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The Noncommutative Phase Space: An Algebraic Approach to Differential Geometry

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Cover design by Martin Helsø

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Preface

Abstract

In this thesis we will study the phase space, $Ph(A)$, for an associative k-algebra A. The phase space can be considered as a noncommutative tangent bundle. We will derive algebraic notions of points, curves, tangent vectors and vector fields, in addition to study differentiation of vector fields, and look at what are called integrable distributions.

Acknowlegdements

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Introduction

In the article *Non-commutative phase spaces, and generalized de rham complexes* [Lau14], Olav Arnfinn Laudal defines the phase space $Ph(A)$ of an associative k-algebra A, where k is a field. The phase space can be considered as the noncommutative tangent bundle; we will see that the collection of maps from the phase space of A into the field k gives us all tangent vectors at all points of A. Oddbjørn Mathias Nødland continued Laudal's work in his master's thesis *Noncommutative tangent bundle: The phase space* [Nø12]. We will in this thesis proceed by exploring how we can use the phase space to give an algebraic variant of certain aspects of differential geometry. The thesis is structured as follows:

In **Chapter 1** we will give a quick walk-through on how to construct the phase space $Ph(A)$ for a k-algebra A. We will, in particular, look at the phase space of rings of the form

$$k < x_1, x_2, \ldots, x_n > /I,$$

where I is an ideal of $k < x_1, x_2, \ldots, x_n >$.

Chapter 2 consists of algebraic definitions of points, curves, infinitesimal curves, tangent vectors and vector fields. In addition we will give an example to show why it is necessary for the phase space to be noncommutative.

Having defined vector fields and tangent vectors, we will in **Chapter 3** see how we can take the derivative of vector fields with respect to tangent vectors and with respect to other vector fields. After that we will generalize the derivative to what is called an affine connection, and look at two applications; torsion fields, and differentiation of vector fields along curves.

In the last chapter, **Chapter 4**, we will study a certain type of subset of the phase space $Ph(A)$, i.e., instead of looking at the collection of all tangent vectors at all points, we will only choose some tangent vectors at each point.

By use of the coproduct of noncommutative rings, we will find a condition for when the phase space of a curve or a subspace of A corresponds to such a subset.

Throughout the thesis, we will assume that k is an algebraically closed field, and homomorphisms are always k-algebra homomorphisms. The reader should be familiar with basic manifold theory and algebraic geometry.

Chapter 1

The phase space

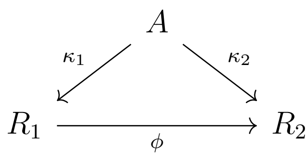
In this section we will give a short introduction to the construction of the phase space, $Ph(A)$, for an associative k-algebra A, following [Nø12] and [Lau14]. The construction involves a certain kind of map called a derivation.

Definition 1.0.1. Assume that A and B are associative k-algebras. A **derivation** $D\colon A \to B$ is a k-linear map satisfying the Leibniz rule

$$D(fg) = D(f)\kappa(g) + \kappa(f)D(g)$$

for some homomorphism $\kappa\colon A \to B$.

Let $A/k - alg$ denote the category where the objects are homomorphisms $\kappa\colon A \to R$ for k-algebras R, and where the morphisms are commutative diagrams,



For simplicity, an object $A \to R$ will often just be referred to as a k-algebra R, and a morphism, i.e. a commutative diagram, as a homomorphism $R_1 \to R_2$.

Assume that $A \to S$ and $A \to Ph(A)$ are two objects in A/k-alg. Let

$$Der_k(A, R)$$

be the set of all derivations from A into R, and

$$Hom_{A/k}(Ph(A), R)$$

the set of all homomorphisms (in A/k-alg) from $Ph(A)$ into R, where R is some k-algebra. Then

$$Der_k(A, -)\colon A/k\text{-alg } \to Sets$$

and

$$Hom_{A/k}(Ph(A), -) \colon A/k\text{-alg} \to Sets$$

are functors from A/k-alg into the category of sets. Our goal is to find an object $A \to Ph(A)$ such that

$$Der_k(A, -) \cong Hom_{A/k}(Ph(A), -).$$

Assume that we have an object $A \to Ph(A)$ and a derivation $d \colon A \to Ph(A)$ satisfying the following universal property: For all derivations $D \in Der_k(A, R)$ there is a unique k-algebra homomorphism $\phi : Ph(A) \to R$ such that $\phi \circ d = D$. Then $Der_k(A, -) \cong Hom_{A/k}(Ph(A), -)$. The implication the other way holds as well: If an object $A \to Ph(A)$ satisfies $Der_k(A, -) \cong Hom_{A/k}(Ph(A), -)$, there is a derivation $d \colon A \to Ph(A)$ (corresponding to the identity map in $Hom_{A/k}(Ph(A), Ph(A))$) satisfying the universal property above. Thus, there is an object $A \to Ph(A)$ such that $Der_k(A, -) \cong Hom_{A/k}(Ph(A), -)$ if and only if there is an object $A \to Ph(A)$ and a derivation $d \colon A \to Ph(A)$ satisfying the universal property. If such an object $A \to Ph(A)$ exists, it is unique up to unique isomorphism (a proof can be found in [Nø12]).

For an associative k-algebra A, $\iota \colon A \to Ph(A)$ can be constructed by letting $Ph(A)$ be the algebra generated by the symbols a and da for all $a \in A$, subject to the two relations

1. k-linearity: $d(ra) = rda$ for all $r \in k$ and $a \in A$,
2. the Leibniz rule: $d(ab) = dab + adb$ for all a,$b \in A$,

and letting $\iota \colon A \to Ph(A)$ be the inclusion map. In the special case where A is of the form

$$A = k < x_1, x_2, \ldots, x_n > /I,$$

we have

$$Ph(A) = k < x_1, x_2, \ldots, x_n, dx_1, dx_2, \ldots, dx_n > /(I, dI),$$

where (dI) is the ideal generated by the elements df for all $f \in I$. To prove this, it is enough to show that there is a derivation $d \colon A \to Ph(A)$ satisfying the universal property. First, note that ι is well-defined since it sends the ideal I to zero. Let $d \colon A \to Ph(A)$ be the map that sends $a \in A$ to $da \in Ph(A)$. The ideal I will then be sent to zero, so it is well-defined. Suppose $D \colon A \to R$ is a derivation,

$$D(fg) = D(f)\kappa(g) + \kappa(f)D(g),$$

where $\kappa\colon A \to R$ is an object in A/k-alg. We define ϕ by letting

$$\phi(x_i) = \kappa(x_i)$$

and

$$\phi(dx_i) = D(x_i).$$

This is well-defined, and $\phi \circ d = D$. It is also unique, because it is the only way we can define ϕ: Let $x_i, x_j \in A$ and $\psi\colon Ph(A) \to R$ any homomorphism satisfying $\psi \circ d = D$. Then

$$\begin{aligned}\psi(d(x_i x_j)) &= \psi(dx_i x_j + x_i dx_j) \\ &= \psi(dx_i)\psi(x_j) + \psi(x_i)\psi(dx_j) \\ &= D(x_i)\psi(x_j) + \psi(x_i)D(x_j).\end{aligned}$$

But since $\psi \circ d = D$, we also have that

$$\begin{aligned}\psi(d(x_i x_j)) &= D(x_i x_j) \\ &= D(x_i)\kappa(x_j) + \kappa(x_i)D(x_j),\end{aligned}$$

so $\psi(x_i) = \kappa(x_i) = \phi(x_i)$ for all $i = 1, 2, \ldots, n$. Since the object $\iota\colon A \to Ph(A)$ with the derivation $d\colon A \to Ph(A)$ satisfies the universal property, we know that $\iota\colon A \to Ph(A)$ is the unique (up to unique isomorphism) object such that

$$Der_k(A, -) \cong Hom_{A/k}(Ph(A), -).$$

For a k-algebra A, we call the unique object $A \to Ph(A)$ satisfying

$$Der_k(A, -) \cong Hom_{A/k}(Ph(A), -)$$

the **phase space of** A. As pointed out earlier, an object in A/k-alg will just be referred to as a k-algebra. Hence whenever $Ph(A)$ or A are mentioned, we mean the homomorphism $\iota\colon A \to Ph(A)$ and the identity map on A, respectively.

Example 1.0.2. The polynomial ring $A = k[x, y]$ may be written as

$$\begin{aligned}A &= k < x, y > /(xy - yx) \\ &= k < x, y > /([x, y]).\end{aligned}$$

Therefore,

$$\begin{aligned}Ph(A) &= k < x, y, dx, dy > /([x, y], dxy + xdy - dyx - ydx) \\ &= k < x, y, dx, dy > /([x, y], [dx, y] + [x, dy]).\end{aligned}$$

♣

Example 1.0.3. Let

$$\begin{aligned} A &= k[x,y]/(y-x^2) \\ &= k<x,y>/([x,y],y-x^2). \end{aligned}$$

The phase space will then be

$$Ph(A) = k<x,y,dx,dy>/I$$

where

$$I = ([x,y], y-x^2, [dx,y]+[x,dy], dy-xdx-dxx).$$

♣

Because $Ph(A)$ is an associative k-algebra, we can construct

$$Ph(Ph(A)) = Ph^2(A).$$

Let the derivation corresponding to the identity homomorphism $Ph(A) \to Ph(A)$ be called d_0 instead of d, and $d_1 \in Der(Ph(A), Ph^2(A))$ be the derivation corresponding to the identity homomorphism $Ph^2(A) \to Ph^2(A)$. Then $Ph^2(A)$ is generated by all symbols f and $d_1 f$, where $f \in Ph(A)$, or, equivalently, by all symbols a, $d_0 a$, $d_1 a$ and $d_1 d_0 a$, where $a \in A$.

Example 1.0.4. Let A be the ring from the previous example:

$$A = k[x,y]/(y-x^2).$$

We saw that

$$Ph(A) = k<x,y,d_0x,d_0y>/I,$$

where

$$I = ([x,y], y-x^2, [d_0x,y]+[x,d_0y], d_0y-xd_0x-d_0xx).$$

$Ph^2(A)$ will then look like

$$k<x,y,d_0x,d_0y,d_1x,d_1y,d_0d_1x,d_0d_1y>/(I,J),$$

where the ideal $J = d_1 I$ is generated by the following elements:

$$\begin{aligned} &[d_1x,y]+[x,d_1y], \\ &d_1y-xd_1x-d_1xx, \\ &[d_1d_0x,y]+[d_0x,d_1y]+[d_1x,d_0y]+[x,d_1d_0y], \\ &d_1d_0y-d_1xd_0x-xd_1d_0x-d_1d_0xx-d_0xd_1x. \end{aligned}$$

♣

We can continue in this way, and construct

$$Ph^3(A) = Ph(Ph^2(A)),$$

$$Ph^4(A) = Ph(Ph^3(A)),$$

and in general

$$Ph^n(A) = Ph(Ph^{n-1}(A))$$

for all $n \in \mathbb{N}$. We let $Ph^0(A) = A$. In addition to the derivations

$$d_n \colon Ph^n(A) \to Ph^{n+1}(A),$$

we have the canonical homomorphisms

$$\iota_0^n \colon Ph^n(A) \to Ph^{n+1}(A).$$

Composing a homomorphism with a derivation gives a new derivation, so $d_{n+1} \circ \iota_0^n$ is a derivation from $Ph^n(A)$ to $Ph^{n+2}(A)$ for all $n \geq 0$. Since $Der(Ph^n(A), Ph^{n+2}(A)) \cong Hom_{A/k}(Ph^{n+1}(A), Ph^{n+2}(A))$, we know that there is a map

$$\iota_1^{n+1} \colon Ph^{n+1}(A) \to Ph^{n+2}(A)$$

such that $d_{n+1} \circ \iota_0^n = \iota_1^{n+1} \circ d_n$. But now we have that for all $n \geq 1$, the composition

$$d_{n+1} \circ \iota_1^n \colon Ph^n(A) \to Ph^{n+2}(A)$$

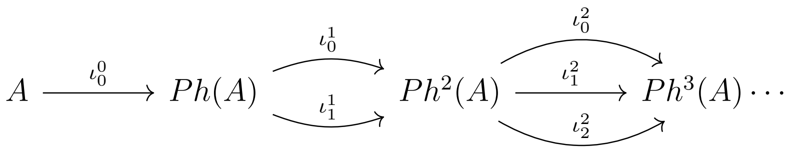
is a derivation. Hence, there exists a homomorphism

$$\iota_2^{n+1} \colon Ph^{n+1}(A) \to Ph^{n+2}(A)$$

such that $d_{n+1} \circ \iota_1^n = \iota_2^{n+1} \circ d_n$. Continuing like this, we get a family of homomorphisms

$$\{\iota_j^n \colon Ph^n(A) \to Ph^{n+1}(A)\}_{j=0}^n$$

for each $n \geq 0$:



In this way we can construct $Ph^\infty(A)$, which is the direct limit of this direct system. We also get an induced derivation $\delta \colon Ph^\infty(A) \to Ph^\infty(A)$. For more details about this construction, see [Lau14].

Chapter 2

Curves, tangent vectors and vector fields

2.1 Curves

In this section we will derive algebraic definitions of points and curves that will correspond to the geometric definitions.

Let $A = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring. By defining a homomorphism

$$p\colon A \to k,$$

we assign to each x_i, where $i = 1, 2, \ldots, n$, an element $a_i \in k$. This gives us a point in k^n. We could for instance let $p(x_i) = i$ for all $i = 1, 2, \ldots, n$, which gives us the point $(1, 2, \ldots, n) \in k^n$. Therefore we will have the following definition of a point:

Definition 2.1.1. Assume A is a k-algebra. A **point** of A is a homomorphism $p\colon A \to k$.

Example 2.1.2. Let $A = k[x, y]/(x^2 + y^2 - 1)$. A homomorphism $p\colon A \to k$ then gives a point on the unit circle. ♣

A curve in $A = k[x_1, x_2, \ldots, x_n]$ can be defined similarly. Let $\gamma\colon A \to k[t]$ be a homomorphism. This corresponds to a morphism between the spectra of these rings,

$$\widetilde{\gamma}\colon Spec(k[t]) \to Spec(A),$$

which is a map between two topological spaces. A curve in a topological space X is a map from a line (or part of line; $I \subset \mathbb{R}$) into X. The set of closed points of $Spec(k[t])$, i.e., the set of maximal ideals, corresponds to the affine line $\mathbb{A}^1$, so the morphism $\widetilde{\gamma}$ actually defines a curve in $X = Spec(A)$. Hence, the homomorphism γ from A into $k[t]$ corresponds to a curve in $Spec(A)$. We will require γ to be surjective, because this implies that $\widetilde{\gamma}$ is injective.

There are, however, other rings than polynomial rings. Consider $A = k[x, x^{-1}]$. It is not possible to find a surjective homomorphism from A into $k[t]$. This is because we would have to map either x or x^{-1} to t, and $xx^{-1} = 1$ has to be mapped to 1, which implies that either x or x^{-1} must be mapped to the inverse of t, but t does not have an inverse in $k[t]$. Again, let us look at the spectrum of A. The ring A is actually isomorphic to $S^{-1}k[x]$, where $S = \{x^l\}_{l \geq 0}$. The prime ideals of this ring are in one-to-one correspondence with the prime ideals of $k[x]$ that do not meet S. The only prime ideal in $k[x]$ that meets S is (x), so

$$Spec(A) \cong \{(0)\} \cup \{(x-a)\colon a \in k, a \neq 0\}.$$

In other words, $Spec(A)$ is the affine line with the point 0 removed; $\mathbb{A}^1 \setminus \{0\}$. Therefore we must remove at least one point from the affine line $Spec(k[t])$ to be able to obtain an injective map. We could for instance let γ be a homomorphism from A onto $T^{-1}k[t]$, where $T = \{t^l\}_{l \geq 0}$, by letting x be mapped to t, and x^{-1} to $\frac{1}{t}$. Of course, $Spec(T^{-1}k[t]) = Spec(A)$, and the morphism corresponding to γ,

$$\widetilde{\gamma}\colon Spec(T^{-1}k[t]) \to Spec(A),$$

is just the identity map. We arrive at the following definition:

Definition 2.1.3. Let A be a k-algebra. A **curve** in A is a surjective homomorphism

$$\gamma\colon A \to B,$$

where B is a smooth k-algebra of dimension 1.

A point on a curve $\gamma\colon A \to B$ is a homomorphism $p\colon A \to k$ that factors through γ, i.e. it is a composition $q \circ \gamma$ for some $q\colon B \to k$.

Example 2.1.4. Let $A = k[x, y]$ and $\gamma\colon A \to k[t]$ the homomorphism that sends x to t and y to 0. The inverse image of the maximal ideal $(t-a) \subset k[t]$, where $a \in k$, is the ideal $(x-a, y) \subset k[x, y]$. Hence, if we identify $Spec(k[t])$ with $\mathbb{A}^1$ and $Spec(k[x, y])$ with $\mathbb{A}^2$, the map $\widetilde{\gamma}\colon \mathbb{A}^1 \to \mathbb{A}^2$ sends $a \in \mathbb{A}^1$ to $(a, 0) \in \mathbb{A}^2$, which gives us the horizontal line through the origin.

Let $q\colon k[t] \to k$ send t to 3. Then $q(\gamma(x)) = 3$ while $q(\gamma(y)) = 0$. ♣

Example 2.1.5. Let A be as in the previous example, but define $\gamma\colon A \to k[t]$ to be the homomorphism such that $\gamma(x) = t$ and $\gamma(y) = t^2$. Again, we identify $Spec(k[t])$ with $\mathbb{A}^1$, and $Spec(A)$ with $\mathbb{A}^2$. A point $a \in \mathbb{A}^1$ then corresponds to the maximal ideal $(t - a) \subset k[t]$, whose inverse image is the ideal $(x - a, y - a^2) \subset Spec(A)$. This ideal corresponds to the point $(a, a^2) \in \mathbb{A}^2$. Thus, γ is a parabola.

Choose a point $q\colon k[t] \to k$ such that $q(t) = 2$. Then $q(\gamma(x)) = 2$ and $q(\gamma(y)) = 4$. ♣

If γ is a curve, it is a map between two k-algebras A and B. Such a map can be extended to a map between the phase spaces of these k-algebras. We define $\gamma_*\colon Ph(A) \to Ph(B)$ as follows: For any $a \in A$, we let

$$\gamma_*(a) = \gamma(a)$$

and

$$\gamma_*(da) = d\gamma(a).$$

This map is well-defined, meaning $\gamma_*(d(ab)) = d\gamma(ab)$, because

$$\begin{aligned}\gamma_*(d(ab)) &= \gamma_*(dab + adb)\\ &= d(\gamma(a))\gamma(b) + \gamma(a)d(\gamma(b))\\ &= d(\gamma(a)\gamma(b))\\ &= d(\gamma(ab)).\end{aligned}$$

Hence, whenever we have a homomorphism $\gamma\colon A \to B$ between two k-algebras A and B (here B does not have to be smooth of dimension 1), we can construct a map $\gamma_*\colon Ph(A) \to Ph(B)$ giving us the following commutative diagram:

$$\begin{array}{ccc} Ph(A) & \xrightarrow{\gamma_*} & Ph(B) \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow \\ A & \xrightarrow{\gamma} & B. \end{array}$$

Sometimes we only want to look at a part of a curve. For instance, we might want to look at the part of the curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, where $\gamma(t) = (t, t^2)$, that starts at $t = -2$ and ends at $t = 2$. In algebra, this is not as easy. Nevertheless, it is possible to define an infinitesimal part of a curve. In the example above, this can be obtained by restricting γ to an interval $(-\epsilon, \epsilon)$, and letting ϵ be arbitrarily close to 0. As we have seen, homomorphisms from a ring A into the ring $k[t]$ correspond to maps from the affine line $\mathbb{A}^1 = Spec(k[t])$ into $Spec(A)$. If we were to look at an infinitesimal part of the curve $\mathbb{A}^1 \to Spec(A)$, we could compose with the inclusion map

$$Spec(k[t]/(t - r)^2) \to Spec(k[t]),$$

where $r \in k$, yielding a map from $Spec(k[t]/(t-r)^2)$ into $Spec(A)$. The only maximal ideal in $k[t]/(t-r)^2$ is $(t-r)$, which corresponds to the point r in $\mathbb{A}^1$. The inclusion map $Spec(k[t]/(t-r)^2) \to Spec(k[t])$ corresponds to the quotient map

$$k[t] \to k[t]/(t-r)^2,$$

so the composition $Spec(k[t]/(t-r)^2) \to Spec(k[t]) \to A$ corresponds to the composition

$$A \to k[t] \to k[t]/(t-r)^2.$$

The next proposition tells us that we can do exactly the same for any smooth k-algebra of dimension 1, not only for $k[t]$.

Proposition 2.1.6. *Let B be a smooth k-algebra of dimension* 1*, and let $\mathfrak{m} \subset B$ be a maximal ideal in B. Then*

$$B/\mathfrak{m}^2 \cong k[t]/(t^2).$$

Proof. Assume that B is a smooth local k-algebra of dimension 1 with maximal ideal $\mathfrak{m}$. Then we have the following short exact sequence:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to B/\mathfrak{m}^2 \to B/\mathfrak{m} \to 0.$$

Since $dim_k(\mathfrak{m}/\mathfrak{m}^2) = dim(B) = 1$, and $B/\mathfrak{m} \cong k$, $dim_k(B/\mathfrak{m}^2) = 2$. Hence, $B/\mathfrak{m}^2 \cong k[t]/(t^2)$.

Now assume that B is any smooth k-algebra of dimension 1, and that $\mathfrak{m}$ is a maximal ideal. Then $B_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{n} = \mathfrak{m}B_{\mathfrak{m}}$. Therefore, by the argument above, $B_{\mathfrak{m}}/\mathfrak{n}^2 \cong k[t]/(t^2)$. But $B_{\mathfrak{m}}/\mathfrak{n}^2 \cong B/\mathfrak{m}^2$ (for a proof, see e.g. [Mil14, p. 21]). Thus, $B/\mathfrak{m}^2 \cong k[t]/(t^2)$. □

Definition 2.1.7. An **infinitesimal curve** in A is a composition of a curve $\gamma\colon A \to B$ with the quotient map $\phi\colon B \to B/\mathfrak{m}^2$, where $\mathfrak{m}$ is a maximal ideal in B.

Example 2.1.8. We let A and γ be as in Example 2.1.5. Composing with the quotient map $\rho\colon k[t] \to k[t]/(t^2)$, we get a homomorphism $A \to k[t]/(t^2)$ sending x to t and y to 0. ♣

If we want to choose a point $p = q \circ \rho \circ \gamma$, where $q\colon k[t]/(t^2) \to k$, on the curve in this example, we have to let $q(t) = 0$. Then $p(x) = p(y) = 0$, so we are looking at the infinitesimal part of the curve around the origin. The following example shows us how to focus on a different part of the same curve.

Example 2.1.9. Assume that we want to look at the curve γ around the point $(2,4)$. Let

$$\rho\colon k[t] \to k[t]/(t-2)^2$$

be the quotient map. In this case, a homomorphism $q\colon k[t]/(t-2) \to k$ has to send t to 2, so

$$p = q \circ \rho \circ \gamma$$

sends x to 2 and y to 4. Hence, we are focusing on the curve around the point $(2,4)$. ♣

Let us look more closely at the definition of an infinitesimal curve; $\gamma\colon A \to k[t]/(t^2)$. An element in $k[t]/(t^2)$ can be written as $d + te$ where $d, e \in k$. Thus, for any $a \in A$, we have

$$\gamma(a) = \gamma_0(a) + tD(a),$$

where γ_0 and D are maps from A into k. Since γ is a homomorphism, we know that:

$$\begin{aligned}\gamma(ab) &= \gamma(a)\gamma(b)\\ &= (\gamma_0(a) + tD(a))(\gamma_0(b) + tD(b))\\ &= \gamma_0(a)\gamma_0(b) + t(\gamma_0(a)D(b) + D(a)\gamma_0(b)) + t^2(D(a)D(b)\\ &= \gamma_0(a)\gamma_0(b) + t(\gamma_0(a)D(b) + D(a)\gamma_0(b)).\end{aligned}$$

But we also have that $\gamma(ab) = \gamma_0(ab) + tD(ab)$. Therefore, $\gamma_0(ab) = \gamma_0(a)\gamma_0(b)$ and $D(ab) = \gamma_0(a)D(b) + D(a)\gamma_0(b)$, and γ_0 is a homomorphism, while D is a derivation.

Example 2.1.10. Let $A = k[x,y]$ and $\gamma\colon A \to k[t]$ the curve where $\gamma(x) = t$ and $\gamma(y) = t^3$. Compose with the quotient map $\rho\colon k[t] \to k[t]/(t-1)^2$ to obtain the homomorphism

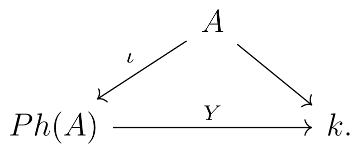
$$\delta = \rho \circ \gamma\colon A \to k[t]/(t-1)^2$$

that maps x to t and y to $3t-2$. A point $q\colon k[t]/(t-1)^2 \to k$ must send t to 1, which gives us the point $p = q \circ \rho \circ \gamma$ on the curve sending both x and y to 1. As we saw above, we can write $\delta = \delta_0 + tD$. In this case $\delta_0(x) = 0$, $\delta_0(y) = -2$, $D(x) = 1$ and $D(y) = 3$. ♣

2.2 Tangent vectors and vector fields

Assume that $A = k[x_1, x_2, \ldots, x_n]/I$ for some ideal $I \subset k[x_1, x_2, \ldots, x_n]$. Let us look at a homomorphism (in the category $A/k - alg$) $Y\colon Ph(A) \to k$.

This is actually a commutative diagram:



Since ι is the inclusion map, Y gives a map from A into k. As we saw in the previous section, this corresponds to a point. The map Y also assigns an element in k to each $dx_i \in Ph(A)$. The n-tuple

$$(Y(dx_1), Y(dx_2), \dots, Y(dx_n)) \in k^n$$

can be interpreted as a tangent vector at the point

$$(Y(x_1), Y(x_2), \dots, Y(x_n)) \in k^n.$$

Example 2.2.1. Let $A = k[x, y]/(y - x^2)$. Then

$$Ph(A) = k < x, y, dx, dy > /I,$$

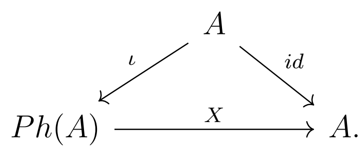
where

$$I = ([x, y], [dx, y] + [x, dy], y - x^2, dy - dxx - xdx).$$

Now we will define a homomorphism $Y : Ph(A) \to k$. Let $Y(x) = a \in k$. Then, because $y - x^2$ should be sent to 0, we must let $Y(y) = a^2$. We have chosen the point $(a, a^2) \in k^2$. To choose a tangent vector, we have to decide where we should send dx and dy. Let $Y(dx) = v \in k$. The relation $dy - dxx - xdx$ forces us to have $Y(dy) = Y(dx)Y(x) + Y(x)Y(dx) = 2av$. Hence, Y is the tangent vector $(v, 2av)$ at the point (a, a^2). If we for instance let $a = 2$ and $v = 1$, we get the vector $(1, 2)$ at the point $(2, 4)$. ♣

Note that the derivative of the function $f : \mathbb{R} \to \mathbb{R}^2$, where $f(x) = (x, x^2)$, is $(1, 2x)$. Multiplying with a $y \in \mathbb{R}$ gives the vector $(y, 2xy)$, or $(v, 2av)$ if we let $x = a$ and $y = v$.

A vector field should assign a tangent vector to each point in a space. Above, we saw that if we choose a homomorphism $Ph(A) \to k$, we have also chosen a point because we get a map from A into k. If we instead look at a homomorphism $X : Ph(A) \to A$, we have the following commutative diagram:



Since the diagram should commute, X must be the identity on A. Hence, X does not make any decisions about the elements of A. Therefore we can choose any point $p\colon A \to k$ we want, and the composition $p \circ X$ restricted to A will just give that exact point. The composition also gives a tangent vector. Hence, for each homomorphism $p\colon A \to k$, the composition $p \circ X$ gives us a point, and a tangent vector at that point.

Example 2.2.2. Let $A = k[x, y]$. Then

$$Ph(A) = k < x, y > /([x, y], [dx, y] + [x, dy]).$$

Define a homomorphism $X\colon Ph(A) \to A$ by sending both x and y to themselves (it has to be the identity on A because of the commutative diagram), and dx to x and dy to y. If we choose the point $p\colon A \to k$ that sends x and y to 2 and 3, respectively, $p(X(x)) = 2$, $p(X(y)) = 3$, $p(X(dx)) = 2$ and $p(X(dy)) = 3$. Thus, $p \circ X = Y$ is the tangent vector $(2, 3)$ at the point $(2, 3)$. For a general point p, where $p(x) = a$ and $p(y) = b$, we get the tangent vector (a, b) at the point (a, b). ♣

Example 2.2.3. Let $A = k[x, y]/(x^2 + y^2 - 1)$. The set $Z(x^2 + y^2 - 1) = Z(f) = \{P \in k^n : f(P) = 0\}$ defines the unit circle (if $k = \mathbb{R}$). We have that

$$Ph(A) = k < x, y, dx, dy > /I,$$

where

$$I = ([x, y], x^2 + y^2 - 1, [dx, y] + [x, dy], dxx + xdx + dyy + ydy).$$

A homomorphism $X : Ph(A) \to A$ that sends both x and y to themselves will always send $[dx, y] + [x, dy]$ to 0 because A is commutative, so we only have to make sure that $dxx + xdx + dyy + ydy$ will be sent to 0. We can for instance let $X(dx) = -y$ and $X(dy) = x$. If we choose a point, let us say $p(x) = \frac{1}{2}$ and $p(y) = \frac{\sqrt{3}}{2}$ (notice that $p(x^2 + y^2 - 1) = 0$), then

$$P = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

and

$$v = (-\frac{\sqrt{3}}{2}, \frac{1}{2}),$$

where $P = (p(X(x)), p(X(y)))$ and $v = (p(X(dx)), p(X(dy)))$. The vector v is actually a tangent vector to the circle at the point P.

Now we choose X a little differently. Let $X(dx) = xy$ and $X(dy) = -x^2$. The element $dxx+xdx+dyy+ydy$ is mapped to 0 by X, so it is a well-defined homomorphism. Keeping our point $p : A \to k$, we see that in this case

$$P = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

and

$$v = (\frac{\sqrt{3}}{4}, -\frac{1}{4}).$$

Again v is tangent to the circle at the point P. ♣

In the case where $A = k[x_1, x_2, \ldots, x_n]/I$, we have seen that we can define a tangent vector to be a homomorphism $Ph(A) \to k$ and a vector field to be a homomorphism $Ph(A) \to A$. Thus, in the general case we will have the following definition:

Definition 2.2.4. Let A be a k-algebra. A **tangent vector** in A is a homomorphism (in $A/k - alg$)

$$Y \colon Ph(A) \to k.$$

A **vector field** on A is a homomorphism (in A/k-alg)

$$X \colon Ph(A) \to A.$$

For each point $p \colon A \to k$, we can possibly have many tangent vectors $Y \colon Ph(A) \to k$ such that $Y \circ \iota = p$, i.e., many tangent vectors at the same point. We will call the set

$$\{Y \in Hom(Ph(A), k) : Y \circ \iota = p\}$$

of all tangent vectors at a point $p \colon A \to k$ the **tangent space to** A **at** p.

Recall from Section 2.1 that an infinitesimal part of a curve, $\gamma \colon A \to k[t]/(t^2)$, can be written as a sum

$$\gamma = \gamma_0 + tD,$$

where $D \colon A \to k$ is a derivation. But we also know that derivations from A correspond to homomorphisms from $Ph(A)$, so $D \colon A \to k$ corresponds to a homomorphism $Y \colon Ph(A) \to k$, which is a tangent vector. So for each infinitesimal curve, there is a corresponding tangent vector.

Example 2.2.5. In Example 2.1.10 we looked at the infinitesimal curve

$$\delta \colon A \to k[t]/(t-1)^2$$

that sends x to t and y to $3t - 2$. We saw that $\delta = \delta_0 + tD$ where $D(x) = 1$ and $D(y) = 3$. The corresponding tangent vector $Y \colon Ph(A) \to k$ sends dx to $D(x) = 1$ and dy to $D(y) = 3$. So at the point $(1, 1)$, we have the tangent vector Y giving the direction of the curve at that point. ♣

2.3 Noncommutativity of the phase space

In this section we will explain why it is necessary for the phase space to be noncommutative. The theorem below is called the inverse function theorem on manifolds, and can be found in [Lee12, p.79]. It states that whenever a map between two manifolds induces an isomorphism of tangent spaces at a point, it is invertible on a neighbourhood around that point.

Theorem 2.3.1. *Let $F\colon M \to N$ be a smooth map between two smooth manifolds M and N. For any $p \in M$, $F_*\colon T_pM \to T_{f(p)}N$ is an isomorphism if and only if there is a neighbourhood $U \subset M$ around p such that $f|_U\colon U \to f(U)$ is a diffeomorphism.*

The next example will show us that when considering the commutativization of the phase space, we get a contradiction to the inverse function theorem.

Example 2.3.2. Let $A = k[y]$ and $B = k[x, y]/(y - x^2)$, and let $\phi\colon A \to B$ be the homomorphism sending y to $y = x^2$. The corresponding morphism between the spectra of A and B is just the map projecting any point on the curve $y = x^2$ to the y-axis. We can, as in the noncommutative case, extend ϕ to a map

$$\phi_*\colon Ph(A)_{com} \to Ph(B)_{com},$$

where $Ph(A)_{com}$ and $Ph(B)_{com}$ denote the commutativizations of $Ph(A)$ and $Ph(B)$:

$$Ph(A)_{com} = k[y, dy]$$

and

$$Ph(B)_{com} = k[x, y, dx, dy]/(y - x^2, dy - 2xdx).$$

Since $\phi_*(dy) = d\phi(y)$, dy is mapped to $dy = 2xdx$. Looking at the morphism between the spectra, we see that we have a one-to-one correspondence between the tangent vectors at points along the curve $y = x^2$ and the tangent vectors at points along the y-axis (except for at the origin). However, if U is any open neighbourhood (in the Zariski-topology) around any of these points, there will always be two points on the curve mapping to the same point on the y-axis. Thus, we can not find an inverse to this map. ♣

Since we can not find an inverse to the map in this example, it gives a contradiction to the inverse function theorem. Consider instead the noncommutative phase space. We can find a homomorphism from $Ph(B)$ into the noncommutative ring $M_2(k)$. We have seen that for a k-algebra A, a homomorphism $\gamma\colon A \to k[t]/(t^2)$ can be written as a sum $\gamma_0 + tD$, where $\gamma_0\colon A \to k$ is a homomorphism and $D\colon A \to k$ is a derivation. The subset

$$\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in k\}$$

of $M_2(k)$ is isomorphic to $k[t]/(t^2)$, so a map $\gamma\colon A \to M_2(k)$ into this subset gives us a point and a tangent vector. If instead the $x_i \in A$, where we let $A = k[x_1, x_2, \ldots, x_n]$, are mapped to elements of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(k),$$

we are looking at two points simultaneously. This is because there are two homomorphisms from the set of such matrices into k; one sending the matrix to a, and the other sending it to b. We can extend $\gamma\colon A \to M_2(k)$ to a homomorphism $\widetilde{\gamma}\colon Ph(A) \to M_2(k)$. The following example shows us that we have non-zero tangent vectors where we in the commutative case can only have zero tangent vectors.

Example 2.3.3. Let B be as in the previous example. Define a map $\gamma\colon B \to M_2(k)$ by letting

$$\gamma(x) = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix},$$

where $a \neq 0$. Then, since $y - x^2 = 0$, we must have

$$\gamma(y) = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}.$$

Thus, we are looking at the points (a, a^2) and $(-a, a^2)$. We can extend this map to a homomorphism $\widetilde{\gamma}\colon Ph(B) \to M_2(k)$. Let

$$\widetilde{\gamma}(dx) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\widetilde{\gamma}(dy) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since

$$\widetilde{\gamma}(dxx + xdx) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$dy - dxx - xdx$ is mapped to 0, so $\widetilde{\gamma}$ is well-defined. Note that if we look at $Ph(A)_{com}$ instead, we must have

$$\widetilde{\gamma}(dy - 2xdx) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so if dy is mapped to 0, we must map dx to 0 as well. ♣

The example tells us that we need the phase space to be noncommutative in order to have enough tangent vectors such that we do not get an isomorphism on differentials. Then the inverse function theorem does not apply, so we do not get a contradiction.

Chapter 3

Differentiation of vector fields

3.1 Affine connections

We will begin this section by looking at differentiation of vector fields in differential geometry, following [Csi98, Unit 11]. Thereafter, we will give corresponding algebraic definitions of differentiation, and generalize these to what is called an affine connection.

In differential geometry, there is a natural way to define what it means to take the derivative of a vector field on $\mathbb{R}^n$ with respect to a tangent vector. Let X be a smooth vector field on a subset U in $\mathbb{R}^n$, i.e., X is a map

$$X : U \to T\mathbb{R}^n.$$

Let $Y \in T_p\mathbb{R}^n$ be a tangent vector at a point $p \in U$, and $\gamma : [-\epsilon, \epsilon] \to U$ a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = Y$. The derivative of X with respect to Y is denoted by $\nabla_Y X$, and it is defined as

$$\nabla_Y X = (X \circ \gamma)'(0).$$

We can write X as a linear combination

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

where each $X_i \colon U \to \mathbb{R}$ is a smooth map, and (x, U) is a smooth chart on U such that

$$x = (x_1, x_2, \ldots, x_n) \colon U \to U$$

is the identity map. Then

$$\begin{aligned}(X \circ \gamma)(t) &= X(\gamma_1(t), \ldots, \gamma_n(t)) \\ &= (X_1(\gamma_1(t), \ldots, \gamma_n(t)), \ldots, X_n(\gamma_1(t), \ldots, \gamma_n(t))).\end{aligned}$$

Since $Y = \gamma'(0) = \sum_{i=1}^n \frac{d\gamma_i}{dt}(0)\frac{\partial}{\partial x_i}\Big|_p$, we get

$$\begin{aligned}(X \circ \gamma)'(0) &= (\sum_{j=1}^n \frac{\partial X_1}{\partial x_j}(p)\frac{d\gamma_j}{dt}(0), \ldots, \sum_{j=1}^n \frac{\partial X_n}{\partial x_j}(p)\frac{d\gamma_j}{dt}(0))\\ &= (\sum_{j=1}^n \frac{d\gamma_j}{dt}(0)\frac{\partial}{\partial x_j}\Big|_p(X_1), \ldots, \sum_{j=1}^n \frac{d\gamma_j}{dt}(0)\frac{\partial}{\partial x_j}\Big|_p(X_n))\\ &= (Y(X_1), \ldots, Y(X_n))\\ &= \sum_{i=1}^n Y(X_i)\frac{\partial}{\partial x_i}\Big|_p.\end{aligned}$$

Hence the derivative is independent of the curve. The derivative of a vector field can also be found with respect to another vector field to yield yet another vector field. If X and Y are vector fields, $\nabla_Y X$ should be a map from $\mathbb{R}^n$ to $T\mathbb{R}^n$, and we just let $\nabla_Y X(p) = \nabla_{Y(p)}X$. For more detail on this, see

Now we want to do this in the algebraic case. In Section 2.2 we saw that for each infinitesimal curve, there is a tangent vector corresponding to the curve giving its direction at a point. We can also start with a tangent vector Y, and get an infinitesimal curve by letting $\gamma = \gamma_0 + t(Y \circ d)$ for any homomorphism $\gamma_0 \colon A \to k$.

Definition 3.1.1. Let X be a vector field and Y a tangent vector. Choose an infinitesimal curve $\gamma = \gamma_0 + tD$ such that $D = Y \circ d$. Then $\gamma \circ X = \gamma_0 \circ X + t(D \circ X)$. The **derivative of X with respect to Y**, $\nabla_Y X$, is defined as

$$\nabla_Y X = D \circ X.$$

Note that the derivative is independent of the curve since $D \circ X = Y \circ d \circ X$ (we do not even have to choose a curve). Also, it is easy to check that $\nabla_Y X : Ph(A) \to k$ is a derivation:

$$\begin{aligned}Y \circ d \circ X(fg) &= Y(d(X(f)X(g)))\\ &= Y(d(X(f))\iota(X(g)) + \iota(X(f))d(X(g)))\\ &= Y(d(X(f)))Y(\iota(X(g))) + Y(\iota(X(f)))Y(d(X(g)))\\ &= (Y \circ d \circ X)(f)(Y \circ \iota \circ X)(g)\\ &\qquad + (Y \circ \iota \circ X)(f)(Y \circ d \circ X)(g)\end{aligned}$$

But $Der(Ph(A), k) \cong Hom(Ph^2(A), k)$, so $\nabla_Y X$ is isomorphic to a homomorphism from $Ph^2(A)$ into k.

To make this definition a little more clear, we should look at an example.

Example 3.1.2. Let $A = k[x, y]/(y - x^2)$. Then

$$Ph(A) = k < x, y, dx, dy > /I,$$

where

$$I = ([x, y], y - x^2, [dx, y] + [x, dy], dy - dxx - xdx).$$

A vector field $X \in Hom_{A/k}(Ph(A), A)$ must act as the identity map on A, so it has to send x and y to themselves. If we let $X(dx) = 1$ and $X(dy) = 2x$, the ideal I will be sent to zero, hence it is a well-defined homomorphism. We choose a tangent vector $Y = p \circ X$ where the point $p : A \to k$ sends (x, y) to $(2, 4)$, so Y sends (dx, dy) to $(1, 4)$. The composition $\nabla_Y X = Y \circ d \circ X$ now sends (dx, dy) to $(0, 2)$, which actually is the rate of change of the vector field with respect to the tangent vector.

Now let us look at another tangent vector, one that does not factor through our vector field. Let

$$Y\colon Ph(A) \to k$$

be the tangent vector $(2, 8)$ at the point $(2, 4)$, i.e., $Y(x) = 2$, $Y(y) = 4$, $Y(dx) = 2$ and $Y(dy) = 8$. In this case the derivative $\nabla_Y X$ sends (dx, dy) to $(0, 4)$. ♣

We should also be able to take the derivative of a vector field with respect to another vector field. The definition is the same as for tangent vectors.

Definition 3.1.3. Let X and Y be vector fields. The **derivative of X with respect to Y** is defined as

$$\nabla_Y X = Y \circ d \circ X : Ph(A) \to A.$$

Again, this is a derivation, thus isomorphic to a homomorphism from $Ph^2(A)$ into A.

Example 3.1.4. Let X be the vector field from the previous example. Define $\hat{X}\colon Ph(A) \to A$ to be another vector field such that $\hat{X}(dx) = x$ and $\hat{X}(dy) = 2y$. Note that the tangent vector Y from Example 3.1.2, mapping x and y to 2 and 4, and dx and dy to 2 and 8, is a tangent vector in this vector field; if we let p be the point from the same example, $p \circ \hat{X} = Y$. The derivative of X with respect to $\hat{X}$, $\nabla_{\hat{X}} X$, maps dx to 0 and dy to $2x$. If we compose with p, we see that

$$\begin{aligned} p \circ \nabla_{\hat{X}} X &= p \circ \hat{X} \circ d \circ X \\ &= Y \circ d \circ X \\ &= \nabla_Y X. \end{aligned}$$

♣

In general, if $X\colon Ph(A) \to B$ and $Y\colon Ph(B) \to C$ for some k-algebras B and C, we can define the derivative of X with respect to Y to be $Y \circ d \circ X \in Der(Ph(A), C)$, where $d\colon B \to Ph(B)$ is the derivation corresponding to the identity homomorphism $Ph(B) \to Ph(B)$. To make it clear what kind of maps X and Y are, we will denote the derivative by $(\nabla_C^B)_Y X$, so $(\nabla_C^B)_Y X = Y \circ d \circ X$. Whenever B or C is equal to A, it will be omitted in the notation ∇_C^B. Thus, if X,$Y \in Hom_{A/k}(Ph(A), A)$, we will just write $\nabla_Y X$. We have the following properties:

Proposition 3.1.5. *Let X, X_1 and X_2 be elements in $Hom_{A/k}(Ph(A), B)$, Y, Y_1 and Y_2 be elements in $Hom_{A/k}(Ph(B), C)$, $c \in C$ and $f \in B$. Then:*

1. $(\nabla_C^B)_{Y_1+Y_2} X = (\nabla_C^B)_{Y_1} X + (\nabla_C^B)_{Y_2} X.$
2. $(\nabla_C^B)_Y (X_1 + X_2) = (\nabla_C^B)_Y X_1 + (\nabla_C^B)_Y X_2.$
3. $(\nabla_C^B)_{cY} X = c(\nabla_C^B)_Y X.$
4. $(\nabla_C^B)_Y (fX) = Y(f)(\nabla_C^B)_Y X + Y(df)(Y \circ \iota \circ X).$

Proof. The first three properties are straightforward to prove, so we will focus on the last one:

$$\begin{aligned}(\nabla_C^B)_Y (fX) &= Y \circ d \circ (fX)\\ &= Y \circ (\iota(f)(d \circ X) + df(\iota \circ X))\\ &= Y(f)(Y \circ d \circ X) + Y(df)(Y \circ \iota \circ X)\\ &= Y(f)(\nabla_C^B)_Y X + Y(df)(Y \circ \iota \circ X).\end{aligned}$$

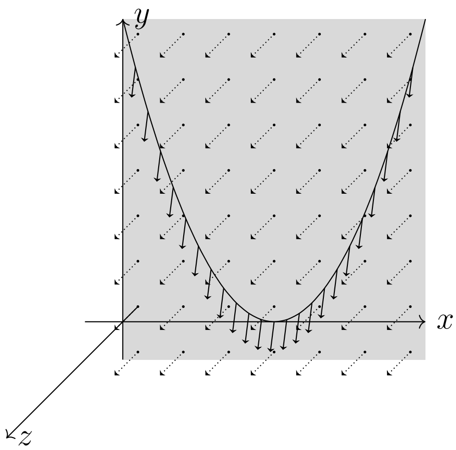
□

If both X and Y are vector fields, that is, if they are homomorphisms from $Ph(A)$ into A, we can simplify the fourth property a little:

$$\begin{aligned}\nabla_Y (fX) &= Y(f)\nabla_Y X + Y(df)(Y \circ \iota \circ X)\\ &= f\nabla_Y X + Y(df)X.\end{aligned}$$

This is because when Y is a vector field, it acts as the identity map on elements of A.

In the case where we have surjective homomorphisms $\gamma\colon A \to B$ and $\psi\colon B \to C$, we may view B as a subspace of A, and C as a subspace of B. This is because the corresponding morphisms between the spectra of the rings will be injective. An element $X \in Hom_{A/k}(Ph(A), B)$ such that $X \circ \iota = \gamma$ will give a tangent vector in A at each point of B, while a $Y \in Hom_{A/k}(Ph(B), C)$ such that $Y \circ \iota = \psi$ gives a tangent vector in B at each point of C. This is depicted in the figure below. We let $A = k[x, y, z]$, $B = k[x, y]$ and $C = k[t]$, with $\gamma\colon A \to B$ sending x and y to themselves, and $\psi\colon B \to C$ sending x to t and y to $(t-2)^2$. The vectors along the curve ψ are the vectors given by $Y\colon Ph(B) \to C$, and the dotted vectors are the vectors given by $X\colon Ph(A) \to B$. Notice that the vectors along the curve do not have to be tangent to the curve. However, they must lie in the plane given by γ. Also, the vectors in the plane may point out of the plane.



The following two properties tell us how we can restrict vector fields from A to B and from B to C when taking the derivative.

Proposition 3.1.6. *Let $\gamma\colon A \to B$ and $\psi\colon B \to C$ be two surjective homomorphisms, and let $X \in Hom_{A/k}(Ph(A), B)$, $Y \in Hom_{A/k}(Ph(B), B)$ and $Z \in Hom_{A/k}(Ph(B), C)$ be maps such that $X \circ \iota = \gamma$ and $Z \circ \iota = \psi$. Then:*

(a) For all $\widetilde{X}\colon Ph(A) \to A$ such that $\gamma \circ \widetilde{X} = X$,

$$(\nabla_B^B)_Y X = (\nabla_B)_{Y\circ\gamma_*}\widetilde{X}.$$

(b) For all $\widetilde{Z}\colon Ph(B) \to B$ such that $\psi \circ \widetilde{Z} = Z$,

$$(\nabla_C^B)_Z X = \psi \circ (\nabla_B^B)_{\widetilde{Z}} X.$$

Proof. (a) Assume that $\gamma \circ \widetilde{X} = X$. Then

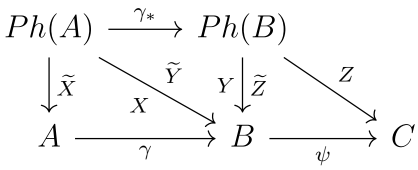
$$\begin{aligned}(\nabla_B^B)_Y X &= Y \circ d \circ X \\ &= Y \circ d \circ \gamma \circ \widetilde{X} \\ &= Y \circ \gamma_* \circ d \circ \widetilde{X} \\ &= (\nabla_B^A)_{Y\circ\gamma_*}\widetilde{X}.\end{aligned}$$

(b) Assume that $\psi \circ \widetilde{Z} = Z$. Then

$$\begin{aligned}(\nabla^B_C)_Z X &= Z \circ d \circ X \\ &= \psi \circ \widetilde{Z} \circ d \circ X \\ &= \psi \circ (\nabla^B_B)_{\widetilde{Z}} X.\end{aligned}$$

□

The following diagram gives an overview of all the different homomorphisms.



Note that since all homomorphisms $\widetilde{X}\colon Ph(A) \to A$ and $\widetilde{Z}\colon Ph(B) \to B$ act as identity maps on A and B, respectively, the conditions $\gamma \circ \widetilde{X} = X$ and $\psi \circ \widetilde{Z} = Z$ imply that $X \circ \iota = \gamma$ and $Z \circ \iota = \psi$.

The maps γ and ψ are surjective, hence they give embeddings of $Spec(B)$ into $Spec(A)$, and $Spec(C)$ into $Spec(B)$. Therefore B and C can be viewed as subspaces of A and B, respectively. When X is a homomorphism from $Ph(A)$ into B such that $X \circ \iota = \gamma$, it has to send every $a \in A$ to the same element in B as γ, thus, it will give us tangent vectors in A at each point of the embedding of B in A. The same reasoning tells us that Z consists of tangent vectors in B at each point of the embedding of C in B.

In the first property of the proposition, we choose any vector field

$$\widetilde{X}\colon Ph(A) \to A$$

such that $\gamma \circ \widetilde{X} = X$. This means that X is the restriction of $\widetilde{X}$ to the embedding of B in A. On the right side of the equality sign, we take the derivative of $\widetilde{X}$ with respect to $Y \circ \gamma_*$. This is a homomorphism from $Ph(A)$ into B. We see that

$$Y(\gamma_*(a)) = Y(\gamma(a))$$

and

$$Y(\gamma_*(da)) = Y(d\gamma(a)),$$

so we can view $Y \circ \gamma_*$ as the embedding of the vector field Y on B into A. It will still only consist of tangent vectors in B, but we view them as part of A. Now the derivative of $\widetilde{X}$ with respect to $Y \circ \gamma_*$ gives us a tangent vector at each point of the embedding of B in A. On the left side of the equality sign, we take the derivative of the restriction of $\widetilde{X}$ to B, X, with respect to Y. Thus, what this property tells us, is that we can choose to restrict a vector field on A to B before taking the derivative.

The second property is very similar. Here we start with two vector fields X and Z, where X is a vector field on B in A, and Z a vector field on C in B. Then we choose any $\widetilde{Z}\colon Ph(B) \to B$, i.e. a vector field on B, such that $\psi \circ \widetilde{Z} = Z$, which means that Z is the restriction of $\widetilde{Z}$ to the embedding of C in B. The element $(\nabla_C^B)_Z X \in Der(Ph(A), B)$ is then the derivative of X with respect to the restriction of $\widetilde{Z}$. On the right side of the equality sign, we do the restriction after taking the derivative: $(\nabla_B^B)_{\widetilde{Z}} X$ is the derivative of X with respect to $\widetilde{Z}$, and by composing with ψ, we restrict ourselves to the points of the embedding of C in B. In other words, we can choose whether to restrict before or after taking the derivative.

In differential geometry, the notion of a derivative of a vector field, X, along a tangent vector or another vector field, Y, can be generalized to what is called an affine connection in the case where Y is a tangent vector, and a global affine connection if Y is a vector field. It is defined to be a map ∇ sending X and Y to either a new tangent vector or a new vector field (according to what Y is) such that ∇ satisfies four properties corresponding to the four properties in Proposition 3.1.5 (see [Csi98, Unit 11]). We will do something similar, but not exactly the same, here.

Definition 3.1.7. Let R be a k-algebra. We say that a k-algebra S is a **projection** of R if there exists a surjective homomorphism $\phi\colon R \to S$.

For a k-algebra R, we let $Pr(R)$ denote the set of all k-algebras S that are projections of R. When S is a projection of R, we have a surjective homomorphism $\phi\colon R \to S$. The corresponding map $\widetilde{\gamma}\colon Spec(S) \to Spec(R)$ is then an inclusion, so $Spec(S)$ is a subspace of $Spec(R)$. That is, a projection corresponds to a subspace.

Definition 3.1.8. Let A be a k-algebra. An **affine connection** on A is a collection of maps, one for each combination of k-algebras $B \in Pr(A)$ and $C \in Pr(B)$,

$$\nabla_C^B\colon Hom_{A/k}(Ph(A), B) \times Hom_{A/k}(Ph(B), C) \to Der(Ph(A), C),$$

such that for all elements X, X_1, $X_2 \in Hom_{A/k}(Ph(A), B)$, Z, Z_1, $Z_2 \in Hom_{A/k}(Ph(B), C)$, $c \in C$ and $f \in B$

1. $(\nabla_C^B)_{Z_1+Z_2} X = (\nabla_C^B)_{Z_1} X + (\nabla_C^B)_{Y_2} X$,
2. $(\nabla_C^B)_Z (X_1 + X_2) = (\nabla_C^B)_Z X_1 + (\nabla_C^B)_Z X_2$,

3. $(\nabla_C^B)_{cZ}X = c(\nabla_C^B)_Z X$,
4. $(\nabla_C^B)_Z(fX) = Z(f)(\nabla_C^B)_Z X + Z(df)(Z \circ \iota \circ X)$,

and such that for all surjective homomorphisms $\gamma\colon A \to B$ and $\psi\colon B \to C$, the following two conditions are satisfied:

(a) For all $Y \in Hom_{A/k}(Ph(B), B)$, and all $\widetilde{X}\colon Ph(A) \to A$ such that $\gamma \circ \widetilde{X} = X$,
$$(\nabla_B^B)_Y X = (\nabla_B)_{Y \circ \gamma_*} \widetilde{X}.$$

(b) For all $\widetilde{Z}\colon Ph(B) \to B$ such that $\psi \circ \widetilde{Z} = Z$,
$$(\nabla_C^B)_Z X = \psi \circ (\nabla_B^B)_{\widetilde{Z}} X.$$

Let us compare this definition to the definition of an affine connection in differential geometry. There, an affine connection on a manifold M is defined to be a mapping, ∇, that to each pair of smooth vector fields X and Y on M assigns a new smooth vector field $\nabla_Y X$, and the mapping should satisfy properties corresponding to the first four properties in Definition 3.1.8. Thus, in differential geometry there is only one mapping, unlike here, where we have one mapping for each pair of k-algebras B and C, B being a projection of A, and C a projection of B. In addition, they do not have conditions corresponding to the last two properties above. However, there is a result in differential geometry saying that if X_1, X_2, Y_1 and Y_2 are vector fields such that X_1 and X_2 agree on some open subset U, and Y_1 and Y_2 agree on some open subset V, then
$$(\nabla_Y X_1)|_U = (\nabla_Y X_2)|_U$$
and
$$(\nabla_{Y_1} X)|_V = (\nabla_{Y_2} X)|_V$$
(a proof can be found in [Csi98, Unit 11]). In other words, the affine connection can be restricted to any open subset, and this yields an affine connection on the open subset. It is also possible, by use of what is called Christoffel symbols, to define affine connections on sets in an open cover of M, and with those define an affine connection on the whole manifold. This is exactly what the last two properties above tell us that we can do here: Let $\gamma\colon A \to B$ and $X \in Hom_{A/k}(Ph(A), B)$ be two maps, and assume that $\widetilde{X_1}$ and $\widetilde{X_2}$ are two elements in $Hom_{A/k}(Ph(A), A)$ such that
$$\gamma \circ \widetilde{X_1} = \gamma \circ \widetilde{X_2} = X.$$

This means that the vector fields X_1 and X_2 agree on the projection B of A. Suppose $Y \in Hom_{A/k}(Ph(B), B)$ is a vector field on B. The first property then states that

$$(\nabla_B)_{Y\circ\gamma_*}\widetilde{X_1} = (\nabla_B)_{Y\circ\gamma_*}\widetilde{X_2} = (\nabla_B^B)_Y X.$$

On the other hand, if $X \in Hom_{A/k}(Ph(A), B)$, and $\widetilde{Z_1}$ and $\widetilde{Z_2}$ are elements in $Hom_{A/k}(Ph(B), B)$ such that

$$\psi \circ \widetilde{Z_1} = \psi \circ \widetilde{Z_2} = Z$$

for some $Z \in Hom_{A/k}(Ph(B), C)$ and $\psi\colon B \to C$, then

$$\psi \circ (\nabla_B^B)_{\widetilde{Z_1}} X = \psi \circ (\nabla_B^B)_{\widetilde{Z_2}} X = (\nabla_C^B)_{\psi\circ Z} X$$

by the second property. Composing with ψ gives a restriction from B to C. If $B = A$ or $B = C$, the restriction is from A to C.

3.2 Torsion fields

In this and the next section we will look at two applications of affine connections; torsion fields and differentiation of vector fields along curves. We start with torsion fields.

Composition of derivations does in general not give new derivations, but it is easy to check that if B is some k-algebra, and $D_1\colon B \to B$ and $D_2\colon B \to B$ are two derivations, then

$$D_1 \circ D_2 - D_2 \circ D_1\colon B \to B$$

is also a derivation.

Definition 3.2.1. Let A be a k-algebra, and let X and Y be two vector fields on A, i.e. they are elements of $Hom_{A/k}(Ph(A), A)$. The **commutator** of X and Y, denoted by $[X, Y]$, is defined to be the derivation

$$[X, Y] = D_X \circ D_Y - D_Y \circ D_X,$$

where D_X and D_Y are the derivations corresponding to X and Y respectively.

Proposition 3.2.2. *Let X and Y be vector fields on A, and ∇ the affine connection on A such that*

$$(\nabla_C^B)_{Z_2} Z_1 = Z_2 \circ d \circ Z_1$$

for all $Z_1 \in Hom_{A/k}(Ph(A), B)$ and $Z_2 \in Hom_{A/k}(Ph(B), C)$. Then

$$[X, Y] = (\nabla_X Y - \nabla_Y X) \circ d.$$

Proof. Since the derivation corresponding to a vector field is obtained by composing the vector field with the derivation d, we get

$$\begin{aligned}[X, Y] &= D_X \circ D_Y - D_Y \circ D_X \\ &= (X \circ d) \circ (Y \circ d) - (Y \circ d) \circ (X \circ d) \\ &= (X \circ d \circ Y - Y \circ d \circ X) \circ d \\ &= (\nabla_X Y - \nabla_Y X) \circ d.\end{aligned}$$

□

Definition 3.2.3. Suppose we have an affine connection on the k-algebra A, and let X and Y be two vector fields on A. Then the **torsion field** of X and Y is defined as

$$T(X, Y) = [X, Y] - (\nabla_X Y - \nabla_Y X) \circ d.$$

If $T(X, Y) = 0$ for all vector fields X and Y on A, the connection is said to be **torsion free**.

Let $\widetilde{\nabla}$ denote the affine connection on A such that

$$(\widetilde{\nabla}_C^B)_{Z_2} Z_1 = Z_2 \circ d \circ Z_1$$

for all $Z_1 \in Hom_{A/k}(Ph(A), B)$ and $Z_2 \in Hom_{A/k}(Ph(B), C)$. Since Proposition 3.2.2 tells us that

$$[X, Y] = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) \circ d$$

for all vector fields X and Y on A, the torsion is actually

$$T(X, Y) = [(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) - (\nabla_X Y - \nabla_Y X)] \circ d.$$

3.3 Vector fields along curves

In this section we will look at vector fields along curves. If we are given a curve γ, a vector field along γ should assign a tangent vector in A to each point on the curve. In Section 3.1 we saw that if $\gamma\colon A \to B$ is a homomorphism and $X\colon Ph(A) \to B$ is such that $X \circ \iota = \gamma$, then X gives a tangent vector in A at each point of the embedding of B in A given by γ. Hence, we have the following defintion.

Definition 3.3.1. Let $\gamma\colon A \to B$ be a curve in A. A **vector field on** A **along** γ is a homomorphism $X\colon Ph(A) \to B$ such that $X \circ \iota = \gamma$.

Recall that a curve is a surjective homomorphism $A \to B$ where B is a smooth k-algebra of dimension 1. Note that the tangent vectors do not have to be tangent to the curve. Also note that since a curve is given by a surjective homomorphism, we can for every vector field X along a curve γ find a vector field $\widetilde{X}\colon Ph(A) \to A$ on A that extends X. That is, we can find an $\widetilde{X} \in Hom_{A/k}(Ph(A), A)$ such that the following diagram commutes:

$$\begin{array}{ccc} Ph(A) & & \\ {\scriptstyle \widetilde{X}}\downarrow & \searrow^{X} & \\ A & \xrightarrow{\gamma} & B. \end{array}$$

Having defined vector fields along a curves, it would be interesting to see how a vector field X along a curve γ changes as we move along γ. In other words; how can we define the derivative of X along the curve?

Assume that $\gamma\colon A \to B$ is a curve in A. Let $u\colon Ph(B) \to B$ be a vector field on B. Composing u with γ_* gives an embedding of the vector field u into A. Therefore we give the following defintion.

Definition 3.3.2. Let $\gamma\colon A \to B$ be a curve in A. A homomorphism $Y\colon Ph(A) \to B$ such that $Y = u \circ \gamma_*$ for some $u\colon Ph(B) \to B$ is a **tangent vector field along** γ.

Example 3.3.3. Let $A = k[x, y]$, $B = k[t]$ and $\gamma\colon A \to B$ be the curve mapping x to t and y to t^2. Define a vector field $u\colon Ph(B) \to B$ by letting $u(dt) = 1$, and let $Y = u \circ \gamma_*$. Then $Y(dx) = 1$ and $Y(dy) = 2t$. ♣

The next definition tells us how we can find the derivative of a vector field along a curve with respect to a tangent vector field along the curve. We will require that the tangent vector field consists not only of zero tangent vectors. That is, if $Y\colon Ph(A) \to B$ is a tangent vector field along a curve, we should have $Y \circ d \not\equiv 0$.

Definition 3.3.4. Suppose we have an affine connection on A. Let X be a vector field along a curve $\gamma\colon A \to B$, and $Y\colon Ph(A) \to B$ a tangent vector field along γ such that $Y \circ d \not\equiv 0$. The **derivative of** X **along** γ **with respect to** Y is denoted by DX, and is defined as

$$DX = (\nabla_B)_Y \widetilde{X},$$

where $\widetilde{X} \in Hom_{A/k}(Ph(A), A)$ is a vector field on A such that $\gamma \circ \widetilde{X} = X$. If for every tangent vector field Y we have that $DX(da) = 0$ for all $a \in A$, we say that X is a **parallel vector field** along γ.

As we saw in Section 3.1, the condition that $\gamma \circ \widetilde{X}$ should equal X means that X is the restriction of $\widetilde{X}$ to γ. Thus, to see how a vector field along a curve changes with respect to a tangent vector field, we use an affine connection on the tangent vector field and an extension of the vector field.

For the derivative to be well-defined, it can not matter which vector field $\widetilde{X}$ we choose. This is true by the next proposition.

Proposition 3.3.5. *Assume that we have an affine connection on A, and let $\gamma\colon A \to B$ be a curve and $Y\colon Ph(A) \to B$ a tangent vector field along γ. If $\widetilde{X_1}$ and $\widetilde{X_2}$ are two vector fields on A such that $\gamma \circ \widetilde{X_1} = \gamma \circ \widetilde{X_2}$, then*

$$(\nabla_B)_Y \widetilde{X_1} = (\nabla_B)_Y \widetilde{X_2}.$$

Proof. Since $Y = u \circ \gamma_*$ for some $u\colon Ph(B) \to B$, we have

$$\begin{aligned}(\nabla_B)_Y \widetilde{X_1} &= (\nabla_B)_{u \circ \gamma_*} \widetilde{X_1} \\ &= (\nabla_B^B)_u \gamma \circ \widetilde{X_1} \\ &= (\nabla_B^B)_u \gamma \circ \widetilde{X_2} \\ &= (\nabla_B)_{u \circ \gamma_*} \widetilde{X_2} \\ &= (\nabla_B)_Y \widetilde{X_2}.\end{aligned}$$

□

Let us look at an example where the affine connection is just the normal derivative.

Example 3.3.6. Let $\gamma\colon A \to B$ and Y be as in Example 3.3.3, and let $X\colon Ph(A) \to k[t]$ be the vector field along γ such that $X(dx) = t$ and $X(dy) = 0$ (vector fields along γ must always send x and y to $\gamma(x)$ and $\gamma(y)$). Assume that the affine connection on A is just the normal derivative, i.e.,

$$(\nabla_C^B)_{Z_2} Z_1 = Z_2 \circ d \circ Z_1$$

for all $Z_1 \in Hom_{A/k}(Ph(A), B)$ and $Z_2 \in Hom_{A/k}(Ph(B), C)$. To find the derivative of X along γ with respect to Y, we have to find a vector field $\widetilde{X}\colon Ph(A) \to A$ such that $\gamma \circ \widetilde{X} = X$. Let $\widetilde{X}(dx) = x$ and $\widetilde{X}(dy) = 0$. Then

$$\begin{aligned}DX(dx) &= Y(d(\widetilde{X}(dx))) \\ &= Y(dx) \\ &= 1\end{aligned}$$

and

$$\begin{aligned} DX(dy) &= Y(d(\widetilde{X}(dy))) \\ &= Y(d(0)) \\ &= 0. \end{aligned}$$

We can also choose $\widetilde{X}$ to be the vector field where $\widetilde{X}(dx) = x$ and $\widetilde{X}(dy) = y - x^2$ and get the same result, because

$$\begin{aligned} \gamma(\widetilde{X}(dx)) &= t \\ &= X(dx) \end{aligned}$$

and

$$\begin{aligned} \gamma(\widetilde{X}(dy)) &= t^2 - t^2 \\ &= 0 \\ &= X(dy), \end{aligned}$$

so $\gamma \circ \widetilde{X} = X$. ♣

When $\gamma\colon A \to B$ is a curve and $X\colon Ph(A) \to B$ is a vector field along γ, we can as usual compose with a point $p\colon B \to k$. This yields the tangent vector $p \circ X\colon Ph(A) \to k$ from the vector field X at the point $p \circ \gamma$ on the curve. We can also start with a tangent vector $X_p\colon Ph(A) \to k$ at a point $p \circ \gamma$ on the curve, which means that $X_p \circ \iota = p \circ \gamma$, and extend this to a vector field along the curve. Just let $X(da) = X_p(da)$ for all $a \in A$. But given an affine connection on A, when is it possible to find a parallel vector field X along γ such that $p \circ X = X_p$, i.e., such that X extends X_p?

Proposition 3.3.7. *Assume that A is a polynomial ring in n indeterminates. Let $\gamma\colon A \to k[t]$ be a curve and $X_p\colon A \to k$ a tangent vector at the point $p \circ \gamma\colon A \to k$ on the curve. If the connection ∇ is the derivative of a vector field with respect to another, there is a unique parallel vector field along γ that extends X_p.*

Proof. Let $X\colon Ph(A) \to k[t]$ be the vector field along γ that sends x_i to $\gamma(x_i)$ and dx_i to $X_p(dx_i) \in k$, and let $Y = u \circ \gamma_*$ be a tangent vector field along γ such that $Y \circ d \not\equiv 0$. Then $p \circ X = X_p$, and $DX(dx_i) = u(d(X(dx_i))) = 0$ because $X(dx_i) = X_p(dx_i) \in k$. Hence there exists a parallel vector field along γ that extends X_p.

Assume now that $\widetilde{X}\colon Ph(A) \to k[t]$ is another vector field along γ such that $p \circ \widetilde{X} = X_p$, and such that for all tangent vector fields Y along γ, we have that $D\widetilde{X} \circ d \equiv 0$. Write $\widetilde{X}(dx_i) = f_i(t) \in k[t]$. Then for any tangent vector field $Y = u \circ \gamma_*$,

$$\begin{aligned} 0 &= D\widetilde{X}(dx_i) \\ &= u(d(\widetilde{X}(dx_i))) \\ &= f_i'(t)u(dt). \end{aligned}$$

Since we require that $Y \circ d \not\equiv 0$, we know that $u(dt) \neq 0$. Therefore $f_i'(t) = 0$, and thus $f_i(t) = f_i \in k$. Then $p(f_i) = f_i$, so

$$\begin{aligned} X_p(dx_i) &= p(\widetilde{X}(dx_i)) \\ &= p(f_i) \\ &= f_i. \end{aligned}$$

We defined X such that $X(dx_i) = X_p(dx_i)$, hence $X(dx_i) = f_i = \widetilde{X}(dx_i)$, and $\widetilde{X} = X$. □

Definition 3.3.8. Let ∇ be an affine connection on a k-algebra A, $\gamma\colon A \to B$ a curve and X a vector field along γ. Two points $p\colon B \to k$ and $q\colon B \to k$ give rise to two tangent vectors $X_p = p \circ X$ and $X_q = q \circ X$. When X is a parallel vector field, two such tangent vectors are said to be **parallel transports** of each other.

Chapter 4

Integrability

4.1 The coproduct of noncommutative rings

In this section we will prove a proposition that tells us that the coproduct of noncommutative rings exists, following [KIBM96, Chapter 1.4]. This will be useful when we in the next section study integrability.

Assume that R and S are two objects in a category $\mathcal{C}$. Then the coproduct of R and S is an object T together with two morphisms, $R \to T$ and $S \to T$, that satisfy the following universal property: For all other objects P with morphisms $R \to P$ and $S \to P$, there is a unique morphism $T \to P$ such that the diagram

$$\begin{array}{ccccc} R & \longrightarrow & T & \longleftarrow & S \\ & \searrow & \downarrow & \swarrow & \\ & & P & & \end{array}$$

commutes.

In the category of commutative algebras, the coproduct over a ring A is just the tensor product. For noncommutative algebras over a commutative ring, we get something a little more complicated.

Proposition 4.1.1. *Assume that R and S are rings, not necessarily commutative, with homomorpshisms $\phi\colon A \to R$ and $\psi\colon A \to S$ that make R a right A-module and S a left A-module, i.e., they are A-algebras. Then there is an A-algebra T and A-algebra homomorphisms $i_R\colon R \to T$ and $i_S\colon S \to T$ satisfying the following universal property: For any other A-algebra P, and any A-algebra homomorphisms $f_R\colon R \to P$ and $f_S\colon S \to P$, there is a unique A-algebra homomorphism $f\colon T \to P$ such that*

$$f \circ i_R = f_R$$

and

$$f \circ i_S = f_S.$$

T is the coproduct of R and S over A.

Proof. Let

$$C = R \oplus S \oplus (R \otimes_A R) \oplus (R \otimes_A S) \oplus (S \otimes_A R) \\ \oplus (S \otimes_A S) \oplus (R \otimes_A R \otimes_A R) \oplus \cdots ,$$

and let D be the ideal in C generated by elements of the form

$$a \otimes b - ab$$

and

$$1_R - 1_S,$$

where a and b are both in R or both in S, and 1_R and 1_S are the identity elements of R and S, respectively. The coproduct of R and S over A is the quotient $T = C/D$ together with the homomorphisms $i_R \colon R \to T$ and $i_S \colon S \to T$ that are just inclusion maps composed with the quotient map $C \to C/D$.

To show that this is the coproduct, we need to give T a ring structure, a map $A \to T$ that makes it an A-algebra, and show that it satisfies the universal property. Let us start with the ring structure. Addition is done componentwise. We define multiplication of two elements $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ and $y_1 \otimes y_2 \otimes \cdots \otimes y_m$, where x_i and y_j are in R or S for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, as follows:

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n)(y_1 \otimes y_2 \otimes \cdots \otimes y_m) \\ = \begin{cases} x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m & \text{if } x_n \in R, y_1 \in S \\ & \text{or } x_n \in S, y_1 \in R \\ x_1 \otimes x_2 \otimes \cdots \otimes x_n y_1 \otimes y_2 \otimes \cdots \otimes y_m & \text{if } x_n, y_1 \in R \\ & \text{or } x_n, y_1 \in S. \end{cases}$$

Multiplication of general elements in T then follows naturally.

We give T an A-module structure by defining a homomorphism $\sigma \colon A \to T$ such that $\sigma(a) = \phi(a) \otimes \psi(a) \in T$.

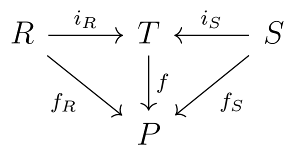
Now we must show that the universal property is satisfied, so let P be an A-algebra with homomorphisms $f_R \colon R \to P$ and $f_S \colon S \to P$. We want to define a map $f \colon T \to P$. Let $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ be an element of T where each x_i is either in R or in S, and let

$$f(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = g_1(x_1) g_2(x_2) \cdots g_n(x_n),$$

where

$$g_i = \begin{cases} f_R & \text{if } x_i \in R \\ f_S & \text{if } x_i \in S. \end{cases}$$

The homomorphism f can be extended linearly to a map from T into P. What we must show, is that the following diagram commutes:



Let $x \in R$ and $y \in S$. Then

$$f(i_R(x)) = f(x) = f_R(x)$$

and

$$f(i_S(y)) = f(y) = f_S(y),$$

hence the diagram commutes. To show that f is unique, assume that $h\colon T \to P$ is any other map such that the diagram above is commutative. Each x_i in an element of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in T$ comes from either R or S, so for each $i = 1, 2, \ldots, n$ we let j_i be the map such that

$$j_i = \begin{cases} i_R & \text{if } x_i \in R \\ i_S & \text{if } x_i \in S. \end{cases}$$

Then

$$\begin{aligned} h(x_1 \otimes x_2 \otimes \cdots \otimes x_n) &= h(j_1(x_1)j_2(x_2)\cdots j_n(x_n)) \\ &= h(j_1(x_1))h(j_2(x_2))\cdots h(j_n(x_n)) \\ &= g_1(x_1)g_2(x_2)\cdots g_n(x_n) \\ &= f(x_1 \otimes x_2 \otimes \cdots \otimes x_n). \end{aligned}$$

Thus, $h = f$. □

We will denote the coproduct of R and S over A by $R *_A S$, and let $r * s$ denote the image of $r \otimes s \in C$ in D. The coproduct $R *_A S$ is then generated by all elements of the form $r * s$ where r is a generator of R, and s a generator of S.

Example 4.1.2. Let $A = k[x, y]$, $R = k < x, y, dx > /(xy - yx)$ and $S = k[x]$ with homomorphisms $\phi\colon A \to R$ and $\psi\colon A \to S$ such that $\phi(x) = x$ and $\phi(y) = y$, and $\psi(x) = x$ and $\psi(y) = 0$. All elements in S are elements in

A, so $1 * s = \phi(s) * 1$ for all $s \in S$. Hence $R *_A S$ is generated by the elements $x * 1$, $y * 1$ and $dx * 1$. Define a map $f \colon R *_A S \to k < x, dx >$ by letting $f(x * 1) = x$, $f(y * 1) = 0$ and $f(dx * 1) = dx$. It is surjective, and since

$$\begin{aligned} y * 1 &= 1 * \psi(y) \\ &= 1 * 0 \\ &= \phi(0) * 0 \\ &= 0, \end{aligned}$$

it is also injective. Therefore

$$R *_A S \cong k < x, dx >= Ph(k[x]).$$

♣

4.2 Integrability

In this section we will assume that A is a polynomial ring,

$$A = k[x_1, x_2, \ldots, x_n].$$

We have seen that $Ph(A)$ can be considered as the tangent bundle of A; it consists of all points and all tangent vectors at each point. But what if we for each point were to pick out only some of the tangent vectors? Could we then find a subspace of A whose tangent space at each point consists of exactly the tangent vectors we have chosen?

First, let us look at how we can choose only some tangent vectors at each point.

Definition 4.2.1. A **distribution** Δ on A is a quotient of $Ph(A)$,

$$\Delta = Ph(A)/J,$$

where J is an ideal of $Ph(A)$, such that the quotient map $\phi \colon Ph(A) \to \Delta$ composed with the inclusion map $\iota \colon A \to Ph(A)$ is injective.

Since $\phi \circ \iota \colon A \to \Delta$ is injective, A lies inside Δ, so in Δ we have all the points of A. Let $X \colon Ph(A) \to A$ be a vector field on A that factors though Δ. This means that there is a homomorphism $\widetilde{X} \colon \Delta \to A$ such that the diagram

$$\begin{array}{ccc} Ph(A) & \xrightarrow{\phi} & \Delta \\ {\scriptstyle X}\downarrow & \swarrow {\scriptstyle \widetilde{X}} & \\ A & & \end{array}$$

commutes. Because $\widetilde{X}$ sends every element in J to 0, and ϕ is the quotient map, X must also send every element in J to 0. Thus, at each point $p\colon A \to k$, the collection of vector fields that factor through a distribution $\Delta = Ph(A)/J$ will only consist of some of the tangent vectors in A at p, i.e., those that send all $f \in J$ to 0. Therefore we will consider a distribution Δ as a subset of the set of all tangent vectors at all points of A.

Example 4.2.2. Let $A = k[x, y]$ and $J = (dy)$, so $\Delta = Ph(A)/(dy)$. A homomorphism from A to k that factors through Δ sends dy to 0, while there is no restriction on dx, so in Δ we have all the tangent vectors where the component in the y-direction is 0. ♣

Example 4.2.3. Again we let $A = k[x, y]$, but let $\Delta = Ph(A)/(dx - x, dy - y)$. In this case we only have one choice of tangent vector Y at each point $p\colon A \to k$, where $p(x) = a$ and $p(y) = b$ for some $a, b \in k$; dx must be sent to a, and dy to b. ♣

In Section 2.1 we defined a curve in A to be a homomorphism $\psi\colon A \to B$, where B is a smooth k-algebra of dimension 1. In general, we can define an **r-dimensional subspace** of A to be a surjective homomorphism $\psi\colon A \to B$, where B is a smooth k-algebra of dimension $r \leq n$. Our goal is to find out when the tangent vectors at a point in a distribution coincide with the tangent vectors in the tangent space to a subspace at the same point, so our question now is how we can restrict the points in a distribution Δ to a subspace of A. In the previous section, we constructed the coproduct of noncommutative A-algebras. The injection

$$\phi \circ \iota\colon A \to \Delta$$

and a subspace

$$\psi\colon A \to B$$

give Δ and B A-module structures. We can therefore form the coproduct $\Delta *_A B$. It is generated by elements $f * b$, where $f \in \Delta$ and $b \in B$. Since A lies inside Δ, among these we will have the elements $x_i * 1$ for $i = 1, 2, \ldots, n$. But $x_i * 1 = 1 * \psi(x_i)$, so the coproduct restricts the points of Δ to the subspace ψ. Thus, the coproduct $\Delta *_A B$ consists of all points of the subspace ψ, and all the tangent vectors of Δ.

Example 4.2.4. Let A and Δ be as in Example 4.2.2, and let $\psi\colon A \to B$, where $B = k[x]$, be the curve that sends x to x and y to 0. Look at $\Delta *_A B$. The element $y * 1$ is 0 because

$$y * 1 = 1 * \psi(y) = 1 * 0 = 0.$$

Therefore, when choosing a point $\Delta *_A B \to k$, we must send $y * 1$ to 0, while $x * 1$ can be sent to any element of k. The coproduct restricts the points of Δ to the x-axis, which is exactly the curve ψ.

If we for instance let ψ send y to 2 instead, a homomorpshism $\Delta *_A B \to k$ must send $y * 1$ to 2 since $y * 1 = 1 * \psi(y) = 1 * 2$. As above, $x * 1$ can be sent to any element in k, so again the coproduct restricts the points of Δ to the curve ψ. ♣

Notice that in this example, we can choose any point $p\colon A \to k$ and find a curve $\psi\colon A \to B$ through that point (meaning $q \circ \psi = p$ for some $q\colon B \to k$) such that the tangent space at each point of the curve is given by the tangent vectors of Δ at that point. These curves are the horizontal lines; if $p(x) = a$ and $p(y) = b$, we let $\psi(x) = x$ and $\psi(y) = b$. The tangent vectors

$$\{Y \in Hom_{A/k}(Ph(B), k)\colon Y \circ \iota = q\}$$

at any point $q \circ \psi$ on this curve are all the tangent vectors with no y-component, which are all the tangent vectors in Δ at that point. Since $Ph(B)$ gives all the tangent vectors at all points of B, we should have

$$\Delta *_A B \cong Ph(B).$$

This is indeed the case with the isomorphism

$$\beta\colon \Delta *_A B \to Ph(B)$$

sending $x * 1$ to x, $y * 1$ to b, $dx * 1$ to dx and $dy * 1$ to 0.

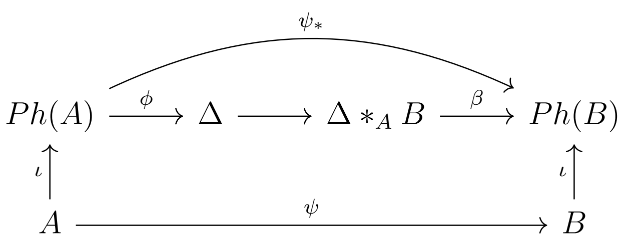
In general, if $\psi\colon A \to B$ is a subspace, we have seen that we can define a homomorphism $\psi_*\colon Ph(A) \to Ph(B)$ by letting $\psi_*(a) = \psi(a)$ and $\psi_*(da) = d\psi(a)$ for all $a \in A$. Since the map ψ is surjective, we know that for all $b \in B$ there is an $a \in A$ such that $\psi(a) = b$. Therefore we have that for all $b \in B$,

$$1 * b = 1 * \psi(a) = a * 1$$

for some $a \in A$. Thus the coproduct is generated by the elements $x_i * 1$ and $\phi(dx_i) * 1$ for all $i = 1, 2, \ldots, n$. We can construct a map

$$\beta\colon \Delta *_A B \to Ph(B)$$

by sending $x_i * 1$ to $\psi_*(x_i) = \psi(x_i)$ and $\phi(dx_i) * 1$ to $\psi_*(dx_i) = d\psi(x_i)$.



This is exactly the homomorphism that gave the isomorphism above between $\Delta *_A B$ and $Ph(B)$. Whenever β is an isomorphism, the tangent vectors of Δ at any point of the subspace given by ψ are the same as the tangent vectors of $Ph(B)$ at that point.

In Example 4.2.3 we can not find a curve ψ through each point such that the map β, as defined above, becomes an isomorphism. There is only one choice of a tangent vector at each point, so the tangent space at any point of a curve $A \to B$ will consist of more tangent vectors than those we have in Δ at the same point. The only possibility we have to get an isomorphism, is if we choose a point instead of a curve. That is, let $B = k$, and let $\psi\colon A \to k$ send x and y to $a \in k$ and $b \in k$, respectively. Then $\psi_*(dx) = d(\psi(x)) = 0$ and $\psi_*(dy) = d(\psi(y)) = 0$, so the map $\beta\colon \Delta *_A B \to Ph(B)$ sends x and y to themselves, while dx and dy are both sent to 0. This works well if $a = b = 0$, because at the origin we have no tangent vectors. At any other point, however, this is not an isomorphism. In fact, it is not even a homomorphism.

Definition 4.2.5. A distribution Δ on a ring A is **integrable** if for each point $p\colon A \to k$ there is a subspace $\psi_p\colon A \to B$ such that

1. $p = q \circ \psi_p$ for some $q\colon B \to k$
2. $\beta\colon \Delta *_A B \to Ph(B)$ is an isomorphism

Note that β depends on ψ_p. The first condition in the definition tells us that the subspace ψ_p contains the point p. The second condition says that for each point $\widetilde{p} = \widetilde{q} \circ \psi_p$ of the subspace, the tangent space of the subspace at $\widetilde{p}$ equals the collection of tangent vectors of Δ at $\widetilde{p}$.

Example 4.2.6. Let $A = k[x, y]$ and $\Delta = Ph(A)/(dy - dxx - xdx)$. Pick an arbitrary point $p\colon A \to k$, i.e., $p(x) = a$ and $p(y) = b$ for some $a, b \in k$. The curve $\psi_p\colon A \to B$, where $B = k[x]$, that sends x to x and y to $x^2 + (b - a^2)$ goes through the point p; if we let $q(x) = a$, we have $q \circ \psi_p = p$.

Let $\beta\colon \Delta *_A B \to Ph(B)$ be defined as above. Then

$$\begin{aligned} \beta(x * 1) &= \psi_p(x) = x, \\ \beta(y * 1) &= \psi_p(y) = x^2 + (b - a^2), \\ \beta(dx * 1) &= d\psi_p(x) = dx, \\ \beta(dy * 1) &= d\psi_p(y) = dxx + xdx. \end{aligned}$$

Both elements $y * 1 - (x^2 + (b - a^2)) * 1$ and $dy * 1 - (dxx + xdx) * 1$ are sent to 0, but since

$$\begin{aligned} y * 1 &= 1 * \psi(y) \\ &= 1 * x^2 + (b - a^2) \\ &= x^2 + (b - a^2) * 1, \end{aligned}$$

and

$$dy * 1 = (dxx - xdx) * 1,$$

these elements are actually 0. Hence β is injective. Because $x * 1$ maps to x and $dx * 1$ maps to dx, the map is also surjective. ♣

As we learned from Example 4.2.3, not all distributions are integrable. We will now look at a condition that is necessary for a distribution to be integrable.

Definition 4.2.7. A distribution $\Delta = Ph(A)/J$ is called **involutive** if for all pairs of vector fields $X,\ Y\colon Ph(A) \to A$ that factor through Δ, the homomorphism corresponding to the commutator $[X, Y]$ also factors through Δ.

Proposition 4.2.8. *All integrable distributions are involutive.*

Proof. Assume that $\Delta = Ph(A)/J$ is an integrable distribution, and let X and Y be two vector fields such that $X(f) = Y(f) = 0$ for all $f \in J$. We must show that the homomorphism $Z\colon Ph(A) \to A$ corresponding to the commutator $[X, Y]$ also factors through Δ.

Since Δ is integrable, there is a k-algebra B such that we for each point $p\colon A \to k$ can find a surjective homomorphism $\psi_p\colon A \to B$ such that

1. $p = q \circ \psi_p$ for some $q\colon B \to k$,
2. the map β is an isomorphism.

The homomorphism ψ_p is surjective, so B is generated by the elements $\psi_p(x_i)$ for $i = 1, 2, \ldots, n$. For each p we can define vector fields $\sigma_p\colon Ph(B) \to B$ and $\tau_p\colon Ph(B) \to B$ as follows:

$$\begin{aligned} \sigma_p\colon Ph(B) &\to B \\ \psi_p(x_i) &\mapsto \psi_p(X(x_i)) = \psi_p(x_i) \\ d\psi_p(x_i) &\mapsto \psi_p(X(dx_i)) \end{aligned}$$

and

$$\begin{aligned} \tau_p\colon Ph(B) &\to B \\ \psi_p(x_i) &\mapsto \psi_p(Y(x_i)) = \psi_p(x_i) \\ d\psi_p(x_i) &\mapsto \psi_p(Y(dx_i)). \end{aligned}$$

Because we have $Ph(B) \cong \Delta *_A B = Ph(A)/J *_A A/ker(\psi_p)$, all $f * 1$ and $1 * a$, where $f \in J$ and $a \in ker(\psi_p)$, should be mapped to 0. That is, we should have $\psi_p(X(f)) = \psi_p(Y(f)) = 0$ and $\psi_p(X(a)) = \psi_p(Y(a)) = 0$. The first equality holds because both X and Y send f to 0. Also, $ker(\psi_p) \subset A$, so $a \in A$ and $X(a) = Y(a) = a$. Therefore

$$\psi_p(X(a)) = \psi_p(Y(a)) = \psi_p(a) = 0.$$

Hence σ_p and τ_p are well-defined.

Let us look at the commutator $[\sigma_p, \tau_p]$. It corresponds to a homomorphism $\xi\colon Ph(B) \to B$ where

$$\begin{aligned}\xi(\psi_p(x_i)) &= \psi_p(x_i) \\ &= \psi_p \circ Z(x_i)\end{aligned}$$

and

$$\begin{aligned}\xi(d\psi_p(x_i)) &= (\sigma_p \circ d \circ \tau_p \circ d - \tau_p \circ d \circ \sigma_p \circ d)(\psi_p(x_i)) \\ &= \sigma_p \circ d \circ \tau_p(d\psi_p(x_i)) - \tau_p \circ d \circ \sigma_p(d\psi_p(x_i)) \\ &= \sigma_p \circ d \circ \psi_p \circ Y(dx_i) - \tau_p \circ d \circ \psi_p \circ X(dx_i) \\ &= \psi_p \circ X \circ d \circ Y(dx_i) - \psi_p \circ Y \circ d \circ X(dx_i) \\ &= \psi_p \circ (X \circ d \circ Y - Y \circ d \circ X)(dx_i) \\ &= \psi_p \circ Z(dx_i).\end{aligned}$$

Thus $\xi = \psi_p \circ Z$. But $Ph(B) \cong \Delta *_A B$, so $\xi(f) = 0$ for all $f \in J$. Therefore

$$\psi_p(Z(f)) = \xi(f) = 0,$$

which implies that $Z(f) \in ker(\psi_p)$ for all $f \in J$. This is true for all $p\colon A \to k$, so for all $f \in J$,

$$Z(f) \in \bigcap_{p\colon A \to k} \ker(\psi_p).$$

If we can show that $\cap ker(\psi_p) = (0)$, we are finished. Assume that $g \in \cap ker(\psi_p)$. Then, by the first condition in Definition 4.2.5, $p(g) = 0$ for all $p\colon A \to k$, which implies that g itself must be equal to 0. Hence,

$$\bigcap_{p\colon A \to k} \ker(\psi_p) = (0).$$

□

The proposition states that if a distribution is not involutive, it can not be integrable either, so it gives us a necessary condition for a distribution to be integrable. In manifold theory, there are similar notions of involutivity and integrability, and the condition is also sufficient, i.e., all involutive distributions are integrable. This result is called the Frobenius Theorem, and can be found in [Lee12, Chapter 19]. The proof of this theorem builds on the existence theorem for ordinary differential equations. It says that if we are given a time-dependent vector field $F\colon \mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n$, we can through each point in the space find a curve $x\colon I \to \mathbb{R}^n$ such that for each $t \in I$, $F(x(t), t)$ is the tangent vector to the curve at the point $x(t)$. That is,

$$\frac{dx}{dt}(\tau) = F(x(\tau), \tau)$$

for all $\tau \in I$ [Lee12, Appendix D]. Let us find similar algebraic notions.

A vector field on a k-algebra A is a homomorphism from $Ph(A)$ into A. If we want it to be dependent on time, we can just add another variable: $A \otimes_k k[t]$. Assume that $A = k[x_1, x_2, \ldots, x_n]$. Then $A \otimes_k k[t]$ is just the polynomial ring $k[x_1, x_2, \ldots, x_n, t]$. A vector field that is dependent on time should be a homomorphism

$$F\colon Ph(A \otimes k[t]) \to A \otimes k[t].$$

However, we only want the vector space to consist of tangent vectors in A (that depend on time), not in $A \otimes k[t]$, so we are only interested in where the dx_i, for $i = 1, 2, \ldots n$, are mapped. Therefore we let $F(dt) = 1$, and write $F(dx_i) = f_i \in A \otimes k[t]$ for all $i = 1, 2, \ldots, n$. For each point of A, we want a curve that goes through that point at a certain time. Let $p\colon A \otimes k[t] \to k$ be a homomorphism such that $p(t) = t_0 \in k$ and $p(x_i) = a_i \in k$ for all $i = 1, 2, \ldots, n$. It consists of two homomorphisms, $p_1\colon A \to k$ and $p_2\colon k[t] \to k$. The first map, p_1, gives us a point in A, while p_2 can be considered a point of time. A map $\psi\colon A \otimes k[t] \to k[t]$ where $\psi(t) = t$ and $p = q \circ \psi$ for some $q\colon k[t] \to k$ is a curve that goes through the point p_1 at the time p_2. If $p = q \circ \psi$, we must have $q(t) = t_0$, so $q = p_2$. The homomorphism ψ maps each $x_i \in A$ to a polynomial in $k[t]$. I.e., for every $i = 1, 2, \ldots, n$, $\psi(x_i) = x_i(t)$ for some $x_i(t) \in k[t]$. Let $\widetilde{F}\colon Ph(k[t]) \to k[t]$ be the homomorphism that maps dt to 1, and assume

$$\psi \circ F = \widetilde{F} \circ \psi_*.$$

Suppose for a moment that all the elements in $Ph(k[t])$ commute. In that case,

$$\psi_*(dx_i) = d\psi(x_i) = \frac{dx_i}{dt} dt.$$

If we now apply $\widetilde{F}$ to this element, we see that $\widetilde{F}(\psi_*(dx_i)) = \frac{dx_i}{dt}$. This is an element in $k[t]$, which is a commutative ring, so even though $Ph(k[t])$ does not commute, $\widetilde{F}(\psi_*(dx_i)) = \frac{dx_i}{dt}$ for all $i = 1, 2, \ldots, n$. Now let us see where $\psi \circ F$ maps dx_i:

$$\begin{aligned}\psi(F(dx_i)) &= \psi(f_i)\\ &= f_i(t, x_1(t), x_2(t), \ldots, x_n(t)).\end{aligned}$$

By the assumption that $\psi \circ F = \widetilde{F} \circ \psi_*$, we have

$$\frac{dx_i}{dt} = f_i(t, x_1(t), x_2(t), \ldots, x_n(t))$$

for all $i = 1, 2, \ldots, n$. Thus, ψ is a curve through the point p such that its derivative is given by the vector field F. In other words, the vector field F consists of tangent vectors in A that are tangent to the curve. However, given a vector field F that depends on time, it is not necessarily the case that there is a curve ψ such that $\psi \circ F = \widetilde{F} \circ \psi_*$. This is illustrated by the following example:

Example 4.2.9. Let $A = k[x, y]$ and

$$F\colon Ph(A \otimes k[t]) \to A \otimes k[t]$$

be the vector field mapping dx to x, dy to y and dt to 1. A curve

$$\psi\colon A \otimes k[t] \to k[t]$$

such that $\psi \circ F = \widetilde{F} \circ \psi_*$ must then satisfy $\frac{dx}{dt} = x(t)$ and $\frac{dy}{dt} = y(t)$, where $\psi(x) = x(t)$ and $\psi(y) = y(t)$. The only possibility is that $x(t) = y(t) = e^t$, but e^t is not an element of $k[t]$. Thus there is no homomorphism $\psi\colon A \otimes k[t] \to k[t]$ such that $\psi \circ F = \widetilde{F} \circ \psi_*$. ♣

This example shows us that we do not have an existence theorem equivalent to that of differential geometry. Hence, we do not have a theorem corresponding to the Frobenius Theorem either.

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