Non-linear convection in a porous medium

by

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Abstract :

The finite amplitude motion in a porous medium is studied using Galerkin's method. The Nusselt number is obtained and shows good agreement with experimental values. Further, the stability of the stationary solution is examined. One finds that two-dimensional motion is stable for Rayleigh numbers and wave numbers inside a closed region in the Ra,α-plane.

1. Introduction.

The non-linear stability of a horizontal fluid layer heated from below has received considerable attention in the recent years. Particular interest has been paid to the classical Bénard problem. But also an increasing number of studies has been performed on convection in a porous medium. This is mainly because of this phenomenon being of considerably geophysical interest, as it may occur within the earth crust. Convection may also effect the motion of oil and gas in permeable rock reservoirs. Mathematically convection in a porous medium is simpler than the ordinary Bénard problem. This is due to the fact that the inertial terms can be neglected when the (particle) Reynold's number is sufficiently small, and that the usual viscosity term is replaced by Darcy's law.

After the first analysis by Horton & Rogers (1945) and Lapwood (1948) where the possibility of free convection in a porous medium was pointed out, a number of both theoretical and experimental papers have appeared in the literature. Schneider (1963), Elder (1967), Buretta (1972) and Bories & Combarnous (1973) have among others performed laboratory experiments , measuring essentially the heat flux across the layer as a function of the vertical temperature difference. Theoretical analyses of finite amplitude convection in a porous medium have been performed among others by Elder (1967) and Strauss (1974) by numerical methods, while Gupta & Joseph (1973) have used an upper bound technique. Palm, Weber & Kvernvold (1972) have analysed the problem by a perturbation method.

The present analysis is divided into two parts. Firstly the equations describing the steady state solution are solved numerically by Galerkin's method. This is analogous to the approach by Veronis

(1967) to the Bénard problem. In connection with the Galerkin procedure, the present paper introduces a time transformation which highly improves the convergence of the solutions. The properties of the steady solutions are discussed, particularly the heat flux as a function of the wave number and the Rayleigh number.

The second part of the paper deals with the stability of the stationary solutions with respect to infinitesimal perturbations.

While this paper was in preparation a similar study was published by Strauss (1974). The present investigation confirms the main results of that paper.

2. Formulation of the problem.

In the present problem we consider free convection in a saturated porous layer of infinite horizontal extent. The layer has a thickness h and is bounded by two parallel impermeable planes. The boundaries are taken to be perfect conductors of heat and to have constant temperature, T_1 and $T_1 + \Delta T$, respectively.

Utilizing the Boussinesque approximation, the equations governing the motion in a porous medium may be written (Palm & Weber (1971)):

$$-\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\vec{g} - \frac{\nu}{k}\vec{v} = 0 \tag{2.1}$$

$$\nabla\cdot\vec{v} = 0 \tag{2.2}$$

$$c\,\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \kappa_m \nabla^2 T \tag{2.3}$$

$$\rho = \rho_0(1-\alpha(T-T_0)) \tag{2.4}$$

Here p is the pressure, ρ_0 a standard density, ρ the density, α the coefficient of expansion, g the acceleration of gravity, ν the viscosity, κ_m the thermal diffusivity for the porous medium, k the permeability, T the temperature, T_0 a standard temperature, $\vec{v} = (u,v,w)$ the velocity, t the time and ∇ is the Laplacian. c is a constant given by $(\rho c_p)_m/(\rho c_p)_f$, where c_p is the heat capasity at constant pressure and the subscripts f and m denote fluid and fluid-solid mixture, respectively. The frame of referance is chosen so that x- and y-axis are horizontal while the z-axis is directed upwards. $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the corresponding unit vectors, and $z = 0$ at the lower plane.

The fields variables may then conveniently be maid dimensionless by choosing h, ΔT, $\mu\kappa_m/k$, κ_m/h, $\frac{ch^2}{\kappa_m}$ as characteristic values of length, temperature, pressure, velocity and time, respectively. Then by eliminating the pressure the equations become :

$$\nabla^2 w - Ra\ \nabla_1{}^2 w = 0 \tag{2.5}$$

$$\frac{\partial\theta}{\partial t} + \vec{v}\cdot\nabla\theta = w + \nabla^2\theta \tag{2.6}$$

$$\nabla\cdot\vec{v} = 0 \tag{2.7}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{2.8}$$

where $Ra = \frac{g\alpha\Delta T k h}{\kappa_m \nu}$ is the Rayleigh number, and θ, being the deviation from the conduction solution, is defined by

$$T = T_0 - \frac{\Delta T}{h} z + \theta \tag{2.9}$$

Equation (2.8) says that the vertical component of vorticity is zero.

The boundary conditions imposed on the system are

$$\theta = w = 0 \qquad \text{for } z = 0,1 \tag{2.10}$$

Equations (2.7) and (2.8) make it possible to introduce a new function ψ, such that (Busse 1967) :

$$\vec{v} = \nabla x(\nabla x\vec{k}\psi) \tag{2.11}$$

Further from (2.5) we find

$$\theta = -\frac{1}{Ra}\nabla^2\psi \tag{2.12}$$

Combining (2.5), (2.6), (2.11) and (2.12) we finally obtain

$$-\frac{\partial}{\partial t}\nabla^2\psi + \nabla^4\psi + Ra\nabla_1{}^2\psi = [\nabla x(\nabla x\vec{k}\psi)]\cdot\nabla\ \nabla^2\psi$$

with the boundary conditions (2.13)

$$\nabla_1{}^2\psi = \frac{\partial^2\psi}{\partial z^2} = 0 \quad \text{for} \quad z = 0,1$$

where $\nabla_1{}^2$ is the two-dimensional Laplacian.

By introducing the scalar function ψ, we only have to deal with one equation, and the field variables $\vec{v}$ and θ are obtained from the linear relations (2.11) and (2.12).

The linear stability problem is well-known (Lapwood, 1948) . One finds a critical Rayleigh number $Ra^c = 4\pi^2$ for a wave number $\alpha = \pi$, where α is the overall horizontal wave number defined by $\nabla_1{}^2\psi = -\alpha^2\psi$.

3. Method of solution.

In this section we will discuss the method of solution of the non-linear equation (2.13), being mainly interested in the stationary state. There are different methods available for this purpose. Palm, Weber & Kvernvold (1972) have used perturbation analysis, and they obtain a solution being valid for Rayleigh numbers up to about five times the critical value. However, for a more careful examination of the problem, especially when the Rayleigh number becomes larger, we have to use numerical methods. In this analysis we will use the Galerkin procedure.

A proof similar to that by Schlüter, Lortz and Busse (1965) shows that two-dimensional motion is the only stable solution for moderate overcritical Rayleigh numbers (Kvernvold, 1972). Accordingly, we assume that the steady solution is periodic in the x-direction and independent of y, and expand ψ in a Fourier series

$$\psi = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{\infty} A_{pq} e^{ip\alpha x} \sin q\pi z \tag{3.1}$$

where each term satisfies the boundary conditions. To assure a real solution, we must have $A_{nm} = A^*_{-nm}$, where the asterisk denotes complex conjugated.

Substituting (3.1) into equation (2.13), multiplying by $e^{-in\alpha x} \sin m\pi z$ and integrating over the layer, we obtain an infinite set of first order non-linear coupled differential equations :

$$\left.\begin{aligned} &(n^2\alpha^2+m^2\pi^2)\dot{A}_{nm} = (-(n^2\alpha^2+m^2\pi^2)^2 + Ra\, n^2\alpha^2)A_{nm} \\ &+ \tfrac{1}{2} \sum_{k,l} s_{m,l}\, A_{n-k,s_{m,l}(m-l)} A_{kl} (k^2\alpha^2+l^2\pi^2)(km-ln)(n-k)\alpha^2\pi \\ &- \tfrac{1}{2} \sum_{k,l} A_{n-k,m+l} A_{kl} (k^2\alpha^2+l^2\pi^2)(km+ln)(n-k)\alpha^2\pi \end{aligned}\right\} \tag{3.2}$$

where

$$s_{m,l} = \begin{cases} 1 & \text{for } m > l \\ 0 & \text{for } m = l \\ -1 & \text{for } m < l \end{cases}$$

Since no method is known to handle an infinite set of equations, it is necessary to truncate the series in order to obtain a finite set.

A commonly used method is to restrict n and m such that $|n| + m \leq N$, where N is a positive integer (Veronis 1966).

To examine the convergency of the solution, we introduce the Nusselt number, Nu, which measures the convective heat transport:

$$Nu = 1 - \left(\frac{\partial \overline{\theta}}{\partial z}\right)_{z=0} \tag{3.3}$$

The over-bar denotes horizontal average.

In our approximation the Nusselt number will be a function of N, and the relation (2.12) together with (3.1) gives

$$Nu_N = 1 - \frac{1}{Ra} \sum_{n=1}^{N} (n\pi)^3 A_{on} \ .$$

We will consider the truncated series as a satisfactorily good approximation if Nu_N differs from Nu_{N+2} by less than 1%.

Systems of equations with $N = 4$ up to $N = 14$ are integrated numerically by the Runga-Kutta method for various wave numbers and Rayleigh numbers.

It turns out that the steady state solution is independent of the initial values of the amplitudes. Accordingly we use the steady solution associated with N as initial values for the computation with $N+2$. This will speed up the convergency for the Runga-Kutta procedure.

For given N, the complex equations (3.2) will result in $N(N+1)$ equations for the unknowns $Re(A_{nm})$ and $I_m(A_{nm})$. The number of equations may, however, be reduced somewhat. Numerical calculations show that all steady state amplitudes are zero, when $|n| + m$ is an odd number. Hence we drop such terms. Furthermore, due to the symmetry of the equations in the x-direction, it follows that for symmetric initial conditions, i.e. $Im(A_{nm}) = 0$, the resulting steady solution will also be symmetric. However, Forster (1969) concludes that for Bénard convection in a layer of infinite horizontal extent, the lateral edges of the convective cells will be slightly inclined. To examine this possibility we have done some numerical calculations with general initial conditions, i.e. both $Re(A_{nm}) \neq 0$ and $Im(A_{nm}) \neq 0$. It is always found that the steady state solution has no tilt. The cells were, however, displaced somewhat horizontally. Then by transforming to a new reference system, we will always have solutions with $Im(A_{nm}) = 0$. We therefore only have to solve $(N/2)(N/2+1)$ equations for a given N.

For the Runga-Kutta method we take as convergency criterion that

$$|\dot{A}_{nm}|/|A_{nm}| < 10^{-4} .$$

It turns out that it is difficult to obtain a convergent solution when the number of equations become sufficiently high. This is due the great difference in the characteristic time scale for the various equations. The equations in (3.2) are essentially of the form

$$\frac{dA_k}{dt} = (-k^2\alpha^2 + Ra)A_k + \text{non-linear terms.}$$

The characteristic time scale is dominated by the factor $k^2\alpha^2$ which very soon becomes large. In fig. 1 we have illustrated the trend towards a steady solution for the two cases $k = 1$ and $k = 10$.

To obtain a steady solution by the Runga-Kutta method we have to choose time step which is smaller than the characteristic period for the fastest oscillating mode. This time step is, however, much too small for the computation of the most important amplitude, A_1, which accordingly leads to quite a lot of numerical iterations. We are, however, only interested in the steady state solution, and not the intermediate time variation. Therefore we may transform the set of equations (3.2) into another set having the same steady solution, but with a different time variation. This is achieved by introducing new time scales in the equations (3.2) given by :

$$t = \frac{n^2 + m^2}{\pi^3} t^*$$

We then obtain:

$$\frac{dA_{nm}}{dt^*} = \frac{\pi^3}{n^2+m^2}\left(-(n^2\alpha^2+m^2\pi^2) + \frac{Ra\ n^2\alpha^2}{n^2\alpha^2+m^2\pi^2}\right)A_{nm} + \frac{\pi^3}{n^2+m^2}\text{(Non-linear term)}. \qquad (3.4)$$

This set of equations have the same stationary solution as (3.2), but now the characteristic period for the various equations is of the same order of magnitude. It turns out that this transformation highly improves the convergency of the method. In most cases the number of iterations needed to obtain a satisfactorily good solution lays between 20 and 40. In some cases (for moderate Rayleigh numbers) less than 10 iterations were needed. This should be compared to the computation by Strauss (1972) for a related problem, where in some cases more than 2000 iterations were required.

4. The non-linear steady state solution.

The numerical calculations were carried out in the following manner. For a given value of Ra, The Fourier coefficients A_{nm} are obtained as functions of N and the horizontal wave number, α. From (3.3) we obtain the corresponding Nusselt number. Fig. 2 shows the maximum Nusselt number (with respect to the wave number) as a function of the Rayleigh number for various values of N. We observe that Nu converges very rapidly with increasing N. It is sufficient to take $N = 2$ for $Ra < 2Ra^c$, which gives 3 equations to solve. At $Ra \simeq 8Ra^c$ it is enough to take $N = 12$ giving 42 equations.

In Fig. 3 the Nusselt number is given as a function of the wave number for different values of Ra.

We observe that the wave number corresponding to maximum heat transport varies considerably. For Ra slightly greater than Ra^c the maximum heat transport occurs for $\alpha \simeq \pi$, while for $Ra = 7.5\ Ra^c$ the corresponding wave number is $\alpha \simeq 2.5\ \pi$.

In Fig. 4 the Nusselt number is displaced as a function of the Rayleigh number. Comparisons are also made with some experimental results obtained by various investigations. For large Rayleigh numbers the present results appear to be somewhat in the lower part of the range of experiments.

Fig. 5 shows the mean temperature profiles for different values of the Rayleigh number. Each profile corresponds to that particular wave number which maximizes the heat transport. We note the development of thermal boundary layers when the Rayleigh number is sufficiently high. It may happen that the boundary layer thickness becomes comparable with the grain diameter. In this case Darcy law will probably be invalid.

5. Stability of the finite amplitude solution.

In this section we discuss the stability of the finite two-dimensional solution obtained in chapter 4. Let $\tilde{\psi}$ be an arbitrary three-dimensional perturbation on the steady state solution. Then, by replacing ψ by $\psi + \tilde{\psi}$ in (2.13), omitting terms quadratic in $\tilde{\psi}$, we obtain the perturbation equation :

$$- \frac{\partial}{\partial t} \nabla^2\tilde{\psi} + \nabla^4\tilde{\psi} + Ra\nabla_1{}^2\tilde{\psi} = \nabla x(\nabla x\vec{k}\tilde{\psi})\cdot\nabla\nabla^2\psi + \nabla x(\nabla x\vec{k}\psi)\cdot\nabla\nabla^2\tilde{\psi} \qquad (5.1)$$

Here $\tilde{\psi}$ must satisfy the same boundary conditions as ψ. For given values of Ra and α, the steady state solution is said to be unstable if there exists a disturbance having positive growth rate. If not the motion is stable. Although equation (5.1) is linear, the analysis is complicated by the fact that arbitrary three-dimensional disturbances must be considered.

Since equation (5.1) has constant coefficients with respect to y and t, and the dependence of x is periodic, the solution can be written as an infinite series in the following way,(Busse 1967)

$$\tilde{\psi} = \sum_{p,q} \tilde{a}_{pq}\, e^{ip\alpha x}\, e^{i(dx+by)} \sin q\pi z\, e^{\sigma t} \qquad (5.2)$$

The linear equations governing the amplitudes $\check{a}_{pq}$ are obtained in the same way as used in section 2. It is usual to take into account only those terms with $|p| + q \leq K$, where $K = N$ (Busse 1967). Strauss (1974), however, has truncated the series so that $K = N + 1$. In the present paper we have determined the stability regions both for $K = N$ and $K = N + 1$. The computations show that when N is sufficiently large there is no significant difference in the results.

The perturbation equations define an eigenvalue problem in $\sigma = \sigma(Ra, \alpha, b, d)$. If for given Ra and α at least one of the eigenvalues has a positive real part for some value of b and d, the system is unstable. Accordingly we only look for the eigenvalue with the largest real part, and find its maximum value, σ_{max}, as a function of b and d.

We know that near $Ra = Ra^c$ there is a region of stable rolls with real growth rate. (Kvernvold 1972). Further, Clever & Busse (1974), show that for Bénard convection, oscillatory instability can occur only if the vertical component of vorticity is different from zero. Since $(\nabla x \vec{v}) \cdot \vec{k} = 0$ for porous convection, σ will be real in this case.

Because of the symmetry of the steady state solution, the stability analysis can be further simplified. Firstly, the amplitudes with even and odd $|p| + q$ separate, and secondly, each of these two systems separate into solutions with symmetric and antisymmetric x-dependence. Still it is quite a lot of numerical work left, because for each α and Ra, both b and d in (5.2) must be varied. Fortunately, the maximum growth rate always occurs for $d = 0$. In most of the Ra, α-plane the cross-roll instability, i.e. $d = 0$, $|p| + q$ odd and symmetric x-dependence is the most critical one. But for $\alpha < \alpha_c$ and small overcritical Rayleigh numbers, the zig-zag instability is the most important. In mathematical sense the zig-zag instability is formed out of disturbances with $d = 0$, $|p| + q$ even and antisymmetric x-dependence.

Fig. 6 shows the results of the numerical stability calculations. Values of Ra and α inside the closed curve correspond to negative values of σ_{max}; which means stable stationary two-dimensional motion. Outside this curve the solution is unstable. The Rayleigh number corresponding to the upper limit, the second critical Rayleigh number, is found to be $Ra_2^c \simeq 335$. This value is about 45 less than that obtained by Strauss (1974). We find no reason for this discrepancy. The main result is, however, that there exists a second critical Rayleigh number above which no steady two-dimensional motion exists.

We observe that for Rayleigh numbers slightly greater than Ra^c steady two-dimensional motion exists for $\alpha \simeq \pi$. At $Ra = 5Ra^c$, for example, steady motion is possible for wave number in the region $\alpha \simeq \pi$ to $\alpha \simeq 2.2\pi$. At the second critical Rayleigh number the corresponding wave number is $\alpha \simeq 2.35\pi$. From this we would expect that the wave length decreases with increasing Rayleigh numbers. This result is qualitatively in agreement with that obtained for ordinary Bénard convection (Busse 1967). However, experimental evidence (Kochschmieder (1969), Busse & Whitehead (1971)) indicates that the wave number is an increasing function of the Rayleigh number. For porous convection similar experiments have not been performed, and one may only speculate about the results.

6. Comparisons and discussions.

In this paper we have studied non-linear convection in a porous medium using Galerkin's method. Two-dimensional finite amplitude solutions have been obtained, and the stability of these solutions with respect to infinitesimal three-dimensional perturbations has been examined.

A comparison between the Nusselt number obtained in the present work and experimental results found by various investigators (Schneider (1963), Elder (1967), Buretta (1972) and Bories & Combarnous (1973)) show good agreement.

The stability analysis shows that for given Rayleigh number larger than the critical value and less than a certain number (the second critical Rayleigh number) there is a possibility for stable two-dimensional motion for a range of wave numbers. The second critical Rayleigh number is given by $Ra_2^c \simeq 335$. For $Ra > Ra_2^c$ the motion will probably be three-dimensional. The most stable wave number is found to increase with increasing Ra, varying from π for $Ra \simeq Ra^c$ to 2.35π for $Ra = Ra_2^c$.

It is worth pointing out, however, that for a given Ra the Nusselt number is practically constant for all wave numbers in the stable region.

The equations and the boundary conditions governing the motion in a porous medium is analogous to those for convection in a fluid layer in the limit of infinite Prandtl number and stress free boundaries. Formally the only difference is that the resistance term is $\frac{\nu}{k}\vec{v}$ for porous convection and $\nu\nabla^2\vec{v}$ for Bénard convection. Surprisingly enough the stability regions for

for these two types of convection are quite different. The computation by Strauss (1972) indicates that for Bénard convection with free boundaries and infinite Prandtl number there is no upper bound for stable two-dimensional motion. For Bénard convection with rigid boundaries, however, an upper bound for the stability of two-dimensional motion is found, Busse (1967).

Strauss (1974) has by a numerical method very similar to that used in the present analysis, discussed the steady motion and the stability conditions for convection in a porous medium. His values for the Nusselt number are in excellent agreement to the present results. The stability region is, however, a little different. Strauss obtains a second critical Rayleigh number of $Ra_2^c \simeq 380$ in contrast to the value 335 of the present study. The reason for this discrepancy is not clear.

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Figure legends :

Fig. 1 Illustration of the trend towards a steady state solution for the amplitudes A_1 and A_{10}.

Fig. 2 Nu vs. Ra/Ra^c for different values of N.

Fig. 3 Nu vs. α^2/α^2_c for different values of N.
- - - maximum value of the Nusselt number.

Fig. 4 Comparison of the Nusselt number with experimental data.
——— present analysis.
Experimental values obtained by Schneider (1963), Elder (1967), Buretta (1972) and Bories & Combarnous (1973) are indicated by the shaded area.

Fig. 5 Mean temperature profiles for different values of Ra/Ra^c.

Fig. 6 Region of stable rolls.
- - - marginal stability.

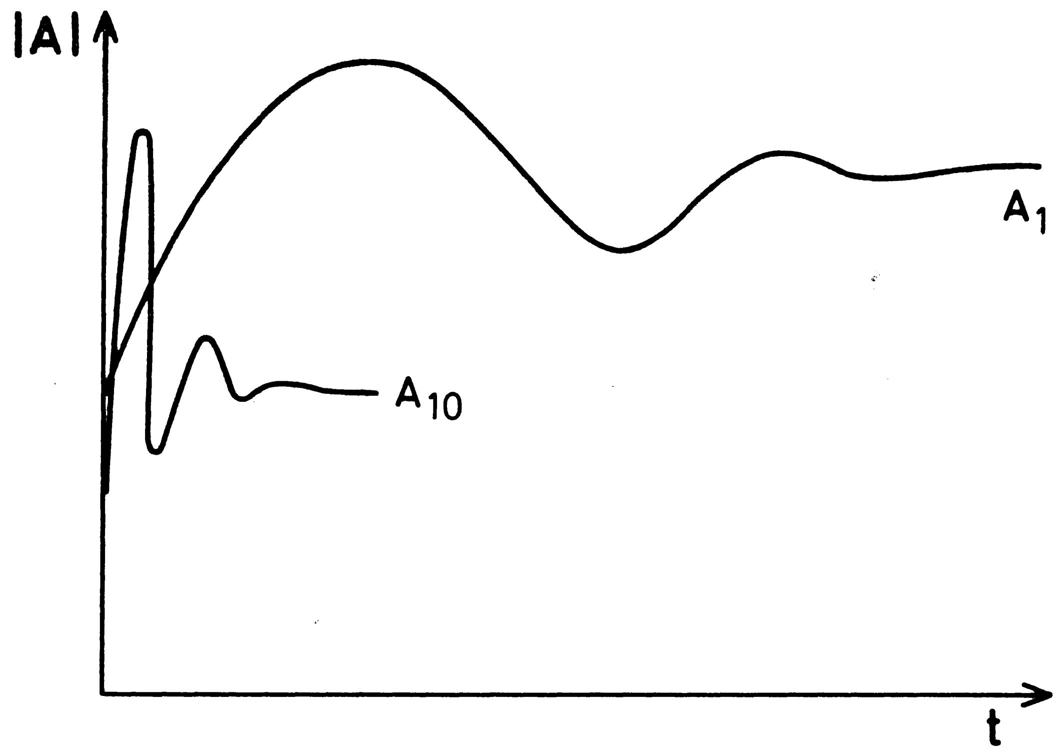

Fig. 1

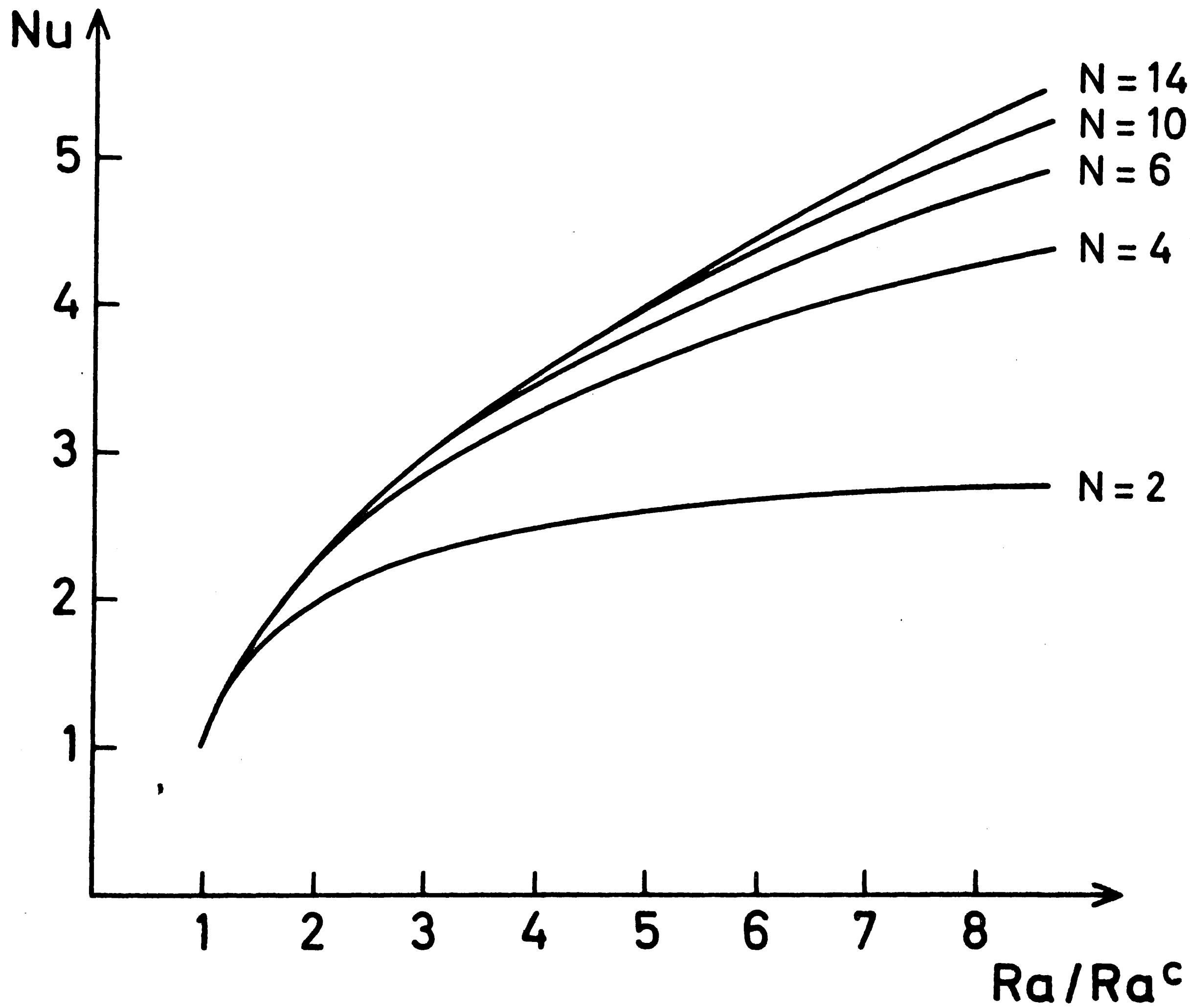

Fig. 2

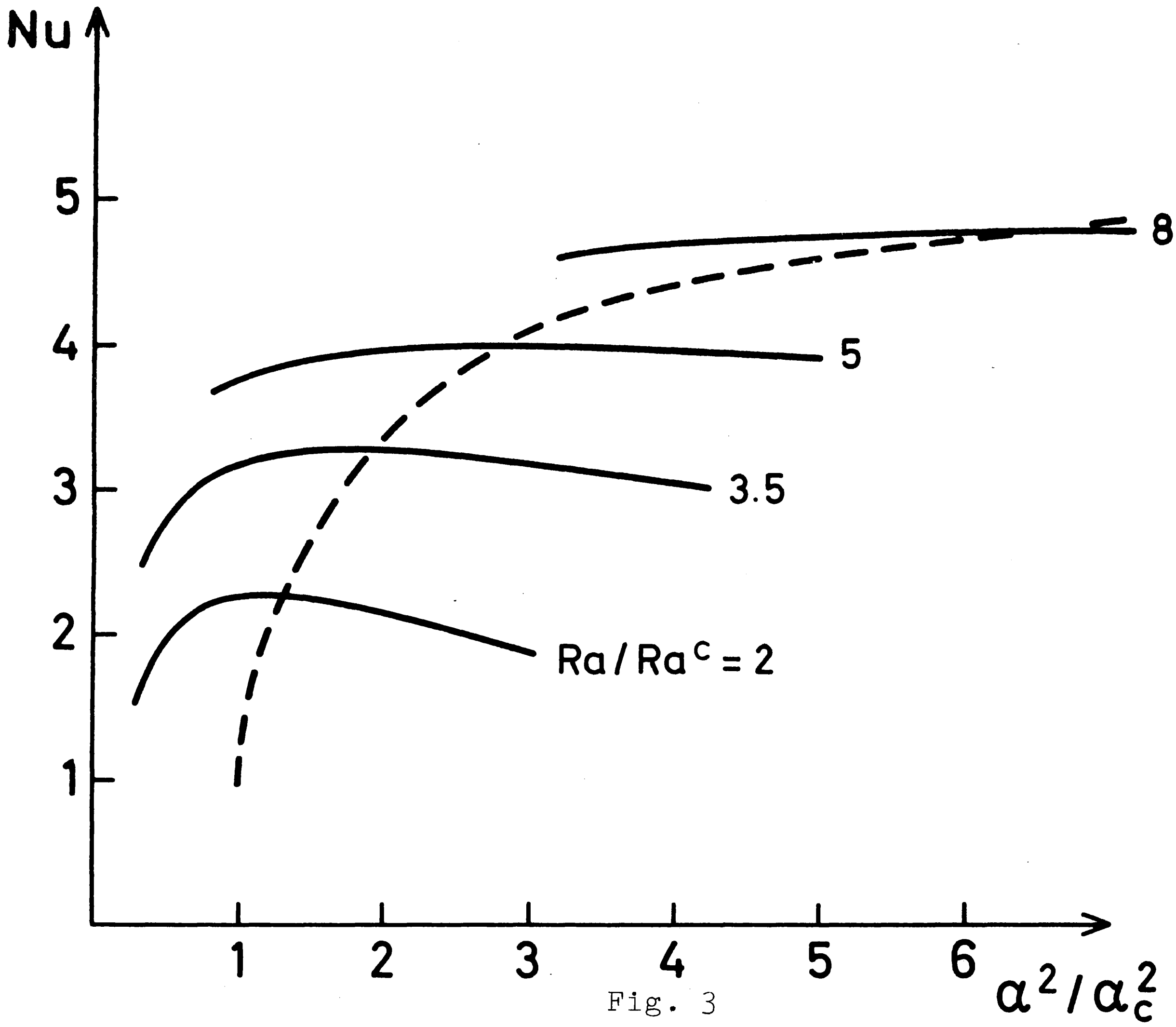
Nu
5
4
3
2
1
8
5
3.5
Ra / Ra^c = 2
1
2
3
4
5
6
α²/α²c

Fig. 3

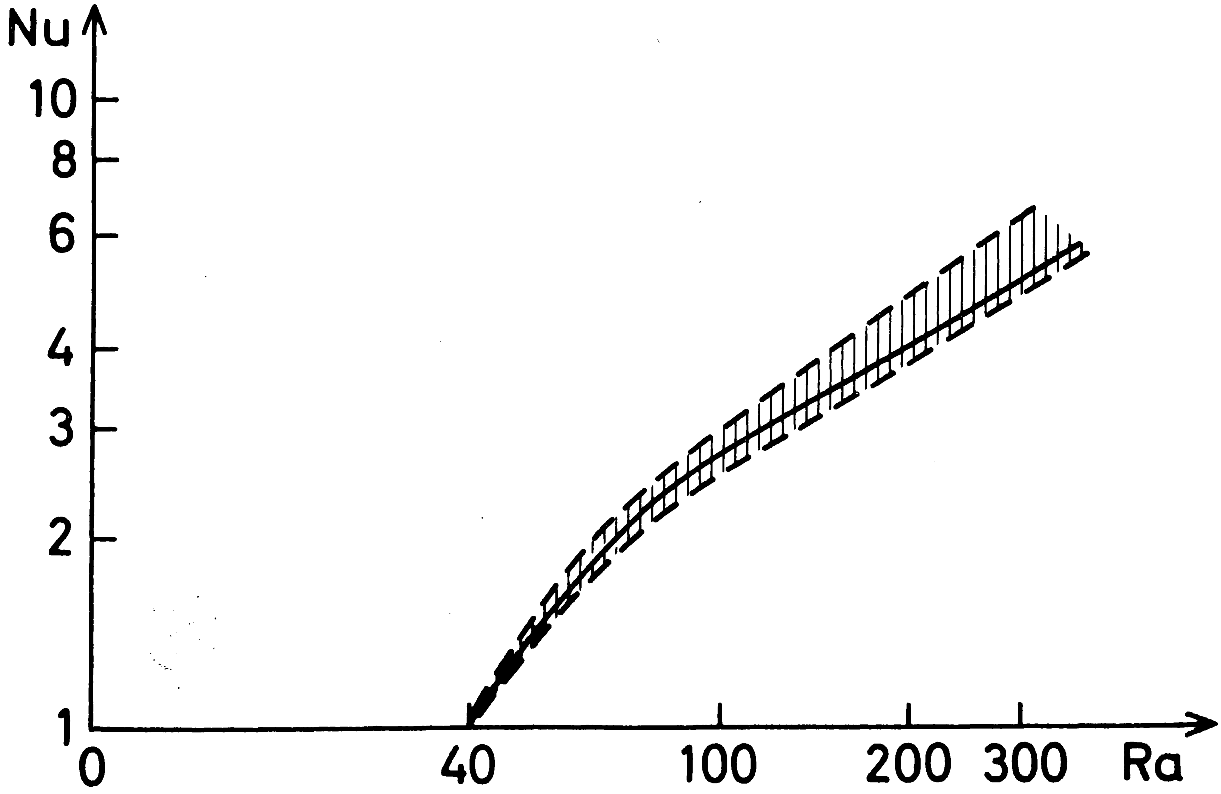
Nu
10
8
6
4
3
2
1
0
40
100
200
300
Ra

Fig. 4

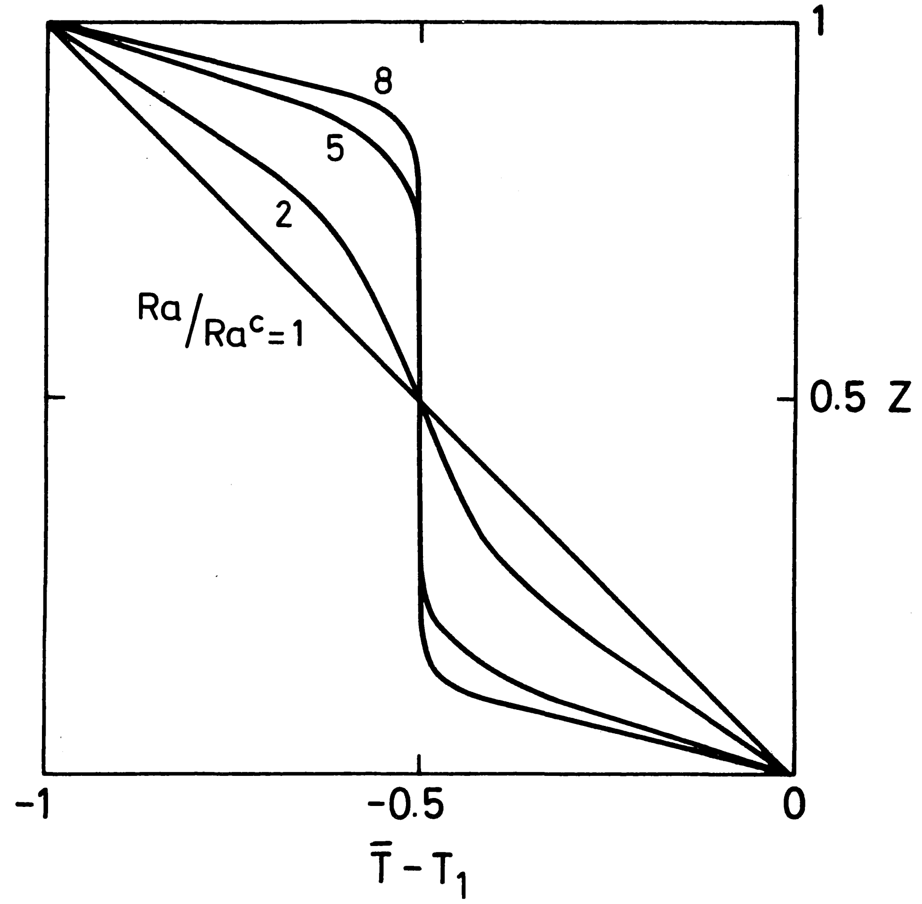

Fig. 5

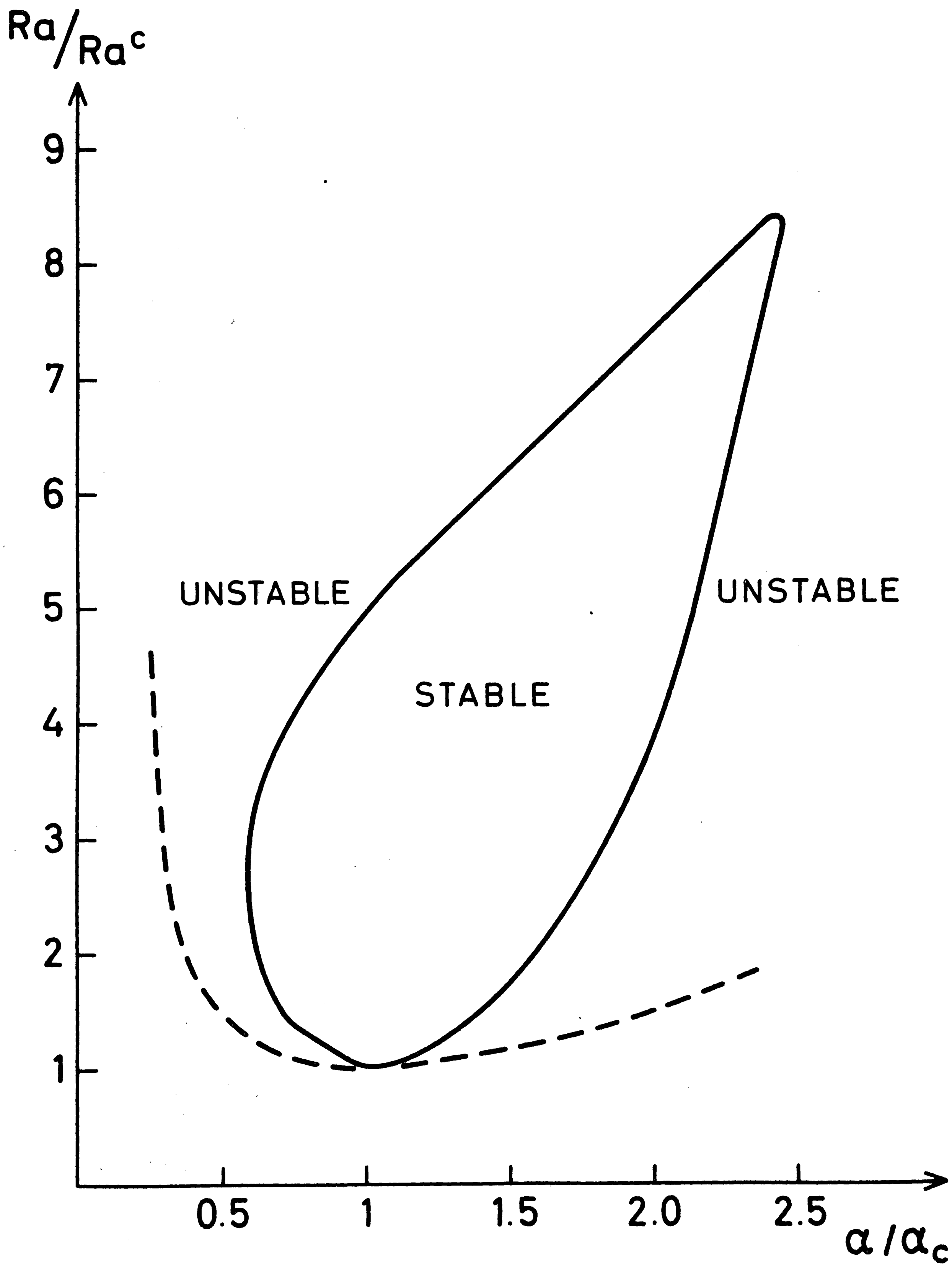

Fig. 6