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THE VELOCITY FIELD INDUCED BY OSCILLATING CYLINDERS IN A VISCOUS INCOMPRESSIBLE FLUID

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Abstract

This report deals with the calculation of the velocity field induced in a viscous incompressible fluid enclosed between two coaxial cylinders of which one is performing transverse, small amplitude, simple harmonic oscillations.

The cases $R_s \ll 1$ and $R_s \gg 1$ are investigated, where R_s is the steady streaming Reynolds number. For $R_s \ll 1$ some new terms of a perturbation expansion of the secondary steady stream function are calculated. The importance of these higher order terms **is** discussed on the basis of numerical examples. In the special case $R_s = 0$ and none outer boundary a comparison with an inner and an outer asymptotic expansion of the theory of Holtsmark & al. (1954) is carried out. Identical term-wise agreement is found. For $R_s \gg 1$ higher order approximations of the steady boundary layer flow close to the inner cylinder are calculated and the effect of the outer cylinder on this boundary layer flow is also estimated. Improved agreement with experimental data is observed.

1. Introduction

This report deals with the velocity field induced in a viscous, incompressible fluid confined by two co-axial cylinders, of which one is performing small amplitude simple harmonic vibrations orthogonal to its generators. The velocity field has a time dependent and a time independent part. We focuse attention upon the steady streaming component. Higher order terms of asymptotic expansions applied by others are calculated subject to $\varepsilon = \frac{U_0}{\omega a} << 1$, $M = a\sqrt{\frac{\varepsilon}{\nu}} >> 1$, $N = \frac{A}{a} >> 1$, $R_s = \varepsilon^2 M^2 << 1$ and $R_s >> 1$, where U_0 is the velocity amplitude of the oscillating cylinder, ω the angular frequency, ν the kinetic viscosity of the fluid, a and A the radius of the inner and the outer cylinders, respectively. The physical relevance of the parameters are: ε, dimensionless amplitude of the oscillating cylinder; M, modified Reynolds number associated with the oscillatory motion; N, dimensionless radius of the outer boundary; R_s, steady streaming Reynolds number.

The knowledge of the steady streaming generated by nonlinear effects in the vicinity of oscillating boundaries of a viscous fluid is quite extensive. Already in the last century Faraday (1831) observed such streaming and Rayleigh (1945) and Schlichting (1931) gave theoretical explanations of the gross features of the flow. In recent years Westervelt (1953), Nyborg (1953), Holtsmark, Johnsen, Sikkeland & Skavlem (1954), Raney, Corelli and Westervelt (1954), Skavlem & Tjøtta (1955), Segel (1961), Stuart (1966), Riley (1965, 1967), Wang (1968), Bertelsen, Svardal & Tjøtta (1973) have paid attention to this streaming problem. But in spite of these efforts Bertelsen (1974) and Riley (1975) observe discrepancies between existing theories and experimental data in the case $R_s >> 1$.

For $R_s \ll 1$ solutions including the effect of the outer boundary on the steady streaming were given by Skavlem & al. (1955) and Svardal (1965, but are formally restricted to $R_s = 0$. We apply the method of matched asymptotic expansions (hereafter abbreviated MAE) to obtain a solution of the stream function including terms due to finite R_s and the importance of the various terms are discussed by means of numerical examples. In the case $N = \infty$ and $R_s = 0$ our results are compared with asymptotic expansions of the theory of Holtsmark & al. (1954) and identical term-wise agreement is observed.

For $R_s \gg 1$ the secondary steady streaming develops its own boundary layers around the inner cylinder. In this case a first order boundary layer solution of stream function associated with this flow were given by Stuart (1966) and Riley (1965). Riley (1975) applied numerical methods to obtain a second order steady boundary layer solution. We calculate second order terms analytically and an estimate of the effect of the outer boundary on the steady streaming in the vicinity of the inner cylinder is also given subject to

$$\frac{\sqrt{R_s}}{[(\;+2)N]^{2/3}} \to 0 \qquad \begin{matrix} R_s \to \infty \\ N \to \infty \end{matrix}$$

This brings theoretical predictions in accordance with observed velocities in the steady boundary layer at the inner cylinder. In spite of this improvement, the theoretical description of the effect of the outer boundary is unsatisfactory with respect to several points which are discussed in section 6.2.

2. Formulation of the problem

2.1 Basic equation

The geometry of the problem is as indicated in figure 1 where a polar coordinate system $(\bar{r},\theta)$ is introduced.

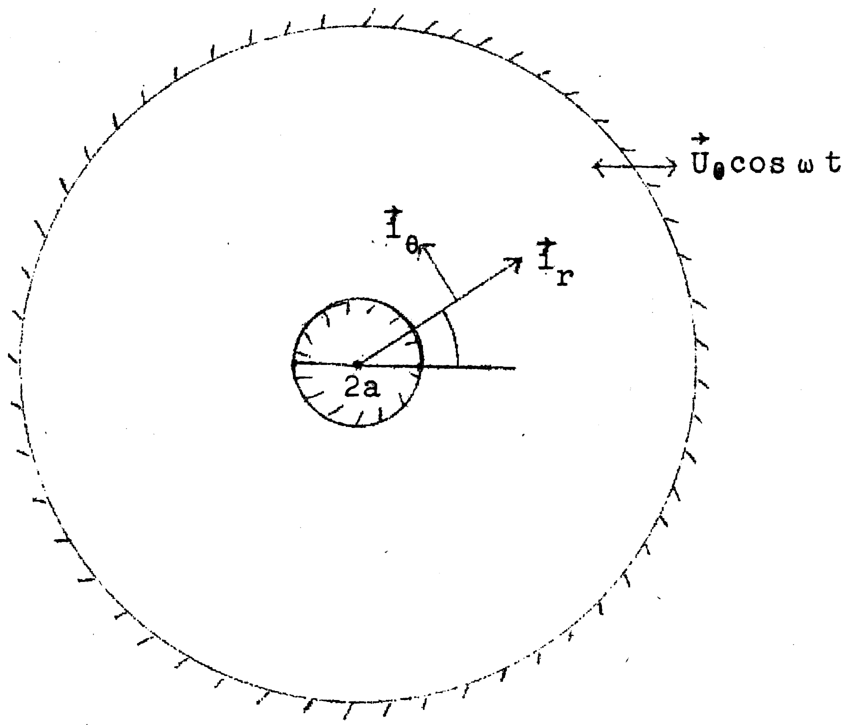

Figure 1. The figure indicates the inner cylinder of radius a and the outer cylinder of radius A which have the constrained motion $\vec{U}_0 \cos \omega t$.

Let $\Psi(\bar{r},\theta,t)$ be the stream function in physical dimensions related to the velocity field by the equation

$$\vec{V}(\vec{r},\theta,t) = - \nabla x(\Psi(\bar{r},\theta,t)\vec{i}_z)$$

where $\vec{i}_z$ is the unit vector orthogonal to the plane of motion, and

$$\nabla x = \vec{i}_r x \frac{\partial}{\partial \bar{r}} + \vec{i}_\theta x \frac{1}{\bar{r}} \frac{\partial}{\partial \theta}$$

The non-dimensional vorticity equation expressed in terms of the non-dimensional stream function $\psi(r,\theta,\tau)$, is,

(1) $$\frac{\partial}{\partial\tau}\nabla^2\psi + \varepsilon\,\frac{1}{r}\,\frac{\partial(\psi,\nabla^2\psi)}{\partial(r,\theta)} = \frac{1}{M^2}\,\nabla^4\psi$$

where $\Psi(\bar{r},\theta,t) = \varepsilon U_0 a\psi(r,\theta,\tau)$, $r = \frac{\bar{r}}{a}$, $\tau = \omega t$, ∇^2 is the two-dimensional Laplacian and $\nabla^4 = \nabla^2\nabla^2$. The outer cylinder is assumed to have the constrained motion,

(2) $$\vec{U}_0 = U_0(-\vec{i}_r\cos\theta + \vec{i}_\theta\sin\theta)\cos\tau\ .$$

We shall mainly be concerned with the calculation of the secondary steady streaming effects subject to the conditions,

$$\varepsilon \ll 1\ ,\quad M \gg 1,\ N = \frac{A}{a} \gg 1,\ R_s = \varepsilon^2 M^2 \ll 1,\ \text{and}\ R_s \gg 1.$$

The formulation of this problem with a view to the application of the method of MAE for $N = \infty$ has been thoroughly investigated by Riley (1965) and (1967). The formulation is easily adjusted to include large, but finite N. In order to linearize equation (1), we put,

(3) $$\psi \sim \psi_0 + \varepsilon(\psi_1(r,\theta) + \psi_1^{(u)}(r,\theta,\tau)) + \cdots$$

which inserted into equation (1), gives the equation for ψ_0,

(4) $$\frac{\partial}{\partial\tau}\,\nabla^2\psi_0 = \frac{1}{M^2}\,\nabla^4\psi_0\ .$$

The general equation for the steady part ψ_0 of the second order stream function is ,

(5) $$\frac{1}{M^2}\,\nabla^4\psi_1 = \Big\langle \frac{1}{r}\,\frac{\partial(\psi_0,\nabla^2\psi_0)}{\partial(r,\theta)} + \frac{\varepsilon}{r}\,\frac{\partial(\psi_0,\nabla^2\psi_1)}{\partial(r,\theta)} + \frac{\varepsilon}{r}\,\frac{\partial(\psi_1,\nabla^2\psi_0)}{\partial(r,\theta)} + \frac{\varepsilon^2}{r}\,\frac{\partial(\psi_1,\nabla^2\psi_1)}{\partial(r,\theta)}\Big\rangle,$$

where the triangle bracket means time averaging.

2.2 Boundary conditions

The boundary conditions must be expanded in powers of ε corresponding to the expansion (3) of the stream-function (see Skavlem and Tjøtta (1954, pp.27). The non-dimensional velocity of the outer cylinder is,

(6) $$\vec{U}_0^* = (-\vec{i}_r\cos\theta + \vec{i}_\theta\sin\theta)\cos\tau$$

which is a Lagrangian velocity. This velocity can be expressed in terms of the Eulerian velocity field by performing a Taylor expansion around the equilibrium position of the oscillating cylinder (see Svardal 1965, pp.8). Thus we have,

(7) $$\vec{v}(\vec{N}+\varepsilon)\int\vec{U}_0^* d\tau,\tau) = \vec{v}(\vec{N},\tau) + \varepsilon\left[\int\vec{U}_0^* d\tau\right]\cdot\left[\nabla\vec{v}\right]_{r=N} + \cdots = \vec{U}_0^*(\theta,\tau) ,$$

where $\vec{N}$ is the position vector of any point on the outer cylinder in its equilibrium position. According to the expansion (3) of the stream function, we know that the solution of the velocity field $\vec{v}$, will be expressed in a power series of ε, it is,

(8) $$\vec{v} = \vec{v}_0 + \varepsilon\vec{v}_1 + \cdots$$

which inserted into equation (7), gives

(9) $$\vec{v}_0(\vec{N},\tau) + \varepsilon\vec{v}_1(\vec{N},\tau) + \varepsilon\int\vec{U}_0^* d\tau\cdot\left[\nabla\vec{v}_0\right]_{r=N} + \cdots = \vec{U}_0^*(\theta,\tau).$$

Equation (9) gives the linearized and second order boundary conditions at the outer cylinder. Since the inner cylinder is at rest, the boundary conditions at this surface are obvious. Thus the linearized boundary conditions are,

(10) $$\begin{cases} \left[\vec{v}_0(r,\theta,\tau)\right]_{r=1} = 0 \\ \left[\vec{v}_0(r,\theta,\tau)\right]_{r=N} = (-\vec{i}_r\cos\theta + \vec{i}_\theta\sin\theta)\cos\tau \end{cases}$$

The second order steady boundary conditions are

$$(11)\qquad \begin{cases} \left[\vec{v}_1^{(s)}(r,\theta)\right]_{r=1} = 0 \\ \\ \left[\vec{v}_1^{(s)}(r,\theta)\right]_{r=N} = -\left< \int \vec{U}_0^{*} d\tau \cdot \left[\nabla \vec{v}_0\right]\right>_{r=N} \end{cases}$$

where the last condition can be expressed explicitly when the linearized solution is known.

3. The linearized solution

The problem of solving equation (4) subject to $M >> 1$, $N >> 1$ and the boundary conditions (10), will be regarded as a singular perturbation problem and treated by the method of MAE. In order to apply this method, we have to construct,

a) an outer asymptotic expansion Φ_0 of ψ_0, valid outside the Stokes layers at the boundaries,

$$(12)\qquad \psi_0(r,\theta,\tau;M;N) \underset{\substack{r\ \text{fixed} \\ M\to\infty \\ N\to\infty}}{\sim} \Phi_0(r,\theta;M;N) = \alpha_{00}(M)\sigma_{000}(N)\Phi_{000}(r,\theta,\tau)$$
$$+ \alpha_{01}(M)\left[\sigma_{010}(N)\Phi_{010}(r,\theta,\tau) + \sigma_{011}(N)\Phi_{011}(r,\theta,\tau) + \cdots\right]$$
$$+ \cdots$$

b) an inner asymptotic expansion X_0 of ψ_0, valid in the Stokes layer at the inner cylinder,

$$(13)\qquad \psi_0(r,\theta,\tau;M;N) \underset{\substack{\eta\ \text{fixed} \\ M\to\infty \\ N\to\infty}}{\sim} X_0(\eta,\theta,\tau;M;N) = \beta_{00}(M)\mu_{000}(N)X_{000}(\eta,\theta,\tau)$$
$$+ \beta_{01}(M)\left[\mu_{010}(N)X_{010}(\eta,\theta,\tau) + \mu_{011}(N)X_{011}(\eta,\theta,\tau) + \cdots\right]$$
$$+ \beta_{02}(M)\left[\mu_{020}(N)X_{020}(\eta,\theta,\tau) + \cdots\right]$$
$$+ \cdots$$

c) an inner asymptotic expansion φ_0 of ψ_0, valid in the Stokes layer at the outer cylinder

$$(14)\quad \psi_0(r,\theta,\tau;M;N) \underset{\substack{\lambda \text{ fixed} \\ M\to\infty \\ N\to\infty}}{\sim} \varphi_0(\lambda,\theta,\tau;M;N) = \kappa_{00}(M)\delta_{000}(N)\varphi_{000}(\lambda,\theta,\tau)$$

$$+ \kappa_{01}(M)\Big[\delta_{010}(N)\varphi_{010}(\lambda,\theta,\tau) + \delta_{011}(N)\varphi_{011}(\lambda,\theta,\tau)\Big]$$

$$+ \kappa_{02}(M)\Big[\delta_{020}(N)\varphi_{020}(\lambda,\theta,\tau) + \delta_{021}(N)\varphi_{021}(\lambda,\theta,\tau) + \cdots\Big]$$

$$+ \cdots ,$$

where $\eta = \frac{M}{\sqrt{2}}(r-1) = \frac{M}{\sqrt{2}}y$ and $= \frac{M}{\sqrt{2}}(N-r) = \frac{M}{\sqrt{2}}Y$ are the sealed boundary layer coordinates in the Stokes layers at the inner and the outer cylinder, respectively. These length scales are determined by claiming viscous and inertia terms in equation (4) to be of the same order of magnitude in the Stokes layers. Most of the other gauge functions are in due course determined from the matching and the boundary conditions, but some of them must also be adapted to the order of magnitude of the inhomogeneity terms of the diffeeential equations in question. In order to obtain the asymptotic expansions listed above, the gauge functions must be asymptotic sequences and fulfil the following requirement, here exemplified by $\{\alpha_{0n}(M)\}$

$$\lim_{M\to\infty}\left[\frac{\alpha_{0n}(M)}{\alpha_{0n-1}(M)}\right] = 0$$

The matching conditions are evaluated from the following equations by equalizing terms of the same order of magnitude;

a) for matching at the inner cylinder,

$$(15)\quad \Big(\alpha_{00}(M)\sigma_{000}(N)\Big\{\Phi_{000}(1,\theta,\tau) + \left[\frac{\partial\Phi_{000}}{\partial r}\right]_{r=1} y + \tfrac{1}{2}\left[\frac{\partial^2\Phi_{000}}{\partial r^2}\right]_{r=1} y^2 + \cdots$$

$$+ \alpha_{01}(M)\Big[\sigma_{010}(N)\Big\{\Phi_{010}(1,\theta,\tau) + \left[\frac{\partial\Phi_{010}}{\partial r}\right]_{r=1} y + \cdots\Big\}$$

$$+ \sigma_{011}(N)\Big\{\Phi_{011}(1,\theta,\tau) + \left[\frac{\partial\Phi_{011}}{\partial r}\right]_{r=1} y + \cdots\Big\}\Big]$$

$$+ \alpha_{020}(M)\sigma_{020}(N)\Phi_{020}(1,\theta,\tau) + \cdots\Big)_{\substack{y \text{ fixed} \\ M\to\infty \\ N\to\infty}}$$

$$= \Big(\beta_{00}(M)\mu_{000}(N)\chi_{000}\big(\tfrac{M}{\sqrt{2}}\, y,\theta,\tau\big)$$

$$+ \beta_{01}(M)\Big[\mu_{010}(N)\chi_{010}\big(\tfrac{M}{\sqrt{2}}\, y,\theta,\tau\big) + \mu_{011}(N)\chi_{011}\big(\tfrac{M}{\sqrt{2}}\, y,\theta,\tau\big) + \cdots\Big]$$

$$+ \beta_{02}(M)\Big[\mu_{020}(N)\chi_{020}\big(\tfrac{M}{\sqrt{2}}\, y,\theta,\tau\big) + \cdots\Big] + \cdots\Big)_{\substack{y \text{ fixed} \\ M\to\infty \\ N\to\infty}}$$

b) for matching at the outer cylinder,

$$(16)\quad \Big(\alpha_{00}(M)\sigma_{000}(N)\Big\{\Phi_{000}(N,\theta,\tau) + \left[\frac{\partial\Phi_{000}}{\partial r}\right]_{r=N} Y + \tfrac{1}{2}\left[\frac{\partial^2\Phi_{000}}{\partial r^2}\right]_{r=N} Y^2 + \cdots\Big\}$$

$$+ \alpha_{01}(M)\Big[\sigma_{010}(N)\Big\{\Phi_{010}(N,\theta,\tau) + \left[\frac{\partial\Phi_{010}}{\partial r}\right]_{r=N} Y + \cdots\Big\}$$

$$+ \sigma_{020}(M)\Big\{\Phi_{020}(N,\theta,\tau) + \left[\frac{\partial\Phi_{020}}{\partial r}\right]_{r=N} Y + \cdots\Big)_{\substack{Y \text{ fixed} \\ M\to\infty \\ N\to\infty}}$$

$$= \Big(\kappa_{00}(M)\delta_{000}(N)\varphi_{000}\Big(\frac{M}{\sqrt{2}}\, Y,\theta,\tau\Big)$$

$$+ \kappa_{01}(M)\Big\{\delta_{010}(N)\varphi_{010}\Big(\frac{M}{\sqrt{2}}\, Y,\theta,\tau\Big) + \delta_{011}(N)\varphi_{011}\Big(\frac{M}{\sqrt{2}}\, Y,\theta,\tau\Big) + \cdots\Big\}$$

$$+ \kappa_{02}(M)\Big\{\delta_{020}(N)\varphi_{020}\Big(\frac{M}{\sqrt{2}}\, Y,\theta,\tau\Big) + \delta_{021}(N)\varphi_{021}\Big(\frac{M}{\sqrt{2}}\, Y,\theta,\tau\Big) + \cdots\Big\}$$

$$+ \cdots\Big)_{\substack{y \text{ fixed} \\ M\to\infty \\ N\to\infty}}$$

The differential equations for the first terms of the outer asymptotic expansion (12) are obtained by inserting this expansion into equation (4) and then performing an outer asymptotic expansion of that equation subject to the condition r fixed, $M \to \infty$ and $N \to \infty$. This gives

$$(17) \qquad \frac{\partial}{\partial \tau} \nabla^2 \Phi_{000} = 0$$

$$(18) \qquad \begin{cases} \frac{\partial}{\partial \tau} \nabla^2 \Phi_{010} = 0 \\ \frac{\partial}{\partial \tau} \nabla^2 \Phi_{011} = 0 \\ \frac{\partial}{\partial \tau} \nabla^2 \Phi_{012} = 0 \end{cases}$$

$$(19) \qquad \begin{cases} \frac{\partial}{\partial \tau} \nabla^2 \Phi_{020} = 0 \\ \frac{\partial}{\partial \tau} \nabla^2 \Phi_{021} = 0 \\ \frac{\partial}{\partial \tau} \nabla^2 \Phi_{022} = 0 \end{cases}$$

Likewise, the differential equations for the first terms of the inner asymptotic (13) are obtained by insertion of this expansion into equation (4) and then performing an inner asymptotic expansion of that equation subject to the conditions η fixed, $M \to \infty$ and $N \to \infty$. This yields,

$$(20) \qquad \frac{\partial^4 \chi_{000}}{\partial \eta^4} - 2 \frac{\partial^3 \chi_{000}}{\partial \tau \partial \eta^2} = 0$$

$$(21) \qquad \frac{\partial^4 \chi_{010}}{\partial \eta^4} - 2 \frac{\partial^3 \chi_{010}}{\partial \tau \partial \eta^2} = 2\sqrt{2} \left[- \frac{\partial^3 \chi_{000}}{\partial \eta^3} + \frac{\partial^2 \chi_{000}}{\partial \tau \partial \eta} \right]$$

$$(22a) \qquad \frac{\partial^4 \chi_{011}}{\partial \eta^4} - 2 \frac{\partial^3 \chi_{011}}{\partial \tau \partial \eta^2} = 0$$

$$(22b) \qquad \frac{\partial^4 \chi_{012}}{\partial \eta^4} - 2 \frac{\partial^3 \chi_{012}}{\partial \tau \partial \eta^2} = 0$$

(23) $$\frac{\partial^4 \chi_{020}}{\partial \eta^4} - 2\frac{\partial^3 \chi_{020}}{\partial \tau \partial \eta^2} = 2\sqrt{2}\Big[-\frac{\partial^3 \chi_{010}}{\partial \eta^3} + \frac{\partial^2 \chi_{010}}{\partial \tau \partial \eta}$$

$$- \sqrt{2}\eta \frac{\partial^2 \chi_{000}}{\partial \tau \partial \eta} + \sqrt{2}\frac{\partial^3 \chi_{000}}{\partial \tau \partial \theta^2} + \sqrt{2}\,\eta \frac{\partial^3 \chi_{000}}{\partial \eta^3}$$

$$- \frac{1}{\sqrt{2}}\frac{\partial^2 \chi_{000}}{\partial \eta^2} - \sqrt{2}\frac{\partial^4 \chi_{000}}{\partial \eta^2 \partial \theta^2}\Big]$$

The differential equations of the four first terms of the asymptotic expansion (14) are obtained by inserting this expansion into equation (4) and then performing an inner asymptotic expansion of that equation subject to the conditions λ fixed, $M \to \infty$ and $N \to \infty$. After some calculations we find,

(24) $$\frac{\partial^4 \varphi_{000}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{000}}{\partial \tau \partial \lambda^2} = 0$$

(25) $$\frac{\partial^4 \varphi_{010}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{010}}{\partial \tau \partial \lambda^2} = 0$$

(26) $$\frac{\partial^4 \varphi_{011}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{011}}{\partial \tau \partial \lambda^2} = 0$$

(27a) $$\frac{\partial^4 \varphi_{020}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{020}}{\partial \tau \partial \lambda^2} = 0$$

(27b) $$\frac{\partial^4 \varphi_{021}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{021}}{\partial \tau \partial \lambda^2} = 2\sqrt{2}\left(\frac{\partial^3 \varphi_{011}}{\partial \lambda^3} - \frac{\partial^2 \varphi_{011}}{\partial \lambda \partial \tau}\right)$$

(28a) $$\frac{\partial^4 \varphi_{030}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{030}}{\partial \tau \partial \lambda^2} = 0$$

(28b) $$\frac{\partial^4 \varphi_{031}}{\partial \lambda^4} - 2\frac{\partial^3 \varphi_{031}}{\partial \tau \partial \lambda^2} = 2\sqrt{2}\left(\frac{\partial^3 \varphi_{020}}{\partial \lambda^3} - \frac{\partial^2 \varphi_{020}}{\partial \tau \partial \lambda}\right)$$

It should be noticed that some of the information that can be drawn from the matching and the boundary conditions already have been used in order to establish the differential equations listed above. The generation and the solving of the differential equations must of course be carried out successively as suggested in appendix A. When carrying out this successive procedure, the gauge functions also are determined. The calculations give in due course,

$$(29)\quad \begin{cases} \alpha_{00}(M) = 1, \quad \alpha_{01}(M) = \frac{1}{M}, \quad \alpha_{02}(M) = \frac{1}{M^2}, \\ \sigma_{000}(N) = \sigma_{010}(N) = \sigma_{020}(N) = \frac{1}{1-N^{-2}}, \\ \sigma_{011}(N) = \sigma_{021}(N) = \frac{1}{1-N^{-2})^2 N^2}, \\ \sigma_{012}(N) = \sigma_{022}(N) = \frac{1}{(1-N^{-2})^2 N^3}. \end{cases}$$

$$(30)\quad \begin{cases} \beta_{00}(M) = \frac{\sqrt{2}}{M}, \quad \beta_{01}(M) = \frac{\sqrt{2}}{M^2}, \quad \beta_{02}(M) = \frac{\sqrt{2}}{M^3}, \\ \mu_{000}(N) = \mu_{010}(N) = \mu_{020}(N) = \frac{1}{1-N^{-2}}, \\ \mu_{011}(N) = \frac{1}{(1-N^{-2})^2 N^2}, \quad \mu_{012}(N) = \frac{11}{(1-N^{-2})^2 N^3}. \end{cases}$$

$$(31)\quad \begin{cases} \kappa_{00}(M) = 1, \quad \kappa_{01}(M) = \frac{\sqrt{2}}{M}, \quad \kappa_{02}(M) = \frac{\sqrt{2}}{M2} \\ \delta_{000}(N) = N, \quad \delta_{010}(N) = 1, \quad \delta_{011}(N) = \delta_{020}(N) = \delta_{030}(N) = \frac{1}{(1-N^{-2})N^2}; \\ \delta_{021}(N) = \delta_{031}(N) = \frac{1}{(1-N^{-2})N^3}. \end{cases}$$

The differential equation (17), (18), (19), ... and (28b) can be solved by elementary methods subject to the boundary conditions (10) and the matching conditions evaluated from equations (15) and (16). The details of these simple calculations are omitted here, but exemplified in appendix A. The final results are, in complex notations, where physical significance is ascribed to the real parts of the solutions, only,

(32) $\Phi_{000}(r,\theta,\tau) = -(r^{-1}-r)\sin\theta\, e^{i\tau}$

(33a) $\Phi_{010}(r,\theta,\tau) = -\sqrt{2}(1-i)r^{-1}\sin\theta\, e^{i\tau}$

(33b) $\Phi_{011}(r,\theta,\tau) = -\sqrt{2}(1-i)(r^{-1}-r)\sin\theta\, e^{i\tau}$

(33c) $\Phi_{012}(r,\theta,\tau) = -\sqrt{2}(1-i)(r^{-1}-r)\sin\theta\, e^{i\tau}$

(34a) $\Phi_{020}(r,\theta,\tau) = i\, r^{-1}\sin\theta\, e^{i\tau}$

(34b) $\Phi_{021}(r,\theta,\tau) = i(5r^{-1}-r)\sin\theta\, e^{i\tau}$

(34c) $\Phi_{022}(r,\theta,\tau) = i(8r^{-1}-4r)\sin\theta\, e^{i\tau}$

(35) $\chi_{000}(\eta,\theta,\tau) = \left\{-(1-i)+2\eta+(1-i)e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\tau}$

(36) $\chi_{010}(\eta,\theta,\tau) = \sqrt{2}\left\{\frac{i}{2}+(1-i)\eta-\eta^2+\left(-\frac{i}{2}+\frac{-1+i}{2}\eta\right)e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\tau}$

(37a) $\chi_{011}(\eta,\theta,\tau) = 2\sqrt{2}\left\{i+(1-i)\eta-i\, e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\tau}$

(37b) $\chi_{012}(\eta,\theta,\tau) = 2\sqrt{2}\left\{i+(1-i)\eta-i\, e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\tau}$

(38)
$$\chi_{020}(\eta,\theta,\tau) = \left\{-\frac{1+i}{8}-i\eta-2(1-i)\eta^2+2\eta^3\right.$$
$$\left.+\left[\frac{1+i}{8}+\frac{5}{4}i\eta+\frac{3}{4}(1-i)\eta^2\right]e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\tau}$$

(39) $\varphi_{000}(\lambda,\theta,\tau) = \sin\theta\, e^{i\tau}$

(40) $\varphi_{010}(\lambda,\theta,\tau) = -\lambda\sin\theta\, e^{i\tau}$

(41) $\varphi_{011}(\lambda,\theta,\tau) = \left\{(1-i)-2\lambda-(1-i)e^{-(1+i)\lambda}\right\}\sin\theta\, e^{i\tau}$

(42a) $\varphi_{020}(\lambda,\theta,\tau) = 2\sqrt{2}\left\{-i-(1-i)\lambda+i\, e^{-(1+i)\lambda}\right\}\sin\theta\, e^{i\tau}$

(42b) $\varphi_{021}(\lambda,\theta,\tau) = \sqrt{2}\left\{-\frac{3}{2}i-(1-i)\lambda-\lambda^2+\left(\frac{3}{2}i-\frac{1-i}{2}\lambda\right)e^{-(1+i)\lambda}\right. \sin\theta\, e^{i\tau}$

(43a) $\varphi_{030}(\lambda,\theta,\tau) = \left\{-(1+i)+2i\lambda+(1+i)e^{-(1+i)\lambda}\right\}\sin\theta\, e^{i\tau}$

(43b) $\varphi_{031}(\lambda,\theta,\tau) = \left\{-3(1+i)+4i\lambda-2(1-i)\lambda^2+\left[3(1+i)+2i\lambda\right]e^{-(1+i)\lambda}\right\}\sin\theta e^{i\tau}$

It should be noticed that $\chi_{000}(\eta,\theta,\tau)$ is identical to a corresponding linearized solution found by Schlichting (1932,pp329). Likewise the term $\chi_{010}(\eta,\theta,\tau)$ was first given by Riley (1967, pp.423). However, the gauge functions $\mu_{000}(N)$ and $\mu_{010}(N)$ were not included in the calculations of Schlichting and Riley.

The main effect of the finite radius of the outer boundary, described by $\mu_{000}(N)$, is to increase the amplitude of the shear wave $\chi_{000}(\eta,\theta,\tau)$. The term $\chi_{010}(\eta,\theta,\tau)$ is to be interpreted as a modification of the asymptotic stream function $\chi_{000}(\eta,\theta,\tau)$ due to the curved boundary (the inner cylinder) and the displacement flow. $\chi_{011}(\eta,\theta,\tau)$ is an effect of the finite radius of the outer boundary. The higher order term $\chi_{020}(\eta,\theta,\tau)$ describes more complicated influences of curvature and displacement flow. The main component of the inner solution at the outer boundary is the potential oscillations given by $N\varphi_{000}(\lambda,\theta,\tau)+\frac{\sqrt{2}}{M}\varphi_{010}(\lambda,\theta,\tau)$. There is also a shear wave $\varphi_{011}(\lambda,\theta,\tau)$, which corresponds to $\chi_{000}(\eta,\theta,\tau)$ in the inner solution at the inner cylinder, but the amplitude of $\varphi_{011}(\lambda,\theta,\tau)$ is reduced with the factor N^{-2} relative to $\chi_{000}(\eta,\theta,\tau)$.

The flow outside the Stokes layers is potential oscillations as indicated by the equations (17), (18) and (19).

4. The non-linear steady streaming effects

It is well known that the Reynolds stresses in an oscillatory boundary layer flow induces secondary steady and oscillatory motion. Referring to equation (3) these motions are described by $\psi_1(r,\theta)$ and $\psi_0^{(u)}(r,\theta,\tau)$, respectively. We pay attention to the second order steady stream function $\psi_1(r,\theta)$, only, which will be discussed in the two cases $R_s \ll 1$ and $R_s \gg 1$.

4.1 The second order steady boundary conditions

The second order steady boundary conditions formally stated by equation (11), can now, with the knowledge of the linearized solution, be given specifically. Inserting into equation (11) $\vec{v}_0$ given by

(44) $$\vec{v}_0 = -\nabla \times (\varphi_0 \vec{i}_z)$$

(where an appropriate inner asymptotic expansion of the curl operator is presupposed), we get,

(45) $$\left[\vec{v}_1^{(s)}(\lambda,\theta;M;N)\right]_{\lambda=0} = \vec{i}_\theta\left\{\frac{M}{\sqrt{2}}\left[\frac{1}{(1-N^{-2})N^2}\left(\tfrac{1}{2}\sin 2\theta\right) + O(N^{-4})\right]\right.$$
$$\left. + \frac{\sqrt{2}}{M}\left[\frac{1}{(1-N^{-2})N^2}\left(-\tfrac{1}{4}\sin 2\theta\right) + \frac{1}{(1-N^{-2}N^2}\left(-\tfrac{3}{4}\sin 2\theta\right) + O(N^{-4})\right] + O(M^{-2})\right\}$$

(46) $$\left[\vec{v}_1^{(s)}(\eta,\theta;M;N)\right]_{\eta=0} = 0 \,.$$

Equation (45) has a noticable property, namely,

$$\left[\vec{v}_1(\lambda,\theta;M;N)\right]_{\substack{\lambda=0 \\ M\to\infty \\ N\ \text{finite}}} \to \infty$$

It should be mentioned, however, that the singularity demonstrated above, and all the other terms of equation (45), are cancelled by the Stokes drift. The time averaged particle velocity is thus zero at the outer boundary, a result which is expected on physical basis also.

4.2 The case $R_s << 1$.

The general equation of the steady part ψ_1 of the second order stream function is given by equation (5). Inspection of this equation with the inner and the outer expansions of the linearized solution in mind, leads us, with a view to the application of the method of MAE, to introduce the following expansions :

(a) an inner asymptotic expansion χ_1 of ψ_1 constructed to be valid in the Stokes layer at the inner cylinder,

(47) $$\psi_1(r,\theta;M;N;R_s) \underset{\substack{\eta \text{ fixed} \\ M\to\infty \\ N\to\infty \\ R_s\to 0}}{\sim} \chi_1(\eta,\theta;M;N;R_s) = \beta_{10}(M)\mu_{100}(N)\chi_{100}(\eta,\theta)$$
$$+\beta_{11}(M)\left\{\mu_{110}(N)\chi_{110}(\eta,\theta;R_s) + \mu_{111}(N)\chi_{111}(\eta,\theta;R_s)\right\}$$
$$+\beta_{12}(M)\mu_{120}(N)\chi_{120}(\eta,\theta;R_s) + \cdots$$

(b) an inner asymptotic expansion φ_1 of ψ_1 constructed to be valid in the Stokes layer at the outer cylinder,

(48) $$\psi_1(r,\theta;M;N;R_s) \underset{\substack{\lambda \text{ fixed} \\ M\to\infty \\ N\to\infty \\ R_s\to 0}}{\sim} \varphi_1(\lambda,\theta;M;N;R_s) = \kappa_{10}(M)\delta_{10}(N)\varphi_{10}(\lambda,\theta)$$
$$+ \kappa_{11}(M)\left\{\delta_{110}(N)\varphi_{110}(\lambda,\theta) + \delta_{111}(N)\varphi_{111}(\lambda,\theta)\right\}$$
$$+ \kappa_{12}(M)\left\{\delta_{120}(N)\varphi_{120}(\lambda,\theta;R_s) + \delta_{121}\varphi_{121}(\lambda,\theta;R_s)+\cdots\right\}$$
$$+ \cdots$$

c) an outer asymptotic expansion Φ_1 of ψ_1 constructed to be valid outside the Stokes layers at the boundaries,

$$(49)\quad \psi_1(r,\theta;M,N;R_s) \underset{\substack{r\ \text{fixed}\\ M\to\infty\\ N\to\infty\\ R_s\to 0}}{\sim} \Phi_1(r,\theta;M,N;R_s) = \gamma_{10}(R_s)\Big\{\alpha_{100}(M)\Big[\sigma_{1000}(N)\Phi_{1000}(r,\theta)$$

$$+ \sigma_{1010}(N)\Phi_{1010}(r,\theta) + \sigma_{1020}(N)\Phi_{1020}(r,\theta) + \cdots \Big]$$

$$+ \alpha_{110}(M)\Big[\sigma_{1100}(N)\ \Phi_{1100}(r,\theta,) + \cdots \Big] + \cdots \Big\}$$

$$+ \gamma_{11}(R_s)\Big\{\alpha_{101}(M)\Big[\sigma_{1001}(N)\Phi_{1011}(r,\theta) + \cdots \Big] + \cdots \Big\}$$

$$+ \gamma_{12}(R_s)\Big\{\alpha_{102}\ (M)\Big[\sigma_{1002}(N)\Phi_{1002}(r,\theta) + \cdots \Big] + \cdots \Big\}$$

Inserting into equation (5), the expansions defined above and taking into account that all the terms of the outer asymptotic expansion Φ_0 describe potential oscillations, we can formally write,

$$(50)\quad \frac{1}{M^2}\ \nabla^4\chi_1 - \frac{1}{r} < \frac{\partial(\chi_0,\nabla^2\chi_0)}{\partial(r,\theta)} > = 0$$

$$(51)\quad \nabla^4\Phi_1 - R_s\,\frac{1}{r}\,\frac{\partial(\Phi_1,\nabla^2\Phi_1)}{\partial(r,\theta)} = 0$$

$$(52)\quad \frac{1}{M^2}\ \nabla^4\varphi_1 - \frac{1}{r} < \frac{\partial(\varphi_0,\nabla^2\varphi_0)}{\partial(r,\theta)} > = 0$$

where of course only the real parts of χ_0 and φ_0 can be used in the calculation of the non-linear terms. The matching conditions for the second order terms χ_1, φ_1 and Φ_1 can also be settled formally now. These conditions can be evaluated from the following equations by claiming terms of the same order of magnitude to be equal,

(a) for matching at the inner cylinder,

$$(53)\quad \left(\Phi_1(1,\theta;M;N;R_s) + \left[\frac{\partial\Phi_1(r,\theta;M;N;R_s)}{\partial r}\right]_{r=1} y + \tfrac{1}{2}\left[\frac{\partial\Phi_1{}^2(r,\theta;M;N;R_s)}{\partial r^2}\right]_{r=1} y^2 + \cdots\right)_{\substack{y\ \text{fixed}\\ M\to\infty\\ N\to\infty\\ R_s\to 0}}$$

$$= \Big(\beta_{10}(M)\mu_{100}(N)\chi_{100}(\frac{M}{\sqrt{2}2}y,\theta)$$

$$+ \beta_{11}(M)\Big[\mu_{110}(N)\chi_{110}(\frac{M}{\sqrt{2}}y,\theta;R_s) + \mu_{111}(N)\chi_{111}(\frac{M}{\sqrt{2}}y,\theta;R_s) + \cdots\Big]$$

$$+ \beta_{12}(M)\mu_{120}(N)\chi_{120}(\frac{M}{\sqrt{2}}y,\theta;R_s) + \cdots\Big)_{\substack{y\ \text{fixed}\\ M\to\infty\\ N\to\infty\\ R_s\to 0}}$$

(b) for matching at the outer cylinder,

$$(54)\quad \left(\Phi_1(N,\theta) - \left[\frac{\partial\Phi_1(r,\theta;M;N;R_s)}{\partial r}\right]_{r=N} Y + \tfrac{1}{2}\left[\frac{\partial^2\Phi_1(r,\theta;M;N;R_s)}{\partial r^2}\right]_{r=N} Y^2 \quad \cdots\right)_{\substack{Y\ \text{fixed}\\ M\to\infty\\ N\to\infty\\ R_s\to 0}}$$

$$= \Big(\kappa_{10}(M)\delta_{100}(N)\varphi_{100}(\frac{M}{\sqrt{2}}Y,\theta)$$

$$+ \kappa_{11}(M)\Big[\delta_{110}(N)\varphi_{110}(\frac{M}{\sqrt{2}}Y,\theta) + \delta_{111}(N)\varphi_{111}(\frac{M}{\sqrt{2}}Y,\theta) + \cdots\Big]$$

$$+ \kappa_{12}(M)\delta_{120}(N)\varphi_{120}(\frac{M}{\sqrt{2}}Y,\theta;R_s) + \cdots\Big)_{\substack{Y\ \text{fixed}\\ M\to\infty\\ N\to\infty\\ R_s\to 0}}$$

The formal expansion (47) are introduced into equation (50). An inner asymptotic expansion of the operator ∇^4 and the non-linear term is performed by putting $r = 1 + \frac{\sqrt{2}}{M}\eta$. Equalizing terms of the same order of magnitude, then leads to

(55) $$\frac{\partial \chi_{100}}{\partial \eta^4} = 2 < - \frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} \quad \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} >$$

(56) $$\frac{\partial^4 \chi_{110}}{\partial \eta^4} = -2\sqrt{2} \frac{\partial^3 \chi_{100}}{\partial \eta^3} + < 2\Big[- \frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{010}}{\partial \eta^3} + \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{010}}{\partial \eta^2 \partial \theta}$$

$$- \frac{\partial \chi_{010}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} + \frac{\partial \chi_{010}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} \Big]$$

$$+ 2\sqrt{2}\eta \Big[\frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} - \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} \Big]$$

$$- 2\sqrt{2} \Big[\frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^2 \chi_{011}}{\partial \eta^2} - \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^2 \chi_{000}}{\partial \eta \partial \theta} \Big] >$$

(57) $$\frac{\partial^4 \chi_{111}}{\partial \eta^4} = < 2\Big[- \frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{011}}{\partial \eta^3} + \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{011}}{\partial \eta^2 \partial \theta}$$

$$- \frac{\partial \chi_{011}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} + \frac{\partial \chi_{011}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} \Big] >$$

(58) $$\frac{\partial^4 \chi_{120}}{\partial \eta^4} = -2\sqrt{2} \frac{\partial^3 \chi_{110}}{\partial \eta^3} + 4\eta \frac{\partial^3 \chi_{100}}{\partial \eta^3} + 2 \frac{\partial^2 \chi_{100}}{\partial \eta^2} - 4 \frac{\partial^4 \chi_{100}}{\partial \eta^2 \partial \theta^2}$$

$$+ < -2\Big[\frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{020}}{\partial \eta^3} + \frac{\partial \chi_{010}}{\partial \theta} \frac{\partial^3 \chi_{010}}{\partial \eta^3} + \frac{\partial \chi_{020}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3}$$

$$- \sqrt{2}\,\eta \frac{\partial \chi_{010}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} + \sqrt{2}\,\eta \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{010}}{\partial \eta^3} + 2\eta^2 \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^3} \Big]$$

$$+ 2\Big[\frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{020}}{\partial \eta^2 \partial \theta} + \frac{\partial \chi_{010}}{\partial \eta} \frac{\partial^3 \chi_{010}}{\partial \eta^2 \partial \theta} + \frac{\partial \chi_{020}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta}$$

$$- \sqrt{2}\eta \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{010}}{\partial \eta^2 \partial \theta} \quad \sqrt{2}\eta \frac{\partial \chi_{010}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} + 2\eta^2 \frac{\partial \chi_{000}}{\partial \eta^2} \frac{\partial^3 \chi_{000}}{\partial \eta^2 \partial \theta} \Big]$$

$$- 2\sqrt{2}\Big[\frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^2 \chi_{010}}{\partial \eta^2} + \frac{\partial \chi_{010}}{\partial \theta} \frac{\partial^2 \chi_{000}}{\partial \eta^2} - 2\sqrt{2}\eta \frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \theta^3} \Big]$$

$$+ 4\Big[\frac{\partial \chi_{000}}{\partial \theta} \frac{\partial \chi_{000}}{\partial \eta} \quad \frac{\partial \chi_{000}}{\partial \theta} \frac{\partial^3 \chi_{000}}{\partial \eta \partial \theta^2} \quad \frac{\partial \chi_{000}}{\partial \eta} \frac{\partial^3 \chi_{000}}{\partial \theta^3} \Big] > ,$$

The asymptotic expansion (48) is inserted into equation (52). An inner asymptotic expansion of the operator ∇^4 and the non-linear term are carried out by putting $r = N - \frac{\sqrt{2}}{M}\lambda$. Equalizing terms of the same order of magnitude then yields,

(59) $$\frac{\partial^4 \varphi_{100}}{\partial \lambda^4} = 2 \left\langle \frac{\partial \varphi_{000}}{\partial \theta} \frac{\partial^3 \varphi_{011}}{\partial \lambda^3} \right\rangle$$

(60a) $$\frac{\partial^4 \varphi_{110}}{\partial \lambda^4} = \sqrt{2} \left\langle \frac{\partial \varphi_{000}}{\partial \theta} \frac{\partial^3 \varphi_{020}}{\partial \lambda^3} \right\rangle$$

(60b) $$\frac{\partial^4 \varphi_{111}}{\partial \lambda^4} = 2 \frac{\partial^3 \varphi_{100}}{\partial \lambda^3} + \sqrt{2} \left\langle \frac{\partial \varphi_{000}}{\partial \theta} \frac{\partial^3 \varphi_{021}}{\partial \lambda^3} \right\rangle$$

(61a) $$\frac{\partial^4 \varphi_{120}}{\partial \lambda^4} = \sqrt{2} \left\langle \frac{\partial \varphi_{000}}{\partial \theta} \frac{\partial^3 \varphi_{030}}{\partial \lambda^3} \right\rangle$$

(61b) $$\frac{\partial^4 \varphi_{121}}{\partial \lambda^4} = 2\sqrt{2} \frac{\partial^3 \varphi_{110}}{\partial \lambda^3} + \sqrt{2} \left\langle \frac{\partial \varphi_{000}}{\partial \theta} \frac{\partial^3 \varphi_{031}}{\partial \lambda^3} \right\rangle$$

Finally, the asymptotic expansion (49) is introduced into equation (51). Equalizing terms of the same order of magnitude, then gives,

(62) $$\nabla^4 \Phi_{10i0} = 0 , \qquad i = 0,1,2$$

(63) $$\nabla^4 \Phi_{11i0} = 0 , \qquad i = 0,1,2,3$$

(64) $$\nabla^4 \Phi_{1200} = 0 ,$$

(65) $$\nabla^4 \Phi_{1001} = \frac{1}{r} \frac{\partial(\Phi_{1000}, \nabla^2 \Phi_{1000})}{\partial(r,\theta)}$$

(66) $$\nabla^4 \Phi_{1002} = \frac{1}{r}\left[\frac{\partial(\Phi_{1000}, \nabla^2 \Phi_{1001})}{\partial(r,\theta)} + \frac{\partial(\Phi_{1001}, \nabla^2 \Phi_{1000})}{\partial(r,\theta)} \right]$$

$$(67) \qquad \nabla^4\Phi_{1011} = \frac{1}{r}\left[\frac{\partial(\Phi_{1000},\nabla^2\Phi_{1010})}{\partial(r,\theta)} + \frac{\partial(\Phi_{1010},\nabla^2\Phi_{1000})}{\partial(r,\theta)}\right]$$

$$(68) \qquad \nabla^4\Phi_{1101} = \frac{1}{r}\left[\frac{\partial(\Phi_{1000},\nabla^2\Phi_{1100})}{\partial(r,\theta)} + \frac{\partial(\Phi_{1100},\nabla^2\Phi_{1000})}{\partial(r,\theta)}\right]$$

It should be noticed that the differential equations listed above and their solutions must be worked out successively. The gauge functions are also determined term by term. Some of the information which can be drawn from the gauge functions have already been used to establish the equations mentioned above. The equations can be solved by elementary methods, but the calculation of some of the particular solutions are rather laborious. The details of the calculations are omitted here, but exemplified in appendix B. The final results of the calculations are,

$$(69) \quad \begin{cases} \beta_{10}(M) = \frac{\sqrt{2}}{M}, \quad \beta_{11}(M) = \frac{\sqrt{2}}{M^2}, \quad \beta_{12}(M) = \frac{\sqrt{2}}{M^3} \\ \mu_{100}(N) = \mu_{110}(N) = \mu_{120}(N) = \frac{1}{(1-N^{-2})^2}, \quad \mu_{111}(N) = \frac{1}{(1-N^{-2})^3N^2} \end{cases}$$

$$(70) \quad \begin{cases} \alpha_{100}(M) = 1, \quad \alpha_{110}(M) = \frac{1}{M}, \quad \alpha_{120}(M) = \frac{1}{M^2}, \quad \alpha_{101} = \alpha_{102} = 1. \\ \gamma_{1n}(R_s) = R_s^n, \quad n = 0,1,2 \\ \sigma_{1000}(N) = \sigma_{1100}(N) = \sigma_{1200}(N) = \frac{1}{(1-N^{-2})^2} \\ \sigma_{1010}(N) = \frac{1}{(1-N^{-2})^4N^2}, \quad \sigma_{1020}(N) = \frac{1}{(1-N^{-2})^4N^4} \\ \sigma_{1110}(N) = \frac{1}{(1-N^{-2})^3N^2}, \quad \sigma_{1120}(N) = \frac{1}{(1-N^{-2})^6N^4} \\ \sigma_{1130}(N) = \frac{1}{(1-N^{-2})^6N^6}, \quad \sigma_{1011}(N) = \frac{1}{(1-N^{-2})^3N^2} \\ \sigma_{1001}(N) = \sigma_{1101}(N) = \sigma_{1002}(N) = \frac{1}{(1-N^{-2})^2} \end{cases}$$

$$(71)\begin{cases} \kappa_{10}(M) = 1, \quad \kappa_{11}(M) = \dfrac{\sqrt{2}}{M}, \quad \kappa_{12}(M) = \dfrac{\sqrt{2}}{M^2} \\ \delta_{100}(N) = \delta_{110}(N) = \dfrac{1}{(1-N^{-2})N^2}, \quad \delta_{111}(N) = \dfrac{1}{(1-N^{-2})N^3} \\ \delta_{120}(N) = \dfrac{1}{(1-N^{-2})N^2}, \quad \delta_{121}(N) = \dfrac{1}{(1-N^{-2})N^3} \end{cases}$$

$$(72) \quad \chi_{100}(\eta,\theta) = \left[\frac{13}{4} - \frac{3}{2}\eta - (2+\eta)e^{-\eta}\sin\eta - 3e^{-\eta}\cos\eta - \frac{1}{4}e^{-2\eta}\right]\sin 2\theta$$

$$(73) \quad \chi_{110}(\eta,\theta;R_s) = \sqrt{2}\Big\{\Big[-\frac{137}{8} + \frac{49}{4}\eta + \frac{9}{4}\eta^2 + \left(-4+\frac{13}{2}\eta + 2\eta^2\right)\sin\eta\, e^{-\eta}$$

$$+\left(\frac{33}{2} + 9\eta\right)\cos\eta\, e^{-\eta} + \left(\frac{5}{8}+\frac{\eta}{2}\right)e^{-2\eta}\Big]\sin 2\theta$$

$$- \frac{3}{32} R_s \eta^2 \sin 4\theta + O(R_s{}^2)\Big\}$$

$$(74) \quad \chi_{111}(\eta,\theta;R_s) = \sqrt{2}\Big\{\Big[\frac{3}{4} - \frac{3}{2}\eta + e^{-\eta}(\sin\eta - \cos\eta) + \frac{1}{4}e^{-2\eta}\Big]\sin 2\theta$$

$$- \frac{3}{16} R_s \eta^2 \sin 4\theta + O(R_s{}^2)\Big\}$$

$$(75) \quad \chi_{120}(\eta,\theta;R_s) = \Big\{\frac{4099}{32} - \frac{1359}{32}\eta - \frac{294}{4}\eta^2 - 6\eta^3$$

$$+ \left[\frac{1078}{8} - \frac{209}{8}\eta - \frac{278}{8}\eta^2 - \frac{25}{4}\eta^3\right]\sin\eta\, e^{-\eta}$$

$$+ \left[-\frac{259}{8} - \frac{779}{8}\eta - \frac{258}{8}\eta^2\right]\cos\eta\, e^{-\eta}$$

$$+ \left[-\frac{263}{32} - \frac{61}{16}\eta - \frac{3}{2}\eta^2\right]e^{-2\eta}\Big\}\sin 2\theta$$

$$+ O(R_s)$$

(76) $\varphi_{100}(\lambda,\theta) = -\frac{1}{2}e^{-\lambda}\sin\lambda\,\sin 2\theta$

(77) $\varphi_{110}(\lambda,\theta) = \frac{1}{2}\left[1 - r^{-\lambda}(\sin\lambda + \cos\lambda)\right]\sin 2\theta$

(78) $\varphi_{111}(\lambda,\theta) = -\lambda\, e^{-\lambda}\sin\lambda\,\sin 2\theta$

(79) $\varphi_{120}(\lambda,\theta) = \sqrt{2}\left[\frac{1}{4} - 3\lambda^{2} - \frac{1}{4}\cos\lambda\, e^{-\lambda}\right]\sin 2\theta$

(80) $\varphi_{121}(\lambda,\theta) = \sqrt{2}\left[\lambda - \frac{1}{4}\lambda(\sin\lambda + \cos\lambda)e^{-\lambda}\right]\sin 2\theta$

(81) $\Phi_{1000}(r,\theta) = \frac{3}{4}(r^{-2} - 1)\sin 2\theta$

(82) $\Phi_{1100}(r,\theta) = \frac{\sqrt{2}}{8}(-49r^{-2} + 75)\sin 2\theta$

(83) $\Phi_{1200}(r,\theta) = \frac{1}{16}(1359r^{-2} - 1907)\sin 2\theta$

(84) $\Phi_{1010}(r,\theta) = \frac{3}{2}(r^{-2} - 2 + r^{2})\sin 2\theta$

(85) $\Phi_{1020}(r,\theta) = \frac{3}{4}(-r^{-2} + 1 + r^{2} - r^{4})\sin 2\theta$

(86) $\Phi_{1110}(r,\theta) = \frac{\sqrt{2}}{8}(-144r^{-2} + 294 - 150r^{2})\sin 2\theta$

(87) $\Phi_{1120}(r,\theta) = \frac{\sqrt{2}}{8}(-129r^{-2} + 333 - 279r^{2} + 75r^{4})\sin 2\theta$

(88) $\Phi_{1130}(r,\theta) = \frac{\sqrt{2}}{8}(224r^{-2} - 560 + 448r^{2} - 112r^{4})\sin 2\theta$

(89) $\Phi_{1001}(r,\theta) = \frac{3}{128}(-r^{-4} + 2r^{-2} - 1)\sin 4\theta$

(90) $$\Phi_{1002}(r,\theta) = \frac{3}{4096}\Big[(-10r^{-4}-28r^{-2}-72r^{-2}\ln r -24\ln r +38)\sin 2\theta + (r^{-6}-3r^{-4}+3r^{-2}-1)\sin 6\theta\Big]$$

(91) $$\Phi_{1101}(r,\theta) = \frac{75\sqrt{2}}{128}(r^{-4}-2r^{-2}+1)\sin 4\theta$$

(92) $$\Phi_{1011}(r,\theta) = \frac{3}{64}(r^{-4}-3+2r^{2})\sin 4\theta$$

The solution χ_{100}, equation (75), was first given by Schlichting (1932). Riley (1965) and Stuart (1966) reconsidered this solution using the method of MAE, and their results agreed identically with Schlichting's. Schlichting also calculated an outer solution corresponding to :

$$\sigma_{1000}\Phi_{1000} + \sigma_{1010}\Phi_{1010} + \sigma_{1020}\Phi_{1020} ,$$

but the gauge functions, σ_{10i0} (i=0,1,2), which are implicitly present in his calculations, differ from ours.

The physical interpretation of the terms mentioned above is: χ_{100} describes the steady flow induced by the Reynolds stresses in the Stokes layer. This flow causes by viscous drag a steady motion outside the Stokes layer which for $M = \infty$, $N = \infty$ and $R_s = 0$ is given by Φ_{1000}. For finite M, displacement flow and curvature corrections to the flow in the Stokes layer described by χ_{110}, propagate to the outer region and are in this region depicted by Φ_{1100}. For R_s finite Φ_{1000} suffers self-interaction and generates the term Φ_{1001}. In this way, by studying the source terms and the boundary conditions, the higher order approximations can be given some sort of physical interpretation. The effect of finite N includes a discussion of the Stokes layer at the outer boundary, but this is most conveniently done in terms of the Lagrangian stream function.

4.3 Calculation of the steady flow in the case $R_s >> 1$.

In this section we seek a solution of equation (5) subject to the conditions $\varepsilon << 1$, $M >> 1$, $N >> 1$ and $R_s >> 1$. The leading term χ_{100} of the second order steady stream function in the Stokes layer is, as pointed out by Riley (1967,pp.428), unchanged relative to the case $R_s << 1$. Therefore, this term still introduces a slip velocity at the outer edge of the Stokes layer, but the Reynolds number R_s associated with this slip velocity, is now assumed to be large. For this case the steady flow develops its own boundary layer just outside the Stokes layer, as emphasized by Stuart (1966). The typical thickness δ_s of the steady boundary layer, is

(93) $$\delta_s \approx \frac{a}{\sqrt{R_s}} = \frac{\delta_{AC}}{\varepsilon}$$

which indicates that the scale δ_s is a factor $\frac{1}{\varepsilon}$ larger than the Stokes layer thickness δ_{AC}. Riley (1965) studied the flow in the steady boundary layer for $N = \infty$. He introduced Blasius series expansions around $\theta = \frac{\pi}{2}$ (and $\theta = -\frac{\pi}{2}$) where the slip velocity at the outer edge of the Stokes layer indicates forward stagnation points in the steady boundary layer. Riley calculated the three first terms of the Blasius series. In the later paper, Riley (1975) incorporated the second order term in the steady boundary layer approximation of the stream function, but still the calculations left some discrepancies between the theory and the experimental data of Bertelsen (1974). Therefore the theory is reconsidered here.

We assume $N >> 1$ which means that the boundary layers at the inner and the outer cylinder do not overlap. Therefore similar Blasius series expansions as used by Riley (1965) are attempted here, but we include gauge functions to take account of finite N.

Accordingly, an asymptotic expansion of the steady stream function, constructed to be valid in the steady boumdary layer, is tentatively written,

$$(94) \qquad \psi_1(r,\xi,M;N;R_s) \underset{\substack{\zeta \text{ fixed} \\ \varepsilon \to 0 \\ M \to \beta \\ N \to \infty \\ R_s \to \infty}}{\sim} F_1(\zeta,\xi;M;N;R_s)$$

$$= a_{00}\left\{\frac{1}{\sqrt{R_s}} p_{100}(N)F_{100}(\zeta,\xi) + p_{101}(N,R_s)F_{101}(\zeta,\xi) + \cdots + \frac{1}{R_s} p_{110}(N)F_{110}(\zeta,\xi) + \cdots\right\}$$

where $\zeta = \sqrt{3R_s}\, y$ is the scaled steady boundarylayer coordinate, and $\xi = \theta + \frac{\pi}{2}$ Since the boundary layers at the inner and the outer cylinder are assumed not to overlap, an outer regiom of approximately inviscid flow is expected. The asymptotic expansion of the steady stream function associated with the flow in this region, can formally be stated as,

$$(95) \qquad \psi_1(r,\xi;M;N;R_s) \underset{\substack{r \text{ fixed} \\ \varepsilon \to 0 \\ M \to \infty \\ N \to \infty \\ R_s \to \infty}}{\sim} G_1(r,\xi;M;N;R_s)$$

$$= a_{00}\left\{\frac{q_{100}(N)}{\sqrt{R_s}} G_{100}(r,\xi) + q_{101}(N;R_s)G_{101}(r,\xi) + \cdots + \frac{1}{R_s} q_{110}(N)G_{110}(r,\xi) + \cdots\right\}$$

The matching of the steady Stokes layer solution with the steady boundary layer solution can be expressed by the following equation,

(96) $$\left[\frac{\sqrt{2}}{M} \mu_{100}(N) \chi_{100}\left(\frac{\zeta}{\sqrt{6\epsilon}}, \theta\right) + \cdots \right]_{\substack{\zeta \text{ fixed} \\ \theta = \xi + \frac{\pi}{2} \\ \epsilon \to 0 \\ M \to \infty \\ N \to \infty \\ R_s \to \infty}}$$

$$\to a_{00}\Big[\frac{1}{\sqrt{R_s}} \Big\{ p_{100}(N) \Big(F_{100}(0,\xi) + \Big[\frac{\partial F_{100}}{\partial \zeta}\Big]_{\zeta=0} \zeta + \cdots \Big)$$

$$+ p_{101}(N;R_s)\Big(F_{101}(0,\xi) + \Big[\frac{\partial F_{101}}{\partial \zeta}\Big]_{\zeta=0}\zeta + \cdots\Big) + \cdots$$

$$+ \frac{1}{R_s} p_{100}(N)\Big(F_{110}(0,\xi) + \Big[\frac{\partial F_{100}}{\partial \zeta}\Big]_{\zeta=0}\zeta + \cdots\Big) + \cdots \Big]_{\substack{\zeta \text{ fixed} \\ \epsilon \to 0 \\ M \to \infty \\ N \to \infty \\ R_s \to \infty}}$$

This equation yields in due course,

(97) $F_{101}(0,\xi) = 0$

(98) $\left[\frac{\partial F_{101}}{\partial \zeta}\right]_{\zeta=0} = a_{00}\left(\xi + a_1\xi^3 + a_2\xi^5 + a_3\xi^7 + \cdots\right)$

(99) $F_{110}(0,\xi) = 0$

(100) $\left[\frac{\partial F_{101}}{\partial \zeta}\right]_{\zeta=0} = 0$

(101) $F_{110}(0,\xi) = 0$

(102) $\left[\frac{\partial F_{110}}{\partial \zeta}\right]_{\zeta=0} = 0$

(103) $p_{100}(N) = \frac{1}{(1 - N^{-2})^{-\frac{1}{2}}}$

where $a_{00} = \sqrt{3}$, $a_1 = -\frac{2}{3}$, $a_2 = \frac{2}{15}$ and $a_3 = -\frac{8}{945}$.

The matching of the steady boundary layer solution with the outer inviscid flow solution, can be formulated as the following,

(104) $$a_{00}\Big[\frac{1}{\sqrt{R_s}} p_{100}(N)F_{100}(\sqrt{3R_s}\,y,\xi) + p_{101}(N,R_s)F_{101}(\sqrt{3R_s}\,y,\xi) + \cdots + \frac{a_{00}}{R_s} p_{110}(N)F_{110}(\sqrt{3R_s}\,y,\xi) + \cdots\Big]_{\substack{y \text{ fixed} \\ \varepsilon \to 0 \\ M \to \infty \\ N \to \infty \\ R_s \to \infty}}$$

$$\to \Big[a_{00}\Big\{\frac{q_{100}(N)}{\sqrt{R_s}}\Big(G_{100}(1,\xi) + \Big[\frac{\partial G_{100}}{\partial r}\Big]_{r=1} y + \cdots\Big) + q_{101}(N,R_s)\Big(G_{101}(1,\xi) + \Big[\frac{\partial G_{101}}{\partial r}\Big] y + \cdots\Big) + \cdots\Big\} + \frac{a_{00}}{R_s} q_{110}(N)G_{110}(1,\xi) + \cdots\Big]_{\substack{y \text{ fixed} \\ \varepsilon \to 0 \\ M \to \infty \\ N \to \infty \\ R_s \to \infty}}$$

Equalizing terms of the same order of magnitude in equation (104) gives,

(105a) $$\lim_{\substack{y \text{ fixed} \\ R_s \to \infty}} F_{100}(\sqrt{3R_s}\,y,\xi) = G_{100}(1,\xi)$$

(105b) $$\lim_{\substack{y \text{ fixed} \\ R_s \to \infty \\ N \to \infty}} \Big[p_{101}(N,R_s)F_{101}(\sqrt{3R_s}\,y,\xi)\Big] = q_{101}(N,R_s)\Big[\frac{\partial G_{101}}{\partial r}\Big]_{r=1} y$$

(105c) $$G_{101}(1,\xi) = 0$$

(106) $$q_{100}(N) = p_{100}(N)$$

The asymptotic expansion (94) is inserted into equation (5) and an asymptotic expansion of the operator ∇^4 and the non-linear term of the equation is performed by putting $r = 1 + \frac{\zeta}{\sqrt{3R_S}}$. Equalizing terms of the same order of magnitude then yields,

$$(107) \qquad \frac{\partial^4 F_{100}}{\partial\zeta^4} = -\frac{\partial F_{100}}{\partial\xi}\frac{\partial^3 F_{100}}{\partial\zeta^3} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^3 F_{100}}{\partial\zeta^2\partial\xi}$$

$$(108) \qquad \frac{\partial^4 F_{101}}{\partial\zeta^4} = -\frac{\partial F_{100}}{\partial\xi}\frac{\partial^3 F_{101}}{\partial\zeta^3} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^3 F_{101}}{\partial^2\partial\xi} - \frac{\partial F_{101}}{\partial\xi}\frac{\partial^3 F_{100}}{\partial\zeta^3} + \frac{\partial F_{101}}{\partial\zeta}\frac{\partial^3 F_{100}}{\partial\zeta^2\partial\xi}$$

$$(109) \qquad \frac{\partial^4 F_{110}}{\partial\zeta^4} + 2\frac{\partial^3 F_{100}}{\partial\zeta^3} = -\frac{\partial F_{100}}{\partial\xi}\frac{\partial^3 F_{110}}{\partial\zeta^3} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^3 F_{110}}{\partial\zeta^2\partial\xi} - \frac{\partial F_{110}}{\partial\xi}\frac{\partial^3 F_{100}}{\partial\zeta^3} + \frac{\partial F_{110}}{\partial\zeta}\frac{\partial^3 F_{100}}{\partial\zeta^2\partial\xi} + \zeta\left[\frac{\partial F_{100}}{\partial\xi}\frac{\partial^3 F_{100}}{\partial\zeta^3} - \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^3 F_{100}}{\partial\zeta^2\partial\xi}\right] - \frac{\partial F_{100}}{\partial\xi}\frac{\partial^2 F_{100}}{\partial\zeta^2} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^2 F_{100}}{\partial\zeta\partial\xi} \, .$$

Partial integration and some manipulation leads to using equation (105),

$$(110) \qquad \frac{\partial^3 F_{100}}{\partial\zeta^3} = -\frac{\partial F_{100}}{\partial\xi}\frac{\partial^2 F_{100}}{\partial\zeta^2} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^2 F_{100}}{\partial\zeta\partial\xi}$$

$$(111) \qquad \frac{\partial^3 F_{101}}{\partial\zeta^3} = -\frac{\partial F_{100}}{\partial\xi}\frac{\partial^2 F_{101}}{\partial\zeta^2} + \frac{\partial F_{100}}{\partial\zeta}\frac{\partial^2 F_{101}}{\partial\zeta\partial\xi} - \frac{\partial F_{101}}{\partial\xi}\frac{\partial^2 F_{100}}{\partial\zeta^2} + \frac{\partial F_{101}}{\partial\zeta}\frac{\partial^2 F_{100}}{\partial\zeta\partial\xi}$$

$$\frac{\partial^3 F_{110}}{\partial \zeta^3} = -\frac{\partial F_{100}}{\partial \zeta}\frac{\partial^2 F_{110}}{\partial \zeta^2} + \frac{\partial F_{100}}{\partial \zeta}\frac{\partial^2 F_{110}}{\partial \zeta \partial \xi} - \frac{\partial F_{110}}{\partial \xi}\frac{\partial^2 F_{100}}{\partial \zeta^2} + \frac{\partial F_{110}}{\partial \zeta}\frac{\partial^2 F_{100}}{\partial \zeta \partial \xi} - \zeta \frac{\partial^3 F_{100}}{\partial \zeta^3} . \tag{112}$$

Riley (1965) calculated an approximate solution of equation (110) by introducing a Blasius series as the following,

$$F_{100}(\zeta,\xi) = \sum_{n=0}^{\infty} a_n F_{100n}(\zeta)\xi^{2n+1} \tag{113}$$

We attempt similar expansions of $F_{101}(\zeta,\xi)$ and $F_{110}(\zeta,\xi)$,

$$F_{101}(\zeta,\xi) = \sum_{n=0}^{\infty} b_n F_{10n}(\zeta)\xi^{2n+1} \tag{114}$$

$$F_{110}(\zeta,\xi) = \sum_{n=0}^{\infty} c_n F_{110n}(\zeta)\xi^{2n+1} \tag{115}$$

The three first terms og $\{F_{100n}\}$ were given by Riley (1965) and his results are quoted in appendix C. For later use we need a solution of $F_{100}(\zeta,\xi)$ with better accuracy than obtained by these terms and therefore $F_{1003}(\zeta)$ is calculated, also in appendix C. A three-term and a four-term Blasius series expansion of $F_{100}(\zeta,\xi)$ are plotted in figure 10 which indicates that the discrepancies between theory and experiment, mentioned introductorily in this section, scarcely can be resolved by higher order terms, of the Blasius series. Therefore, the effect of the outer boundary is investigated more closely.

The main features of the steady flow field are sketched in figure 9 and can be characterized as follows: The steady boundary layers, outside the Stokes layers at the inner cylinder, impinge on each other at $\xi = \frac{\pi}{2}$ and $\xi = -\frac{\pi}{2}$ and form narrow outgoing jets. For the presence of an outer boundary, these jets collide with this boundary and form boundary layer flows resembling the essentials of wall jets. The wall jets impinge on each other in pairs at $\xi = 0$ and $\xi = \pi$ and two returning jets are created. It turns out that these returning jets affect the boundary layer flow at the inner cylinder appreciably. This influence will be estimated analytically.

The outgoing jets.

Following Riley (1974) the outgoing jets are approximated by the two-dimensional jet solution. The free constants of the solution are determined assuming the volume and the momentum flux of the impinging steady boundary layers at the inner cylinder to be transferred to the jets. Riley used numerical integration of the boundary layer equations to obtain sufficient accuracy of the volume and the momentum flux at $\xi = \pm \frac{\pi}{2}$. However, a four term Blasius series expansion of $F_{100}(\zeta,\xi)$ gives these quantities with satisfactory precision. Referring to the results in appendix C, the volume flow in a steady boundary layer at $\xi = \frac{\pi}{2}$ is, in physical dimensions,

(116) $$Q_b = \varepsilon U_0 a \frac{a_{00}}{\sqrt{R_s}} \left\{ \left[\sum_{n=0}^{3} a_n F_{100n}(\zeta)\xi^{2n+1} \right]_{\substack{\zeta = \infty \\ \xi = \frac{\pi}{2}}} - \left[\sum_{n=0}^{3} a_n F_{100n}(\zeta)\xi^{2n+1} \right]_{\substack{\zeta = 0 \\ \xi = \frac{\pi}{2}}} \right\} = \varepsilon U_0 a \frac{a_{00}}{\sqrt{R_s}} q_b$$

where $q_b \approx 1,239$.

Likewise, the momentum flux in the boundary layer in question is, for $\xi = \frac{\pi}{2}$,

(117) $$M_b = \rho\varepsilon^2 U_0^2\, a \frac{a_{00}^3}{\sqrt{R_s}} \int^{\infty} \left\{ \frac{\partial}{\partial\zeta} \left[\sum_{n=0}^{3} a_n F_{100}(\zeta)\xi^{2n+1} \right]_{\xi=\frac{\pi}{2}} \right\}^2 d\zeta$$

$$= \rho\varepsilon^2 U_0^2\, a \frac{a_{00}^3}{\sqrt{R_s}} m_b$$

where $m_b \approx 0{,}3284$.

The two-dimensional jet solution represented by the stream function (see for example Batchelor 1967,pp.345) is in physical dimensions

(118) $$\Psi_g(x,z;R_s) = 6\alpha(R_s)\nu(x+x_0)^{1/3} \tanh\left[\alpha(R_s)(x+x_0)^{-2/3} z\right]$$

where x is measured along and z orthogonal to the jet. The virtual momentum source of the jet is situated at $x = -x_0$, $z = 0$. The free constants are α and x_0. The volume and the momentum flux in the z-direction of one of the symmetrical halves of the jet (a half-jet) are, respectively,

(119) $$Q_g(x_0) = 6\alpha\,\nu\,x_0^{1/3}$$

(120) $$M_g = 24\,\rho\,\nu^2\alpha^3$$

Claiming,

(121) $$Q_g(x_0) = Q_b(\tfrac{\pi}{2})$$

(122) $$M_g = M_b(\tfrac{\pi}{2})$$

yields,

(123) $$\alpha = c_0\left[\frac{m_b}{24\,a}\right]^{1/3} R_s^{1/2}$$

(124) $$x_0 = \frac{q_b{}^3}{9m_b}\, a\ .$$

This completes the jet-solution in question.

The steady boundary layers at the outer cylinder.

The outgoing jets impinge on the outer cylinder for $x = (N-1)a$. Each of them are in regions of stagnating flow (see figure 9), split into two equivalent halves which are the sources of the wall jets. The details of the flow in the stagnating regions are not considered, but it is supposed that the volume and the momentum flux of each half-jet are transferred to the associated wall jet. These assumptions are used to determine the free constants of the wall jet solution (see Glauert 1956). The general wall jet solution, represented by the stream function in physical dimensions, is,

(125) $$\Psi_W(S,Y) = \left[40\,\beta\,\nu(S+S_0)\right]^{1/4} f(\Lambda)$$

where

$$\Lambda = \left[\frac{5\beta}{32\nu^3(S+S_0)^3}\right]^{1/4} Y$$

S is measured along and Y orthogonal to the wall. $S = -S_0$, $Y = 0$ are the position of the virtual source of the jet. $S = 0$, $Y = 0$ correspond to the stagnation point of the associated outgoing jet. The function $f(\Lambda)$ is given implicitly as follows,

(126) $$f(\Lambda) = (g(\Lambda))^2$$

(127) $$\Lambda = \log \frac{\sqrt{1+g+g^2}}{1-g} + \sqrt{3}\ \mathrm{Arctan}\ \frac{\sqrt{3}g}{2+g}$$

The volume and the momentum flux along the wall for $S = 0$ are, respectively,

(128) $$Q_w(S_0) = \left[40\,\beta\,\nu S_0\right]^{1/4}$$

(129) $$M_w(S_0) = \rho\left[\frac{250\beta^3}{\nu S_0}\right]^{1/4}$$

The outgoing jets impinge on the outer cylinder with the following volume and momentum flux in each half jet,

(130) $$Q_g((N-1)a) \approx Q_g(Na \approx 6\alpha\nu(Na)^{1/3}$$

(131) $$M_g = 24\,\rho\,\alpha^3\nu^2$$

Claiming,

(132) $$Q_w(S_0) = Q_g(Na)$$

(133) $$M_w(S_0) = M_g \quad ,$$

gives

(134) $$S_0 = \frac{1}{2}Na$$

(135) $$\beta = \frac{324}{5}\alpha^4\nu^3(Na)^{1/3}$$

which completes the wall jet solution in question. Of course, the solution has physical significance in this case for $S \in (0,\frac{\pi}{2}\,Na)$, only.

The returning jets.

The wall jets originating at, say $\xi = \frac{\pi}{2}$ and $\xi = -\frac{\pi}{2}$, impinge on each other at $\xi = 0$ and a new jet, flowing towards the inner cylinder, is formed. An equivalent returning jet develops at $\xi = \pi$

The returning jets are approximated by the two-dimensional jet solution which for this application is written,

$$(136) \qquad \Psi_c(X,Z) = 6\gamma\nu(X+X_o)^{1/3}\tanh\left[\gamma(X+X_o)^{-2/3}Z\right].$$

Here X is measured along and Z orthogonal to the jet. The position of the virtual source of the jet is $X = -X_o$, $Z = 0$.
The returning jets are created with the following volume and momentum flux (in each symmetrical half), respectively,

$$(137) \qquad Q_c(X_o) = 6\gamma\nu X_o^{1/3}$$

$$(138) \qquad M_c = 24\rho\gamma^3\nu^2$$

Claiming,

$$(139) \qquad Q_c(X_o) = Q_w(\tfrac{\pi}{2}Na)$$

$$(140) \qquad M_c = M_w(\tfrac{\pi}{2}Na)$$

we find,

$$(141) \qquad X_o = (\pi+1)Na$$

$$(142) \qquad \gamma = \frac{\alpha}{(\pi+1)^{1/12}}$$

and this completes the jet solution in question.

The returning jets impinge on each other and a stagnating flow develops around the inner cylinder. In this region the jet solution referred to above is not valid, of course. Nevertheless, an approximate solution of the stagnation flow must be known before the effect og the returning jets on the steady boundary layer can be estimated. This approximate solution is established subject to the condition,

$$(143) \qquad \frac{[(\pi+2)N]^{2/3}}{\sqrt{R_s}} \gg 1$$

which physically means that the width of the returning jets is much larger than the diameter of the inner cylinder. The convective inertia term of the momentum equation, which is balanced by viscous forces in the returning jets must in the region of stagnation flow be balanced by pressure forces. Therefore, the stagnation flow can be characterized as an effectively inviscid flow with vorticity. The general equation for this type of motion in two dimensions is (see Batchelor 1967, pp. 536)

$$\nabla^2\psi = -f(\psi)$$

where ψ is the stream function and the vorticity distribution $f(\psi)$, which is arbitrary so far as inviscid fluid theory is concerned, must be known. The history of the establishment of the steady flow determines the vorticity distribution. In accordance with the approximations already introduced into the problem considered here, the vorticity distribution in question is supposed to be generated by the returning jets and transported essentially unchanged through the region of stagnation flow. The solution of the returning jets (equation 136) gives,

$$(144) \qquad \nabla^2\Psi_c = -12\gamma^3\nu(\tilde{X}+X_1)^{-1}\left\{\tanh\left[\gamma(X+X_1)^{-2/3}Z\right] - \tanh^3\left[\gamma(X+X_1)^{-2/3}Z\right]\right\} + O(\gamma^3\nu X_1^{-3})$$

where $X_1 = X_o + Na = (\pi+2)Na$. Since $X_1 \gg a$, we put ,

$$(145) \qquad \Psi_c \approx 6\gamma\nu X_1^{1/3}\tanh(\gamma X_1^{-2/3}Z_1)$$

which is valid for $|\tilde{X}| \ll X_1$.

Introducing,

(146) $$H_o = \tanh(\gamma X_1^{-2/3} Z)$$

we get,

(147) $$\nabla^2 H_o = -2\gamma^2 X_1^{-4/3}(H_o - H_o^3)$$

where terms of order $\gamma^2 X_1^{-8/3}$ has been neglected. According to the assumptions stated above, the flow in the stagnation region must be governed by an equation formally equal to equation (147), i.e.,

(148) $$\nabla_\rho^2 H_1(\rho,\xi) = -H_1(\rho,\xi) + H_1(\rho,\xi)^3$$

where $$\rho = \sqrt{2}\,\gamma\, X_1^{-2/3}\,\bar{r}$$

and $$\nabla_\rho^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\xi^2}$$

Unfortunately, no linearization parameter appeared in equation (148). Therefore, the following iteration procedure is used to generate an approximate solution of the equation. Let,

(149) $$H_1(\rho,\xi) = H_{10}(\rho,\xi) + H_{11}(\rho,\xi) + H_{12}(\rho,\xi) + \dots$$

and provided,

(150) $$H_{10}(\rho,\xi) \gg H_{11}(\rho,\xi) \gg H_{12}(\rho,\xi) \gg \dots$$

then, approximately,

(151) $$\nabla_\rho^2 H_{10}(\rho,\xi) = H_{10}(\rho,\xi)$$

(152) $$\nabla_\rho^2 H_{11}(\rho,\xi) = -H_{11}(\rho,\xi) + H_{10}^3(\rho,\xi)$$

(153) $$\nabla_\rho^2 H_{12}(\rho,\xi) = -H_{12}(\rho,\xi) + 3H_{10}^2(\rho,\xi)H_{11}(\rho,\xi)$$

$$\dots$$

In analogy to

(154) $$H_o(\rho,\xi) = \frac{1}{\sqrt{2}}\left[\rho \sin\xi - \frac{1}{6}\rho^3 \sin^3\xi + \frac{1}{30}\rho^5 \sin^5\xi - \dots\right]$$

the above iteration is expected to give useful results if,

(155) $$\left|\frac{\rho \sin\xi}{\sqrt{2}}\right| \ll \frac{\pi}{2} .$$

With a view to the specification of the matching conditions, the non-dimensional velocity components of the returning jet along $\xi = 0$ are calculated,

(156) $$\begin{cases} v_r = -\dfrac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}}\left[1 - \frac{1}{2}(1+\rho^2)\xi^2 + \dots\right] \\ v_\theta = \dfrac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}}\left[\xi - \frac{1}{2}(\frac{1}{3} + \rho^2)\xi^3 + \dots\right] \end{cases}$$

which lead to

(157) $$\left[\frac{\sqrt{2}}{\rho}\frac{\partial H_1}{\partial\xi}\right]_{\substack{\rho \text{ fixed} \\ \frac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}} \to 0 \\ \xi \to 0}} \to 1 - \frac{1}{2}(1+\rho^2)\xi^2 + \dots$$

(158) $$\left[\sqrt{2}\frac{\partial H_1}{\partial\rho}\right]_{\substack{\rho \text{ fixed} \\ \frac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}} \to 0 \\ \xi \to 0}} \to \xi - \frac{1}{2}(\frac{1}{3} + \rho^2)\xi^3 + \dots$$

Obviously, $H_1(\rho,\xi)$ must be regular for finite r which gives,

(159) $$\lim_{\substack{r \text{ fixed} \\ \frac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}} \to 0}} H_1(\rho,\xi) = [H_1(\rho,\xi)]_{\rho=0} \quad \text{regular.}$$

The interaction between the inviscid stagnation flow and the outgoing jets along $\xi = \frac{\pi}{2}$ and $\xi = -\frac{\pi}{2}$ can be neglected to leading order of magnitude $H_1(\rho,\xi)$ of the inviscid flow for $\rho \lesssim 1$. Therefore $H_1(\rho,\xi)$ is an approximate solution of the stagnation flow at least for $\xi \in [0;\pi]$ and $\rho \lesssim 1$ and the following symmetry exists,

(160) $$H_1(\rho,\xi) = - H_1(\rho,\pi-\xi)$$

(161) $$H_1(\rho,\xi) = - H_1(\rho,-\xi) \, .$$

Therefore,

$$H_{10}(\rho,\xi) = K_{00}(\rho) \sin 2\xi$$

$$H_{11}(\rho,\xi) = K_{10}(\rho) \sin 2\xi + K_{11}(\rho) \sin^3 2\xi$$

which inserted into equations (151) and (152), respectively, give,

(162) $$K_{00}''(\rho) + \frac{1}{\rho}K_{00}'(\rho) + (1 - \frac{4}{\rho^2})K_{00}(\rho) = 0$$

(163) $$K_{10}''(\rho) + \frac{1}{\rho}K_{10}'(\rho) + (1 - \frac{4}{\rho^2})K_{10}(\rho) = - \frac{24}{\rho^2}K_{11}(\rho)$$

(164) $$K_{11}''(\rho) + \frac{1}{\rho}K_{11}'(\rho) + (1 - \frac{36}{\rho^2})K_{11}(\rho) = K_{00}^3(\rho) \, .$$

The general solutions are,

(165) $$K_{00}(\rho) = A_{02}J_2(\rho) + B_{02}Y_2(\rho)$$

(166) $$K_{10}(\rho) = A_{12}J_2(\rho) + B_{12}Y_2(\rho) + P_{10}(\rho)$$

(167) $$K_{11}(\rho) = A_{16}J_6(\rho) + B_{16}Y_6(\rho) + P_{11}(\rho)$$

where $J_n(\rho)$ and $Y_n(\rho)$ are Bessel functions of the first and second kind, respectively, and order n $(n = 2,6)$. Owing to the regularity condition (159),

(168) $$B_{02} = B_{12} = B_{16} = 0 .$$

$P_{10}(\rho)$ and $P_{11}(\rho)$ are particular solutions. With reference to assumption (150)

(169) $$H_1(\rho,\xi) \approx A_{02}\, J_2(\rho) \sin 2\xi$$

is used for later calculations. The matching conditions (157, 158) give,

(170) $$A_{02} \approx 1{,}96 ,$$

(171) $$\rho_{fixed} = \rho_o \approx 2{,}30$$

which show that no overlap-region between the returning jet solution and the stagnation flow solution exists, as required by the method of MAE. This is a very unsatisfactory result, but not surprising since there in the theoretical approach is an abrupt change of the governing forces at $\rho = \rho_o$.

According to condition (143) the radius of the inner cylinder is vanishingly small compared to the length scale in the stagnating flow region and therefore the cylinder does not affect the gross features of this flow. However, close to the cylinder, i.e. at distances of the order of its radius, the flow is obviously affected by the cylinder. In order to study this effect the inviscid flow solution is written in physical dimensions,

$$(172)\quad \Psi_d(r,\xi) = 0{,}0846\,\epsilon\, U_o\, a \frac{a_{o\infty}\sqrt{R_s}}{(\pi+2)N}\left[r^2 - \frac{1}{6}\frac{\gamma^2 a^2}{X_1^{4/3}} r^4 + \dots\right]\sin 2\xi\,,$$

which shows that for r fixed and $\frac{\sqrt{R_s}}{(\pi+2)N} \to 0$ the inviscid flow is asymptotically potential. Therefore, the presence of the inner cylinder can be accounted for by introducing a region of potential flow where the function $G_{101}(r,\xi)$ (see equation 95) have to satisfy the following boundary and matching conditions,

$$(173)\qquad \left[q_{101}(N,R_s)G_{101}(r,\xi)\right]_{r\to\infty} \longrightarrow 0.0846\,\frac{\sqrt{R_s}}{(\pi+2)N}\, r^2 \sin 2\xi$$

$$\frac{\sqrt{R_s}}{[(\pi+2)N]^{2/3}} \to 0$$

$$(174)\qquad \left[G_{101}(r,\xi)\right]_{r=1} = 0\,.$$

The solution is,

$$(175)\qquad G_{101}(r,\xi) = 0.0846(r^2 - r^{-2})\sin 2\xi$$

which give the gauge function $p_{101}(N,R_s)$ of equation (94),

$$(176)\qquad p_{101}(N,R_s) = q_{101}(N,R_s) = \frac{\sqrt{R_s}}{(\pi+2)N}$$

and the constants $\{b_n\}$ of equation (114) are also determined,

$$(177)\qquad \begin{cases} b_o \approx 0.3907 \\ b_1 \approx -0.2605 \\ b_2 \approx 0.05210 \\ \vdots \end{cases}$$

The term proportional to r^{-2} of the solution (175) can be matched to the inviscid stagnation flow by including Bessel functions of the second kind in the higher order approximations of this flow.

<u>The outer potential flow induced by the boundary layers.</u>

The steady boundary layers at the walls of the container and the jet flows discribed above, induce a potential velocity field outside these boundary layers. The purpose is to estimate the influence of this velocity field on the steady boundary layer at the inner cylinder and the potential velocity field is therefore approximated with a solution valid for $r \ll N$. In this approximation the suction in the wall jets at the outer boundary can be neglected as the contribution from here to the potential flow for $r \to 1$ is of order $N^{-5/3}$. Consistent with assumptions already utilized, the potential velocity field is therefore calculated as if $N = \infty$.

Since the width of the outgoing jets is of the order of magnitude $R_s^{-1/2}$ and,

$$a\, r \sin \xi = x + a \;, \quad 0 \leq \xi \leq \pi$$

the coordinate approximation,

(178) $$a\, r = x + a$$

is valid inside the width of the jet flowing out along $\xi = \frac{\pi}{2}$. Referring to equations (118), (123) and (178), we find,

$$\Psi_g(x,z;R_s) \approx \epsilon U_o\, a \frac{a_{oo}}{\sqrt{R_s}} (9m_b)^{1/3} (r+r_o)^{1/3}$$

~~x & z fixed~~

$x = (r-1)a$

$R_s \to \infty$

where $r_o = \frac{x_o}{a} - 1 \approx -0.35647$ (x_o by equation 124). Since $r_o \ll 1$, we put

(179)
$$\begin{array}{l}\Psi_g(x,z;R_s) \\ x \,\&\, z \text{ fixed} \\ x = (r-1)a \\ R_s \to \infty\end{array} \approx \epsilon U_o a \frac{a_{oo}}{\sqrt{R_s}} (9m_b)^{1/3}\left[r^{1/3} + \frac{r_o}{3} r^{-2/3} - \frac{r_o^{\,2}}{9} r^{-5/3} + \frac{5r_o^{\,3}}{81} r^{-8/3} - \ldots\right]$$

which is a boundary condition on the outer potential flow in question. Owing to the symmetry of the problem, it is sufficient to state a solution valid for $-\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}$. Letting $G_{100}(r,\xi)$ (see equation 95) be this solution, equation (179) gives the following boundary condition,

(180)
$$\begin{cases} G_{100}(r,\frac{\pi}{2}) = (9m_b)^{1/3}\left[r^{1/3} + \frac{r_o}{3} r^{-2/3} - \frac{r_o^{\,2}}{9} r^{-5/3} + \frac{5r_o^{\,3}}{81} r^{-8/3} - \ldots\right] \\ G_{100}(r,-\frac{\pi}{2}) = -G_{100}(r,\frac{\pi}{2}) \end{cases}$$

while the matching condition evaluated from equation (105a) is,

(181) $$G_{100}(1,\xi) = \xi - \frac{2}{17}\xi^3 + 0.012817\,\xi^5 + 0.000052\,\xi^7 + \ldots$$

and also,

(182) $$q_{100}(N) = 1 \,.$$

Since, from equation (125)

$$\left[\frac{\partial \Psi_w}{\partial S}\right]_{\Lambda \to \infty} \longrightarrow O(N^{-5/3})$$

we get

(183) $$\left[\frac{1}{r}\frac{\partial G_{100}}{\partial \xi}\right]_{r=N\to\infty} \longrightarrow 0 \,.$$

Introducing,

(184) $$G_{100}(r,\xi) = G_g(r,\xi) + G_b(r,\xi)$$

and claiming

(185) $$\begin{cases} G_g(r,\pm\frac{\pi}{2}) = G_{100}(r,\pm\frac{\pi}{2}) \\ G_b(1,\xi) = G_{100}(1,\xi) - G_g(1,\xi) \end{cases}$$

and also noticing,

(186) $$\nabla^2 G_{100}(r,\xi) = 0 \ ,$$

the solution of $G_g(r,\xi)$ is,

(187) $$G_g(r,\xi) = (72m_b)^{1/3}\Big[r^{1/3}\sin\frac{\xi}{3} + \frac{r_o\sqrt{3}}{9}\,r^{-2/3}\sin\frac{2}{3}\xi - \frac{r_o^{\,2}}{9}\,r^{-5/3}\sin\frac{5}{3}\xi - \frac{5r_o^{\,3}\sqrt{3}}{243}\,r^{-8/3}\,\xi + \ldots\Big] .$$

This leads to,

(188) $$G_b(1,\xi) \approx \Big[0.242077\,\xi - 0.140922\,\xi^3 + 0.017278\,\xi^5 - 0.000237\,\xi^7 + \ldots\Big]$$

Since, by equations (184) and (185)

$$G_b(r,\pm\frac{\pi}{2}) = 0$$

$G_b(1,\xi)$ expressed by the leading terms of its Fourier series, is

(189) $$G_b(1,\xi) \approx 0.12692\sin 2\xi - 0.00290\sin 4\xi$$

Therefore, an approximate solution of $G_b(r,\xi)$, is,

(190) $$G_b(r,\xi) = 0.12692\,r^{-2}\sin 2\xi - 0.00290\,r^{-4}\sin 4\xi$$

The constants $\{c_n\}$ of equation (115) are hereby determined,

$$(191) \quad \begin{cases} c_o = -\ 0.01252 \\ c_1 = \ \ 0.1607 \\ c_2 = -\ 0.03535 \\ \vdots \end{cases}$$

where the term $0.0290\ r^{-4}\sin 4\xi$ of $G_b(r,\xi)$ has been neglected since,

$$|0.00290\, r^{-4}\sin 4\xi| \ll |0.12692\, r^{-2}\sin 2\xi|$$

and moreover, if only three terms of the power series of $\sin 4\xi$ is included, meaningful accuracy of this term is not obtained for $\xi = 1$ for which comparison with experimental data will be carried out.

Higher order boundary layer approximations.

Since it was possible to estimate numerical calues of $b_o, b_1, b_2, \ldots$ and $c_o, c_1, c_2, \ldots$, the leading terms of the Blasius series of $F_{101}(\zeta,\xi)$ and $F_{110}(\zeta,\xi)$ will be given explicitly. Introducing the expansions (113) and (114) into equation (111) and by equalizing terms of the same order of magnitude the following equations for the leading terms of $F_{101}(\zeta,\xi)$ are obtained,

$$(192) \quad F_{1010}'''(\zeta) + (1-e^{-\zeta})F_{1010}''(\zeta) - 2e^{-\zeta}F_{1010}'(\zeta) - e^{-\zeta}F_{1010}(\zeta) = 0$$

$$(193) \quad F_{1011}'''(\zeta) + (1-e^{-\zeta})F_{1011}''(\zeta) - 4e^{-\zeta}F_{1011}'(\zeta) - 3e^{-\zeta}F_{1011}(\zeta)$$

$$= \frac{a_1 b_o}{b_1}\left[-F_{1001}''(\zeta)F_{1010}(\zeta) + 4F_{1001}'(\zeta)F_{1010}'(\zeta) - 3F_{1001}(\zeta)F_{1010}''(\zeta)\right]$$

$$(194)\quad F_{1012}'''(\zeta) + (1-e^{-\zeta})F_{1012}''(\zeta) - 6e^{-\zeta}F_{1012}'(\zeta) - 5e^{-\zeta}F_{1012}(\zeta)$$
$$= \frac{3a_1b_1}{b_2}\Big[-F_{1001}''(\zeta)F_{1011}(\zeta) + 2F_{1001}'(\zeta)F_{1011}'(\zeta) - F_{1001}(\zeta)F_{1011}''(\zeta)\Big]$$
$$+ \frac{a_2b_o}{b_2}\Big[-F_{1002}''(\zeta)F_{1010}(\zeta) + 6F_{1002}'(\zeta)F_{1010}'(\zeta) - 5F_{1002}(\zeta)F_{1010}''(\zeta)\Big].$$

Higher order terms of the Blasius series given by equation (114) is not considered. The equations (192), (193) and (194) are more closely treated in Appendix D where the following approximate solutions of the equations, subject to the relevant boundary and matching conditions, are obtained,

$$(195)\quad F_{101}(\zeta) = \frac{1}{72}\Big\{-188 + 72\zeta + (205+100\zeta+36\zeta^2)e^{-\zeta} - 18e^{-2\zeta} + e^{-3\zeta}\Big\}$$

$$(196)\quad F_{1011}(\zeta) = f_{11}(\zeta) + \frac{a_1b_o}{b_1}g_{11}(\zeta)$$

where

$$(196a)\quad f_{11}(\zeta) = \frac{1}{204}\Big\{-348 + 204\zeta + (98-48\zeta+306\zeta^2)e^{-\zeta}$$
$$- (741 + 306\zeta)e^{-2\zeta} - (61 + 17\zeta)e^{-3\zeta}\Big\}$$

$$(196b)\quad g_{11}(\zeta) = \frac{1}{83232}\Big\{156180 + (-544151+49660\zeta - 109548\zeta^2 - 22032\zeta^3)e^{-\zeta}$$
$$+ (366102 + 193392\zeta + 22032\zeta^2)e^{-2\zeta}$$
$$+ (21869 + 10608\zeta + 1836\zeta^2)e^{-3\zeta}\Big\}$$

$$(197)\quad F_{1012}(\zeta) = f_{12}(\zeta) + \frac{3a_1b_1}{b_2}g_{12}(\zeta) + \frac{3a_1{}^2b_o}{b_2}h_{12}(\zeta) + \frac{a_2b_o}{b_2}i_{12}(\zeta)$$

where

$$(197a)\quad f_{12}(\zeta) = \frac{1}{456}\Big\{-732 + 456\zeta + (4765-1836\zeta+1140\zeta)e^{-\zeta}$$
$$- (3294 + 2280\zeta)e^{-2\zeta} - (739 + 380\zeta)e^{-3\zeta}\Big\},$$

while the algebra involved prohibit solving the equations of $g_{12}(\zeta)$, $h_{12}(\zeta)$ and $i_{12}(\zeta)$. The typical magnitude of $F_{1012}(\zeta)$

relative to $F_{101}(\zeta,\xi)$ subject to

$$\zeta \to \infty\,,\ \xi \approx 1$$

is given by $\left|\frac{b_2}{b_1}\right|$ and since,

$$\left|\frac{b_2}{b_1}\right| << 1$$

in our case, no more effort is spent at the Blasius series of $F_{101}(\zeta,\xi)$.

Insertion of the expansions (113) and (115) into equation (112 leads to,

$$(198)\quad F_{1100}'''(\zeta) + (1-e^{-\zeta})F_{1100}''(\zeta) - 2e^{-\zeta}F_{1100}'(\zeta) - e^{-\zeta}F_{1100}(\zeta) = \\ = \frac{1_o}{c_o}\zeta\, F_{1000}'''(\zeta)$$

$$(199)\quad F_{1101}'''(\zeta) + (1-e^{-\zeta})F_{1101}'' - 4e^{-\zeta}F_{1101}'(\zeta) - 3e^{-\zeta}F_{1101}(\zeta) \\ = \frac{a_1c_o}{c_1}\left[-F_{1001}''(\zeta)F_{1100}(\zeta) + 4F_{1001}'(\zeta)F_{1100}'(\zeta) - 3F_{1001}(\zeta)F_{1100}''(\zeta)\right], \\ - \frac{a_1}{c_1}\zeta\, F_{1001}'''(\zeta)$$

$$(200)\quad F_{1102}'''(\zeta) + (1-e^{-\zeta})F_{1102}''(\zeta) - 6e^{-\zeta}F_{1102}'(\zeta) - 5e^{-\zeta}F_{1102}(\zeta) \\ = \frac{3a_1c_1}{c_2}\left[-F_{1001}''(\zeta)F_{1101}(\zeta) + 2F_{1001}'(\zeta)F_{1101}'(\zeta) - F_{1001}(\zeta)F_{1101}''(\zeta)\right] \\ + \frac{a_2c_o}{c_2}\left[-F_{1002}''(\zeta)F_{1100}(\zeta) + 6F_{1002}'(\zeta)F_{1100}'(\zeta) - 5F_{1002}(\zeta)F_{1100}''(\zeta)\right] \\ - \frac{a_2}{c_2}\zeta\, F_{1002}'''(\zeta)\ .$$

Higher order terms of the Blasius series given by equation (115) is not considered. The equations (198), (199) and (200) are dealt with more closely in appendix D, where the following approximate solutions are obtained,

$$(201)\quad F_{1100}(\zeta) = f_{20}(\zeta) + \frac{1}{c_o} g_{20}(\zeta)$$

where,

$$(201a)\quad f_{20}(\zeta) = F_{1010}(\zeta) \quad (F_{1010}(\zeta) \text{ is given by equation (195)).}$$

$$(201b)\quad g_{20}(\zeta) = \frac{1}{72}\left\{80 - (97+64\zeta+36\zeta^2)e^{-\zeta} + 18e^{-2\zeta} - e^{-3\zeta}\right\}$$

$$(202)\quad F_{1101}(\zeta) = f_{21}(\zeta) + \frac{a_1 c_o}{c_1} g_{21}(\zeta) + \frac{a_1}{c_1} h_{21}(\zeta)$$

where,

$$(202a)\quad f_{21}(\zeta) = \frac{1}{204}\left\{-969 + 204\zeta + (1498-48\zeta+306\zeta^2)e^{-\zeta}\right.$$

$$\left. + (-741-306\zeta)e^{-2\zeta} + (-61-17\zeta)e^{-3\zeta}\right\}.$$

$$(202b)\quad g_{21}(\zeta) = \frac{1}{83232}\left\{156180 + (-544151+49660\zeta\right.$$

$$- 109548\zeta^2 - 22032\zeta^3)e^{-\zeta}$$

$$+ (366102+193392\zeta+22032\zeta^2)e^{-2\zeta}$$

$$\left. + (21869+10608\zeta+1836\zeta^2)e^{-3\zeta}\right\}$$

$$(202c)\quad h_{21}(\zeta) = \frac{1}{2996352}\left\{-966092 + (5772641-486996\zeta+220325\zeta^2\right.$$

$$+ 793152\zeta^3)e^{-\zeta}$$

$$+ (-4828452-3222792\zeta-798660\zeta^2)e^{-2\zeta}$$

$$\left. + (21903-108766\zeta-66402\zeta^2)e^{-3\zeta}\right\}$$

$$(203)\quad F_{1102}(\zeta) = f_{22}(\zeta) + \frac{a_1 c_1}{c_2} g_{22}(\zeta) + \frac{a_1^2 c_o}{c_2}(\zeta)\, h_{22}(\zeta)$$

$$+ \frac{a_1^2}{c_o} i_{22}(\zeta) + \frac{a_2}{c_2} j_{22}(\zeta)$$

where,

$f_{22}(\zeta) = f_{12}(\zeta)$ ($f_{12}(\zeta)$ is given by equation 197a)
while $g_{22}(\zeta)$, $h_{22}(\zeta)$, $i_{22}(\zeta)$ and $j_{22}(\zeta)$ are not given explicitly for the same reasons as the calculation of the corresponding terms of $F_{101}(\zeta,\xi)$ was dropped. The typical **magnitude** of $F_{1102}(\zeta)$ relative to $F_{110}(\zeta,\xi)$ subject to

$$\zeta \to \infty , \quad \xi \approx 1$$

is depicted by $\left|\frac{c_2}{c_1}\right|$ and since

$$\left|\frac{c_2}{c_1}\right| << 1 \qquad (204)$$

in our case (see equation 191), no more attention is paid to the Blasius series of $F_{101}(\zeta,\xi)$. The steady boundary layer solution which now is known, is discussed more closely in the later section "Numerical results and discussion".

5. The Stokes drift.

The reason to consider the Stokes drift is two-fold. Primarily, the steady streaming motion considered in this paper is of practicel interest chiefly because it leads to extensive migration of fluid particles, and secondly, the available experimental data is Lagrangean **velocities** (i.e. particle velocities). The Stokes ~~drift introduced as an additial correction term $\Delta\psi_1$, to the~~ Eulerian stream function is (see Raney, Corelli & Westervelt 1954) to order ϵ ,

$$\Delta\psi_1 = \frac{1}{2}\underline{i}_z \cdot \langle (\int \underline{v}_o d\tau) x \underline{v}_o \rangle \qquad (205)$$

where $\underline{v}_o$ is the linearized solution calculated in section 3.

Equation (205) gives;

a) in the Stokes layer at the inner cylinder,

$$(206) \quad \left[\Delta\psi_1\right]_{\substack{\eta \text{ fixed} \\ M \to \infty \\ N \to \infty \\ R_s \to 0}} \sim \Delta\chi_1(\eta,\theta;M;N;R_s)$$

$$= \frac{1}{(1-N^{-2})^2}\left\{\frac{\sqrt{2}}{M}\left[-\frac{1}{2}+e^{-\eta}(\eta\sin\eta-\cos\eta)-\frac{1}{2}e^{-2\eta}\right]\right.$$

$$+\frac{1}{M^2}\left[-\frac{1}{2}+2\eta+e^{-\eta}\cos\eta+\eta\, e^{-\eta}(\sin\eta-4\cos\eta)\right.$$

$$\left.\left.-4\eta^2e^{-\eta}\sin\eta+e^{-2\eta}\left(-\frac{1}{2}+2\eta\right)\right]+\dots\right\}\sin 2\theta$$

b) in the outer region,

$$(207) \quad \left[\Delta\psi_1\right]_{\substack{r \text{ fixed} \\ M \to \infty \\ N \to \infty \\ R_s \to 0}} \sim \Delta\Phi_1(r,\theta;M;N;R_s) = \frac{1}{(1-N^{-2})^2}\left\{\frac{1}{2}\left[\frac{\sqrt{2}}{M}+\frac{1}{M^2}\right]r^{-2}+\dots\right\}\sin 2\theta$$

c) in the Stokes layer at the outer cylinder

$$(208) \quad \left[\Delta\psi_1\right]_{\substack{\lambda \text{ fixed} \\ M \to \infty \\ N \to \infty \\ R_s \to 0}} \sim \Delta\varphi_1 = \frac{1}{(1-N^{-2})N^2}\left\{\frac{1}{2}e^{-\lambda}\sin\lambda\right.$$

$$\left.+\frac{\sqrt{2}}{M}\frac{1}{2}\left[-1+e^{-\lambda}(\cos\lambda+\sin\lambda)\right]+\dots\right\}\sin 2\theta$$

where the Stokes drifts (a,b & c) match passing from one region to another. Comparing equations (76) and (195) it is easily verified that the singularity in the Eulerian velocity at $\lambda = 0$ is cancelled by the Stokes drift. The effect of Stokes layer flow at the outer cylinder on the time averaged particle velocity is at most of order (N^{-3}).

The results above is valid for $R_s = 0$. In the case $R_s \to \infty$, the leading term of the Stokes drift given by equation (206) is unchanged because the linearized solution $\chi_{000}(\eta,\theta,\tau)$ (equation 35) is so. The Stokes drift in the steady boundary layer can not be calculated unless more accurate time-dependant solutions are known. It can be verified, however, by considering the order of magnitude of the terms involved that the Stokes drift in this layer is at most of the order of magnitude $\epsilon^2\sqrt{R_s}$ relative to the typical steady velocity in the layer.

6. Comparison with other calculations.

6.1 The case $R_s \ll 1$.

Holtsmark & al. (1954) studied the problem treated here by solving equation (4) on closed form subject to the condition $N = \infty$. This solution of ψ_0 was inserted into equation (5) and the equation solved subject to the formal restriction $R_s = 0$. Thus the results of Holtsmark & al. (1954) is a good approximation when $\epsilon \ll 1$, $N \gg 1$, $R_s \ll 1$ and almost arbitrary M, only restricted by the condition $\epsilon^2 M^2 \ll 1$. Since the results in section (4.2) are valid subject to similar conditions, these results and the theory of Holtsmark & al. (1954) can be compared by performing inner and outer asymptotic expansions of the Holtsmark-theory.

Raney, Corelli and Westervelt (1954) carried out an inner asymptotic expansion of this theory. The inner asymptotic expansion of the steady part of the second order $(O(\epsilon))$ stream function is given by their equation (21). The comparable result achieved by the method of MAE is a two-term inner expansion with $R_s = 0$ and $N = \infty$. Referring to our equations (69), (72) and (73) we can readily write down this expansion in physical dimensions,

$$(209)\quad \Psi_1^{(i)}(\eta,\theta;R_s=0) = \epsilon U_o a\Big\{\frac{\sqrt{2}}{M}\Big[\frac{13}{4}-\frac{3}{2}\eta-(\frac{1}{4}+\eta)\sin\eta\, e^{-\eta} - 3\cos\eta\, e^{-\eta}-\frac{1}{4}e^{-2\eta}\Big] + (\frac{\sqrt{2}}{M})^2\Big[-\frac{137}{8}+\frac{49}{4}\eta+\frac{9}{4}\eta^2 + (-4+\frac{13}{2}\eta+2\eta^2)\sin\eta\, e^{-\eta} + (\frac{33}{2}+9\eta)\cos\eta\, e^{-\eta}+(\frac{5}{8}+\frac{\eta}{2})e^{-2\eta}\Big\}\sin 2\theta\ .$$

It is easily verified that this solution of $\Psi_1^{(i)}$ is identical with that given by equation (21) in the paper of Raney & al. (1954) (notice their sligtly different definition of η) .

In order to compare the outer asymptotic expansion $\Phi_1(r,\theta)$ (obtained by the method of MAE) with the solution of Holtsmark & al., it is necessary to carry out a corresponding expansion of equation (3.12) in their paper. The equation can be written,

$$(210)\quad \Psi_{1H}(r,\theta) = \Big\{-\frac{r^4}{48}\int_r^\infty\sigma(x)dx+\frac{r^2}{16}\int_r^\infty x\sigma(x)dx + \frac{1}{16}\Big[\int_1^\infty\frac{1}{x}\sigma(x)\,dx-2\int_1^\infty x\sigma(x)dx+\int_1^r x^3\sigma(x)dx\Big] + \frac{r^2}{48}\Big[-2\int_1^\infty\frac{1}{x}\sigma(x)dx+3\int_1^\infty x\sigma(x)dx-\int_1^r x^5\sigma(x)dx\Big]\Big\}\sin 2\theta$$

where r and x are the non-dimensional radial position and,

$$(211)\quad \sigma(x) = \frac{iU_o\epsilon M^4}{a^3}\Big\{\Big[\frac{H_o^{(1)}(z_o)^*H_2^{(1)}(z)+\frac{1}{r^2}H_2^{(1)}(z_o)H_o^{(1)}(z)^*+2H_o^{(1)}(z)H_2^{(1)}(z)}{H_o^{(1)}(z_o)H_o^{(1)}(z_o)^*}\Big]-\Big[\cdots\Big]^*\Big\}$$

where $z = xM\sqrt{i}$ and $H_m^{(1)}(z)$, $m = 0,2$), are Hankel functions. Standard asymptotic series of the Hankel functions are used to obtain an asymptotic expansion of $\sigma(x)$. Repeated partial integration of the integrals involved in equation (95) was used to generate their asymptotic series. The important feature of the expressions then achieved, is that they consist of polynomials of $(\frac{1}{r})$ multiplicated by the exponential factors

$$e^{-(r-1)\frac{M}{\sqrt{2}}} \quad \text{and} \quad e^{-(r-1)M\sqrt{2}} .$$

The outer asymptotic expansion $\Psi_{1H}^{(\sigma)}$ of Ψ_{1H} is defined as,

$$(212) \qquad \Psi_{1H} \xrightarrow[\substack{y\ \text{fixed} \\ M \to \infty}]{} \Psi_{1H}^{(\sigma)}$$

where $y = r-1$. Referring to the main feature of the outer asymptotic series mentioned above, this means that only the limit of integration $r=1$ contributes to the final result. This turns out to be

$$(213) \qquad \Psi_{1H}^{(\sigma)}(r,\theta) = \epsilon U_o a\left\{\tfrac{3}{4}(r^{-2}-1) + \tfrac{1}{M}\tfrac{\sqrt{2}}{8}(-49r^{-2}+75)\right\}\sin 2\theta .$$

Comparison with the equations (82) and (87), and the gauge functions (73) with $N=\infty$, (it is a two-term outer expansion and $R_s = 0$ and $N=\infty$) shows identical agreement with equation (213). Thus, two different methods generate identical two-term inner and two-term outer expansions of the steady part of the second order stream function. This agreement we interprete as a justification of applying the matching principle of the method of MAE.

6.2 The case $R_s >> 1$.

None analytical results for the case $R_s >> 1$ and suitable for comparison with our section 4 are published, at least to the authors knowledge.

6. Numerical results and discussion.

6.1 The case $R_s << 1$.

The main features of the steady streaming pattern in our problem are known to be an inner and an outer vortex system in each quadrant (see Holtsmark & al. 1954, figure 7). In order to discuss the validity of the method of MAE, we first investigate the inner asymptotic expansion subject to $R_s = 0$ and $N = \infty$,

$$(214) \qquad \chi_1(\eta,\theta;R_s=0;\ N=\infty) = \frac{\sqrt{2}}{M}\chi_{100}(\eta,\theta) + \frac{\sqrt{2}}{M^2}\chi_{110}(\eta,\theta;R_s=0) + \frac{\sqrt{2}}{M^2}\chi_{120}(\eta,\theta;R_s=0) ,$$

where the functions χ_{100}, χ_{110} and χ_{120} are given by equations (75), (76) and (78), respectively. The thickness δ_{DC} of the inner vortex system, defined by the zero-point of the radial velocity component for $\theta = 0$, say, is an important quantity with respect to the validity of the theory. Let the location of this zero-point be η_o. Then we have,

$$(215) \qquad \frac{\delta_{DC}}{\delta_{AC}} = \sqrt{2}\,\eta_o$$

where $\delta_{AC} = \sqrt{\frac{\nu}{\omega}}$. Figure 2 shows δ_{DC}/δ_{AC} versus M for a one- two- and three-term inner asymptotic expansion. We have also plotted the corresponding curve from the closed form solution of Holtsmark & al. (1954). This solution is of course the best approximation in the case $R_s = 0$ and $N = \infty$, and is therefore used as a reference for further conclusions. Thus figure 2 indicates that a one-, two- or three-term inner asymptotic expansion predicts the thickness δ_{DC} within 10% accuracy for $M \geq 120$, $M \geq 60$, $M \geq 40$, respectively. Figure 2 in the paper of Raney & al. (1954) should also be referred to in this connection.

Since there is identical agreement between the asymptotic expansions of the solution of Holtsmark & al. (1954) treated in the previous section and the inner and outer expansions generated by the method of MAE ($R_s = 0,\ N = \infty$), we may use a composite solution, including R_s-terms, to estimate the region of validity of the solution of Holtsmark & al. (1954) with respect to R_s. The uniformly valid composite solution is,

$$(216)\quad \psi_1^{(c)} = \frac{\sqrt{2}}{M}\chi_{100} + \frac{\sqrt{2}}{M^2}\chi_{110} + \Phi_{1000} + \frac{1}{M}\Phi_{1100} + \frac{1}{M^2}\Phi_{1200} + R_s\left[\Phi_{1001} + \frac{1}{M}\Phi_{1101}\right] + R_s^2\Phi_{1002} - C_p(\eta,\theta;R_s)$$

where the functions χ_{100}, χ_{110}, Φ_{1000}, Φ_{1100}, Φ_{1200}, Φ_{1001} and Φ_{1101} are given by the equations (75), (76), (82), (83), (84), (87), (88) and (90), respectively. The common part $C_p(\eta,\theta;R_s)$ of the inner and outer symptotic expansions is,

$$(217)\quad C_p(\eta,\theta;R_s) = \frac{\sqrt{2}}{M}\left[\frac{13}{4} - \frac{3}{2}\eta\right]\sin 2\theta + \left(\frac{\sqrt{2}}{M}\right)^2\left[\left(-\frac{137}{8} + \frac{49}{4}\eta + \frac{9}{4}\eta^2\right)\sin 2\theta - \frac{3}{32}R_s\eta^2\sin 4\theta\right]$$

The steady radial velocity component is given by,

$$(218)\quad v_{1r}^{(s)} = -\frac{1}{r}\frac{\partial\psi_1^{(c)}}{\partial\theta}$$

and the tangential component,

$$(219)\quad v_{1\theta}^{(s)} = \frac{\partial\psi_1^{(c)}}{\partial r}\,.$$

Figure 3 and 4 show the radial velocity at $\theta = 0$ for $M = 50$

and 100, respectively. The three different curves in each figure, represent different values of R_s . The argument $\theta = 0$ is an angular position where the R_s-terms should be important. Figure 3 and 4 indicate very little influence of these terms for $R_s \leq 1$, but for $R_s = 5$, they are of significant importance, The plot of the tangential velocity component at $\theta = 45^o$ in figure 5 shows only a minor difference between the curves for $R_s = 0$ and $R_s = 5$. This is reasonable since terms of magnitude $O(R_s)$ are identical zero for $\theta = 45^o$, while terms of magnitude $O(R_s^2)$ contribute. The angular position $\theta = 45^o$ is thus a very favourable position in order to observe good agreement between experiment and a theory formally valid for $R_s = 0$, only. On the other hand, such a theory is expected to be a good approximation for $R_s \leq 1$.

In order to illustrate the influence of the outer boundary on the flow field, the following composite solution of the stream-function is used for numerical examples,

$$(220) \qquad \psi_1^c = \beta_{10}\mu_{100}\chi_{100} + \beta_{11}\mu_{110}\chi_{110}$$
$$+ \sum_{i=0}^{2} \alpha_{1io} \sum_{j=0}^{2} \sigma_{1ijo}\Phi_{1ijo} + \alpha_{110}\sigma_{1130}\Phi_{1130}$$
$$+ R_s\{\alpha_{101}[\sigma_{1001}\Phi_{1001} + \sigma_{1011}\Phi_{1011}]$$
$$+ \alpha_{111}\sigma_{1101}\Phi_{1101}\}$$
$$+ R_s^2\alpha_{102}\sigma_{1002}\Phi_{1002} - C_p$$

where C_p is the common part of the inner and the outer expansions and,

$$(221) \quad C_p = \beta_{10}\mu_{100}(\tfrac{13}{4} - \tfrac{3}{2}\eta)\sin 2\theta$$
$$+ \beta_{11}\mu_{110}\sqrt{2}\left[(-\tfrac{137}{8} + \tfrac{49}{4}\eta + \tfrac{9}{4}\eta^2)\sin 2\theta - \tfrac{3}{32}R_s\,\eta^2 \sin 4\theta\right]$$

The gauge functions and the other terms involved in equations (220) and (221) are given at the end of section 4.2. The solution (220) is correct to $O(M^{-i}N^{-j}R_s^{\ k})$ where $i+j+k \geq 2$. Numerical examples based on (220) and on the theory of Svardal (1965, equation 4.53, formally valid for $R_s = 0$) are given in figures 6 and 7. For $R_s = 0$, figure 6 indicate that these solutions give qualitatively the same increase of δ_{DC}/δ_{AC} versus decreasing $(A-a)/\delta_{AC}$. Moreover, figure 6 shows that for $R_s = 1$, the solution (220) predicts the thickness of the inner vortex systems to increase considerably from $\theta = \pm\frac{\pi}{2}$ to $\theta = 0$ or $\theta = \pi$. In figure 7 the radial position r_c/δ_{AC} of the core of the outer vortex systems (defined by $v_r = v_\theta = 0$) is plotted versus $(A-a)/\delta_{AC}$. The main features of the figure are decreasing r_c/δ_{AC} with decreasing $(A-a)/\delta_{AC}$, but for $(A-a)/\delta_{AC} \approx 25$ the core begins to move outwards for decreasing $(A-a)/\delta_{AC}$. This unexpected effect is amplified for $R_s > 0$. The angular position of the core is also, for $R_s > 0$, dependant on M and N. This can be recognized from the solution (220), and the effect is depicted in figure 8. Referring to this figure a resonable question is why the flow field observed by Bertelsen & al. (1973, figure 7) is so symmetrical for $R_s = 0{,}75$. An evaluation of the various terms of the solution (220) reveals that there are mainly two competing effects, described by the terms Φ_{1001} and Φ_{1101}, which cause distortion in opposite directions and thus preserve the symmetry. Unfortunately, the solution (220) is not expected to give sufficiently precise results unless $M > 40$, and a quantitative comparison with the experimental results of Bertelsen & al. (1973) is therefore meaningless.

6.2. The case $R_s >> 1$.

The main features of the flow field in this case is shown stylisticly in figure 9, except for the Stokes layers at the boundaries. Details of the flow in the steady boundary layer at the inner cylinder is depicted in figure 10 where a three- and a four-t term approximation of the Blasius series of $F_{100}(\zeta,\xi)$ (see appendix C) is the bases of the velocity profiles. Comparison with the numerical solutions given by Riley (1975, figure 2) indicate fair accuracy of a Blasius series of $F_{100}(\zeta,\xi)$ including four terms for $|\xi| \leq \frac{\pi}{2}$. This statement is also supported by the good agreement between the terminal momentum flux in the steady boundary layers at $|\xi| = \frac{\pi}{2}$ (Riley 1975, p.p. 807, his $M = 0.964$ corresponding to $3m_b \approx 0.9852$ here given on page 32).

The velocity profiles in figure 11 is based on the following solution of the streamfunction,

$$
(222)\quad \Psi_1^{(c)} = \epsilon U_o a\Big\{\frac{\sqrt{2}}{M}\chi_{100} + \frac{a_{oo}}{\sqrt{R_s}}\sum_{n=0}^{3} a_n F_{100n}(\zeta)\xi^{2n+1}
+ \frac{a_{oo}}{(\pi+2)N}\left[b_o F_{1010}(\zeta)\xi + b_1 F_{1011}(\zeta)\xi^3\right]
+ \frac{a_{oo}}{R_s}\left[c_o F_{1100}(\zeta)\xi + c_1 F_{1101}(\zeta)\xi^3\right]
- \frac{a_{oo}\zeta}{\sqrt{R_s}}\sum_{n=0}^{3} a_n \xi^{2n+1}\Big\}
$$

where the solutions involved are given by equations (72), (C1), (C2), (C3), (C5), (195), (196),(201) and (202). The Stokes drift is negligible in the steady boundary layers. The measurements of Bertelsen (1974) are also plotted in figure 11 and good agreement between these measurements and the tangential velocity based upon

(222) is observed. The figure also indicate that the correction due to the suction velocity of the first order steady boundary layer solution and of the outgoing jet are of about the same magnitude as the correction forced upon the boundary layer solution by the returning jets. A stream line diagram based upon the results of section 43 is shown in figure 12. The diagram resembles the essentials of figure 4b in the paper of Bertelsen (1974).

In spite of the improved agreement between theory and experiment mentioned above, the theory is unsatisfactory with respect to several points. The following points are emphasized here: The model adopted is not unique (see Riley 1975, section 3) and the crude matching of the steady boundary layer flows with the regions of stagnating flows. But nevertheless the improved agreement between theory and experiment indicate that the main influence of the outer boundary on the steady flow close to the inner cylinder is preserved in the calculations of section 4.3.

Appendix A: Calculation of the linearized solution.

The calculation procedure for the linearized solution ψ_o is here demonstrated in some detail. The equations (17), (20), (24) and (25) is first established as suggested on page 10 and 11. Using that the stream function according to the boundary conditions (10) is proportional to $\sin\theta\cos\tau$, the formal solution in complex notations can readily be written down,

(A 1) $\chi_{ooo}(\eta,\theta,\tau) = \left[A_o + A_1\eta + A_2 e^{-(1+i)\eta} + A_3 e^{(1+i)\eta}\right]\sin\theta\, e^{i\tau}$

(A 2) $\Phi_{ooo}(r,\theta,\tau) = \left[B_o r + B_1 r^{-1}\right]\sin\theta\, e^{i\tau}$

(A 3) $\varphi_{ooo}(\lambda,\theta,\tau) = \left[C_o + C_1\lambda + C_2 e^{-(1+i)\lambda} + C_3 e^{(1+i)\lambda}\right]\sin\theta\, e^{i\tau}$

(A 4) $\varphi_{o1o}(\lambda,\theta,\tau) = \left[D_o + D_1\lambda + D_2 e^{-(1+i)\lambda} + D_3 e^{(1+i)\lambda}\right]\sin\theta\, e^{i\tau}$

The boundary conditions at the inner cylinder give,

(A 5) $A_o + A_2 + A_3 = 0$

(A 6) $A_1 - (1+i)A_2 + (1+i)A_3 = 0$.

The matching as expressed by equation (15) yields,

(A 7) $\alpha_{oo}(M)\sigma_{ooo}(N)\left\{B_o + B_1 + (B_o - B_1)y + \dots\right\} + \dots$

$= \beta_{oo}(M)\mu_{ooo}(N)\left\{A_o + A_1\frac{M}{\sqrt{2}}y\right\} + \dots$

where necessarily $A_3 = 0$ as there are no terms in the outer solution that can match an exponential growing term. The r-component of the boundary conditions (10) for $r = N$ now give

(A 8) $\kappa_{oo}(M)\delta_{ooo}(N)\{C_o + C_2 + C_3\} = N$

which leads to,

(A 9) $\varkappa_{oo}(M) = 1$

(A 10) $\delta_{ooo}(N) = N$

Now the θ-component of the same boundary conditions yields,

(A 11) $C_1 - (1+i)C_2 + (1+i)C_3 = 0$.

The matching as expressed by equation (16) give,

(A 12) $$\alpha_{oo}(M)\sigma_{ooo}(N)\left\{B_o N + \frac{B_1}{N} + (-B_o + \frac{B_1}{N^2})Y + \dots\right\} + \dots$$

$$= N\left\{C_o + C_1\frac{M}{\sqrt{2}}Y\right\} + \varkappa_{01}(M)\delta_{010}(N)\left\{D_o + D_1\frac{M}{\sqrt{2}}Y\right\} + \dots$$

where it has been used that,

(A 13) $C_3 = D_3 = 0$

of the same reasons as $A_3 = 0$ above. Matching to leading order, equation (A 12) give,

(A 14) $\alpha_{oo}(M) = 1$

Inserting (A 14) into equation (A 7) we find by term-wise matching,

(A 15) $\sigma_{ooo}(N) = \mu_{ooo}(N)$

(A 16) $\beta_{oo}(M) = \frac{\sqrt{2}}{M}$

(A 17) $B_1 = -B_o$

Putting $B_1 = -B_o$ into equation (A 12) reveals,

(A 18) $\sigma_{ooo}(N) = \frac{1}{1-N^{-2}}$

(A 19) $C_1 = 0$.

Combination of the equations (A 11), (A 13) and (A 19) leads to,

(A 20) $C_o = 1$

It is,

(A 21) $\varphi_{000}(\lambda,\theta,\tau) = \sin\theta\, e^{i\tau}$.

It should be remembered that the θ-component of the boundary conditions (10) for $r = N$ was not fullfilled by equation (A 11). Therefore we have to claim,

(A 22) $\kappa_{01}(M)\delta_{010}(N)[D_1 - (1+i)D_2]\frac{M}{\sqrt{2}} = -1$,

giving,

(A 23) $\kappa_{01}(M) = \frac{\sqrt{2}}{M}$

(A 24) $\delta_{010}(N) = 1$

(A 25) $D_1 - (1+i)D_2 = -1$.

Since the r-component of the boundary conditions (10) for $r = N$ was exactly fullfilled by equation (A 8), we have to demand,

(A 26) $D_o + D_2 = 0$.

Inserting (A 17) and (A 19) into equation (A 12) and matching to leading order, gives,

(A 27) $B_o = 1$,

and then from equation (A 17)

(A 28) $B_1 = -1$.

Introducing the known gauge functions and constants into equation

(A 12), this equation can be written,

(A 29) $N - Y - \frac{1}{1-N^{-2}} \frac{2}{N^2} Y - \dots = N + \frac{\sqrt{2}}{M} D_o + D_1 Y + \dots$.

Equalizing terms of the same order of magnitude in equation (A 29), gives,

(A 30) $D_1 = -1$

which inserted into equation (A 25)

(A 31) $D_2 = 0$.

Equation (A 26) now yields,

(A 32) $D_o = 0$.

Thus we have,

(A 33) $\varphi_{010}(\lambda, \theta, \tau) = -\lambda \sin\theta \, e^{i\tau}$

and also by equations (A 27) and (A 28),

(A 34) $\Phi_{000}(r, \theta, \tau) = (r - \frac{1}{r}) \sin\theta \, e^{i\tau}$.

Equation (A 7) can now be written,

(A 35) $2y + \dots = \frac{\sqrt{2}}{M} A_o + A_1 y + \dots$.

Matching to leading order reveals,

(A 36) $A_1 = 2$.

The boundary conditions (A 5) and (A 6) then give using $A_3 = 0$;

(A 37) $A_o = -(1-i)$

(A 38) $A_1 = 1 - i$.

Thus we have

(A 39) $$\chi_{ooo}(\eta,\theta,\tau) = 2\left\{-\frac{1}{2}(1-i)+\eta+\frac{1}{2}(1-i)e^{-(1+i)\eta}\right\}\sin\theta\, e^{i\theta}$$

The calculation procedure demonstrated above can be continued systematically to arbitrary order of magnitude of the asymptotic expansions, but further details are omitted here.

Appendix B: Calculation of the second order steady stream function ψ_1 , $R_s \ll 1$.

The viscous and the inertia forces are presupposed to be of the same order of magnitude in the Stokes layers. Subject to this assumption, the generation of equation (55), leads to,

(B 1) $\beta_{10}(M) = \frac{\sqrt{2}}{M}$

(B 2) $\mu_{100}(N) = \frac{1}{(1-N^{-2})^2}$

and equation (55) is then established. The general solution of this equation is

(B 3)
$$\chi_{100}(\eta,\theta) = A_o(\theta) + A_1(\theta)\eta + A_2(\theta)\eta^2 + A_3(\theta)\eta^3 + \left[-2e^{-\eta}\sin\eta - 3e^{-\eta}\cos\eta - \eta\, e^{-\eta}\sin\eta - \frac{1}{4}e^{-2\eta}\right]\sin 2\theta \ .$$

In the Stokes layer at the outer cylinder the same assumption yields,

(B 4) $\kappa_{10}(M) = 1$

(B 5) $\kappa_{11}(M) = \frac{\sqrt{2}}{M}$

(B 6) $\delta_{100}(N) = \frac{1}{(1-N^{-2})N^2}$

(B 7) $\delta_{110}(N) = \frac{1}{(1-N^{-2})N^2}$

and the equations (59) and (60) are then established. The general solutions of these equations are,

(B 8)
$$\varphi_{100}(\lambda,\theta) = B_o(\theta) + B_1(\theta)\lambda + B_2(\theta)\lambda^2 + B_3(\theta)\lambda^3 - \frac{1}{2}e^{-\lambda}\sin\lambda\,\sin 2\theta$$

(B 9) $$\varphi_{110}(\lambda,\theta) = C_o(\theta) + C_1(\theta)\lambda + C_2(\theta)\lambda^2 + C_3(\theta)\lambda^3 - \frac{1}{2}e^{-\lambda}(\sin\lambda + \cos\lambda)\sin 2\theta \,.$$

The outer expansions of the Stokes layer solutions are now investigated. Thus we find,

(B 10) $$\left[\frac{\sqrt{2}}{M}\chi_{100}\left(\frac{M}{\sqrt{2}}y,\theta\right)\right]_{\substack{y \text{ fixed} \\ M \to \infty}} \longrightarrow \frac{\sqrt{2}}{M}A_o(\theta) + A_1(\theta)y + \frac{M}{\sqrt{2}}A_2(\theta)y^2 + \frac{M^2}{2}A_3(\theta)y^3$$

and claiming no singular term in this expansion, give

(B 11) $$A_2 \equiv 0$$

(B 12) $$A_3 \equiv 0$$

The boundary condition (46) demands,

(B 13) $$\chi_{100}(0,\theta) = 0$$

(B 14) $$\left[\frac{\partial\chi_{100}}{\partial\eta}\right]_{\eta=0} = 0 \,,$$

which yield,

(B 15) $$A_o(\theta) = \frac{13}{4}\sin 2\theta$$

(B 16) $$A_1(\theta) = -\frac{3}{2}\sin 2\theta \,.$$

A similar treatment of $\varphi_{100}(\lambda,\theta)$ and $\varphi_{110}(\lambda,\theta)$ leads to,

(B 17) $$B_o(\theta) = B_1(\theta) = B_2(\theta) = B_3(\theta) \equiv 0$$

(B 18) $$C_1(\theta) = C_2(\theta) = C_3(\theta) \equiv 0$$

(B 19) $$C_o(\theta) = \frac{1}{2}\sin 2\theta \,.$$

Matching to leading order in the equations (53) and (54), give,

(B 20) $\alpha_{100}(M) = 1$

(B 21) $\sigma_{1000}(N) \underset{N \to \infty}{\longrightarrow} 1$

Since all the functions Φ_{10i0}, $i = 0,1,2$ satisfy the same differential equations and the matching conditions are proportional to $\sin 2\theta$, it turns out be convenient to introduce,

(B 22) $$L_{100}(r;N)\sin 2\theta = \sigma_{1000}(N)\Phi_{1000}(r,\theta) + \sigma_{1010}(N)\Phi_{1010}(r,\theta) + \sigma_{1020}(N)\Phi_{1020}(r,\theta)\ ,$$

which give

(B 23) $\nabla^4[L_{100}(r;N)\sin 2\theta] = 0$.

The general solution of (B 23) is,

(B 24) $L_{100}(r;N)\sin 2\theta = [D_0(N)r^{-2} + D_1(N) + D_2(N)r^2 + D_3(N)r^4]\sin 2\theta$.

The matching conditions evaluated from equations (53) and (54) yield,

(B 25) $L_{100}(1,N) = 0$

(B 26) $\left[\frac{\partial L_{100}(r;N)}{\partial r}\right]_{r=1} = -\frac{3}{2}\frac{1}{(1-N^{-2})^2}$

(B 27) $L_{100}(N;N) = 0$

(B 28) $\left[\frac{\partial L_{100}(r;N)}{\partial r}\right]_{r=N} = 0$

These equations leads to,

(B 29) $D_o(N) = \frac{3}{4(1-N^{-2})^4}$

(B 30) $D_1(N) = -\frac{3}{4(1-N^{-2})^4} - \frac{3}{2(1-N^{-2})^4 N^2}$

(B 31) $D_2(N) = \frac{3}{2(1-N^{-2})^4 N^2} + \frac{3}{4(1-N^{-2})^4 N^4}$

(B 32) $D_3(N) = -\frac{3}{4(1-N^{-2})^4 N^4}$

The solution $L_{100}(r;N)\sin 2\theta$ can now be reorganized by using equation (B 22). Thus we find,

(B 33) $\sigma_{1000}(N) = \frac{1}{(1-N^{-2})^2}$

(B 34) $\sigma_{1010}(N) = \frac{1}{(1-N^{-2})^4 N^2}$

(B 35) $\sigma_{1020}(N) = \frac{1}{(1-N^{-2})^4 N^4}$

(B 36) $\Phi_{1000}(r,\theta) = \frac{3}{4}(r^{-2} - 1)\sin 2\theta$

(B 37) $\Phi_{1010}(r,\theta) = \frac{3}{2}(r^{-2} - 2 + r^2)\sin 2\theta$

(B 38) $\Phi_{1020}(r,\theta) = \frac{3}{4}(- r^{-2} + 1 + r^2 - r^4)\sin 2\theta$

The calculation procedure demonstrated above can be continued systematically to arbitrary order of magnitude, but further details are not given here.

Appendix C . The steady boundary layer solution in the case $R_s \gg 1$.

As mentioned in section 4.3, Riley (1965 pp. 167) gave the three leading terms of the Blasius series (113). His results are quoted here in our notations,

(C 1) $$F_{1000}(\zeta) = 1 - e^{-\zeta}$$

(C 2) $$F_{1001}(\zeta) = \frac{1}{68}\left[12 + (7+36\zeta)e^{-\zeta} - 18e^{-2\zeta} - e^{-3\zeta}\right]$$

(C 3) $$F_{1002}(\zeta) = f_2(\zeta) + \frac{3a_{01}^2}{a_{02}} g_2(\zeta)$$

where,

$$f_2(\zeta) = \frac{1}{18552}\left[1440 + (7063+7200\zeta)e^{-\zeta} - 7200e^{-2\zeta} - 1200e^{-3\zeta} - 100e^{-4\zeta} - 3e^{-5\zeta}\right]$$

$$\begin{aligned} g_2(\zeta) = \frac{1}{53615280}\Big[&99216 - (5392720+4779000\zeta + 2504520\zeta^2)e^{-\zeta} \\ &+ (5486940+5009040\zeta)e^{-2\zeta} \\ &+ (-140655+417420\zeta)e^{-3\zeta} \\ &- 53270e^{-4\zeta} + 489e^{-5\zeta}\Big] \end{aligned}$$

The equation of the fourth term is,

(C 4) $$\begin{aligned} F_{1003}''' + (1-e^{-\zeta})F_{1003}'' - 8e^{-\zeta} F_{1003}' - 7e^{-\zeta} F_{1003} \\ = \frac{a_{01}a_{02}}{a_{03}}\left[-5F_{1001}'' f_2 + 8F_{1001}' f_2' - 3F_{1001} f_2''\right] \\ + \frac{3a_{01}^3}{a_{03}}\left[-5F_{1001}'' g_2 + 8F_{1001}' g_2' - 3F_{1001} g_2''\right] . \end{aligned}$$

Putting,

(C 5) $$F_{1003}(\zeta) = f_3(\zeta) + \frac{a_{01}a_{02}}{a_{03}} g_3(\zeta) + \frac{3a_{01}^3}{a_{03}} h_3(\zeta)$$

the equations of $f_3(\zeta)$, $g_3(\zeta)$ and $h_3(\zeta)$ will read,

(C 6) $$f_3''' + (1-e^{-\zeta})f_3'' - 8e^{-\zeta}f_3' - 7e^{-\zeta}f_3 = 0$$

(C 7) $$g_3''' + (1-e^{-\zeta})g_3'' - 8e^{-\zeta}g_3' - 7e^{-\zeta}g_3 = -5F_{1001}''f_2 + 8F_{1001}'f_2' - 3F_{1001}f_2''$$

(C 8) $$h_3''' + (1-e^{-\zeta})h_3'' - 8e^{-\zeta}h_3' - 7e^{-\zeta}h_3 = -5F_{1001}''g_2 + 8F_{1001}'g_2' - 3F_{1001}g_2''$$

The relevant term of the boundary conditions (98) is claimed satisfied by f_3 , and thus g_3 and h_3 must fulfil homogeneous boundary conditions, i.e.,

(C 9) $$f_3'(0) = 1$$
$$f_3(0) = f_3'(\infty) = 0$$

(C 10) $$g_3'(0) = g_3(0) = g_3'(\infty) = 0$$

(C 11) $$h_3'(0) = h_3(0) = h_3'(\infty) = 0$$

The solutions can formally be written as,

(C 12) $$\sum_{m=o} \sum_{n=o} A_{m,n}\zeta^m e^{-n\zeta} ,$$

which leads to,

(C 13) $$f_3(\zeta) = \frac{1}{3927660}\{151200 + (1957814 + 1058400\zeta)e^{-\zeta} - 1587600e^{-2\zeta} - 441000e^{-3\zeta} - 73500e^{-4\zeta} - 6615e^{-5\zeta} - 294e^{-6\zeta} - 5e^{-7\zeta}\}$$

Putting,

(C 14) $$g_3(\zeta) = g_{3h}(\zeta) + g_{3p}(\zeta)$$

(C 15) $$h_3(\zeta) = h_{3h}(\zeta) + h_{3p}(\zeta) ,$$

where index "h" and "p" indicate homogeneous and particular solutions, respectively, we find subject to the conditions (C 10) and (C 11),

(C 16) $$g_{3h}(\zeta) = A_{3g}e^{-\zeta} + B_{3g}\{151200 + 1058400\zeta e^{-\zeta} - 1587600e^{-2\zeta} - 441000e^{-3\zeta} - 73500e^{-4\zeta} - 6615e^{-5\zeta} - 294e^{-6\zeta} - 5e^{-7\zeta}\}$$

(C 17) $$g_{3p}(\zeta) = \frac{1}{210256}\{7254 - 43200\zeta^2 e^{-\zeta} + (112611 + 129600\zeta)e^{-2\zeta} + (-7426 + 25200\zeta)e^{-3\zeta} + (-\frac{46315}{12} + 2400\zeta)e^{-4\zeta} + (-\frac{30299}{80} + 90\zeta)e^{-5\zeta} - \frac{2581}{200}e^{-6\zeta} + \frac{39}{560}e^{-7\zeta}\}$$

(C 18) $$A_{3g} \approx -0.704077008$$
$$B_{3g} \approx -0.96801791 \cdot 10^{-7}$$

(C 19) $$h_{3h}(\zeta) = A_{3h}e^{-\zeta} + B_{3h}\{151200 + 1058400\zeta e^{-\zeta} - 1587600e^{-2\zeta} - 441000e^{-3\zeta} - 73500e^{-4\zeta} - 6615e^{-5\zeta} - 294e^{-6\zeta} - 5e^{-7\zeta}\}$$

(C 20) $$h_{3p}(\zeta) = \frac{1}{91145976}\{-\frac{4766634}{7} + (-8181\zeta^2 + 751356\zeta^3)e^{-\zeta} - (\frac{19164177}{4} + 9430020\zeta + 4508136\zeta^2)e^{-2\zeta} + (\frac{23985321}{8} + \frac{833949}{2}\zeta - 563517\zeta^2)e^{-3\zeta} + (\frac{1690601}{4} + 191772\zeta)e^{-4\zeta} - (\frac{14325}{400} + \frac{4401}{2}\zeta)e^{-5\zeta} + \frac{10074}{40}e^{-6\zeta} - \frac{2301}{112}e^{-7\zeta}\}$$

$$(C\ 21)\ \begin{cases} A_{3h} \approx 0.0880122017 \\ B_{3h} \approx 0.334608072 \cdot 10^{-7} \end{cases}$$

The general homogeneous solution of (C 6), (C 7) and (C 8) consists of three linear independant solutions, one of which increasing proportionally to ζ as $\zeta \to \infty$. Subject to the conditions at infinity (see equations C 9, C 10 and C 11) this solution must be dropped.

Appendix D. The higher order steady boundary layer approximations.

Using test solutions of the form

(D 1) $A_{mn}\zeta^m e^{-n\zeta}$

the following general solution of equation (192) can be constructed,

(D 2) $$F_{1010}(\zeta) = B_{oo}e^{-\zeta} + B_{01}(1+\zeta e^{-\zeta}) + B_{02}\left(-4+\zeta+\frac{1}{2}\zeta^2 e^{-\zeta} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} e^{-n\zeta}}{(n-1)!n^2(n-2)^2}\right).$$

For the further calculations an approximate form of (D 2) is chosen, i.e.,

(D 3) $$F_{1010}(\zeta) = B_{oo}e^{-\zeta} + B_{01}(1+\zeta e^{-\zeta}) + B_{02}\left(-4+\zeta+\frac{1}{2}\zeta^2 e^{-\zeta} - \frac{1}{4}e^{-2\zeta} + \frac{1}{72}e^{-3\zeta}\right)$$

which subject to the boundary conditions (99) and (100), and the matching condition (104) give $F_{1010}(\zeta)$ as presented by equation (195). This approximate form of $F_{1010}(\zeta)$ and the solution (C 2) of $F_{1001}(\zeta)$ yield inserted into equation (193),

(D 4) $$F_{1011}'''(\zeta) + (1-e^{-\zeta})F_{1011}''(\zeta) - 4e^{-\zeta}F_{1011}'(\zeta) - 3e^{-\zeta}F_{1011}(\zeta)$$
$$= \frac{a_1 b_o}{b_1}\cdot\frac{1}{4896}\Big\{(-6640+2664\zeta-3888\zeta^2)e^{-\zeta}$$
$$+ (-1048+8784\zeta+2592\zeta^2)e^{-2\zeta}$$
$$+ (7164+4680\zeta-648\zeta^2)e^{-3\zeta}$$
$$+ (344-144\zeta)e^{-4\zeta} + 180e^{-5\zeta}\Big\},$$

$F_{1011}(\zeta)$ is decomposed by writing, (D 5) $$F_{1011}(\zeta) = f_{11}(\zeta) + \frac{a_1 b_o}{b_1} g_{11}(\zeta)$$

which is used to construct a homogeneous equation of $f_{11}(\zeta)$ and an inhomogeneous equation of $g_{11}(\zeta)$. The solutions are claimed to satisfy the following conditions,

(D 6) $$\begin{cases} f_{11}(0) = f_{11}'(0) = 0 \\ f_{11}'(\infty) = 1 \end{cases}$$

(D 7) $$g_{11}(0) = g_{11}'(0) = g_{11}'(\infty) = 0$$

which are found from the equations (99), (100) and (104). The general solution of $f_{11}(\zeta)$ is,

(D 8) $$f_{11}(\zeta) = B_{10}e^{-\zeta} + B_{11}(1 + 3\zeta e^{-\zeta} - \tfrac{3}{2}e^{-2\zeta} - \tfrac{1}{12}e^{-3\zeta})$$
$$+ B_{12}\Big\{\tfrac{35}{9} - \tfrac{2}{3}\zeta + (5\zeta-\zeta^2)e^{-\zeta} + \zeta e^{-2\zeta}$$
$$+ (\tfrac{7}{108} + \tfrac{1}{18}\zeta)\, e^{-3\zeta} + \tfrac{4}{3}\sum_{n=o}^{\infty} (-1)^n \frac{(n-4)!}{(n!)^2} e^{-n\zeta}\Big\}.$$

The general solution of $g_{11}(\zeta)$ can be written,

(D 9) $$g_{11} = g_{11h}(\zeta) + g_{11p}(\zeta)$$

where index "h" and "p" indicate homogeneous and particular solutions, respectively, and

(D 10) $$g_{11h}(\zeta) = f_{11}(\zeta)$$

as far as the general solutions are concerned. Since an approximation already is introduced by $F_{1010}(\zeta)$, the following form of (D 8) is used,

$$\text{(D 11)} \quad f_{11}(\zeta) = B_{10}e^{-\zeta} + B_{11}(1 + 3\zeta e^{-\zeta} - \tfrac{3}{2}e^{-2\zeta} - \tfrac{1}{12}e^{-3\zeta})$$
$$+ B_{12}\left\{\tfrac{35}{9} - \tfrac{2}{3}\zeta + (5\zeta - \zeta^2)e^{-\zeta} + \zeta e^{-2\zeta} + (\tfrac{7}{108} + \tfrac{1}{18}\zeta)e^{-3\zeta}\right\}$$

and a corresponding approximation of the particular solution $g_{11p}(\zeta)$ is ,

$$\text{(D 12)} \quad g_{11p}(\zeta) = \tfrac{1}{4896}\left\{- \tfrac{18320}{3} + 4296\zeta - 1296\zeta^3 e^{-\zeta}\right.$$
$$\left. + (6886 + 4932\zeta + 1296\zeta^2)e^{-2\zeta} + (\tfrac{494}{9} + 266\zeta + 108\zeta^2)e^{-3\zeta}\right\}.$$

The solution (D 11) subject to the conditions (D 6), and the solution of $g_{11}(\zeta)$, (constructed as the sum of (D 11) and (D 12), subject to the conditions (D 7), leads to the specific solutions of $f_{11}(\zeta)$ and $g_{11}(\zeta)$ given by equations (196a) and (196b). The typical magnitude of the error introduced by the approximations mentioned above, is $O(10^{-3} \cdot e^{-4\zeta})$ relative to the solution in question, which is regarded as sufficient accuracy for our purposes.

A general homogeneous solution of equation (194) is calculated using test solutions of the type (D 1) and subject to the boundary conditions,

$$\text{(D 13)} \quad \begin{cases} f_{12}(0) = f_{12}'(0) = 0 \\ f_{12}'(\infty) = 1 \end{cases} ,$$

where $f_{12}(\zeta)$ is defined by equation (197), an approximate solution of $f_{12}(\zeta)$ is that given by equation (197a). The particular solutions of equation (194) are not considered in detail for reasons mentioned before.

The same procedure as sketched above is used to obtain the solutions (201a), (201b), (202a), (202b) and (202c) of equations (198) and (199).

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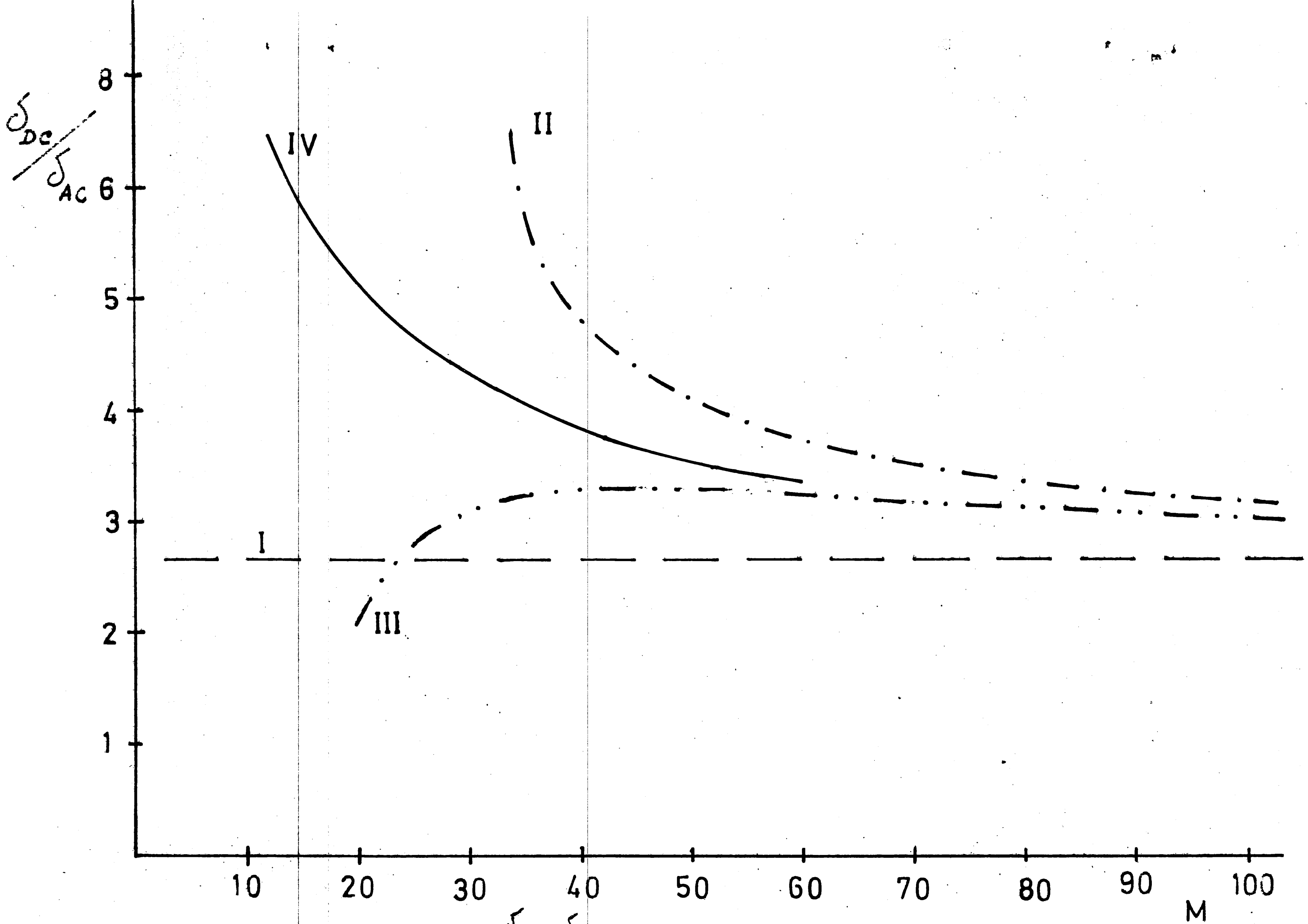

Figure 2. The thickness δ_{DC}/δ_{AC} of the inner vortex system versus M based on different approximations: Curve I, one-term inner expansion; Curve II, two-term inner expansion; Curve III, three-term inner expansion; Curve IV, Holtsmark & al. (1954) closed form solution.

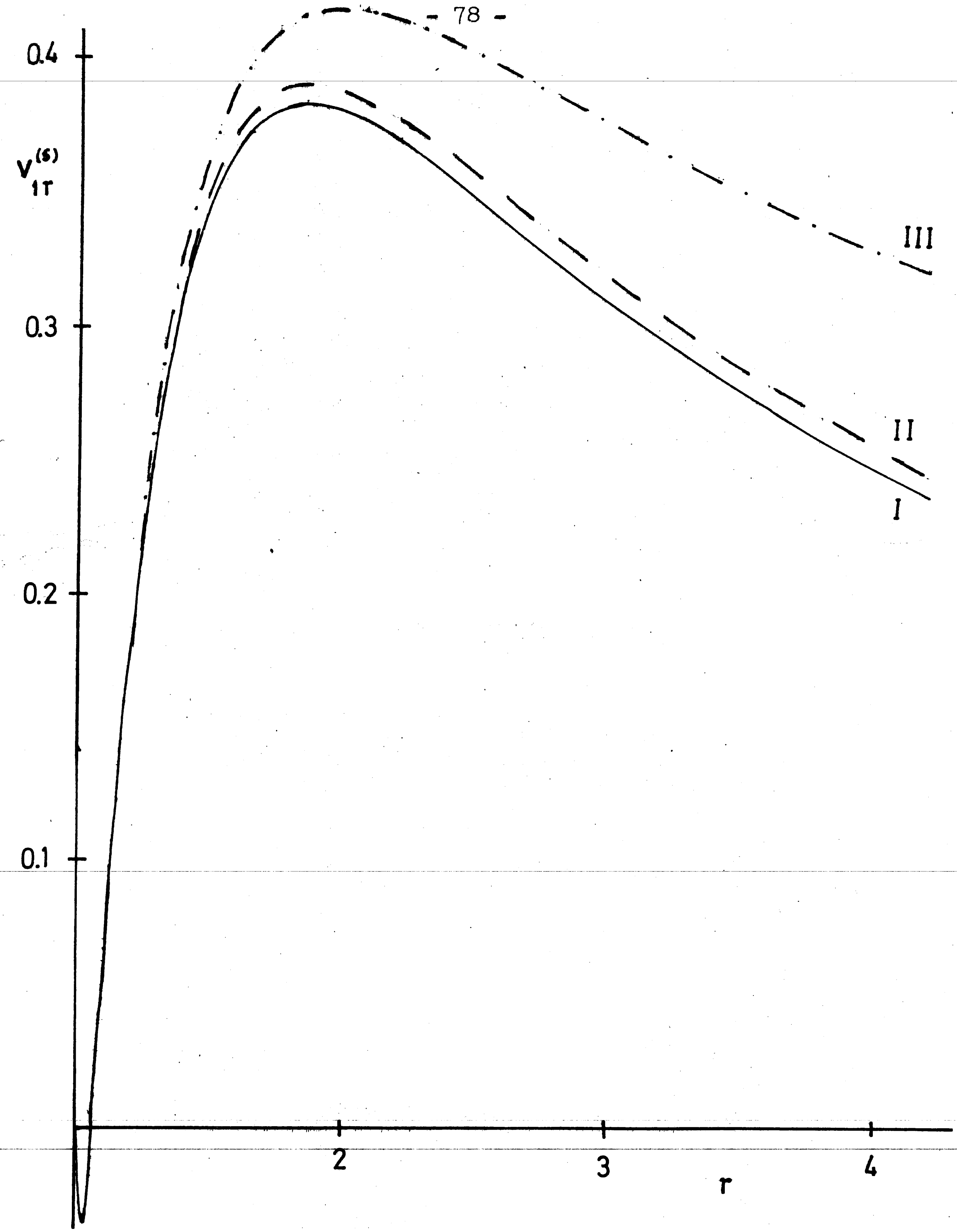

Figure 3. Dimensionless radial velocity versus r for $\theta = 0^\circ$ and M = 50. Curve I, $R_s = 0$; Curve II, $R_s = 1$; Curve III, $R_s = 5$.

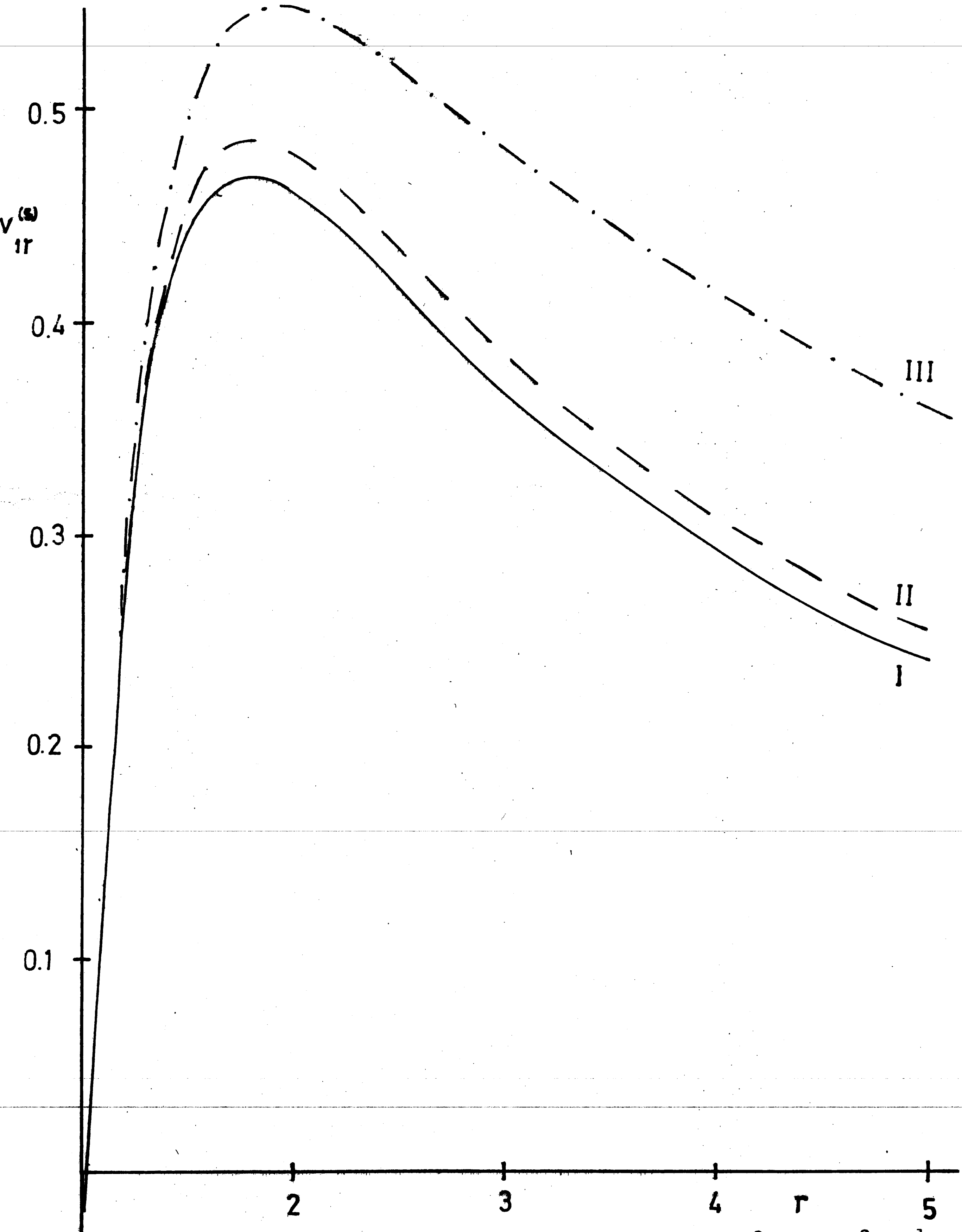

Figure 4. Dimensionless radial velocity versus r for θ = 0 and M = 100. Curve I, $R_s = 0$; Curve II, $R_s = 1$; Curve III, $R_s = 5$.

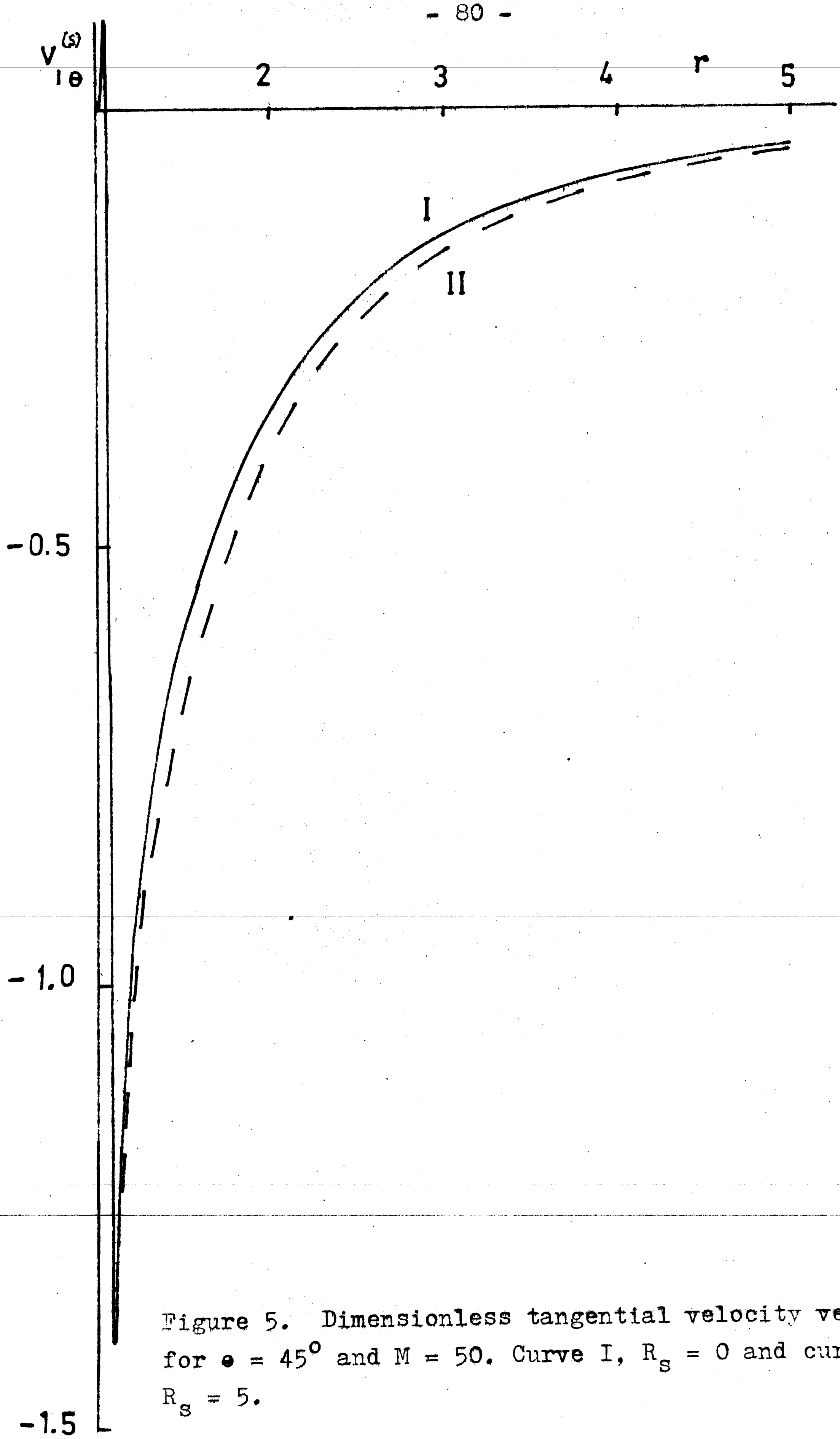

Figure 5. Dimensionless tangential velocity versus r for $\theta = 45^{o}$ and M = 50. Curve I, $R_s = 0$ and curve II, $R_s = 5$.

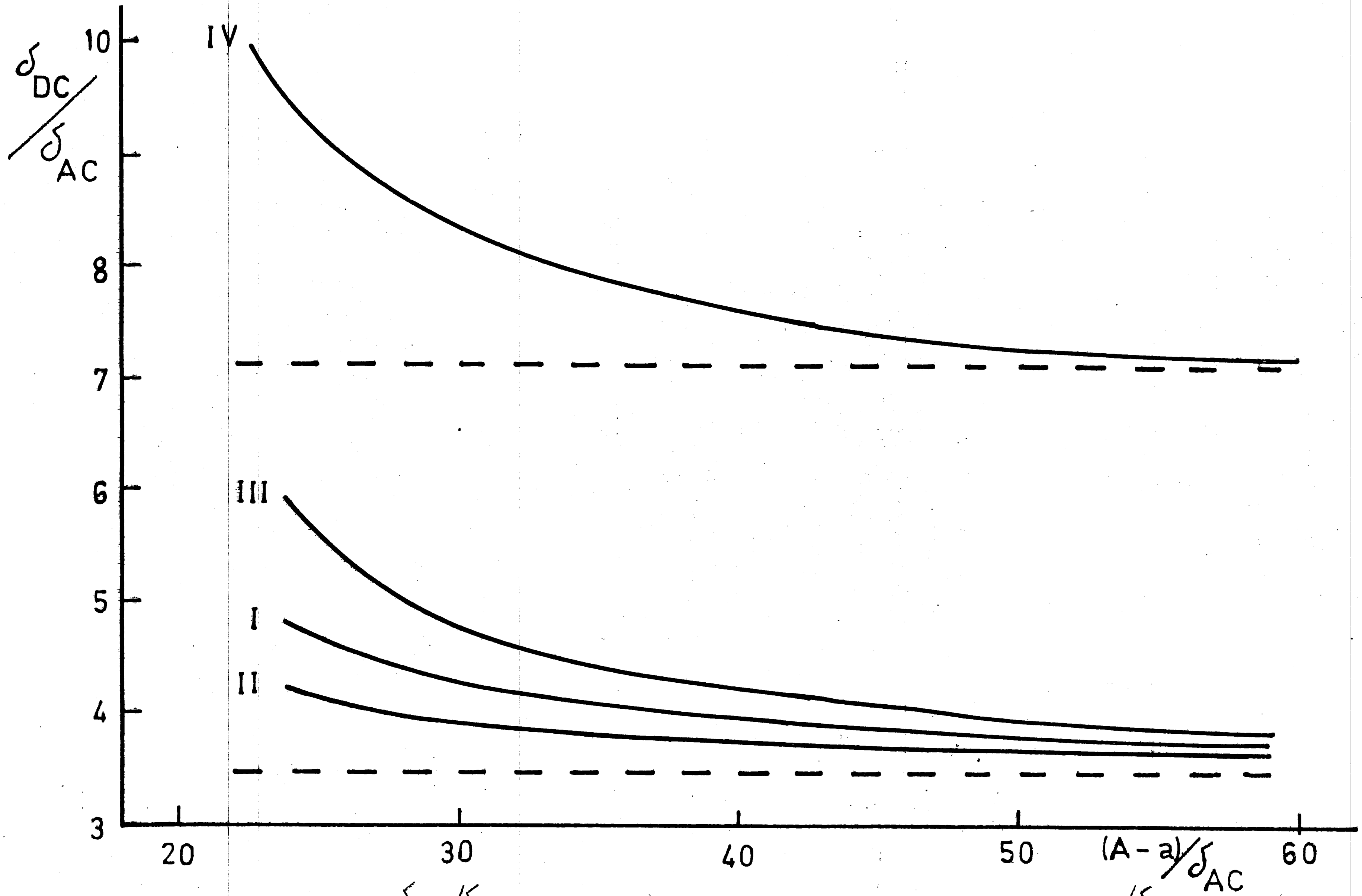

Figure 6. The tickness δ_{DC}/δ_{AC} of the inner vortex system versus $(A - a)/\delta_{AC}$. Curve I, II and III are based on equation (220) with M = 50 and the other parameters are: Curve I, $R_s = 0$; Curve II, $R_s = 1$ and $\theta = 90^\circ$; Curve III, $R_s = 1$ and $\theta = 0^\circ$. Equation (4.53) of Svardal (1965) with M = 9.5 was used to obtain curve IV. The dashed lines indicate asymptotic values of δ_{DC}/δ_{AC} with respect to $R_s = 0$ and $N = \infty$.

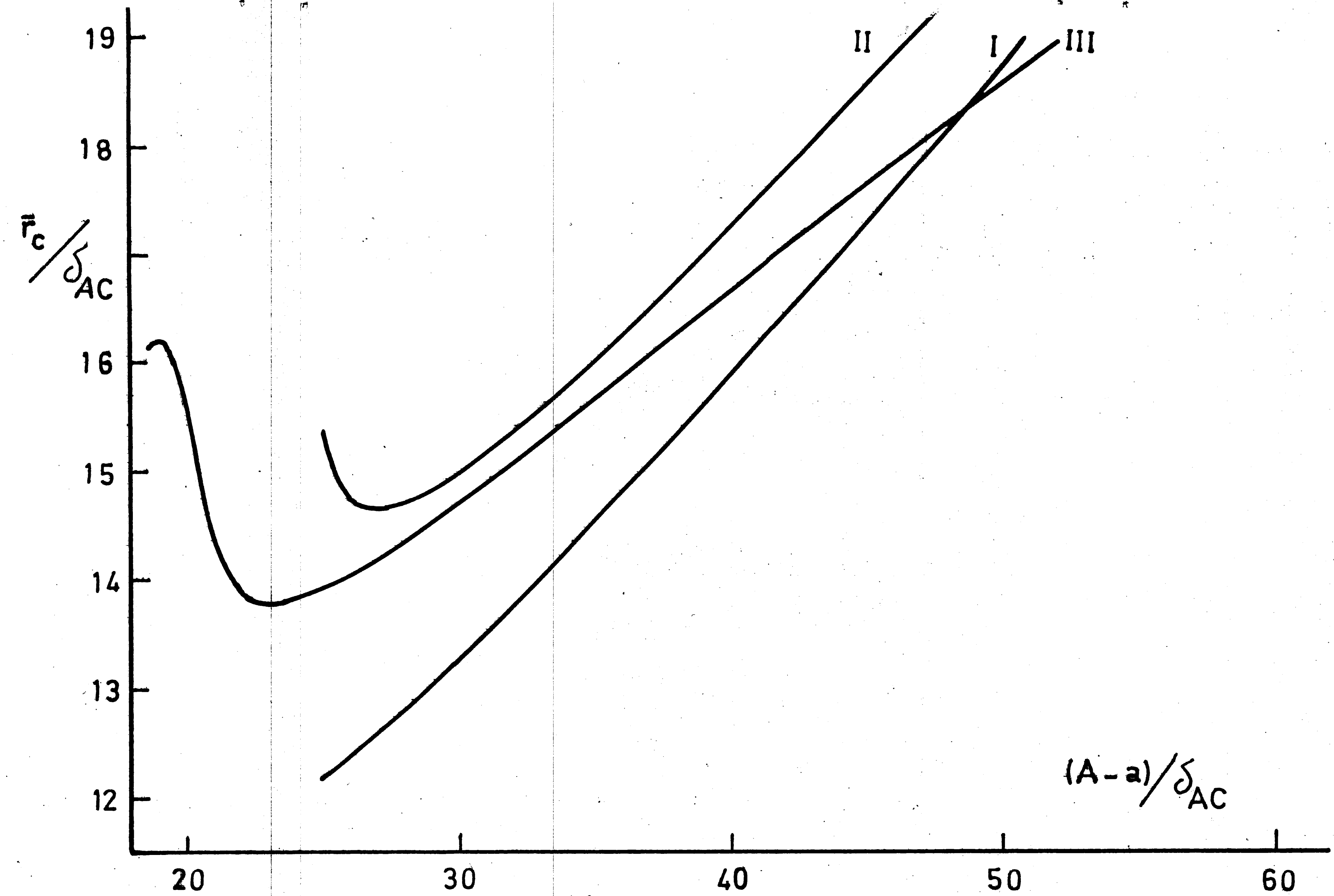

Figure 7. The radial position $\bar{r}_c/\delta_{AC}$ of the core of the outer vortex system versus $(A - a)/\delta_{AC}$. Curve I and II are based on equation (220), both curves with $M = 50$, but $R_s = 0$ in I and $R_s = 1$ in II. Equation (4.53) of Svardal (1965) is the basis of curve III where $M = 9.5$.

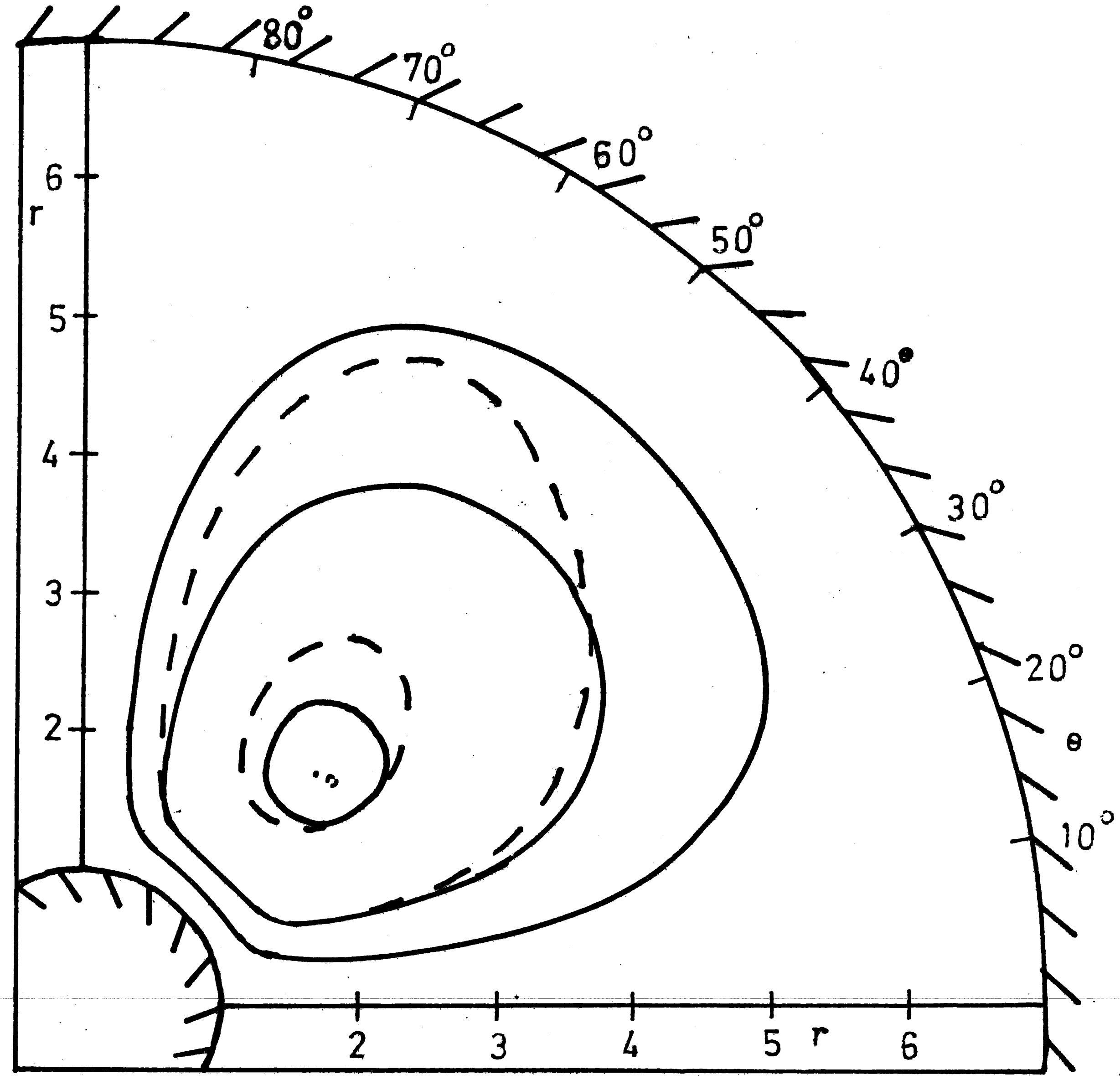

Figure 8. Stream line diagram in one quadrant based on equation (220) and the parameters are: Full lines; M = 50, R_s = 0. Dashed lines; M = 50, R_s = 1.

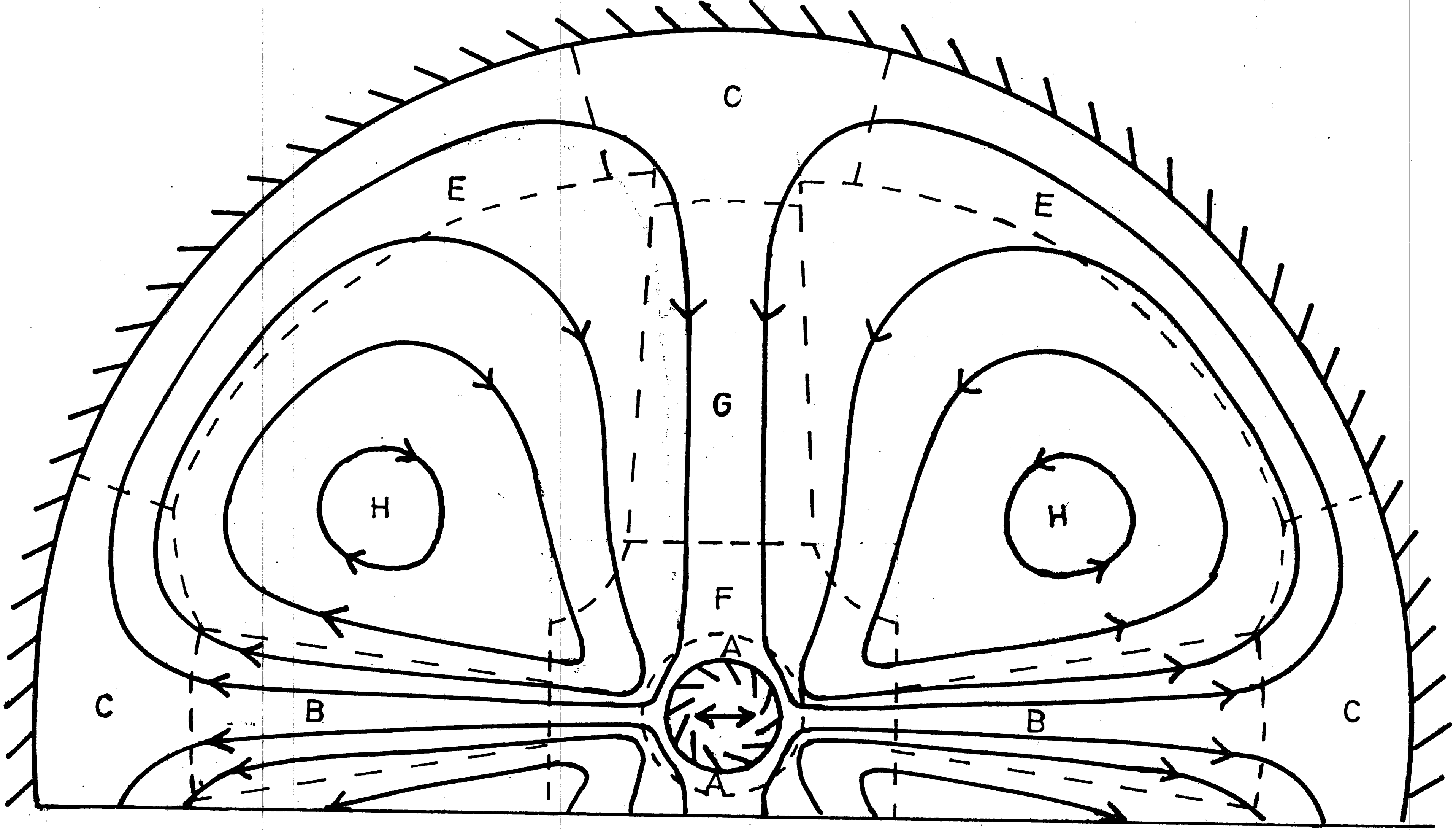

Figure 9. Sketch of the stream lines in the case $R_s \gg 1$ where regions with special features are bounded by dashed lines: A, steady boundary layer at the inner cylinder; B, outgoing jets; C, regions of stagnating flows at the outer boundary; E, wall jets; E, stagnating flow around the inner cylinder owing to the collision of the returning jets; G, returning jet; H, outer inviscid flow.

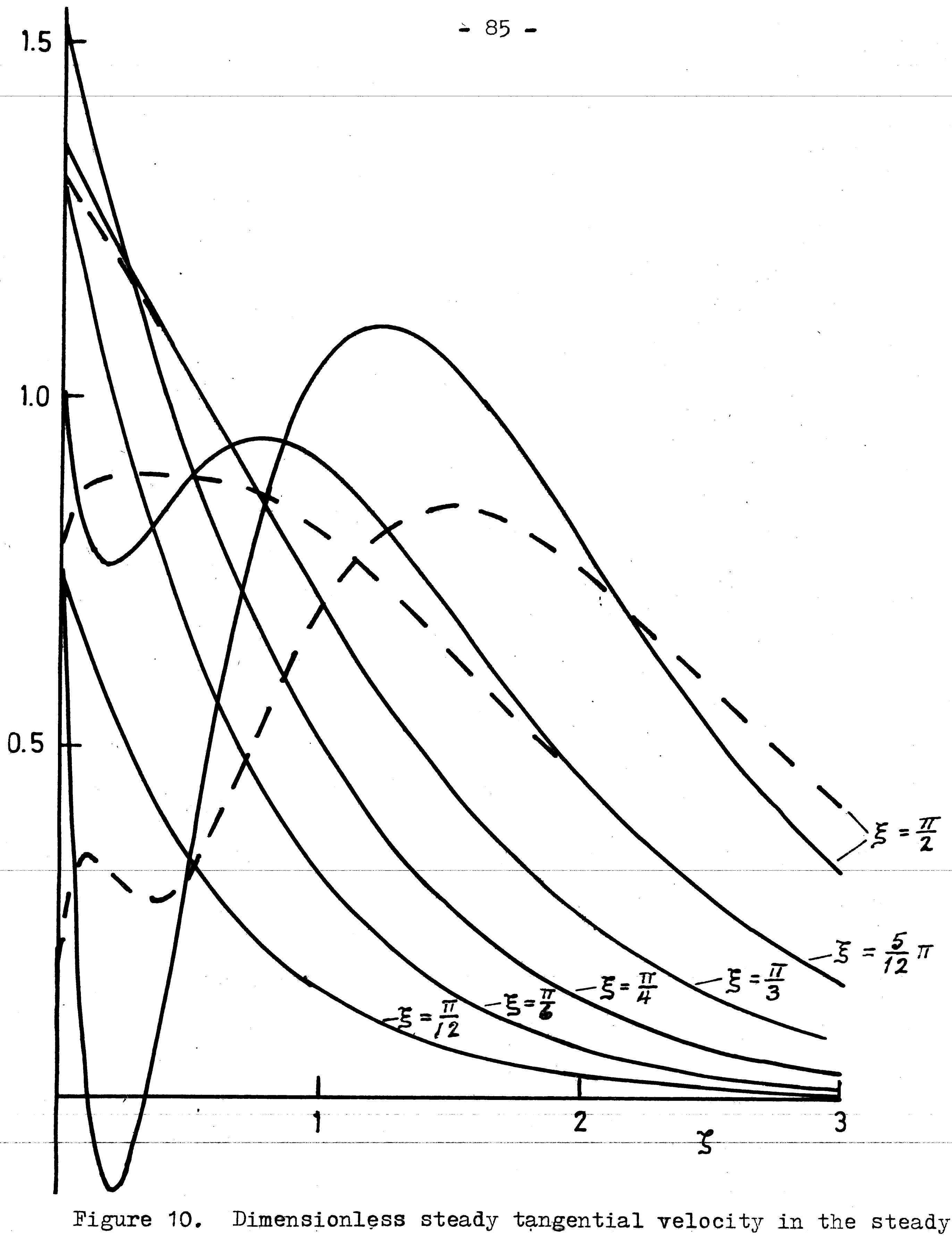

Figure 10. Dimensionless steady tangential velocity in the steady boundary layer based on first order boundary layer theory: Full lines; three-terms Blasius series expansion (Riley 1965). Dashed lines; four- terms Blasius series expansion.

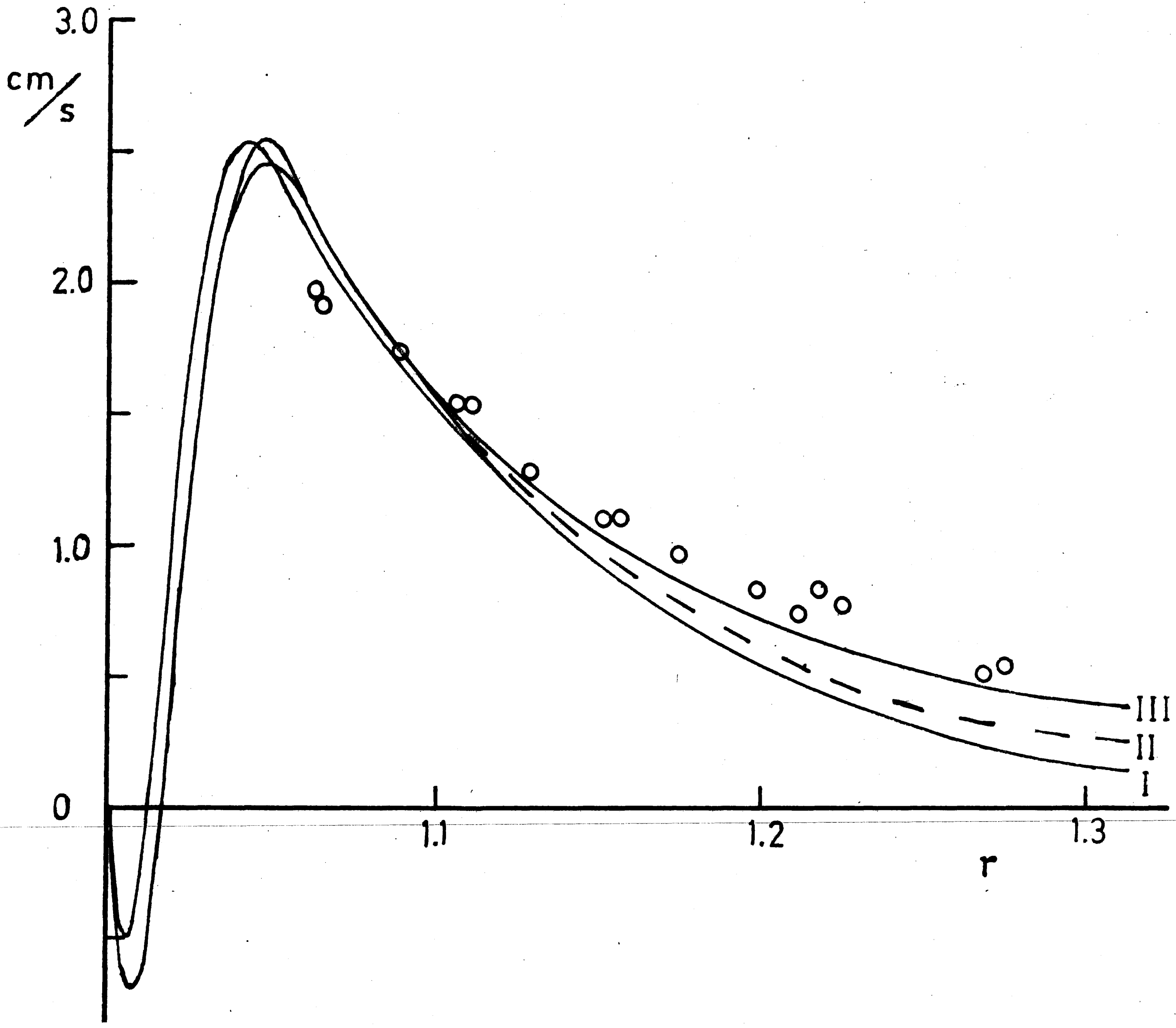

Figure 11. Steady tangential velocity in the boundary layer at the inner cylinder: Curve I, solutions due to Stuart (1966) and Riley (1965); Curve II, numerical solution of Riley (1975); Curve III, based on equation (222) here. Circles indicate measured velocities of Bertelsen (1974).

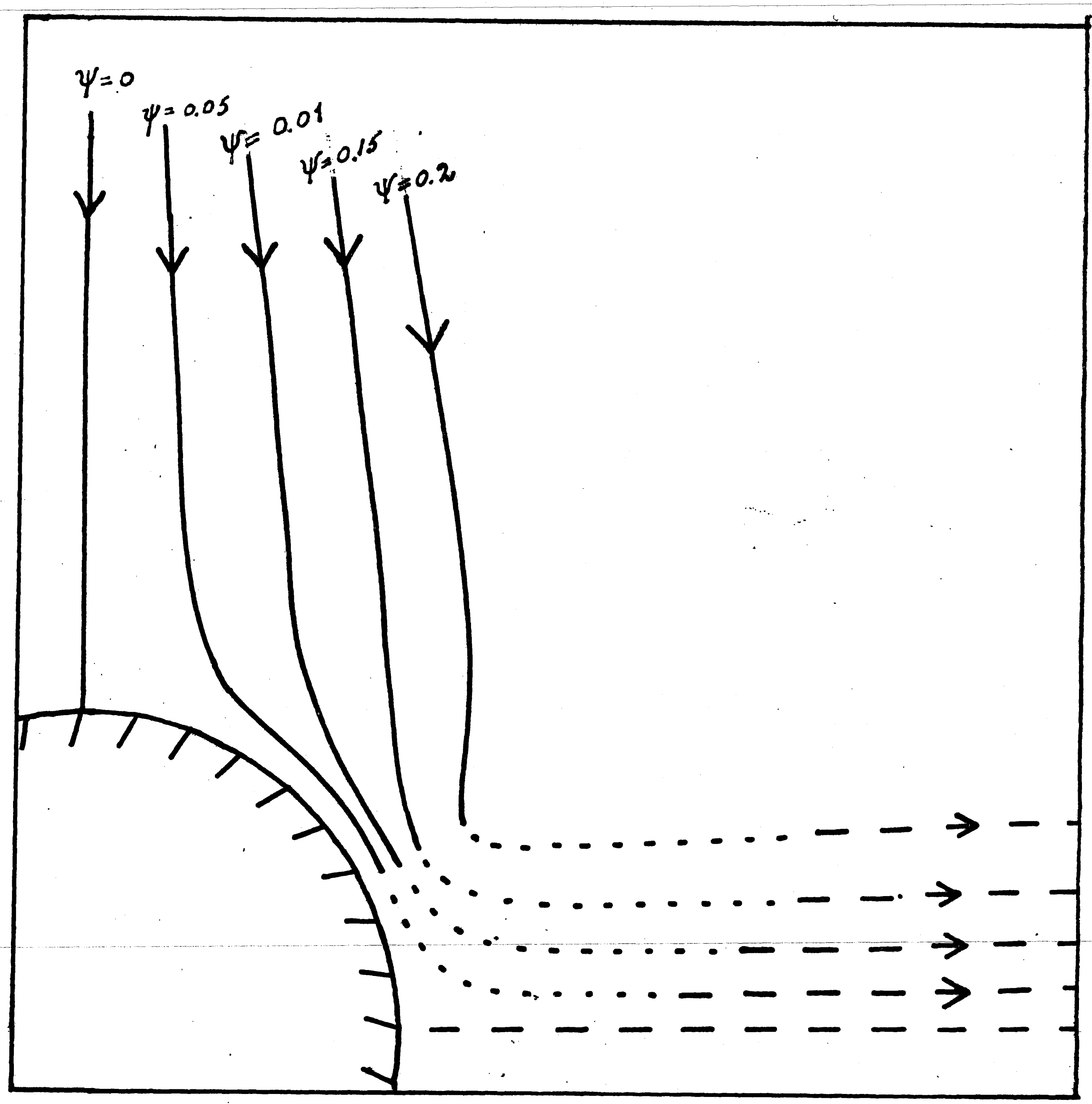

Figure 12. Stream line diagram based on composite solutions constructed of: Full lines, steady boundary layer solution and outer solution; Dashed lines, outgoing jet solution and outer solution. Dotted lines indicate possible matching of the stream lines through a region in which the stream function is not known.