uniformly bounded, but clearly this condition is unreasonable for fields on unbounded operators. We instead suggest a slightly modified version of Hausdorff continuity.

Definition 4.12. For every $k \in \mathbb{N}$, define the map $\Theta_k : \mathscr{C}(\mathbb{R}) \to \mathscr{C}(\mathbb{R})$ by

$$\Theta_k(F) = \Big(F \cup \{\pm k\}\Big) \cap [-k, k]$$

Definition 4.13. Given a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ and any $t_0 \in T$, we say that *the map $t \mapsto F_t$ is $\mathbb{N}$-Hausdorff continuous at t_0* if for every $k \in \mathbb{N}$, the map

$$t \in T \mapsto \Theta_k(F_t) \in \mathscr{C}(\mathbb{R})$$

is Hausdorff continuous at t_0, that is, if for every $k \in \mathbb{N}$ we have

$$\lim_{t \to t_0} \text{dist}\Big(\big(F_{t_0} \cup \{\pm k\}\big) \cap [-k,k]\ ,\ \big(F_t \cup \{\pm k\}\big) \cap [-k,k]\Big) = 0$$

Theorem 4.14. *Assume that we are given some map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ and some $t_0 \in T$. Then $t \mapsto F_t$ is Fell continuous at $t_0 \in T$ if and only if it is $\mathbb{N}$-Hausdorff continuous at t_0.*

Taking the intersection of F_t and $[-k,k]$ intuitively makes sense, as we are trying to "enforce" local uniform boundedness. The addition of the two points $\{\pm k\}$ is less intuitive. However, we do need them, as the following example demonstrates.

Example 4.15. We slightly modify Example 1.8. Set $T = \mathbb{R}$ with its standard topology, and consider the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ given by

$$F_t = (-\infty, -t] \cup [t, \infty)$$

This map is clearly both Fell continuous and Hausdorff continuous at all $t \in T$. However, we claim that *for any $k \in \mathbb{N}$, the map $t \in T \mapsto F_t \cap [-k,k] \in \mathscr{C}(\mathbb{R})$ is neither Fell continuous nor Hausdorff continuous at $t_0 = k$.* Proving this claim is rather simple. We have

$$F_{t_0} = (-\infty, -k] \cup [k, \infty)$$

and so clearly

$$F_{t_0} \cap [-k,k] = \Big((-\infty,-k] \cup [k,\infty)\Big) \cap [-k,k] = \{\pm k\}$$

However, for any $t > t_0$ we see that $F_t \cap [-k,k] = \emptyset$. Thus

$$\text{dist}(F_{t_0}, F_t) = \text{dist}\left(\{\pm k\}, \emptyset\right) = \infty$$

for all $t > t_0$, so $t \mapsto F_t \cap [-k,k]$ cannot be Hausdorff continuous at t_0. Furthermore, clearly we have $F_t \cap O = \emptyset \cap O = \emptyset$ for all $t > t_0$, even if $F_{t_0} \cap O \neq \emptyset$, so $t \mapsto F_t \cap [-k,k]$ cannot be Fell continuous either at t_0.

Proposition 4.16. *Assume two maps $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ and $t \in T \mapsto F'_t \in \mathscr{C}(\mathbb{R})$ are both Fell continuous at some point $t_0 \in T$. Then the map $t \in T \mapsto F_t \cup F'_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous at t_0.*

Proof. Write $F_0 := F_{t_0}$ and $F_0' := F_{t_0}'$, and choose any compact subset $K \subset \mathbb{R}$ and any finite family $\mathscr{F}$ of open subsets of $\mathbb{R}$ such that $F_0 \cup F_0' \in \mathscr{U}(K, \mathscr{F})$. We must find a neighbourhood of t_0 such that $F_t \cup F_t' \in \mathscr{U}(K, \mathscr{F})$ for all t in this neighbourhood.

1) We will first find a neighbourhood $U \subset T$ of t_0 such that

$$(F_t \cup F_t') \cap K = \emptyset$$

for all $t \in U$.

Note that $(F_0 \cup F_0') \cap K = \emptyset$ implies that $F_0 \cap K = \emptyset$ and $F_0' \cap K = \emptyset$. By the Fell continuity of both maps $t \mapsto F_t$ and $t \mapsto F_t'$, there exist two neighbourhoods $U_1, U_2 \subset T$ of t_0 such that $F_t \cap K = \emptyset$ for all $t \in U_1$ and $F_t' \cap K = \emptyset$ for all $t \in U_2$. It follows that if we set $U := U_1 \cap U_2$ then U is a neighbourhood of t_0 such that

$$(F_t \cup F_t') \cap K = \emptyset$$

for all $t \in U$.

2) For the second part, we want to find a neighbourhood $V \subset T$ of t_0 such that

$$(F_t \cup F_t') \cap O \neq \emptyset$$

for all $t \in V$ and all $O \in \mathscr{F}$. Start by fixing $O \in \mathscr{F}$.

As $(F_0 \cup F_0') \cap O \neq \emptyset$, there must exist some $x \in (F_0 \cup F_0') \cap O$, and we must have that either $x \in F_0$ or $x \in F_0'$. Assume the former. Then $F_0 \cap O \neq \emptyset$, and by the Fell continuity of the map $t \mapsto F_t$ we can find some neighbourhood $V_O \subset T$ of t_0 such that $F_t \cap O \neq \emptyset$ for every $t \in O$. If we instead had $x \in F_0'$, then analogously we can find some neighbourhood $V_O \subset T$ of t_0 such that $F_t' \cap O \neq \emptyset$ for every $t \in O$.

It follows that if we define $V := \bigcap_{O \in \mathscr{F}} V_O$, then for each $t \in V$ and each $O \in \mathscr{F}$ we have either $F_t \cap O \neq \emptyset$ or $F_t' \cap O \neq \emptyset$, and so

$$(F_t \cup F_t') \cap O \neq \emptyset$$

for all $t \in V$.

3) We now see that for all $t \in U \cap V$, $(F_t \cap F_t') \cap K = \emptyset$ and $(F_t \cap F_t') \cap O \neq \emptyset$ for all $O \in \mathscr{F}$, so $(F_t \cap F_t') \in \mathscr{U}(K, \mathscr{F})$ for all $t \in U \cap V$. □

Lemma 4.17. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous at some point $t_0 \in T$. Then for any $k \in \mathbb{N}$, the map*

$$t \in T \mapsto \Theta_k(F_t) \in \mathscr{C}(\mathbb{R})$$

is Fell continuous at t_0.

Proof. Fix any $k \in \mathbb{N}$. Write $F_0 := F_{t_0}$, and choose any compact subset $K \subset \mathbb{R}$ and any finite family $\mathscr{F}$ of open subsets of $\mathbb{R}$ such that $\Theta_k(F_0) \in \mathscr{U}(K, \mathscr{F})$. We must find a neighbourhood of t_0 such that $\Theta_k(F_t) \in \mathscr{U}(K, \mathscr{F})$ for all t in this neighbourhood.

1) We will first find a neighbourhood $U \subset T$ of t_0 such that $\Theta_k(F_t) \cap K = \emptyset$ for all $t \in U$.

Note that $[-k, k] \cap K$ is a compact subset of $\mathbb{R}$, and by assumption we have

$$\Big(F_0 \cup \{\pm k\}\Big) \cap \Big([-k, k] \cap K\Big) = \Big(\Big(F_0 \cup \{\pm k\}\Big) \cap [-k, k]\Big) \cap K$$
$$= \Theta_k(F_0) \cap K = \emptyset$$

As the map $t \mapsto F_t$ at t_0 is Fell continuous at t_0 by assumption, and as the map $t \in T \mapsto \{\pm k\} \in \mathscr{C}(\mathbb{R})$ is (rather trivially) Fell continuous, we have from Proposition 4.16 that the map $t \mapsto F_t \cup \{\pm k\}$ is Fell continuous at t_0. It follows that we can find some neighbourhood $U \subset T$ of t_0 such that

$$\Big(F_t \cup \{\pm k\}\Big) \cap \Big([-k, k] \cap K\Big) = \emptyset$$

for every $t \in U$. But

$$\Big(F_t \cup \{\pm k\}\Big) \cap \Big([-k, k] \cap K\Big) = \Big(\Big(F_t \cup \{\pm k\}\Big) \cap [-k, k]\Big) \cap K$$
$$= \Theta_k(F_t) \cap K$$

for every $t \in T$, and so $\Theta_k(F_t) \cap K = \emptyset$ for every $t \in U$.

2) For the second part, we want to find a neighbourhood $V \subset T$ of t_0 such that

$$\Big((F_t \cup \{\pm k\}) \cap [-k, k]\Big) \cap O \neq \emptyset$$

for all $t \in V$ and all $O \in \mathscr{F}$. Start by fixing $O \in \mathscr{F}$.

By assumption, we have

$$\Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap O \neq \emptyset$$

In particular, there exists some $x \in \Big(F_0 \cup \{\pm k\}\Big) \cap O$ with $|x| \leq k$, and as B is open there exists some $r > 0$ such that $B_r(x) \subset O$. Accordingly,

$$\Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap B_r(x) \supset \{x\} \neq \emptyset$$

and so we have

$$\Big(F_0 \cup \{\pm k\}\Big) \cap B_r(x) \supset \Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap B_r(x) \neq \emptyset$$

Again, we use the fact that the map $t \mapsto F_t \cup \{\pm k\}$ is Fell continuous at t_0. It follows that there exists some neighbourhood $V_O \subset T$ of t_0 such that

$$\Big(F_t \cup \{\pm k\}\Big) \cap B_r(x) \neq \emptyset$$

for every $t \in V_O$; in particular, for each $t \in V_O$ there exists some $x_t \in \mathbb{R}$ with

$$x_t \in \Big(F_t \cup \{\pm k\}\Big) \cap B_r(x)$$

Fix $t \in V_O$. If $|x_t| \leq k$, then clearly

$$x_t \in \Big(F_t \cup \{\pm k\} \cap [-k, k]\Big) \cap B_r(x)$$

and so $\Theta_k(F_t) \cap O \supset \Theta_k(F_t) \cap B_r(x) \neq \emptyset$.

If $|x_t| > k$, then we must have $k < |x_t| < k + r$ since $x_t \in B_r(x)$. But this implies that $k - r < |x| \leq k$, and so in particular we have either $k \in B_r(x)$ or $-k \in B_r(x)$. In the first case, we have

$$k \in \Big(F_t \cup \{\pm k\} \cap [-k, k]\Big) \cap B_r(x)$$

while in the other case we have

$$-k \in \Big(F_t \cup \{\pm k\} \cap [-k, k]\Big) \cap B_r(x)$$

and so again, $\Theta_k(F_t) \cap O \supset \Theta_k(F_t) \cap B_r(x) \neq \emptyset$.

As $t \in V_O$ was arbitrary, we have $\Theta(F_t) \cap O \supset \Theta(F_t) \cap B_r(x) \neq \emptyset$ for all $t \in V_O$. Setting $V := \bigcap_{O \in \mathscr{F}}$, we get that

$$\Theta_k(F_t) \cap B_r(x) \neq \emptyset$$

for all $O \in \mathscr{F}$ and all $t \in V$.

3) We see that $\Theta_k(F_t) \in \mathscr{U}(K, \mathscr{F})$ for every $t \in U \cap V$, and we are done.

□

Note the necessity of adding in the two singletons $\{\pm k\}$; part 2 of the above proof would not have gone through without them.

Corollary 4.18. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous at some point $t_0 \in T$. Then the map $t \mapsto F_t$ is $\mathbb{N}$-Hausdorff continuous at t_0.*

Proof. This follows immediately from Lemmas 4.17 and 1.27, as we clearly have that for each $k \in \mathbb{N}$, the map

$$t \mapsto \Theta_k(F_t) = \Big(F_t \cup \{\pm k\}\Big) \cap [-k, k]$$

is locally uniformly bounded at t_0.

□

Lemma 4.19. *Assume that for some point $t_0 \in T$ and for any $k \in \mathbb{N}$, the map*

$$t \in T \mapsto \Big(F_t \cup \{\pm k\}\Big) \cap [-k, k] \in \mathscr{C}(\mathbb{R})$$

is Fell continuous at t_0. Then $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is Fell continuous at t_0.

Proof. Choose any compact subset $K \subset \mathbb{R}$ and any finite family $\mathscr{F}$ of open subsets of $\mathbb{R}$ such that $F_0 := F_{t_0} \in \mathscr{U}(K, \mathscr{F})$. We must find a neighbourhood of t_0 such that $F_t \in \mathscr{U}(K, \mathscr{F})$ for all t in this neighbourhood.

1) We will first find a neighbourhood $U \subset T$ of t_0 such that $F_t \cap K = \emptyset$ for all $t \in U$.
By assumption, we have $F_0 \cap K = \emptyset$. As K is compact, there exists some $k \in \mathbb{N}$ such that $K \subset [-k, k]$. It follows that

$$F_0 \cap K = \Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap K$$

By the Fell continuity of $t \mapsto \Big(F_t \cup \{\pm k\}\Big) \cap [-k, k]$ at t_0, there exists some neighbourhood $U \subset T$ of t_0 such that

$$\Big((F_t \cup \{\pm k\}) \cap [-k, k]\Big) \cap K = \emptyset$$

for all $t \in U$. But

$$\Big((F_t \cup \{\pm k\}) \cap [-k, k]\Big) \cap K = F_t \cap K$$

for all $t \in T$, so we have $F_t \cap K = \emptyset$ for all $t \in U$, and so the first part is done.

2) For the second part, we want to find a neighbourhood $V \subset T$ of t_0 such that $F_t \cap O \neq \emptyset$ for all $t \in V$ and all $O \in \mathscr{F}$. Start by fixing $O \in \mathscr{F}$.
By assumption, we have $F_0 \cap O \neq \emptyset$. In particular, there exists some $x \in F_0 \cap O$. Take any $k > |x|$. We see that $x \in \Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap O$ and so in particular,

$$\Big(F_0 \cup \{\pm k\} \cap [-k, k]\Big) \cap O \neq \emptyset$$

By the Fell continuity of $t \mapsto \Big(F_t \cup \{\pm k\}\Big) \cap [-k, k]$ at t_0, there exists some neighbourhood $V_O \subset T$ of t_0 such that

$$\Big((F_t \cup \{\pm k\}) \cap [-k, k]\Big) \cap O \neq \emptyset$$

for all $t \in V_O$. It follows that

$$F_t \cap O \supset \Big((F_t \cup \{\pm k\}) \cap [-k, k]\Big) \cap O \neq \emptyset$$

for all $t \in V_O$. Since $\mathscr{F}$ is finite, it follows that $V := \bigcap_{O \in \mathscr{F}} V_O$ is the required neighbourhood of t_0 such that

$$F_t \cap O \neq \emptyset$$

for all $t \in V$ and all $O \in \mathscr{F}$.

3) It now follows that $F_t \in \mathscr{U}(K, \mathscr{F})$ for every $t \in U \cap V$, with $U \cap V$ being an open neighbourhood of t_0, so we are done.

□

Corollary 4.20. *Assume that a map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{R})$ is $\mathbb{N}$-Hausdorff continuous at some point $t_0 \in T$. Then the map $t \mapsto F_t$ is Fell continuous at t_0.*

Proof. This follows immediately from Lemmas 4.19 and 1.28, as for each $k \in \mathbb{N}$, the map

$$t \mapsto \Theta(F_t) = \Big(F_t \cup \{\pm k\}\Big) \cap [-k, k]$$

is locally uniformly bounded at t_0.

□

Proof of Theorem 4.14: This is just Corollaries 4.18 and 4.20. □

4.3 R-continuity for unbounded operators

Recall that the definition of R-continuity, specifically Definition 2.23, did not rely on boundedness. Now that we have defined the inverse, resolvent and spectrum of unbounded operators, it is natural to ask what the consequences of R-continuity are in the context of fields of unbounded operators. The main idea comes from [7], Problem 6.16 on page 177.

Proposition 4.21. *Let B be a self-adjoint operator. Then for any $\lambda \in \rho(B)$, $(\lambda - B)^{-1}$ is a normal operator.*

Proof. With $\langle \cdot, \cdot \rangle$ as the inner product on H, we see that for any $x \in D(B)$ and any $y \in H$ we have

$$\begin{aligned}\left\langle \left[(\lambda - B)^{-1}\right]^* \left(\bar{\lambda} - B\right) x, y \right\rangle &= \left\langle \left(\bar{\lambda} - B\right) x, (\lambda - B)^{-1} y \right\rangle \\ &= \left\langle x, \left(\bar{\lambda} - B\right)^* (\lambda - B)^{-1} y \right\rangle \\ &= \left\langle x, (\lambda - B)(\lambda - B)^{-1} y \right\rangle \\ &= \langle x, y \rangle\end{aligned}$$

where all the products and operations are well defined. Similarly, for any $x \in H$ and any $y \in D(B)$ we have

$$\begin{aligned}\left\langle \left[(\lambda - B)^{-1}\right]^* \left(\bar{\lambda} - B\right) x, y \right\rangle &= \left\langle \left(\bar{\lambda} - B\right) x, (\lambda - B)^{-1} y \right\rangle \\ &= \left\langle x, \left(\bar{\lambda} - B\right)^* (\lambda - B)^{-1} y \right\rangle \\ &= \left\langle x, (\lambda - B)(\lambda - B)^{-1} y \right\rangle \\ &= \langle x, y \rangle\end{aligned}$$

This implies that $\left[(\lambda - B)^{-1}\right]^* = \left(\bar{\lambda} - B\right)^{-1}$. By Theorem 4.8, the resolvents of B commute with each other, so

$$\begin{aligned}\left[(\lambda - B)^{-1}\right]^* (\lambda - B)^{-1} &= \left(\bar{\lambda} - B\right)^{-1} (\lambda - B)^{-1} \\ &= (\lambda - B)^{-1} \left(\bar{\lambda} - B\right)^{-1} \\ &= (\lambda - B)^{-1} \left[(\lambda - B)^{-1}\right]^*\end{aligned}$$

and so $(\lambda - B)^{-1}$ is normal.

$\square$

Proposition 4.22. *Let B be a self-adjoint operator. Then for any $\lambda \in \rho(B)$, we have*

$$\left\|(\lambda - B)^{-1}\right\| = \frac{1}{\mathit{dist}(\lambda, \sigma(B))}$$

Proof. As $\lambda \in \rho(B)$, we know by Proposition 4.21 that $(\lambda - B)^{-1}$ is a bounded, normal operator, so we have

$$\left\|(\lambda - B)^{-1}\right\| = \sup\left|\sigma\left((\lambda - B)^{-1}\right)\right|$$

We claim that for any $\zeta \in \rho(B)\setminus\{\lambda\}$, the operator

$$(\lambda - \zeta) + (\lambda - \zeta)^2 (\zeta - B)^{-1}$$

is the (bounded) inverse of $(\lambda - \zeta)^{-1} - (\lambda - B)^{-1}$.

To see this, we first show that the operator

$$S_{\lambda,\zeta} := (\lambda - B)^{-1} - (\zeta - B)^{-1} + (\lambda - \zeta)(\lambda - B)^{-1}(\zeta - B)^{-1}$$

satisfies $S_{\lambda,\zeta}(H) \subset D(B)$ and is the zero operator. For the first part, note that by assumption we have that $(\lambda - B)^{-1}(H) = (\zeta - B)^{-1}(H) = D(B)$, which implies that $(\lambda - B)^{-1}(\zeta - B)^{-1}(H) \subset D(B)$; since $D(B)$ is a linear subspace of H, it follows that $S_{\lambda,\zeta}(H) \subset D(B)$.

For the second part, note that since $S_{\lambda,\zeta}(H) \subset D(B)$, the product $(\lambda - B)\, S_{\lambda,\zeta}$ is well-defined, and we have

$$\begin{gathered}(\lambda - B)\, S_{\lambda,\zeta} = \\ (\lambda - B)\left[(\lambda - B)^{-1} - (\zeta - B)^{-1} + (\lambda - \zeta)(\lambda - B)^{-1}(\zeta - B)^{-1}\right] = \\ I - \lambda(\zeta - B)^{-1} + B(\zeta - B)^{-1} + (\lambda - \zeta)(\zeta - B)^{-1} = \\ I - (\zeta - B)(\zeta - B)^{-1} = 0\end{gathered}$$

As $\lambda - B$ is by assumption injective on $D(B)$, being invertible, the above implies that we must necessarily have that $(\lambda - B)^{-1} - (\zeta - B)^{-1} + (\lambda - \zeta)(\lambda - B)^{-1}(\zeta - B)^{-1}$ is the zero operator.

It follows that

$$\begin{gathered}\left((\lambda - \zeta)^{-1} - (\lambda - B)^{-1}\right)\left((\lambda - \zeta) + (\lambda - \zeta)^2 (\zeta - B)^{-1}\right) = \\ I - (\lambda - \zeta)(\lambda - B)^{-1} + (\lambda - \zeta)(\zeta - B)^{-1} - (\lambda - \zeta)^2(\lambda - B)^{-1}(\zeta - B)^{-1} = \\ I - (\lambda - \zeta)\left[(\lambda - B)^{-1} - (\zeta - B)^{-1} + (\lambda - \zeta)(\lambda - B)^{-1}(\zeta - B)^{-1}\right] = \\ I - (\lambda - \zeta)S_{\lambda,\zeta} = I - 0 = I\end{gathered}$$

By Theorem 4.8, all the involved operators commute, so it follows that we also have

$$\left((\lambda - \zeta) + (\lambda - \zeta)^2 (\zeta - B)^{-1}\right)\left((\lambda - \zeta)^{-1} - (\lambda - B)^{-1}\right) = I$$

Being the sum of (by assumption) bounded operators, $(\lambda - \zeta) + (\lambda - \zeta)^2 (\zeta - B)^{-1}$ must be bounded as well, and thus is the inverse of $(\lambda - \zeta)^{-1} - (\lambda - B)^{-1}$. So we have $(\lambda - \zeta)^{-1} \in \rho\big((\lambda - B)^{-1}\big)$ for all $\zeta \in \rho(B)\setminus\{\lambda\}$.

We claim that this implies that

$$\sup\left|\sigma\Big((\lambda - B)^{-1}\Big)\right| \leq \frac{1}{\text{dist}(\lambda, \sigma(B))} \tag{1}$$

For take any $z \in \mathbb{C}$ with $|z| > \frac{1}{\text{dist}(\lambda,\sigma(B))}$. Clearly there exists some unique $\zeta \in \mathbb{C}\setminus\{\lambda\}$ such that $z = \frac{1}{\lambda-\zeta}$, which implies that

$$\frac{1}{|\lambda - \zeta|} > \frac{1}{\text{dist}(\lambda, \sigma(B))}$$

or equivalently, $|\lambda - \zeta| < \text{dist}(\lambda, \sigma(B))$. It follows by the definition of dist that we cannot have $\zeta \in \sigma(B)$, so by our earlier calculations we must necessarily have that $z = \frac{1}{\lambda-\zeta} \in \rho\Big((\lambda - B)^{-1}\Big)$ for all $z \in \mathbb{C}$ with $|z| > \frac{1}{\text{dist}(\lambda,\sigma(B))}$, which is equivalent to (1).

Next, we claim that

$$\sup\left|\sigma\Big((\lambda - B)^{-1}\Big)\right| \geq \frac{1}{\text{dist}(\lambda, \sigma(B))} \tag{2}$$

As was mentioned in 4.8, the map $\lambda \in \rho(B) \mapsto (\lambda - B)^{-1} \in B(H)$ is analytic in λ; as is argued in [10], we have for each $\lambda \in \rho(B)$ that $\zeta \in \rho(B)$ whenever we have $|\lambda - \zeta| < \|(\lambda - B)^{-1}\|^{-1}$. Thus if we instead had $\zeta \in \sigma(B)$, then we would necessarily have

$$|\lambda - \zeta| \geq \left\|(\lambda - B)^{-1}\right\|^{-1}$$

As this holds for every $\zeta \in \sigma(B)$, we see that

$$\begin{aligned} \text{dist}(\lambda, \sigma(B)) &= \inf_{x\in\sigma(B)} |\lambda - x| \\ &\geq \left\|(\lambda - B)^{-1}\right\|^{-1} \end{aligned}$$

or equivalently

$$\sup\left|\sigma\Big((\lambda - B)^{-1}\Big)\right| = \left\|(\lambda - B)^{-1}\right\| \geq \frac{1}{\text{dist}(\lambda, \sigma(B))}$$

Combining (1) and (2), we see that

$$\sup\left|\sigma\Big((\lambda - B)^{-1}\Big)\right| = \frac{1}{\text{dist}(\lambda, \sigma(B))}$$

□

Note that we could not simply use the bounded Borel functional calculus to prove Proposition 4.22, unlike in the bounded cases, since - as we remarked earlier - we generally only have

$$\|f(B)\| \leq \sup_{x\in\sigma(B)} |f(x)|$$

for unbounded operators B (and valid choices of f), unlike the case where B is bounded, where we can obtain an equality by using the continuous functional calculus.

Theorem 4.23. *Let $A = (A_t)_{t \in T}$ be a field of self-adjoint operators, and assume we are given some point $t_0 \in T$. Then A is R-continuous at t_0 if and only if A is Fell continuous at t_0.*

Proof. Proposition 4.22 implies that A is R-continuous at t_0 if and only if the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is Wijsman continuous. The Lemmas 1.22 and 1.23 imply that Wijsman continuity at t_0 is equivalent to Fell continuity at t_0, so the result follows. □

5 Normal Operators

5.1 The problem with normal operators

The article [3] concerned itself entirely with self-adjoint operators; this simplified matters significantly, as self-adjoint operators always have spectra belonging to the real line, allowing for very simple characterizations of gaps as just intervals. However, genereally speaking, the spectra of operators are considered as subsets of $\mathbb{C}$, and given a not necessarily self-adjoint operator, its spectrum may be any arbitrary closed subset of $\mathbb{C}$.

In this chapter, we will attempt to consider fields of *normal, bounded operators*. However, while boundedness still gives us many good properties, it turns out that normal operators are significantly more troublesome than self-adjoint operators.

We will start by redefining some of the concepts used earlier in the setting of normal, bounded operators.

Definition 5.1. Let F be a closed subset of $\mathbb{C}$. A *gap* of F is a connected component $G \subset \mathbb{C}\backslash F$; each G is necessarily open. A gap G is a *bounded gap* if G is a bounded set, and an *unbounded gap* otherwise.

This definition is significantly harder to work with than the definition of gaps we had in the first section. While the gaps of an element of $\mathscr{C}(\mathbb{R})$ can be characterized very easily as open intervals, there is generally no such easy characterization of gaps of an element of $\mathscr{C}(\mathbb{C})$ - gaps are still open sets, but their shape can be almost entirely arbitrary.

One small advantage of elements F of $\mathscr{C}(\mathbb{C})$ is that if F is bounded (i.e., compact), then F has exactly one unbounded gap; for since F is bounded, there exists some $r > 0$ such that $F \subset B_r(0)$. Now clearly $\mathbb{C}\backslash\overline{B}_r(0)$ is connected (as a subset of F^c) and thus lies in exactly one connected component of F^c. Clearly no other connected component can be unbounded. For unbounded F, however, there is no such result - there can be any number of unbounded gaps, including none and infinitely many.

Another problem is that when we were working in $\mathbb{R}$, gaps behaved "nicely" in the sense of the first and second conditions of gap edge continuity in $\mathbb{R}$ - "convergent families of gaps" either become a single gap or close. In $\mathbb{C}$, however, things are not so simple, as the following example serves to illustrate.

Example 5.2. Set $T = [0, 1]$ with the standard topology inherited from $\mathbb{R}$, and define the map $t \in T \mapsto F_t \in \mathscr{C}(\mathbb{C})$ by

$$F_t := \{z \in \mathbb{C} \mid |\Im(z)| = 1\} \cup \bigcup_{k\in\mathbb{Z}} \{z \in \mathbb{C} \mid |\Re(z)| = k \text{ and } t \leq |\Im(z)| \leq 1\}$$

It can be checked quite easily that we indeed have $F_t \in \mathscr{C}(\mathbb{C})$ for every $t \in T$.

For $t \neq 0$, F_t has three gaps - one "above", one "below" and one "between". However, the number of gaps of F_0 is countably infinite - one "above", one "below" and countably infinite many gaps "between". While the construction of $t \mapsto F_t$ intuitively seems "continuous", it still exhibits disturbing behaviour at $t_0 = 0$.

In particular, this example demonstrates that we cannot hope to find anything resembling Proposition 1.12 for the complex case, which severely complicates any attempt at working with "continuity of gaps" in this setting.

The topological concepts of Hausdorff continuity and Fell continuity fortunately still make sense in the complex setting.

Definition 5.3. Let A be a field of operators. We say that *the spectrum function of A is Hausdorff continuous at a point $t_0 \in T$* (or just that *A is Hausdorff continuous at t_0*) if the function $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{C})$ is Hausdorff continuous at t_0; that is, if the map

$$t \in T \mapsto \text{dist}(\sigma(A_{t_0}), \sigma(A_t)) \in \mathbb{R}^+ \cup \{\infty\}$$

is continuous at t_0.

If the A is Hausdorff continuous at every point $t \in T$, we will say that *the spectrum function of the field A is Hausdorff continuous* (or just that *A is Hausdorff continuous*.

Definition 5.4. Let a A be a field of operators. We say that *the spectrum of A is Fell continuous at a point $t_0 \in T$* (or just that *A is Fell continuous at t_0*) if the spectrum function $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{C})$ is Fell continuous at t_0.

If the spectrum of A is Fell continuous at every point $t \in T$, we say that *the spectrum of A is Fell continuous* (or just that *A is Fell continuous*).

From Lemmas 1.27 and 1.28, we know the relation between these two forms of continuity:

Theorem 5.5. *Let $A = (A_t)_{t \in T}$ be a field of normal, bounded operators, and take $t_0 \in T$. Then A is Fell continuous and locally uniformly bounded at t_0 if and only if it is Hausdorff continuous at t_0.*

However, as before, these forms of continuity are not really tied to the nature of the operators A_t involved, and so verifying either type of continuity cannot be done without geometric study of their spectra. With bounded, self-adjoint operators, we worked with $p2$-continuity and proper-continuity in order to link continuity of the spectra to the behaviour of each operator A_t. Recall that the main "trick" in using $p2$-continuity was that for any gap (a, b) we could define a polynomial taking its maximum value inside that gap; see e.g. Example 2.7 or Lemma 3.13. However, a result from complex analysis tells us that we cannot do anything similar in $\mathbb{C}$ - the famous *maximum principle*, here cited (very slightly paraphrased) from [11]:

Theorem 5.6. *If a function $f : \mathbb{C} \to \mathbb{C}$ is analytic in an open set $D \subset \mathbb{C}$ and $|f(z)|$ achieves its maximum value at a point $z_0 \in D$, then f is constant in D.*

As polynomials are a rather famous example of functions that are analytic on all of $\mathbb{C}$, it follows that no matter which polynomial p and which gap G we choose, we will always have

$$\sup_{z \in G} |p(z)| = |p(z_0)|$$

for some $z_0 \in \partial G$, rendering our "trick" unusable in $\mathbb{C}$. As it is not hard to find functions that are both analytic and proper, we must similarly abandon any hopes of utilizing proper-continuity.

There does exist a rather large class of more or less well-behaved functions that are not analytic on all of $\mathbb{C}$ - the *meromorphic functions*. However, rather than delving into complex analysis, we will simply note that any results that could be gained through studying meromorphic functions can be obtained much more easily by simply studying the behaviour of the inverse, as we shall do in the next section.

5.2 R-continuity for normal operators

In Section 2.3, we stated that a field A of self-adjoint operators is called R-continuous if for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$, the function

$$t \in T \mapsto \left\|(\lambda I - A_t)^{-1}\right\| \in \mathbb{R}^+$$

is continuous. However, for normal operators, this definition does not make sense - while self-adjoint operators have spectra that are always subsets of $\mathbb{R}$, in general normal operators have spectra that can be arbitrary closed subsets of $\mathbb{C}$. Thus while some arbitrary $\lambda \in \mathbb{C}$ may be in the resolvent set of A_{t_0} for some $t_0 \in T$, there is no guarantee that λ will lie in the resolvent set of any other A_t.

However, we can still define a form of R-continuity which will make sense for general fields of bounded operators.

Definition 5.7. Let A be a field of bounded operators, and let t_0 be a point in T. Then A is R-continuous at t_0 if, for every $\lambda \in \mathbb{C}$, the function $d_\lambda : t \in T \to \mathbb{R}^+ \cup \{\infty\}$ defined by

$$d_\lambda(t) := \begin{cases} \|(\lambda I - A_t)^{-1}\| & \text{if } \lambda \in \rho(A_t) \\ \infty & \text{if } \lambda \in \sigma(A_t) \end{cases}$$

is continuous at t_0. If A is R-continuous at every point $t \in T$, we say that A is R-continuous.

The main result of this section is the following:

Theorem 5.8. *Let A be a field of bounded, normal operators, and take any $t_0 \in T$. Then A is Fell-continuous at t_0 if and only if it is R-continuous at t_0.*

Proof. The proof of is largely the same as in Section 2.3, with only slight modification. We start by noting that Lemma 2.25 still holds, so for any bounded, normal operator B we have

$$\left\|(\lambda - B)^{-1}\right\| = \frac{1}{\text{dist}(\lambda, \sigma(B))}$$

for any $\lambda \in \rho(B)$. In particular, it follows that the map $d_\lambda : T \to \mathbb{R}^+$ is continuous at t_0 if and only if the map $t \in T \mapsto \text{dist}(\lambda, \sigma(A_t)) \in \mathbb{R}^+$ is continuous at t_0. Now we just re-use the proof from Section 2.3 - for the map $t \in T \mapsto \text{dist}(\lambda, \sigma(A_t)) \in \mathbb{R}^+$ is continuous at t_0 exactly when the map $t \in T \mapsto \sigma(A_t) \in \mathscr{C}(\mathbb{R})$ is Wijsman continuous at t_0, and Lemmas 1.22 and 1.23 imply that this happens if and only if it is Fell continuous at t_0, so the result follows.

□

Closing thoughts

There are still many interesting questions that remain. In particular, our treatment of normal operators was extremely superficial; however, Beckus and Bellisard point out that many strong results can be obtained through the use of C^*-algebras, an approach that is certainly worth pursuing further. In a similar vein, it is likely that our results about unbounded, self-adjoint operators could have been strengthened considerably through the use of alternative approachs. The author also suspects that, if done correctly, one can construct something similar to $p2$-continuity for unbounded operators, and possibly use it to give certain Hölder estimates even in the unbounded case.

When it comes to Hölder estimates, the author also spent some time trying to find a version of Theorem 3.31 that would hold even when the closed gap c is not an interior point; although the results so far are too fragmentary to include in this thesis, it is quite possibly a problem worth revisiting.

A very interesting idea, suggested by Professor Christian Pötzsche in personal conversation, is extending our definitions and results to encompass fields where instead of working on Hilbert spaces, we work on Banach spaces. Unfortunately there was not enough time and space to deal with this approach, but it is another promising future project.

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