Interaction of Progressing Waves in Solutions of $U_{xx} - U_{tt} + V'(U) = 0$

by

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Introduction.

The Klein-Gordon equation $U_{xx} - U_{tt} + V'(U) = 0$ has solutions in the form of progressing waves. Here I describe a technique to deal with the case of two such waves superimposed, in which each wave is fully nonlinear, and a perturbation expansion is made which assumes that the interaction between them is less strong. The technique is based upon that used in Grant (1972,73), and the particular problem was inspired by Ablowitz (1972).

1. Linearized Solutions.

Taking $V'(U) \sim U$ for U small, the linearized equation is

$$U_{xx} - U_{tt} + U = 0 \tag{1}$$

with the solution $U = \varepsilon \cos(kx-\omega t)$

$k^2 = \omega^2 + 1$ the dispersion relation

or $U = \varepsilon \cos \varphi$ $\varphi = kx - \omega t$.

2. Two Waves.

To study the interaction of two waves, with wave numbers and frequencies (ω_1, k_1) and (ω_2, k_2) respectively, it is best to transform to the phase coordinates.

Let $\varphi = k_1 x - \omega_1 t$

$\theta = k_2 x - \omega_2 t$

The equation is now

$$g_1 U_{\varphi\varphi} + g_2 U_{\theta\theta} + 2\lambda U_{\theta\varphi} + V'(U) = 0 \tag{2}$$

$$g_1 = k_1^2 - \omega_1^2$$

$$g_2 = k_2^2 - \omega_2^2$$

$$\lambda = k_1 k_2 - \omega_1 \omega_2$$

Once again, there is the linearized solution

$$U = \alpha \cos\varphi + \beta \cos\theta$$

with the dispersion relation:

$$g_1 = g_2 = 1$$

This solution can be iterated upon generating an expansion in powers of α and β:

$$\begin{aligned} U = {} & \alpha \cos\varphi + \beta \cos\theta \\ & +\alpha\beta(\cos\varphi\cos\theta, \sin\varphi\sin\theta) \\ & +\alpha^2(\cos 2\varphi, 1) + \beta^2(\cos 2\theta, 1) + \alpha^2\beta(\quad) + \alpha\beta^2 \\ & \qquad + \alpha^3(\quad) + \beta^3(\quad) \\ & + \cdots \end{aligned} \tag{3}$$

$$g_1 = 1 + c_1\alpha^2 + c_1{}^1\beta^2 + \cdots$$
$$g_2 = 1 + c_1{}^1\alpha^2 + c_1\beta^2 + \cdots$$

($\alpha\beta(\cos\varphi\cos\theta, \sin\varphi\sin\theta)$ means $\alpha\beta \times$ some linear combination of $\cos\varphi\cos\theta$ & $\sin\varphi\sin\theta$).

All coefficients also depend on λ.

3. First, consider the problem of only one wave, say $\beta = 0$. Then

$$U = \alpha\cos\varphi + \alpha^2(\cos^2\varphi, 1) + \alpha^3(\cos^3\varphi, \cos\varphi) + O(\alpha^4) \tag{4}$$

The terms at each order in α can just as well be written as a polynomial in $\cos\varphi$ as in harmonics of φ.

Since U is a function of φ only, the equation can in fact be integrated:

$$\begin{aligned} & g_1 U_{\varphi\varphi} + V'(U) = 0 \\ & \tfrac{1}{2} g_1 U_\varphi{}^2 + V(U) = E(\alpha) \qquad \text{etc.} \end{aligned} \tag{5}$$

This implies that the expansion (4) is known

$$U = F(\alpha\cos\varphi;\alpha^2) \tag{6}$$

The expansion (4) has, at order α^n, a polynomial of degree n in $\cos\varphi$. It can be enlightening to restrict attention to those terms that are most rapidly varying at each order - the highest power of $\cos\varphi$:

$$U_o \sim \alpha\cos\varphi + \alpha^2(\cos^2\varphi) + \alpha^3(\cos^3\varphi) \quad = \mathcal{f}(\alpha\cos\varphi) \tag{7}$$

(in fact, $\mathcal{f}(\alpha\cos\varphi) = F(\alpha\cos\varphi;0)$)

These terms depend only upon each other, and are independent of the omitted terms in the expansion. They consequently form the basis for an expansion of the solution F, whose first approximation $\mathcal{f}$ is nonlinear. This follows the reasoning used on water waves.

$\mathcal{f}$ depends on α & $\cos\varphi$ only in the single combination $\mu = \alpha\cos\varphi$. The equation for U_o can be found by making this change of variables :

$$U \sim U_o = U_o(\mu) = \mathcal{f}(\mu)$$

$$U_\theta \quad = -\alpha\sin\varphi\, \mathcal{f}'$$

$$U_{\theta\theta} \quad = -\alpha\cos\varphi\, \mathcal{f}'' + \alpha^2\sin^2\varphi\, \mathcal{f}''$$

$$g_1[\alpha^2\sin^2\varphi\, \mathcal{f}'' - \alpha\cos\varphi\, \mathcal{f}'] + V(\mathcal{f}) = 0$$

but

$$\alpha^2\sin^2\varphi = \alpha^2 - \alpha^2\cos^2\varphi = \alpha^2 - \mu^2 \tag{8}$$

which, to highest order (i.e. $\alpha \to 0$ with μ fixed)

$$\sim -\mu^2 \tag{9}$$

also $\quad g_1 = 1$

$$-\mu^2 f'' - \mu f' + V(f) = 0 \tag{10}$$

This is the equation for the solution (4).

Here it is easy to recover the full solution, simply by retaining $\alpha^2 - \mu^2$ instead of $-\mu^2$ in the coefficient of U'' :

$$g_1[(\alpha^2 - \mu^2)U'' - \mu U'] + V(U) = 0 \tag{11}$$

$g_1(\alpha^2)$ is determined by the condition that the solution be analytic in μ as $\mu \to 0$ (i.e. periodic in φ)

(10) is the equation for $f(\mu) = F(\mu;0)$

(11) is the equation for $F(\mu;\alpha^2)$

An iteration based upon f would proceed:

$$U = f + \alpha^2 f_1 + \alpha^4 f_2 \cdots$$

with all the f_i being functions of μ alone. Substitution and collection of powers of α^2 gives equations for the f_i.

It is worthy to note that $f(\mu)$ provides a nonlinear approximation to $F(\mu;\alpha^2)$, since it contains some terms from all orders in the perturbation expansion. However, it is not a wholly satisfactory approximation, and will fail when U becomes singular. For the coefficient of U'' in (11) is $\alpha^2 - \mu^2$, which vanishes at $\alpha = \mu$. (I assume a singularity first accurs at maximum μ); and in (10) it is $-\mu^2$, which does not vanish. (Apart from the awkward fact of estimating the positive definite $\alpha^2 - \mu^2$ by the negative definite $-\mu^2$). So (10) will get the order of the singularity wrong. So much for one wave.

4. Two interacting waves.

$\alpha \neq 0, \quad \beta \neq 0.$

$$U = \alpha\cos\varphi + \beta\cos\theta + \alpha\beta(\cos\varphi\cos\theta, \sin\varphi\sin\theta) + $$
$$+ \alpha^2(\cos^2\varphi, 1) + \beta^2(\cos^2\theta, 1)$$

when $\beta \to 0$

$$U \sim \alpha\cos\varphi + \alpha^2(\cos^2\varphi, 1) = F(\alpha\cos\varphi; \alpha^2) \qquad (12)$$

when $\alpha \to 0$

$$U \sim F(\beta\cos\theta; \beta^2)$$

I define an expansion, whose first term is valid in both limits :

$$U = F(\alpha\cos\varphi + \beta\cos\theta; \varepsilon^2) + O(\alpha\beta)$$
$$\varepsilon^2 = \alpha^2 + \beta^2 + O(\alpha\beta) \qquad (13)$$

This choice is not uniquely determined. Another would be the obvious

$$U = F(\alpha\cos\varphi; \alpha^2) + F(\beta\cos\theta; \beta^2) + O(\alpha\beta) \qquad (14)$$

The advantage of (13) is the convenient form. It involves $\alpha\cos\varphi$ and $\beta\cos\theta$, at lowest order, only in the combination

$$\mu = \alpha\cos\varphi + \beta\cos\theta \qquad (15)$$

This functional form also enables comparison with the expansion technique based upon the most rapidly varying terms. In the case of one wave, it was unnecessary. But now the problem cannot be immediately integrated, and some approximation is needed.

It is also more convenient because forcing terms at higher orders are always of the form: Sum of terms like :

$$\cos^n\theta\cos^m\varphi F_1(\mu)F_2(\mu)$$

$$\sin\theta\sin\varphi\cos^n\theta\cos^n\varphi F_1(\mu)\, F_2(\mu)$$

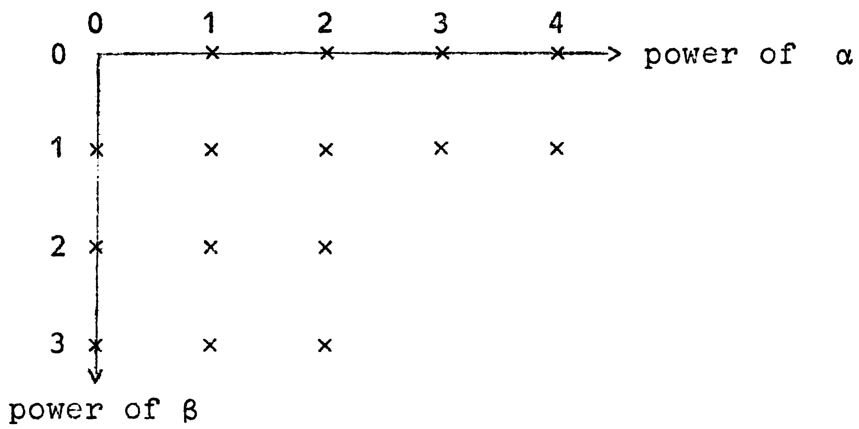
and hence generate terms of a similar form. Whereas (14) will generate forcing terms like:

$$\cos^n\theta\cos^m\varphi F_1(\alpha\cos\varphi)F_2(\beta\cos\theta)$$

and it will not be so easy to solve for the functions so forced.

The expansion based on this will, at all orders, discard terms that are "more mixed" than the dominant ones. "More mixed" means more cross products of terms from the two waves - i.e. $\alpha^2\beta\cos^2\varphi\cos\theta$ is more mixed than $\alpha^3\cos^3\varphi$ or $\beta^3\cos^3\varphi$. Thus, at lowest order, I have correct only those terms in the expansion (3) that involve only α or only β. The next iteration includes terms of the form $\alpha^n\beta$ and $\alpha\beta^n$ - one order of interaction between the two waves. And so on. The expansion is essentially an expansion in powers of the interaction between the two waves, each wave being itself fully nonlinear.

Another way of looking at it is to array the terms of (3) in a chart of (power of α) vs. (power of β):



(16)

The complete solution is the sum of all these terms. If I assume β is small, I can iterate upon a nonlinear α-wave :

$$U = F(\alpha\cos\varphi;\alpha^2)+\beta\{\cos\theta\overline{F}_1(\alpha\cos\varphi;\alpha^2) + \sin\theta\alpha\sin\varphi\overline{F}_2(\alpha\cos\varphi;\alpha^2)\}+ O(\beta^2) \quad (17)$$

This is an iteration, whose first term collects all the terms in the first row of (16), and then iterates,collecting row after row. Likewise, the corresponding β-approximation

$$U = F(\beta\cos\theta;\beta^2) + \alpha\{\cos\varphi\overline{F}_1(\beta\cos\theta;\beta^2 + \sin\varphi\beta\sin\theta\overline{F}_2(\beta\cos\theta;\beta^2) + O(\alpha^2) \qquad (18)$$

iterates column by column.

The form (14), at its first approximation, has correct both the first row and the first column, and moves inwards with higher order. 'Higher order' means higher order of interaction, of the two waves with each other. It is not true, for example, that at n^{th} order all terms from the first n rows and columns have been collected. For example a term of $(\alpha^2\beta)$ occurs both at second and third order :

$$U = F(\mu) + \alpha\beta\{\cos\theta\cos\varphi F_1 + \sin\theta\sin\varphi F_2\} + (\alpha\beta)^2 [\cos^2\theta\cos^2\varphi F_3 + \cos\theta\cos\theta\sin\theta\sin\varphi F_4] + (\alpha^2\beta\cos\theta + \beta^2\alpha\cos\varphi)F_5 + \cdots$$

$F(\mu)$	is first order
F_1 & F_2	are second order
F_3, F_4, F_5	are third order

A term of order $(\alpha^2\beta)$ can occur in F_1, F_2 and F_5. But only at F_5 is any correction made to $g^{(1)}$. Correction to the dispersion relation occur only at every second order, as they do in the expansions (17) and (18), because they require a wave to interact with something else, and then back again to itself. So terms of order $(\alpha^2\beta)$ in F_1 and F_2 represent, say an (α^2) term from

F interaction with a (β) term from F. An $(\alpha^2\beta)$ term in F_5 is generated $(\alpha) \to (\alpha\beta) \to (\alpha^2\beta)$. This is the last time such a term can arise in the expansion. At order p, terms of order $\alpha^n\beta^m$ are included with $n+m \geq p$.

First, for comparison, I do the expansion (17)

$$U = F_0(\alpha\cos\varphi) + \beta[\cos\theta\bar{F}_1(\alpha\cos\varphi) + \sin\theta\alpha\sin\varphi\bar{F}_2(\alpha\cos\varphi)] + \cdots$$

$$= F_0(\mu) + \beta\cos\theta\bar{F}_1(\mu) + \alpha\beta\sin\theta\sin\varphi\bar{F}(\mu) . \qquad \mu = \alpha\cos\varphi$$

$$U_\varphi = -\alpha\sin\varphi(F_0' + \beta\cos\theta\bar{F}_1') - \beta\alpha^2\sin^2\varphi\sin\theta\bar{F}_2' + \beta\alpha\sin\theta\cos\varphi F_2$$

$$U_{\varphi\varphi} = -\alpha\cos\varphi(F_0' + \beta\cos\theta\bar{F}_1') + \alpha^2\sin^2\varphi(F_0'' + \beta\cos\theta\bar{F}_1'')$$
$$- 2\beta\alpha^2\sin\varphi\cos\varphi\sin\theta\bar{F}_2' - \beta\alpha\sin\theta\sin\varphi\bar{F}_2$$
$$+\beta\alpha^3\sin^3\varphi\sin\theta\bar{F}_2'' - \beta\alpha^2\sin\theta\cos\varphi\sin\varphi\bar{F}_2'$$
$$= -\mu(F_0' + \beta\cos\theta\bar{F}_1') + (\alpha^2-\mu^2)(F_0'' + \beta\cos\theta\bar{F}_1'')$$
$$- 3\beta\alpha\sin\theta\sin\varphi\mu\bar{F}_2' - \beta\alpha\sin\theta\sin\varphi\bar{F}_2 + \beta\alpha\sin\theta\sin\varphi(\alpha^2-\mu^2)\bar{F}_2''$$

$$U_{\theta\theta} = -\beta\cos\theta\bar{F}_1 - \beta\sin\theta\alpha\sin\varphi\bar{F}_2$$

$$U_{\theta\varphi} = -\alpha\sin\varphi(-\beta\sin\theta\bar{F}_1') - \beta(\alpha^2-\mu^2)\cos\theta\bar{F}_2' + \beta\alpha\cos\theta\cos\varphi\bar{F}_2$$

also

$$g_1 = g_1^{(o)}(\alpha^2) + O(\beta^2)$$

$$g_2 = g_2^{(o)}(\alpha^2) + O(\beta^2)$$

$$g_1U_{\varphi\varphi} + g_2U_{\theta\theta} + 2\lambda U_{\theta\varphi} + V'(U) = 0 \qquad \text{becomes}$$

$$g^{(o)}\{-\mu F_0' + (\alpha^2-\mu^2)F_0'' + \beta[-\mu\cos\theta\bar{F}_1' + (\alpha^2-\mu^2)\cos\theta\bar{F}_1'']$$
$$+ \beta\alpha\sin\theta\sin\varphi[(\alpha^2-\mu^2)\bar{F}_2'' - 3\mu\bar{F}_2' - F_2]\}$$
$$+ g_2^{(o)}\{-\beta\cos\theta\bar{F}_1 - \beta\alpha\sin\theta\sin p\,\bar{F}_2\}$$
$$+ 2\lambda\{-\beta(\alpha^2-\mu^2)\cos\theta\bar{F}_2' + \beta\cos\theta\mu\bar{F}_2 + \alpha\beta\sin\theta\sin\varphi\bar{F}_1'\}$$
$$+ V'(F_0) + V''(F_0)\{\beta\cos\theta\bar{F}_1 + \alpha\beta\sin\theta\sin\varphi\bar{F}_2\} = 0$$

at $O(\beta^0)$: $$\boxed{g_1^{(o)}\{-\mu F_o' + (\alpha^2-\mu^2)F_o''\} + V'(F_o) = 0}$$

- as usual.

at $O(\beta\cos\theta)$:

$$g_1^{(o)}[-\mu\overline{F}_1' + (\alpha^2-\mu^2)\overline{F}_1''] - g_2^{(o)}\overline{F}_1 + 2\lambda[-(\alpha^2-\mu^2)\overline{F}_2' + \mu\overline{F}_2] + \overline{F}_1 V''(F_o) = 0 \qquad (19)$$

at $O(\beta\sin\theta \circ \alpha\sin\varphi)$

$$g_1^{(o)}[(\alpha^2-\mu^2)\overline{F}_2'' - 3\mu\overline{F}_2' - \overline{F}_2] - g_2^{(o)}\overline{F}_2 + 2\lambda\overline{F}_1' + \overline{F}_2 V''(F_o) = 0 \qquad (20)$$

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And now for the next: with two waves. The method mimics the preceeding cases, and the rules for collecting terms, or discarding them to higher orders are thus :

Any term that is more mixed belongs at a higher order.

Any term that is more slowly varying belongs at a higher order - except that I attempt to retain some terms, that can easily be incorporated, as is the one wave case - retaining α^2 in $\alpha^2-\mu^2$. This is a purely ad hoc improvement.

$$U = F_o(\alpha\cos\varphi + \beta\cos\theta;\varepsilon^2) + \alpha\beta[\cos\theta\cos\varphi F_1(\mu;\varepsilon^2) + \sin\theta\sin\varphi F_2(\mu;\varepsilon^2)] + \cdots \qquad (21)$$

$\mu = \alpha\cos\varphi + \beta\cos\theta$

ε^2 to be suitably chosen

To lowest order :

$U \sim F_o(\mu)$.

$$
\begin{aligned}
U_{\varphi} &\sim -\alpha\sin\varphi F_o' \\
U_{\varphi\varphi} &\sim -\alpha\cos\varphi F_o' + \alpha^2\sin^2\varphi F_o'' \\
U_{\varphi\theta} &\sim \alpha\beta\sin\varphi\sin\theta F_o'' \\
U_{\theta\theta} &\sim -\beta\cos\theta F_o' + \beta^2\sin^2\theta F_o'' \\
g_1 &= g^{(o)}(\varepsilon^2) + \beta^2 g^{(1)}(\varepsilon^2) \\
g_2 &= g^{(o)}(\varepsilon^2) + \alpha^2 g^{(1)}(\varepsilon^2)
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
g^{(o)}[(\alpha^2\sin^2\varphi + \beta^2\sin^2\theta)F_o'' - (\alpha\cos\varphi + \beta\cos\theta)F_o'] \\
+ 2\lambda\alpha\beta\sin\varphi\sin\theta F_o'' + V'(F_o) = 0
\end{aligned} \tag{23}
$$

A) $2\lambda\alpha\beta\sin\varphi\sin\theta F_o''$ is mixed - an $O(\alpha\beta)$ term - out it goes

B) $\alpha^2\sin^2\varphi + \beta^2\sin^2\theta = \alpha^2+\beta^2-\alpha^2\cos^2\varphi - \beta'\cos^2\theta$

$= \alpha^2+\beta^2-(\alpha\cos\varphi+\beta\cos\theta)^2 + O(\alpha\beta)$

Let $\varepsilon^2 = \alpha^2+\beta^2$, & there results :

$$
g^{(o)}[(\varepsilon^2-\mu^2)F_o'' - \mu F_o'] + V'(F_o) = 0 \tag{24}
$$

familar equation with a familar solution.

The choice of ε^2 is correct only to order $(\alpha\beta)$ - i.e. correct to all orders of α & β alone. Another choice could be

$\varepsilon^2 = (\alpha+\beta)^2$ (if $\alpha > 0$, $\beta > 0$ is prescribed)

This can be more reasonable. For example, for a standing wave, where $\alpha = \beta$, it turns out to be required. This is because $\mu_{max} = |\alpha| + |\beta|$, and it is convenient to have $\varepsilon = \mu_{max}$, so that $\varepsilon^2 - \mu^2$ vanishes at μ_{max} - as does the original expression $(\alpha^2\sin^2\varphi + \beta^2\sin^2\theta)$.

Carried to $O(\alpha\beta)$ from (24) is

$$2\lambda\alpha\beta\sin\varphi\sin\theta F_o'' + 2g^{(o)}\alpha\beta\cos\varphi\cos\theta F_o'' \tag{25}$$

At order $\alpha\beta$:

$$U \sim \alpha\beta[\cos\theta\cos\varphi F_1 + \sin\theta\sin\varphi F_2]$$

$$U_\theta \sim \alpha\beta[-\sin\theta\cos\varphi F_1 + \cos\theta\sin\varphi F_2 - \beta\sin\theta\cos\theta\cos\varphi F_1' - \beta\sin^2\theta\sin\varphi F_2']$$

$$U_{\theta\theta} \sim \alpha\beta[-\cos\theta\cos\varphi F_1 - \sin\theta\sin\varphi F_2 - \beta(2\cos^2\theta-1)\cos\varphi)F_1' - 2\beta\cos\theta\sin\theta\sin\varphi F_2' + \beta\sin^2\theta\cos\varphi F_1' - \beta\cos\theta\sin\theta\sin\varphi F_2' + \beta^2\sin^2\theta\cos\theta\cos\varphi F_1'' + \beta^2\sin^3\theta\sin\varphi F_2'']$$

$$= \alpha\beta\{\cos\theta\cos\varphi[-F_1 - 2\beta\cos\theta F_1' - \beta\cos\theta F_1' - \beta^2\cos^2\theta F_1'' + \beta^2 F_1''] + \sin\theta\sin\varphi[-F_2 - 3\beta\cos\theta F_2' + \beta^2 F_2'' - \beta^2\cos^2\theta F_2''] - \beta\cos\varphi F_1' - \beta\cos\varphi F_1'\}$$

The last two terms, $-\alpha\beta[-2\beta\cos\varphi F_1']$ belong at a higher order, since they are more slowly varying (in θ) than the retained terms. The terms $\beta^2 F_1''$, which are also higher order, are retained because the corresponding terms are retained at lowest order.

$$U_{\varphi\varphi} \sim \alpha\beta\{\cos\theta\cos\varphi[-F_1 - 3\alpha\cos\varphi F_1' + \alpha^2 F_1'' - \alpha^2\cos^2\varphi F_1''] + \sin\theta\sin\varphi[-F_2 - 3\alpha\cos\varphi F_2' + \alpha^2 F_2'' - \alpha^2\cos^2\varphi F_2''] + \cdots\}$$

$$U_{\theta\varphi} \sim \alpha\beta\{+\sin\theta\sin\varphi F_1+\cos\theta\cos\varphi F_2+\beta\sin\theta\cos\theta\sin\varphi F_1'-\beta\sin^2\theta\cos\varphi F_2'$$
$$+\alpha\sin\varphi\sin\theta\cos\varphi F_1'-\alpha\sin^2\theta\cos\theta F_2'+\alpha\beta\sin\theta\cos\theta\sin\varphi\cos\varphi F_1''$$
$$+\alpha\beta\sin^2\theta\sin^2\varphi F_2''\}$$

$$= \alpha\beta\{\sin\theta\sin\varphi[F_1+\beta\cos\theta F_1'+\alpha\cos\varphi F_1']$$
$$+\cos\theta+\cos\varphi[F_2+\beta\cos\theta F_2'+\alpha\cos\varphi F_2']$$
$$+\sin\theta\sin\varphi\cdot\alpha\beta\cos\theta\cos\varphi F_2''+\alpha\beta\cos^2\theta\cos^2\varphi F_2''$$
$$-\beta\cos\varphi F_2'-\alpha\cos\theta F_2'+\alpha\beta(1-\cos^2\theta-\cos^2\varphi)F_2''\}$$

and the third & fourth lines contain terms belonging at higher order. Collecting all the $O(\alpha\beta)$ terms gives:

$$0 = \cos\theta\cos\varphi\{g^{(o)}[-F_1-3\beta\cos\theta F_1'+\beta^2F_1''-\beta^2\cos^2\theta F_1'']$$
$$+g^{(o)}[-F_1-3\alpha\cos\varphi F_1'+\alpha^2F_1''-\alpha^2\cos^2\varphi F_1'']]$$
$$+2\lambda[F_2+(\alpha\cos\varphi+\beta\cos\theta)F_2']+V'(F_0)F_1\}$$

$$+\sin\theta\sin\varphi\{g^{(o)}[-F_2-3\beta\cos\theta F_2'+\beta^2F_2''-\beta^2\cos^2\theta F_2'']$$
$$+g^{(o)}[-F_2-3\alpha\cos\varphi F_2'+\alpha^2F_2''-\alpha^2\cos^2\varphi F_2'']$$
$$+2\lambda[F_1+\mu F_1']+V'(F_0)F_2\}$$
$$+2g^{(o)}\cos\theta\cos\varphi F''+2\lambda\sin\theta\sin\varphi F_0''$$

& remembering $\alpha^2+\beta^2-\alpha^2\cos^2\varphi-\beta^2\cos^2\theta = \varepsilon^2-\mu^2$ + higher order ,

the equation for F_1 & F_2 are :

$$g^{(o)}[-2F_1-3\mu F_1'+(\varepsilon^2-\mu^2)F_1'']+2\lambda[F_2+\mu F_2']+V'(F_0)F_1 = -2g^{(o)}F_0'' \qquad (26)$$

$$g^{(o)}[-2F_2-3\mu F_2'+(\varepsilon^2-\mu^2)F_2'']+2\lambda(F_1+\mu F_1')+V')F_o)F_2 = -2\lambda F_o'' \qquad (27)$$

and if I define $\quad F_\pm = F_1 \pm F_2$

$$g^{(o)}\{(\varepsilon^2-\mu^2)F_\pm''-3\mu F_\pm'-2F_\pm\}\pm 2\lambda\{F_\pm+\mu F_\pm' +V'(F_o)F_\pm = -2(g^{(o)}\pm\lambda)F_o'' \qquad (28)$$

i.e. F_+ & F_- are independent of each other.

F is fully specified, given the addition condition that it is analytic in both μ and ε when these are small.

The mechanism of this expansion can be regarded as summing the terms of the expansion (16) "in the μ-direction" - each step adds another layer. Comparing with the expansion for the one wave case, it can be seen

A) In finding each $F_i(\mu)$, I am able to retain the ε^2 term as well - i.e. it is better than just a simple "most rapidly varying" approximation.

B) But in terms of the expansion in the orthogonal direction $\Sigma(\alpha\beta)^n \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\ \theta,\varphi\ F_i(\mu)$, it is not possible to retain the analogous terms, and hence it is just a simple most rapidly varying approximation.

Summary.

The exact solution is a function generated by an "improved most rapidly varying approximation" crossed with another such. I approximate by an "improved approximation" crossed with a "simple approximation".

N.B. The expansion is not an expansion in powers of $\alpha\beta$. The next terms are:

$$(\alpha\beta)^2[\cos^2\theta\cos^2\varphi F_3 + \cos\theta\sin\theta\cos\varphi\sin\varphi F_4] \div (\alpha^2\beta\cos\theta + \beta^2\alpha\cos\varphi)F_5$$

all of these are of comparable order, and it is the equation for F_5 that gives the next term $g^{(1)}$ in the expansion of the g_i.

Most unfortunately, Eqn(28), although only an O.D.E., can't be solved in closed form. And since it contains two parameters, λ & ε, it isn't even suitable for numerical computation - doesn't represent an improvement over doing the original eqn.by Fourier series, a la Ablowitz.

This is a great shame, since it would be reasonable to truncate at this point (or the next, in order to get another approximation for the g_i). The basic solution F_0 contains two waves, each described exactly as far as interactions with itself go. The order $\alpha\beta$ term contains the first order interactions. That is, if β is small, we have a nonlinear α-wave with β linearized about it. This gives now the amplitude and phase of the β-wave changes, bec .se of the effect of the "background" - the α-wave. So the results, to order $\alpha\beta$, represent a solution with two nonlinear waves, each with amplitude and phase changing in accordance with the other wave, but with the other wave in its free form for the purposes of computing this interaction. Second-order effects come at next order - e.g. effects on the β-wave caused by what the β-waves changes in the α-waves.

The main advantage here then is the very convenient functional form for expressing the solution.

The disadvantage is the insolubility of (28). (28) is very similar to (19) and (20) - the equations of motion linearized about a nonlinear wave. If one is explicitly soluble, so is the other. So the fact that (28) can't be neatly solved, reflects that the problem of a linearized wave riding on a non-linear one, also cannot be solved.

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