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ON THERMAL CONVECTION BETWEEN NON-UNIFORMLY HEATED PLANES

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On thermal convection between non-uniformly heated planes

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Abstract

The stability of natural convection in a thin, horizontal layer subjected to horizontal as well as vertical temperature gradients is investigated on the basis of linear theory. The boundaries are taken to be stress-free and perfectly conducting, and the horizontal temperature gradient is assumed to be small. The analysis shows that the critical Rayleigh number is always larger than that for the ordinary Bénard problem. The preferred mode of disturbance is stationary, and will be a transverse or longitudinal roll depending on whether the Prandtl number is less or larger than 5.1. Finally, some calculations are made of the converted energy associated with the unstable perturbations, indicating that the mechanism of instability is of thermal (convective) origin.

Nomenclature

d	depth of layer
k,m	wave numbers in the x and z direction
$\vec{i},\vec{j},\vec{k}$	unit vectors
$\vec{v}$	velocity vector
u,v,w	velocity components
t	time
$U(y)$	basic flow velocity
p	pressure
$P(x,y)$	basic flow pressure
T	temperature
T_0	standard temperature
ΔT	temperature difference between lower and upper plane
∇^2	Laplacian operator
L	operator defined by (4.3)
K,I	defined by (5.2) and (5.6), respectively
Pr	Prandtl number $\frac{\nu}{\kappa}$
Ra	Rayleigh number $\frac{g\bar{\alpha}\Delta T d^3}{\kappa\nu}$ (g = acceleration of gravity, $\bar{\alpha}$ = coefficient of volume expansion)

Greek letters:

α	overall wave number
β	horizontal temperature gradient
θ,Θ	temperatures
κ	thermal diffusivity
ν	kinematic viscosity
ρ	density
ρ_0	standard density
σ	amplification factor of disturbance
ξ,ζ	defined by (4.9) and (4.27), respectively

Superscripts ' non-dimensional quantities,
^ perturbation quantities

1. Introduction.

Thermal convection in thin, horizontal fluid layers uniformly heated from below is quite well described in the literature (the Bénard problem); see the review article by Brindley [1] for references. In many practical problems, however, non-uniformly heating may occur, and thus the layer will be subjected to horizontal as well as vertical temperature variations. Few theoretical attempts have been made to analyse the stability of thin layers under such conditions. Zierep [2] has approached the problem by investigating a model with a discontinuous jump in the bottom temperature. Further, Koschmieder [3] has performed a laboratory experiment on convection between circular planes, the upper at constant temperature, the lower non-uniformly heated. At subcritical conditions a density gradient roll was observed, breaking up into axially symmetric rolls of different sizes and rotation when the vertical temperature difference was sufficiently increased. Theoretically Müller [4] has given a two-dimensional linear analysis of this problem.

In the present paper we investigate the effect of horizontal temperature variation on ordinary Bénard convection, assuming that the fluid is unlimited in the lateral directions. Due to the horizontal density gradient thus produced, a shear flow develops,and when the temperature difference between the bottom and top plane exceeds a certain critical value, this flow becomes unstable.

It is well known that in the absence of shear, a non-linear analysis must be applied to obtain the final flow structure, being two-dimensional rolls if the fluid properties are constant (Schlüter et.al.[5]), or hexagons if the properties vary with temperature (Palm [6], Busse [7]).

In stability problems involving a basic flow, a preferred direction is introduced into the system, and a unique flow pattern may be predicted from linear theory, (Liang and Acrivos [8]). The selected type of disturbance will depend on the instability mechanisms involved. For non-stratified shear flows, the mechanism is purely hydrodynamical, and by Squire's theorem it can be proved that instability first occurs for rolls having axes normal to the mean flow (transverse rolls). For shear flow with unstable vertical stratification due to heating from below, the instability will be of thermal origin if the basic flow Reynolds number is sufficiently small, and then rolls having axes aligned in the direction of the mean flow (longitudinal rolls) will be preferred ([8],[9],[10]).

In the present problem we shall assume a small horizontal temperature variation which implies a small basic flow velocity. For moderate (or large) Prandtl numbers then,the hydrodynamical instability mechanism will not seriously affect the problem, and thermal instability will dominate. Hence longitudinal rolls would be expected. It is therefore a little surprising,at least to the author, that the final flow pattern may be transverse or longitudinal rolls depending on whether the Prandtl number is smaller or larger than 5.1.

The reason for this, however, is purely of thermal origin. This is indicated in the last part of the paper where we consider the conversion of energy between the mean flow and the perturbation. There we show that the horizontal transfer of vertical momentum cannot account for the change of mode about $Pr = 5.1$, while the release of potential energy may do so.

2. Basic flow.

Consider natural three-dimentional convection of a viscous fluid confined between horizontal planes, see fig. 1. For mathematical simplicity we shall assume the planes to be stress-free and perfectly conducting, and the lateral temperature variation to be linear in the x-direction. For a given x-coordinate, the temperature difference between the planes is constant, ΔT, and the lower plane is the warmer. We then may write $T = T_0 - \frac{\Delta T}{2} - \beta x$ and $T = T_0 + \frac{\Delta T}{2} - \beta x$ at the top and bottom plane, respectively, where β is a positive constant.

To avoid infinite temperatures on the boundaries, we must limit the model in the x-direction, but we assume that the ratio of the depth to the characteristic horizontal dimension is so small that the lateral boundaries will not affect the motion.

Introducing non-dimensional (primed) quantities by

$$(x,y,z) = (x',y',z')d \,, \qquad t = t'\frac{d^2}{\kappa}$$

$$(u,v,w) = (u',v',w')\frac{\kappa}{d} \,, \qquad p = p'\frac{\rho_0 \kappa \nu}{d^2}$$

$$T - T_0 = T'\Delta T$$

and making the Boussinesq approximation, we may write the governing equations in vector notation

$$Pr^{-1}\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \nabla^2\vec{v} + RaT\vec{j} \qquad (2.1)$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \nabla^2 T \qquad (2.2)$$

$$\nabla\cdot\vec{v} = 0 \qquad (2.3)$$

where the primes have been dropped.

We now consider a particular solution of these equations. Setting

$$\begin{aligned} \frac{\partial}{\partial t} &= v = w = 0 \\ u &= U(y) \\ T &= T(y) - \beta x \end{aligned} \tag{2.4}$$

where β now is dimensionless, and eliminating the pressure from (2.1), the governing equations reduce to

$$\begin{aligned} D^3U(y) &= -\beta Ra \\ D^2T(y) &= -\beta U \end{aligned} \tag{2.5}$$

where $D = \frac{d}{dy}$.

These are subject to the boundary conditions

$$DU(\pm\tfrac{1}{2}) = 0 , \qquad T(\pm\tfrac{1}{2}) = \mp\tfrac{1}{2} \tag{2.6}$$

and to the continuity condition

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} U(y)dy = 0 \tag{2.7}$$

The solution of this system is easily obtained, being

$$\begin{aligned} U(y) &= \tfrac{1}{2}\beta Ra(\frac{y}{4} - \frac{y^3}{3}) \\ T(y) &= \frac{1}{24}\beta^2 Ra(\frac{9y}{80} - \frac{y^3}{2} + \frac{y^5}{5}) - y \end{aligned} \tag{2.8}$$

For sufficiently large values of Ra (or β), the solution (2.8) may become unstable, and this possibility is investigated in the next sections.

It is worth pointing out that this type of flow is solely caused

by a horizontal density gradient, and exists even when no vertical temperature difference is present ($\Delta T = 0$).

3. Stability analysis.

Following the usual approach of linear stability theory, infinitesimal perturbations (denoted by carets) are introduced into the governing equations. Setting

$$u = U(y) + \hat{u}(x,y,z,t), \qquad v = \hat{v}(x,y,z,t), \qquad w = \hat{w}(x,y,z,t)$$
$$\theta = T(y) - \beta x + \hat{\theta}(x,y,z,t), \qquad p = P(x,y) + \hat{p}(x,y,z,t) \tag{3.1}$$

and neglecting the non-linear terms, we obtain:

$$Pr^{-1}\left(\frac{\partial u}{\partial t} + U(y)\,\frac{\partial u}{\partial x} + v\,DU(y)\right) = -\frac{\partial p}{\partial x} + \nabla^2 u$$
$$Pr^{-1}\left(\frac{\partial v}{\partial t} + U(y)\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y} + \nabla^2 v + Ra\theta \tag{3.2}$$
$$Pr^{-1}\left(\frac{\partial w}{\partial t} + U(y)\frac{\partial w}{\partial x}\right) = -\frac{\partial p}{\partial z} + \nabla^2 w$$

$$\frac{\partial \theta}{\partial t} + U(y)\frac{\partial \theta}{\partial x} - \beta u + v\,DT(y) = \nabla^2 \theta \tag{3.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.4}$$

where the carets have been dropped.

We assume solutions of the form

$$\begin{Bmatrix} u \\ v \\ w \\ \theta \end{Bmatrix} = \begin{Bmatrix} u(y) \\ v(y) \\ w(y) \\ \theta(y) \end{Bmatrix} \exp(i(kx + mz - \sigma t)) \tag{3.5}$$

for the system (3.2)-(3.4), where the real part is considered

to have physical significance. The wave numbers, k and m, are real and the growth rate $\sigma(\equiv \sigma^r + i\sigma^i)$ is complex. Eliminating the pressure from (3.2), and utilizing (3.5), we finally obtain

$$\{Pr(D^2-\alpha^2)-ik\beta Ra\bar{U}+i\sigma\}(D^2-\alpha^2)v+ik\beta RaD^2\bar{U}v-\alpha^2 PrRa\theta = 0 \qquad (3.6)$$

$$\{D^2-\alpha^2-ik\beta Ra\bar{U}+i\sigma\}\theta+\beta u+v-\beta^2 RavD\bar{\theta} = 0 \qquad (3.7)$$

$$\{Pr(D^2-\alpha^2)-ik\beta Ra\bar{U}+i\sigma\}(-\alpha^2 u+ikDv)+m^2\beta RaD\bar{U}v = 0 \qquad (3.8)$$

subject to the boundary conditions

$$v = D^2v = Du = \theta = 0 \quad \text{for} \quad y = \pm\tfrac{1}{2} \qquad (3.9)$$

Here α is the overall wave number defined by $\alpha^2 \equiv k^2 + m^2$. Furthermore, we have introduced

$$\bar{U} \equiv \frac{U(y)}{\beta Ra} = \tfrac{1}{2}\left(\frac{y}{4} - \frac{y^3}{3}\right)$$

$$\bar{\theta} \equiv \frac{T(y)+y}{\beta^2 Ra} = \frac{1}{24}\left(\frac{9}{80}\,y - \frac{y^3}{2} + \frac{y^5}{5}\right) \qquad (3.10)$$

Equation (3.6) is the Orr-Sommerfeld equation which is coupled with the energy equation (3.7). Equation (3.8) is the vertical component of the vorticity equation combined with the equation of continuity.

4. Method of solution.

To simplify the problem, we shall assume that β is small. Then the equations (3.6)-(3.8) may be solved by a perturbation technique using β as a small perturbation quantity. This is a method similar to that introduced in [8]. The solutions are expanded into the form

$$\begin{aligned}(u,v,\theta,k,m,\sigma,Ra) = {} & (u_0,v_0,\theta_0,k_0,m_0,\sigma_0 R_0) \\ & + \beta(u_1,v_1,\theta_1,k_1,m_1,\sigma_1,R_1) \\ & + \beta^2(u_2,v_2,\theta_2,k_2,m_2,\sigma_2,R_2) \\ & + - - - -\end{aligned} \tag{4.1}$$

and the different orders are obtained by inserting these expressions into (3.6)-(3.8), equating equal powers of β and utilizing the solvability condition. For this procedure to be valid, the Prandtl number must be a zeroth-order quantity.

The zeroth-order system corresponds to thermal convection without shear (the Bénard problem), and the equations are

$$\begin{aligned}\{Pr(D^2-\alpha_0^2) + i\sigma_0\}(D^2-\alpha_0^2)v_0 - \alpha_0^2 PrR_0\theta_0 &= 0 \\ \{D^2 - \alpha_0^2 + i\sigma_0\}\theta_0 + v_0 &= 0 \\ -\alpha_0^2 u_0 + ik_0 Dv_0 &= 0\end{aligned} \tag{4.2}$$

It is well known [11] that the principle of exchange of stabilities is valid for this system. Hence $\sigma_0^r = 0$. At the neutral state ($\sigma_0^i = 0$), the governing equation may be stated

$$L(v_0) \equiv \{(D^2 - \alpha_0^2)^3 + \alpha_0^2 R_0\}v_0 = 0 \tag{4.3}$$

with boundary conditions

$$v_0 = D^2 v_0 = D^4 v_0 = 0 , \qquad y = \pm \tfrac{1}{2} \tag{4.4}$$

The solutions are readily obtained, being

$$\begin{aligned} v_0 &= A \cos \pi y \\ u_0 &= -i \frac{2k_0}{\pi} A \sin \pi y \\ \theta_0 &= \frac{2}{3\pi^2} A \cos \pi y \end{aligned} \tag{4.5}$$

and $R_0 = \frac{27}{4} \pi^4$ for $\alpha_0^2 = k_0^2 + m_0^2 = \frac{\pi^2}{2}$.

The amplitude A, which can not be determined from the linear, homogeneous system, may be equated to unity without loss of generality.

Next, for the first-order equations we obtain

$$\begin{aligned} Pr(D^2-\alpha_0^2)^2 v_1 - \alpha_0^2 Pr R_0 \theta_1 = \{ & 2\xi Pr(D^2-\alpha_0^2) + ik_0 R_0 (\bar{U}(D^2-\alpha_0^2) - D^2\bar{U}) \\ & - i\sigma_1 (D^2-\alpha_0^2) \} v_0 + \{ \xi Pr R_0 + \alpha_0^2 Pr R_1 \} \theta_0 \end{aligned} \tag{4.6}$$

$$(D^2-\alpha_0^2)\theta_1 + v_1 = \{ \xi + ik_0 R_0 \bar{U} - i\sigma_1 \} \theta_0 - u_0 \tag{4.7}$$

$$Pr(D^2-\alpha_0^2)\{ -\alpha_0^2 u_1 + ik_0 Dv_1 \} = -Pr(D^2-\alpha_0^2)(-\xi u_0 + ik_1 Dv_0) - m_0^2 R_0 D\bar{U} v_0 \tag{4.8}$$

where

$$\xi = 2(k_0 k_1 + m_0 m_1) . \tag{4.9}$$

Eliminating θ_1 from (4.6) by using (4.7), we finally get

$$\begin{aligned}\mathrm{Pr}L(v_1) = {} & \{2\xi\mathrm{Pr}(D^2-\alpha_0^2)^2+ik_0R_0(D^2-\alpha_0^2)\left[\bar{U}(D^2-\alpha_0^2)-D^2\bar{U}\right] \\ & - i\sigma_1(D^2-\alpha_0^2)^2\}v_0+\{\xi\mathrm{Pr}R_0(D^2-\alpha_0^2)+\alpha_0^2\mathrm{Pr}R_1(D^2-\alpha_0^2) \\ & + \alpha_0^2\mathrm{Pr}R_0\xi+ik_0\alpha_0^2\mathrm{Pr}R_0^2\bar{U}-i\sigma_1\alpha_0^2\mathrm{Pr}R_0\}\theta_0-\alpha_0^2\mathrm{Pr}R_0u_0 \qquad (4.10)\end{aligned}$$

subject to the boundary conditions

$$v_1 = D^2v_1 = D^4v_1 = 0\ , \qquad y = \pm\tfrac{1}{2} \qquad (4.11)$$

The operator L has been defined by (4.3).

Under the present conditions, L is easily seen to be self-adjoint. Then a necessary condition for (4.10) to have a solution is that the right hand side be orthogonal to v_0.

Defining the inner-product

$$\langle f,g\rangle \equiv \int_{-\frac{1}{2}}^{+\frac{1}{2}} fg\ dy \qquad (4.12)$$

the condition for solvability may be stated as

$$\langle L(v_1),v_0\rangle = 0 \qquad (4.13)$$

From (4.13) we obtain

$$R_1 = -i\sigma_1\frac{9\pi^2}{2}(1+\mathrm{Pr}^{-1}) \qquad (4.14)$$

Since R_1 must be purely real, this equation gives at the marginal state $(\sigma_1^i = 0)$

$$\begin{aligned}\sigma_1^r &= 0 \\ R_1 &= 0\ .\end{aligned} \qquad (4.15)$$

Thus we have no oscillatory instability to first order.

Concerning the Rayleigh number, it is clear from physical considerations that Ra generally can not contain any term involving odd powers of β, since the only effect of changing sign in β, is to reverse the basic velocity, which of course can not alter the stability conditions. Hence

$$R_{2n+1} = 0 , \quad n = 0,1,2, \text{---} \tag{4.16}$$

For the sake of simplicity, we shall represent the basic velocity by a sine-profile, which is indeed a good approximation. This may be seen from table 1, where the difference between the sine-profile $\bar{U}_1 = \frac{1}{24} \sin \pi y$ and the exact profile $\bar{U}_2 = \frac{1}{2}(\frac{y}{4} - \frac{y^3}{3})$ is given in percent for several values of y.

y	0	0.1	0.2	0.3	0.4	0.5
Diff. in %	0	4.2	3.4	2.1	0.7	0

Table 1: Difference between $\bar{U}_1$ and $\bar{U}_2$.

Throughout the remainder of this analysis, we then take as basic velocity

$$U(y) = \frac{\beta Ra}{24} \sin \pi y \tag{4.17}$$

The corresponding basic temperature is obtained from (2.5), being

$$T(y) = \frac{\beta^2 Ra}{12\pi^2} (-y + \tfrac{1}{2} \sin \pi y) - y = \Theta(y) - y \tag{4.18}$$

Introducing $\bar{U} \equiv U/\beta Ra$ and $\bar{\Theta} \equiv \Theta/\beta^2 Ra$ into (4.10) and utilizing $R_1 = \sigma_1 = 0$, we get

$$L(v_1) = ik_0 R_0 \{\frac{3}{64} \pi^4 (1+Pr^{-1}) \sin 2\pi y + \pi \sin \pi y\} \tag{4.19}$$

with boundary conditions $v_1 = D^2 v_1 = D^4 v_1 = 0 , \quad y = \pm \tfrac{1}{2}$.

Setting

$$v_1 = ik_0R_0 \frac{3}{64}\pi^4(1+Pr^{-1})\bar{v}_1 + ik_0R_0\pi\tilde{v}_1, \tag{4.20}$$

$\bar{v}_1$ is immediately obtained, while $\tilde{v}_1$ is found most conveniently by Galerkin's method, giving

$$\tilde{v}_1 = \sum_{n=1}^{\infty} A_{2n}\sin 2n\pi y \tag{4.21}$$

where

$$A_{2n} = \frac{64n(-1)^n}{\pi^7(4n^2-1)[(8n^2+1)^3-27]}$$

The solution of (4.19) may then be written

$$v_1 = ik_0\{-\frac{3\pi^2}{13\cdot 64}(1+Pr^{-1})\sin 2\pi y + \pi R_0\tilde{v}_1\}. \tag{4.22}$$

From (4.7) and (4.8) we now obtain the solutions for θ_1 and u_1, namely,

$$\theta_1 = ik_0\{-\frac{(27+Pr^{-1})}{32\cdot 39}\sin 2\pi y - \frac{4}{3\pi^3}\sin\pi y + \frac{4}{3\pi^3}\frac{\sinh(\pi y/\sqrt{2})}{\sinh(\pi/2\sqrt{2})} + \frac{2R_0}{\pi}\tilde{\theta}_1\} - \frac{4}{9\pi^4}\xi\cos\pi y \tag{4.23}$$

where

$$\tilde{\theta}_1 = \sum_{n=1}^{\infty} \frac{A_{2n}}{8n^2+1}\sin 2n\pi y,$$

and

$$u_1 = \pi\{\frac{3}{13\cdot 16}(1+Pr^{-1})k_0^2 - \frac{Pr^{-1}}{16}m_0^2\}\cos 2\pi y - \frac{2k_0^2}{\pi}R_0D\tilde{v}_1 + \frac{4i}{\pi^3}\{k_0\xi - \frac{\pi^2}{2}k_1\}\sin\pi y - \frac{9\pi Pr^{-1}}{16}m_0^2 \tag{4.24}$$

Here $\tilde{v}_1$ is defined by (4.21).

To determine R_2 and σ_2, we must apply the solvability condition to the second-order equations. These are given by

$$\Pr\{(D^2-\alpha_0^2)^2 v_2-\alpha_0^2 R_0\theta_2\} = \{2\Pr\xi(D^2-\alpha_0^2)+ik_0R_0\left[\bar{U}(D^2-\alpha_0^2)-D^2\bar{U}\right]\}v_1$$
$$+ \xi\Pr R_0\theta_1+\{2\Pr\zeta(D^2-\alpha_0^2)-\Pr\xi^2+ik_1R_0\left[\bar{U}(D^2-\alpha_0^2)-D^2\bar{U}\right]$$
$$- ik_0\xi R_0\bar{U}-i\sigma_2(D^2-\alpha_0^2)\}v_0+\{\zeta\Pr R_0+\alpha_0^2\Pr R_2\}\theta_0 \qquad (4.25)$$

and

$$(D^2-\alpha_0^2)\theta_2+v_2 = (\xi+ik_0R_0\bar{U})\theta_1-u_1+(ik_1R_0\bar{U}+\zeta-i\sigma_2)\theta_0+R_0D\bar{\Theta}v_0, \qquad (4.26)$$

where

$$\zeta = k_1^2+2k_0k_2+m_1^2+2m_0m_2. \qquad (4.27)$$

The equation for u_2 is not relevant for the present purpose, and is thus not stated.

We now multiply (4.25) and (4.26) by $(D^2-\alpha_0^2)$ and $\alpha_0^2\Pr R_0$, respectively, summing and take the innerproduct with v_0. The solvability condition for this system is then expressed by

$$\langle L(v_2), v_0\rangle = 0 \qquad (4.28)$$

After some algebra we obtain from (4.28)

$$R_2 = \frac{3}{16}\pi^2R_0 + \frac{R_0}{4\cdot13}\left[8-8\cdot13\Sigma_1+\frac{81\pi^4}{16\cdot32}+\left(\frac{4}{3}+\frac{3\pi^4}{64}\right)\Pr^{-1}\right.$$
$$\left.+\frac{21\pi^4}{16\cdot32}\Pr^{-2}\right]k_0^2+\frac{7}{3}R_0\Pr^{-1}m_0^2+36(k_0k_1+m_0m_1)^2 \qquad (4.29$$
$$- i\sigma_2\frac{9\pi^2}{2}(1+\Pr^{-1}),$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty}\frac{2\cdot27\cdot64n^2}{\pi^4(4n^2-1)^2\left[(8n^2+1)^3-27\right]} \qquad (4.30)$$

This series converges very rapidly, giving approximately

$$\Sigma_1 = 5\cdot63\cdot10^{-3} \tag{4.31}$$

At the neutral state, σ_2 must be real. Since R_2 is real, the real and imaginary parts of equation (4.29) reduce, respectively, to

$$R_2 = \frac{3}{16}\pi^2 R_0 + \frac{1}{4\cdot13} R_0\Big[8-8\cdot13\Sigma_1 + \frac{81\pi^4}{16\cdot32} + (\frac{4}{3} + \frac{3}{64}\pi^4)Pr^{-1} + \frac{21\pi^4}{16\ 32} Pr^{-2}\Big]k_0^2 + \frac{7}{3}R_0 Pr^{-1} m_0^2 + 36(k_0k_1+m_0m_1)^2 \tag{4.32}$$

and $$\sigma_2^r \frac{9\pi^2}{2}(1+Pr^{-1}) = 0 \tag{4.33}$$

The last relation implies that $\sigma_2^r = 0$ at the marginal state, and hence we have no oscillatory instability to second order.

From (4.32) we immediately conclude that R_2 is positive and greater than zero for all kinds of disturbances, which means that the onset of convection in the present problem will occur for a Rayleigh number larger than the critical value corresponding to the Bénard problem. This is to be expected since the basic flow convects warmer fluid in the upper part of the layer and colder in the lower part, thus opposing the destabilizing effect of the temperature difference between the lower and upper plane.

The preferred mode of disturbance will make R_2 a minimum. If we introduce $h = m_0/k_0$ and utilize $k_0^2 + m_0^2 = \pi^2/2$, R_2 may be written

$$R_2(h) = \frac{3}{16}\pi^2 R_0 + \{A(Pr^{-1}, Pr^{-2}) + B(Pr^{-1})h^2 + 36(k_1+m_1h)^2\}\frac{\pi^2}{2(1+h^2)} \tag{4.34}$$

where the expressions for A and B easily follow from (4.32).

It is then seen that R_2 has an absolute minimum either for $h = 0$, $k_1 = 0$ (transverse rolls) or $h = \infty$, $m_1 = 0$ (longitudinal rolls) depending on the values of A and B, i.e. the Prandtl number.

In fig. 2, R_2 is displayed for the two kind of rolls, and we observe that for $Pr < 5.1$ transverse rolls are preferred, while we get longitudinal rolls when $Pr > 5.1$.

It should be noted that formally (4.32) also gives one more point of intersection between the transverse and longitudinal roll curves, namely for $Pr = 0.03$. Whether this reflects any real change of mode, however, is doubtful, since our perturbation method is not supposed to be valid at such a small Prandtl number. In fact, we suggest that the tendency to select transverse rolls will be strengthened at small Pr by the increased importance of shear on the mechanism of instability.

When Pr increases towards infinitely, the critical Rayleigh number assumes the asymptotic value

$$Ra = R_0(1 + \frac{3}{16}\pi^2\beta^2) + O(\beta^4) \qquad (4.35)$$

5. Exchange of energy between the mean flow and the perturbation.

In this section we shall be concerned with the perturbation energy. Taking the real parts of the component equations in (3.2), multiplying by the real parts of u, v and w, respectively, averaging over a wavelength in the x- and z-direction, adding, and integrating from $y = -\frac{1}{2}$ to $y = +\frac{1}{2}$, using the boundary conditions, we finally obtain the familiar equation for the kinetic energy of the perturbation

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\overline{u^2}+\overline{v^2}+\overline{w^2}\rangle = -\langle DU(y)\overline{uv}\rangle + PrRa\langle\overline{v\theta}\rangle - Pr\langle(\overline{\nabla u})^2+(\overline{\nabla v})^2+(\overline{\nabla w})^2\rangle \tag{5.1}$$

where the bars and the brackets denote mean and vertical integrations, respectively.

Here the term $-\langle DU\ \overline{uv}\rangle$ represents the conversion of kinetic energy between the perturbation and the mean flow through vertical transfer of horizontal momentum, while the second and the third term on the right represent conversion of potential energy to kinetic energy and viscous dissipation, respectively

We define

$$K \equiv -\langle DU\ \overline{uv}\rangle \tag{5.2}$$

Here $\overline{uv} = \frac{1}{2}\left[u^r(y)v^r(y) + u^i(y)v^i(y)\right]$ where the superscripts r and i denote real and imaginary parts of the velocities defined by (3.5).

We will consider the marginal stable solutions. Since the solution denoted by subscript zero corresponds to pure convection, it is obvious that $\overline{u_0v_0} = 0$. The lowest order contribution to the

Reynolds stresses is then given by

$$\overline{uv} = \beta(\overline{u_1 v_0} + \overline{u_0 v_1}) + O(\beta^2) \tag{5.3}$$

For the expression (5.2) we then obtain to second order

$$K = -3.6(1.6 + Pr^{-1})\beta^2 \tag{5.4}$$

and

$$K = 198.1\, Pr^{-1}\beta^2 \tag{5.5}$$

for transverse and longitudinal rolls, respectively. Accordingly, transverse rolls always lose kinetic energy to the mean flow, while longitudinal rolls always gain energy. Similar results were obtained by Asai [12] for convection in Couette flow, from which it was concluded that longitudinal rolls were preferred.

In the present problem a similar conclusion is obviously incorrect. The fastest growing mode will depend on the conversion of potential energy as well as viscous dissipation. This dependence will not fully be explored in this paper. At large Prandtl numbers, however, it is immediately clear from the equation for the kinetic energy,(5.1), that the processes mentioned above will dominate. For Pr of about unity, we shall consider one important second order term of the released potential energy.

We define

$$P \equiv PrR_2\, \tfrac{1}{2}\langle v_0\theta_0\rangle\beta^2 = \frac{1}{6\pi^2}\, PrR_2\beta^2 \tag{5.6}$$

where R_2 is given by (4.32).

From the graph of R_2 in fig. 2 it follows that the value of P for transverse rolls will be less than its value for longitudunal

rolls when Pr is less than five, while for Pr greater than five the opposite is true. Further it can be shown that for Pr = 5 the release of potential energy (P) for a longitudinal roll is about six times larger than the energy converted from the mean flow through vertical momentum transfer. This indicates that the process of conversion of potential energy will dominate for Pr about unity, and may account for the change of mode at Pr = 5.1. Since we consider marginally stable solutions, the left hand side of (5.1) is zero. To satisfy this condition, the viscous dissipation must also be important for Pr of about unity.

6. Summary and discussion.

When the Prandtl number is less than 5.1, we find that the Rayleigh number at the neutral state has a minimum for steady, transverse rolls, i.e. rolls with axes normal to the mean flow. For Prandtl numbers greater than 5.1, the Rayleigh number is smallest for steady, longitudinal rolls having axes aligned in the direction of the mean flow.

Our conclusions are, in some respects, similar to those reached by Liang and Acrivos [8] for convection in a tilted slot. As in the present case, the neutral state remains stationary for all disturbance wave numbers, i.e. the principle of exchange of stabilities applies, and the critical Rayleigh number decreases with increasing Pr to an asymptotic value independent of Pr. In the present problem this limit is given by (4.35).

Two important differences may be noted, however. In [8] the most unstable mode was found to be a longitudinal roll, and the critical Rayleigh number the same as for pure convection without shear. In our case, the most unstable mode may be either transverse or longitudinal depending on whether Pr is smaller than 5.1 or not. The critical Rayleigh number will always be larger than that corresponding to convection without horizontal density gradients. Physically this is due to the upward convection of warm fluid and downward convection of cold fluid in the basic flow.

The last section has been devoted to energy considerations. We have shown that, analogous to [12], a longitudinal roll always gains kinetic energy from the mean flow through vertical transfer of horizontal momentum, while a transverse roll always loses energy by this process. This does not explain the change of mode at Pr = 5.1 in the present problem. By computing one particular term in the released potential energy, it is indicated that the mechanism of instability is primarily of convective origin.

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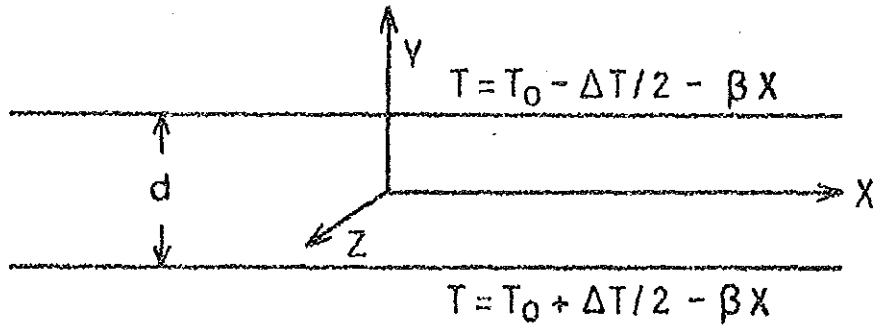

Figure 1. The coordinate system.

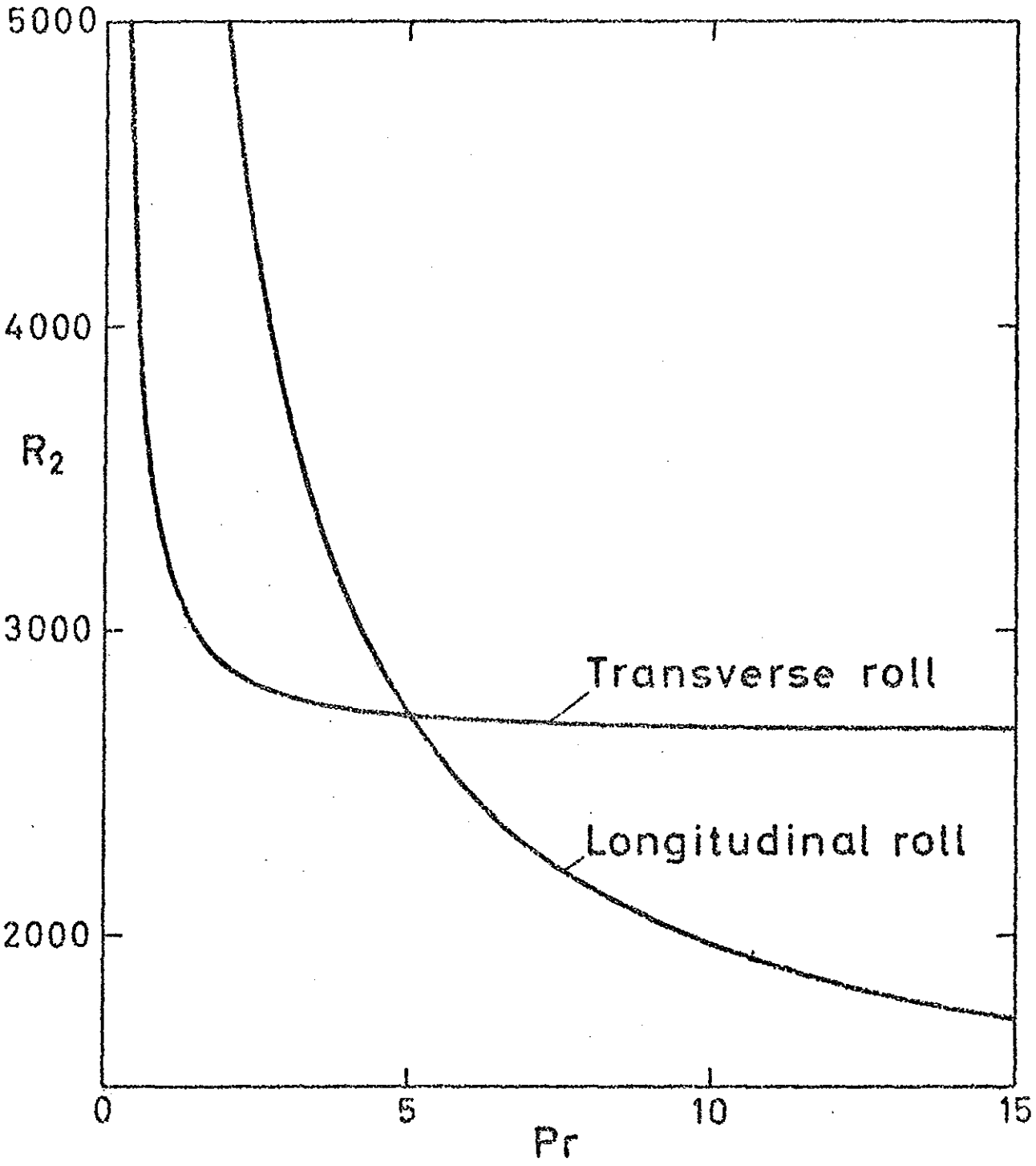

Figure 2. $R_2 = (Ra - R_0)/\beta^2$ versus Pr for transverse and longitudinal rolls.