A NOTE ON DOUBLE-DIFFUSIVE INSTABILITY IN A POROUS MEDIUM

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Abstract.

The onset of convection in a differentially heated porous vertical layer is investigated theoretically. The medium is filled with a solution which is stably stratified in the vertical by temperature and solute. For moderate and large values of the solute Rayleigh number Rs, it is found that the presence of a stable temperature gradient acts stabilizing, while for sufficiently small values of Rs this gradient has a destabilizing effect. Analogous to Hart [6], nonlinear resonance is found to occur in the limit of large R_s at a smaller wave number than that predicted by linear theory. Finally the effects of bounding side walls on the onset of convection in a porous medium with vertical gradients of heat and solute is considered. The results conform to those of Beck [11] for the analogous one-component problem

Nomenclature.

L, D, H,	width, depth and height, respectively, of the model;
t,	dimensionless time;
x, y, z,	dimensionless Cartesian coordinates;
$\vec{i}, \vec{j}, \vec{k}$,	unit vectors;
K,	permeability of porous medium;
g,	acceleration of gravity;
c_p,	specific heat at constant pressure;
ΔT,	horizontal temperature difference;
S_0,	initial vertical solute distribution;
$\vec{v}(=u, v, w)$,	dimensionless velocity vector;
T, S, p,	dimensionless temperature, solute, and pressure, respectively;
$W(x), T(x), S(x)$,	dimensionless basic solution, Eq. (2.8);
C, R,	defined by (2.9);
l, m,	dimensionless wave numbers in the y- and z-direction;
a,	overall wave number;
c,	defined by (2.4);
Ra,	thermal Rayleigh number $Kg\alpha\Delta TL/\nu\kappa_{mT}$;
Rs,	solute Rayleigh number $Kg\beta(-\partial S_0/\partial z_*)L^2/\nu\kappa_{mS}$;
$\Delta T_A, \Delta S_A$;	vertical temperature and solute differences, respectively;
Ra^*, Rs^*,	defined by (A.2);
A, B,	defined by (A.4).

Greek letters.

α, β,	thermal and volumetric expansion coefficients, respectively;
γ,	dimensionless temperature gradient along $x = \pm \frac{1}{2}$;
δ,	defined by (A.7), (A.8);
θ,	dimensionless perturbation temperature;
κ_{mT}, κ_{mS},	diffusivities of heat and solute in the porous medium;
$\mu_{1,2}$,	defined by (4.5);
ν,	kinematic viscosity;
ρ,	density;
ρ_0,	reference density;
σ,	amplification factor of disturbance;
ψ,	stream function.

Subscripts.

*,	dimensional quantities;
f,	fluid;
m,	solid-fluid mixture;
c,	critical;
r,	resonant;
e,	even;
o,	odd.

Superscripts.

~,	complex conjugate.

1. Introduction.

Convective instability in a fluid with two (or more) diffusive components has lately received considerable interest in the literature; see the review by Turner [1]. This phenomenon, known as double-diffusive convection, has several important geophysical applications. Especially in the ocean, where the relevant components are heat and salt, double-diffusive (or thermohaline) convection is of importance, as this phenomenon in general contributes to the vertical mixing process, and in particular may account for the formation of certain layers observed in the sea [2].

Published studies on double-diffusive convection in porous media are not so numerous, although this phenomenon may be of importance in connection with hot, salty springs or fertilizer migration in saturated heated soil [3].

The main part of the present note is concerned with convection in a differentially heated infinite vertical porous layer filled with a stably stratified solution. Similar studies in Newtonian fluids have been performed by Thorpe et.al. [4] and Hart [5,6]. In addition to their models, the present study also includes the effect of a stable vertical temperature gradient.

As in the Newtonian fluid case, the parallel flow set up by sideways heating results in a slanting of the original horizontal isosolutal lines. Accordingly, a parcel of fluid being displaced sideways, experiences a net buoyancy force due to the differential diffusion of heat and solute in the porous medium. The motion resulting from this diffusive destabilization, has been called sideways diffusive convection.

It is thought that a porous medium may provide a convenient means for studying sideways diffusive convection. Due to the lack of inertial terms in the equation of motion, shear instability does not occur in porous media. Accordingly the sideways diffusive mechanism is the only one acting in the present problem. Also when performing laboratory experiments, which can be done by using a Hele-Shaw cell, the porous model may have some advantages. The velocities will now be very small, and hence the initial vertical concentration gradient will last for quite a long period of time.

In the appendix we study the onset of convection in a limited porous medium when the side heating is abolished, and vertical gradients of temperature and solute are imposed. This is an extension of the analysis by Taunton et.al. [3], and corrects some of their results.

2. Basic solution.

Consider the motion in a rectangular porous medium bounded by impermeable planes. A Cartesian coordinate system, with the z-axis directed upwards, is placed in the middle of the model such that the bounding planes are situated at $x_* = \pm L/2$, $y_* = \pm D/2$ and $z_* = \pm H/2$. In this part of the analysis we assume that H/L and D/L are infinite. The medium is filled with a stably stratified solution i.e. $\partial S_0/\partial z_* < 0$. The vertical boundaries at $x_* = \pm L/2$ are taken to be insulating to solute, and kept at temperatures $\mp \Delta T/2 + \gamma_* z_*$, such that the left-hand wall is the warmer.

Dimensionless (unstarred) variables may now be introduced by taking

$$L, \quad \frac{L\nu}{Kg\alpha\Delta T}, \quad \frac{Kg\alpha\Delta T}{\nu}, \quad \Delta T, \quad \frac{\alpha\Delta T}{\beta}, \quad \rho_0 g\alpha\Delta TL \tag{2.1}$$

as units of length, time, velocity, temperature, solute concentration and pressure.

Utilizing the Boussinesq approximation, the governing equations may be written in dimensionless form

$$\nabla p = -\vec{v} + (T-S)\vec{k} \tag{2.2}$$

$$\nabla \cdot \vec{v} = 0 \tag{2.3}$$

$$Ra(c \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T) = \nabla^2 T \tag{2.4}$$

$$Ra\tau^{-1}(\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S) - Rsw = \nabla^2 S \tag{2.5}$$

where $c = (\rho c_p)_m / (c_p)_f$ and $\tau = \kappa_{mS}/\kappa_{mT}$. The parameters $Ra = Kg\alpha\Delta TL/\nu\kappa_{mT}$ and $Rs = Kg\beta(-\partial S_0/\partial z_*)L^2/\nu\kappa_{mS}$ are the thermal and solute Rayleigh numbers, respectively.

Eqs. (2.2)-(2.5) permit a steady parallel flow solution of the form

$$\begin{aligned} u &= v = 0 \\ w &= W(x) \\ T &= T(x) + \gamma z \\ S &= S(x) \end{aligned} \tag{2.6}$$

where $\gamma = \gamma_* L/\Delta T$ isthe prescribed dimensionless temperature gradient at $x = \pm \frac{1}{2}$. By inserting these expressions into eqs.(2.2)-(2.5) and utilizing the boundary and continuity conditions

$$T(\pm\tfrac{1}{2}) = \mp\tfrac{1}{2}, \qquad \frac{dS}{dx}(\pm\tfrac{1}{2}) = 0$$
$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} W dx = 0 \tag{2.7}$$

we obtain

$$W(x) = -\frac{C\ \sinh(\sqrt{R}x)}{\sqrt{R}\cosh(\sqrt{R}/2)}$$

$$T(x) = -\ C(1 - \frac{\gamma Ra}{R})x + \frac{\gamma Ra}{R}\ W(x) \qquad (2.8)$$

$$S(x) = T(x) - W(x)$$

where

$$C = 1/[1-\gamma Ra/R + (2\gamma Ra/R\sqrt{R})\tanh(\sqrt{R}/2)],$$

$$(2.9)$$

$$R = Rs + \gamma Ra$$

In a real problem the model will always be bounded vertically. If, however, the ratio L/H is sufficiently small, the solution (2.8) will be adequate in the central part of the layer.

Analogous to the studies by Elder [7], Vest and Arpaci [8], Hart [9] a.o. in a differentially heated fluid slot with isothermal vertical boundaries, we may let γ represent the building up of a stable temperature gradient in the interior of the model due to the presence of horizontal end walls. For L/H given, γ will then be a function of Ra. For large Ra, γ approaches an asymptotic limit, being 0.67L/H for the vertical porous layer problem (Weber [10]). In the present case, with a stably stratified solution, this process is accompanied by a gradual destruction of the initial solute gradient.

Since heat diffuses much faster than solute in a fluid, however, ($\tau \approx 1/100$ for heat and salt in water), it is believed that, in a laboratory experiment, a stable temperature gradient may manifest itself in the central part of the layer before the solute distribution is noticeably reduced there.

3. Linear perturbation analysis.

Perturbating the solution (2.8) by infinitesimal disturbances of the form

$$\{u,w,\theta,s\} = \{\hat{u}(x),\hat{w}(x),\hat{\theta}(x),\hat{s}(x)\}\exp\{i(ly+mz) + \sigma t\} \qquad (3.1)$$

where l and m are real dimensionless wave numbers in the y- and z-direction, respectively, and σ the (complex) growth rate, the linearized perturbation equations are

$$(d^2 - a^2)u = -im\, d\,\theta + im\, ds \qquad (3.2)$$

$$Ra(c\sigma\theta + imW(x)\theta + udT(x) + \gamma w) = (d^2 - a^2)\theta \qquad (3.3)$$

$$\tau^{-1}Ra(\sigma s + imW(x)s + udS(x)) - Rsw = (d^2 - a^2)s \qquad (3.4)$$

where $d \equiv d/dx$, $a^2 \equiv l^2 + m^2$, and the carets have been dropped. The boundary conditions are

$$u = \theta = ds = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \qquad (3.5)$$

In this problem we are able to demonstrate that two-dimensional disturbances with axes aligned along the basic flow (longitudinal rolls) are stable.

By assuming that $\partial/\partial z = im = 0$, it follows immediately from (3.2) that $u = 0$. Multiplying (3.3) and (3.4) by the complex conjugates $\tilde{\theta}$ and $\tilde{s}$, respectively, integration across the layer (denoted by brackets) yields

$$\sigma\langle cRa|\theta|^2 + \frac{\gamma Ra^2}{\tau Rs}|s|^2\rangle = -\langle\gamma Ra|w|^2 + \frac{\gamma Ra}{Rs}(|ds|^2 + l^2|s|^2)$$

$$+ |d\theta|^2 + l^2|\theta|^2\rangle \qquad (3.6)$$

where we have utilized that $\langle w\tilde{\theta} - w\tilde{s}\rangle = \langle|w|^2\rangle$ from the vorticity equation. Hence, since γ, Ra, Rs > 0, σ is real and negative i.e. longitudinal rolls can not grow. According to this we study the stability of the solution (2.8) with respect to disturbances having $\partial/\partial y = 0$ (transverse rolls). We then may introduce a streamfunction ψ such that $w = d\psi$ and $u = -\,im\,\psi$. The stability analysis is performed by Galerkin's method. Since numerical computations for the analogous problem in a fluid (Hart [5]), and the asymptotic result presented later in this section show no oscillatory behaviour at the marginal state, we consider stationary rolls only.

Our solutions may then be written

$$\begin{aligned} \theta &= \theta_e + i\theta_o \\ s &= s_e + is_o \\ \psi &= \psi_o + i\psi_e \end{aligned} \qquad (3.7)$$

where θ_e, θ_o, s_e, s_o, ψ_o, ψ_e are real functions of x. The subscripts e and o denote even and odd with respect to $x = 0$, respectively. Equivalently, the combination $\theta = \theta_o + i\theta_e$ etc. might have been chosen (see Vest and Arpaci [8]). In (3.7) the functions θ_e, θ_o, s_e, ... are developed in trigonometric series satisfying the boundary conditions at the vertical walls. By inserting these expansions into Eqs. (3.2)-(3.4) and using the orthogonalization conditions, the resulting algebraic problem reduces to finding zero of a $6N \times 6N$ determinant, where N is the number of terms in the exapnsions. The result will be discussed in section 5.

4. Asymptotic results for strong concentration gradients.

When Rs is large, and $\gamma Ra \ll Rs$, the situation is analogous to that treated in detail by Hart [5,6]. We therefore just outline the main steps in the analysis. From (2.8) it now follows that the vertical velocity effectively vanishes except in thin boundary layers of order $Rs^{-\frac{1}{2}}$ at the vertical walls. Here also the velocity is of order $Rs^{-\frac{1}{2}}$. In the interior, then, the horizontal gradients of heat and solute are almost linear. Assuming that the strong solute stratification forces the disturbance into thin, wide cells, the lowest order interior equations follow from Eqs. (3.2)-(3.4) by assuming $W = 0$, $dS = dT = -1$ and neglecting d^2 compared to m^2. The solution of this system is readily obtained. By applying $\psi = 0$ at $x = \pm \frac{1}{2}$, which, according to Hart, are the proper boundary conditions for the interior problem, an eigenvalue equation is obtained from which it follows that σ is real. The corresponding marginal state is governed by

$$(\tau^{-1}-1)Ra = 2(m^2 Rs + \frac{\pi^2 Rs^2}{m^2})^{\frac{1}{2}} \tag{4.1}$$

giving a critical Rayleigh number

$$Ra_c = \frac{\sqrt{8\pi}\, Rs^{\frac{3}{4}}}{(\tau^{-1}-1)} \tag{4.2}$$

for $m_c = \sqrt{\pi} Rs^{\frac{1}{4}}$ (4.3

The normalized marginal solution may then be written

$$\psi = \cos \pi x \, \cos\{\tfrac{1}{2}(\mu_1+\mu_2)x + mz\} \tag{4.4}$$

where

$$\mu_{1,2} = \frac{m}{2Rs}\left\{(\tau^{-1}-1)Ra \pm \sqrt{(\tau^{-1}-1)^2 Ra^2 - 4m^2 Rs}\right\} \tag{4.5}$$

At the critical point, the equation for the tilt lines can be obtained from (4.2)-(4.4). We find

$$z = -\sqrt{2\pi}\, Rs^{-\frac{1}{4}} x + \text{const.} \tag{4.6}$$

This shows that the cells are sloping down from the hot wall. We further see that the slope decreases with increasing vertical stability, as would be expected.

Further comments on the asymptotic case will be given at the end of next section, where some nonlinear results are presented.

5. Results and discussion.

In figure 1 we have displayed the results from the stability analysis. The critical thermal Rayleigh number is plotted as a function of the solute Rayleigh number. In figure 2 the corresponding critical wave numbers are presented. In the computations we have chosen $\tau = 1/100$ which is a reasonable value for the salt-heat problem in a porous medium. The number of terms needed for convergence varied along the curves. The criterion was chosen such that the change in the eigenvalue should be less than 0.5 % when increasing the number of terms from N to $N+1$. For $Rs \gtrsim 40$ five terms were sufficient, while for $Rs < 30$ nine or more terms had to be used.

Figure 1 quite strikingly exhibits the effectiveness of the sideways diffusive mechanism. The minimum value of Ra is found to be 2.9 for $Rs = 70$ and $m = 4.3$, which, for many geophysical problems, is obtained for very small temperature and solute gradients.

As mentioned before, however, one advantage by working with a porous medium, or a Hele-Shaw cell, is that experiments may be conducted even at this minimum point, and not only in the limit of large Rs as in [4] or [6]. For experiments in progress with $H = 100$ cm,

$L = 1$ cm, $D = 0.1$ cm and salt dissolved in 60 % glycerine, it is found that $W_{max} \approx 5 \cdot 10^{-4}$ cm/s at this particular point. Hence it will take a particle about 28 h to travel from the center of the cell to one end, which puts an upper limit to the time one can run an experiment under quasi-static asumptions. The required horizontal temperature difference will be 2.8 deg.C in this case.

We further observe from figure 1 that the presence of a stable temperature gradient γ acts stabilizing for moderate and large values of Rs. This is due to the fact that a positive γ effectively diminishes the horizontal basic solute gradient, as seen from (2.8). Hence the sideways diffusive mechanism is weakened.

For sufficiently small values of Rs, however, we find that γ acts destabilizing. This may be explained as follows: For marginal stability at small Rs and non-zero γ, the vertical density stratification is mainly due to temperature and governed by the parameter γRa. Even though this parameter, as before, results in a reduced horizontal basic solute gradient, it also limits the vertical cell height, as seen from figure 2. Obviously thinner cells are more efficient in providing the solute and heat balance. Accordingly, for sufficiently small values of Rs, the configuration with a positive vertical temperature gradient will be the most preferable.

If γ is taken to represent the interior temperature gradient caused by the finiteness of the model, as discussed in section 2, this implies that γ would be of order L/H. If now $L \ll H$, as we have assumed for (2.8) to be valid, figure 1 shows that this has a negligible effect on the stability problem.

Although no experiments have been performed yet in the limit of large Rs, it seems unlikely that a disturbance with wavelength

$\lambda_c = 2\pi/m_c$ should be observed at the marginal state. This is due to the fact that a nonlinear resonant instability may occur in the present problem,analogeously to the case discussed by Hart [6]. If one tries the usual normal mode cascade process for the interior problem, starting with the solution (4.4) as a first approximation, and writing the nonlinear solutions as power series in a small parameter (which essentially is $|Ra - Ra_c|^{\frac{1}{2}}$) it is found that the second order solution contains a part

$$\psi_2 \propto \frac{1}{(\pi^2 Rs - 3m^4)} \cos\{(\mu_1 + \mu_2)x + 2mz\} \tag{5.1}$$

This part expresses the respons from the forcing of the second harmonic of the fundamental, and becomes infinite when

$$m = \left(\frac{\pi^2}{3} Rs\right)^{\frac{1}{4}} = m_r$$

or (5.2)

$$m_r = 0.76\ m_c .$$

As noted by Hart [6], the occurrence of resonans suggests that also the time dependence of the second harmonic must be taken into account in the nonlinear procedure. Hence one should start with the fundamental and the second harmonic as primary waves. This will be attempted in a forthcoming paper.

APPENDIX: Double-diffusive convection in a bounded porous medium.

Assume now that H/L and D/L are finite. We abolish the side heating, and take the vertical walls to be insulating to heat and solute. Let the horizontal boundaries have constant temperature and solute concentration; the differences between the bottom and top plane being ΔT_A and ΔS_A, respectively.

Analogous to Beck [11] it is easily shown that the perturbation solutions are separable in space and time, i.e.

$$w,\theta,s \propto \cos(p\pi \frac{H}{L} x)\cos(q\pi \frac{H}{D} y)\sin(r\pi z)e^{\sigma t} \qquad (A.1)$$

where we, for simplicity, have placed the origo in a corner of the box and used the height H as length scale. In (A.1) $r = 1,2,3\cdots$ and $p,q = 0,1,2,3$ (but $p+q \neq 0$). The appropriate Rayleigh numbers are now

$$Ra^* = \frac{Kg\alpha\Delta T_A H}{\nu\kappa_{mT}}$$
$$Rs^* = \frac{K\beta\Delta S_A H}{\nu\kappa_{mT}} \qquad (A.2)$$

(note that here κ_{mT} is used in the definition of solute Rayleigh number). The governing equations then reduce to

$$cA^2\sigma^2+\{A^4(1+c\tau)-B^2Ra^*+cB^2Rs^*\}\sigma+A^2\{\tau A^4-\tau B^2Ra^*+B^2Rs^*\} = 0 \qquad (A.3)$$

where

$$B^2 = \pi^2(p^2H^2/L^2 + q^2H^2/D^2) \ ,$$
$$A^2 = B^2 + \pi^2r^2 \qquad (A.4)$$

We look for marginal stable solutions, i.e. $\sigma_r = 0$. Equation (A.3) then yields

$$Ra^* = A^4/B^2 + Rs^*/\tau \tag{A.5}$$

$$Ra^* = (1+c\tau)A^4/B^2 + cRs^* \tag{A.6}$$

for stationary and oscillatory modes, respectively. It is easily seen that Ra^* has a minimum for $r = 1$. Accordingly the critical Rayleigh numbers for the stationary and oscillatory case are given, respectively, by

$$Ra_c^* = \pi^2 \min(\delta+\delta^{-1})^2 + Rs^*/\tau \tag{A.7}$$

$$Ra_c^* = \pi^2(1+c\tau)\min(\delta+\delta^{-1})^2 + cRs^* \tag{A.8}$$

where

$$\delta^2 = B^2/\pi^2 = p^2H^2/L^2 + q^2H^2/D^2$$

We observe that the function $I = (\delta+\delta^{-1})^2$ has an absolute minimum for $\delta = 1$, i.e. $I_{min} = 4$. Beck [11] has discussed the minimum of I for general values of the box geometry parameters L/H and D/H, and we refer to his work for details. We just mention that when L/H or L/D becomes large, I_{min} rapidly approaches 4. Hence for an unbounded horizontal layer (A.7) and (A.8) reduce to

$$Ra_c^* = 4\pi^2 + Rs^*/\tau \tag{A.9}$$

$$Ra_c^* = 4\pi^2(1+c\tau) + cRs^* \tag{A.10}$$

In both cases the preferred disturbance has a horizontal overall wave number that equals π, as in ordinary one-component porous convection.

From (A.9) and (A.10) it follows that the transition from stationary to oscillatory convection occurs when Rs^* exceeds the value

$$Rs_0^* = \frac{4\pi^2 c\tau^2}{1-c\tau} \qquad (A.11)$$

which is a very small quantity for the heat-salt problem. The corresponding value of Ra_c^* is then slightly above $4\pi^2$.

The result (A.9) has previously been obtained by Nield [12] and Taunton et.al. [3], while the result (A.10) for convection via oscillatory modes differs from that given in [3]. This is due to the fact that Taunton et.al. keep the time derivative of the velocity in the equation of motion when they at the same time neglect the convective acceleration. When the (pore) Reynolds number is small, however, both these terms are small and can be neglected, such that the equation of motion expresses a balance between pressure, friction and buoyancy (see eq. (2.2)).

As in Beck's work, the main conclusions to be drawn from this part of the study is that the presence of side walls in general makes the system more stable. Further, when two-dimensional disturbances (rolls) occur, they will be orientated in such a way that each roll has the closest possible approximation to a square cross section.

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Figure legends.

Figure 1 Marginal stability curves for various values of the stable vertical temperature gradient γ when $\tau = 1/100$.
The broken line represents the asymptotic result (4.2).

Figure 2 Wave numbers for marginal stability when $\tau = 1/100$ and $\gamma = 0,1,10$. The upper and lower broken lines represent the linear asymptotic result (4.3) and the resonant wave number (5.2), respectively.

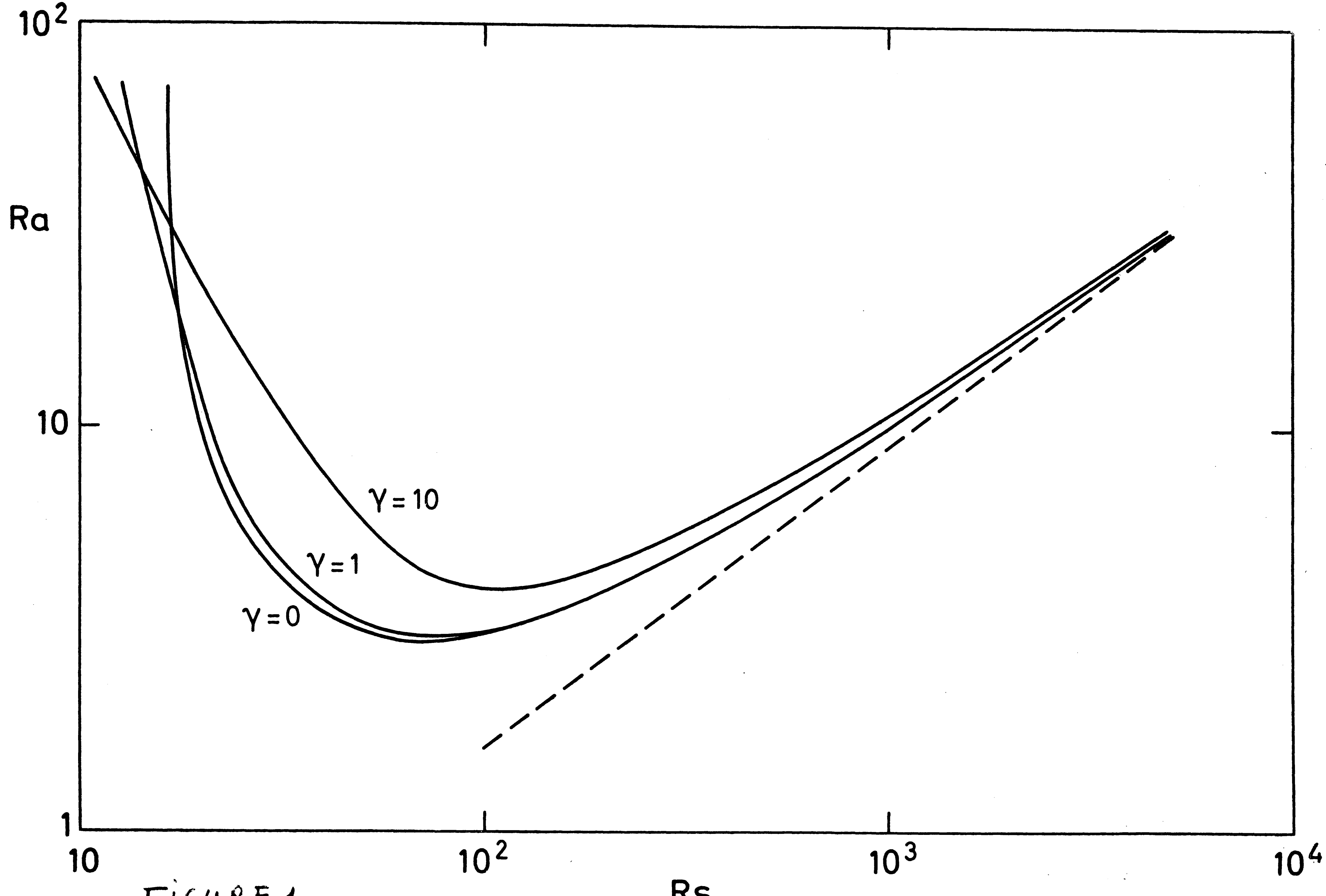

FIGURE 1.

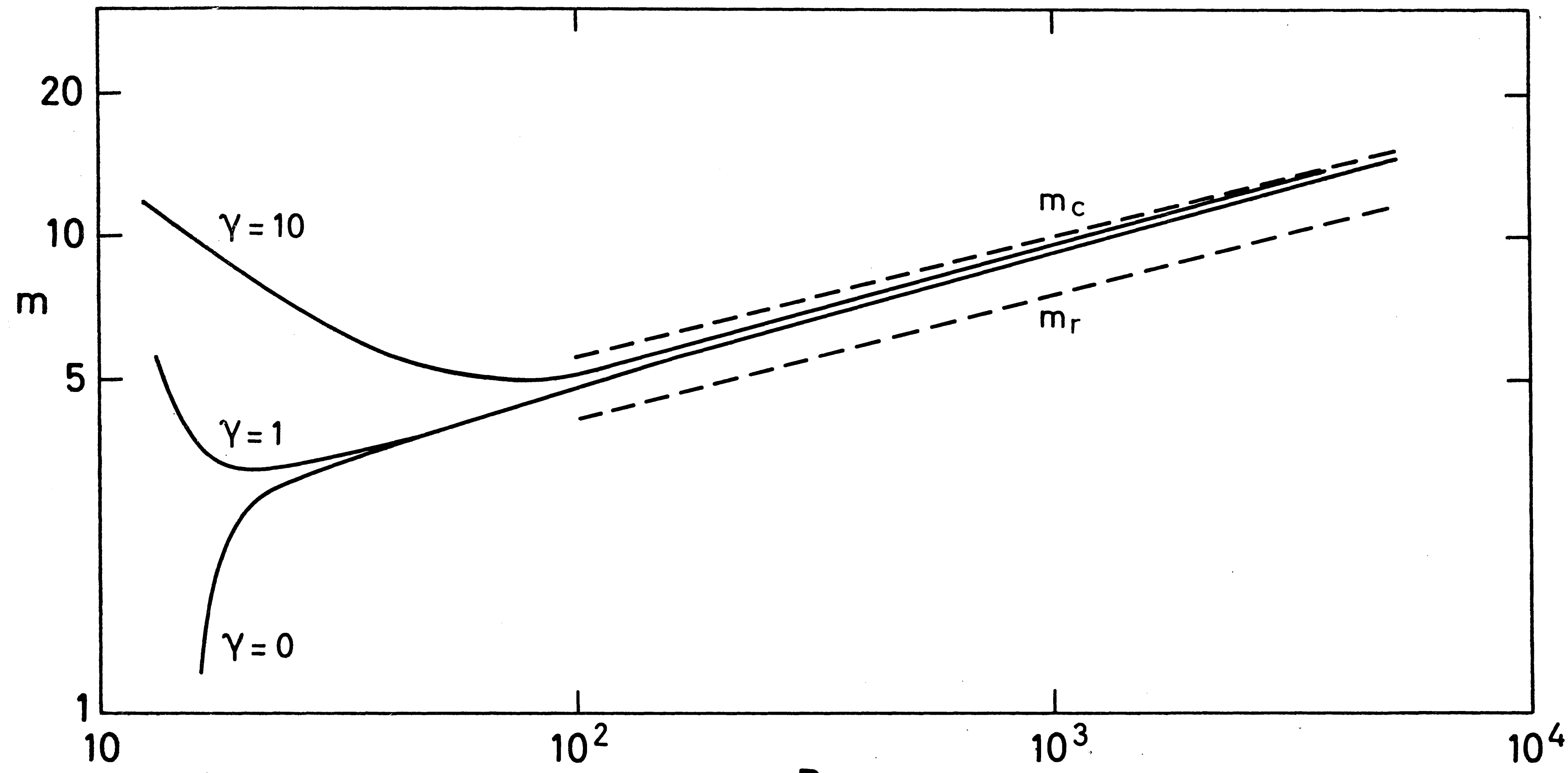

FIGURE 2.