A note on the solution of the two-dimensional slip-boundary layer problem

by

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Abstract

The general homogeneous solution of the differential equation associated with an arbitrary term of the Blasius series expansion of the stream function, is given. This result is used to establish the solution of higher order terms of the Blasius series, than earlier published. A numerical example is also given.

I. Introduction and formulation of the problem

The two-dimensional flow induced in a viscous fluid (kinematic viscosity ν) around a circular cylinder (radius a) by the following slip velocity on the cylinder surface,

$$(1) \qquad v(a,\theta) = V_0 c_0^2 [\theta + \sum_{N=1}^{\infty} a_N \theta^{2N+1}]$$

is considered (r, θ denote two-dimensional polar coordinates V_0 characteristic velocity).

The velocity field (u,v) is related to the stream function Ψ by,

$$(2) \qquad (u,v) = \left(-\frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \frac{\partial \Psi}{\partial r}\right)$$

Introducing the following dimensionless quantities,

$$(3) \qquad \begin{cases} \zeta = c_0\sqrt{R}\,\dfrac{r-a}{a} = c_0\sqrt{R}\,\dfrac{y}{a} \\ \Psi = \dfrac{V_0 a}{\sqrt{R}} c_0\,\psi(\zeta,\theta;R)\,, \quad R = \dfrac{V_0 a}{\nu} \end{cases}$$

into the vorticity equation, it is well known that the following asymptotic expansion of $\psi(\zeta,\theta;R)$ can be carried out,

$$(4) \qquad \psi(\zeta,\theta;R) \underset{\substack{\zeta \text{ fixed} \\ R \to \infty}}{\sim} \psi_0(\zeta,\theta) + \frac{1}{\sqrt{R}}\psi_1(\zeta,\theta) + \cdots$$

This expansion leads to after partial integration and some manipulations are carried out,

$$(5) \qquad \frac{\partial^3 \psi_0}{\partial \zeta^3} = -\frac{\partial \psi_0}{\partial \theta}\frac{\partial^2 \psi_0}{\partial \zeta^2} + \frac{\partial \psi_0}{\partial \zeta}\frac{\partial^2 \psi_0}{\partial \zeta \partial \theta}$$

$$(6) \qquad \frac{\partial^3 \psi_1}{\partial \zeta^3} = -\frac{\partial \psi_0}{\partial \theta}\frac{\partial^2 \psi_1}{\partial \zeta^2} + \frac{\partial \psi_0}{\partial \zeta}\frac{\partial^2 \psi_1}{\partial \zeta \partial \theta} - \frac{\partial \psi_1}{\partial \theta}\frac{\partial^2 \psi_0}{\partial \zeta^2} + \frac{\partial \psi_1}{\partial \zeta}\frac{\partial^2 \psi_0}{\partial \zeta \partial \theta} - \zeta\frac{\partial^3 \psi_0}{\partial \zeta^3}$$

(higher order terms of ψ are not considered) with the boundary conditions,

$$(7)\qquad \begin{cases} \psi_0(0,\theta) = 0 \\ \left[\dfrac{\partial\psi_0}{\partial\zeta}\right]_{\zeta=0} = \theta + \sum\limits_{N=1}^{\infty} a_N\theta^{2N+1} \\ \left[\dfrac{\partial\psi_0}{\partial\zeta}\right]_{\zeta=\infty} = 0 \end{cases}$$

$$(8)\qquad \begin{cases} \psi_1(0,\theta) = 0 \\ \left[\dfrac{\partial\psi_1}{\partial\zeta}\right]_{\zeta=0} = 0 \\ \left[\dfrac{\partial\psi_1}{\partial\zeta}\right]_{\zeta=\infty} = \sum\limits_{N=0}^{\infty} b_N\theta^{2N+1} \end{cases} ,$$

where of course the flow outside the boundary layer must be calculated before $\{b_N\}$ is known $(N = 0,1,2,\cdots)$.

The problem stated by (5, 6, 7 & 8) appears for example in connection with oscillatory boundary layers where the time averaged Reynolds stresses induce the slip velocity (1) on the cylinder (see Stuart 1966). Riley (1965) studied the problem and solved the three first terms in the Blasius series expansions,

$$(9)\qquad \psi_0(\zeta,\theta) = \psi_{0,0}(\zeta)\theta + \sum_{N=1}^{\infty} a_N\psi_{0,N}(\zeta)\,\theta^{2N+1}$$

We attempt a similar expansion of $\psi_1(\zeta,\theta)$,

$$(10)\qquad \psi_1(\zeta,\theta) = \sum_{N=0}^{\infty} b_N\,\psi_{1,N}(\zeta)\,\theta^{2N+1}$$

These expansions give the following equations

$$(11)\qquad \psi_{0,0}''' + \psi_{0,0}\,\psi_{0,0}'' - \psi_{0,0}'^{\,2} = 0$$

$$(12)\qquad L_N\psi_{0,N} = \begin{cases} 0\ , & N=1 \\ \dfrac{1}{a_N}\sum\limits_{k=1}^{N-1} a_k a_{N-k}(2k+1)\{-\psi_{0,k}\psi''_{0,N-k}+\psi'_{0,k}\psi'_{0,N-k}\}, & N=2,3,4,\cdots \end{cases}$$

$$(13)\qquad L_N\psi_{1,N} = \begin{cases} -\dfrac{\zeta}{b_0}\psi'''_{0,0}\ , & N=0 \\ \dfrac{1}{b_N}\sum\limits_{k=1}^{N} a_k b_{N-k}\{-(2k+1)\psi_{0,k}\psi''_{1,N-k}+2(N+1)\psi'_{0,k}\psi'_{0,N-k} \\ \qquad -[2(N-k)+1]\psi''_{0,k}\psi_{1,N-k}\}-\dfrac{a_N}{b_N}\zeta\psi'''_{0,N}, & N=1,2,3,\cdots \end{cases}$$

$$(14)\qquad L_N \equiv \frac{d^3}{d\zeta^3}+(1-e^{-\zeta})\frac{d^2}{d\zeta^2}-2(N+1)e^{-\zeta}\frac{d}{d\zeta}-(2N+1)e^{-\zeta}$$

where the solution $\psi_{0,0} = 1-e^{-\zeta}$ (Riley 1965, eq. 27, but deviant notations) has been used to obtain (12), (13) and (14). The boundary conditions are,

$$(15)\qquad \begin{cases} \psi_{0,N}(0) = 0 \\ \left[\psi'_{0,N}\right]_{\zeta=0} = 1 \\ \left[\psi'_{0,N}\right]_{\zeta=\infty} = 0 \end{cases}$$

$$(16)\qquad \begin{cases} \psi_{1,N}(0) = 0 \\ \left[\psi'_{1,N}\right]_{\zeta=0} = 0 \\ \left[\psi'_{1,N}\right]_{\zeta=\infty} = 1 \end{cases}$$

Inspection of equations (12) and (13) reveals that $\psi_{0,N}$ and $\psi_{1,N}$ have identical general homogeneous solutions for $N \geq 1$.

II. Solution

An important step in achieving the general solution of an arbitrary term of the Blasius series is to establish the general homogeneous solution of the equation concerned. The general homogeneous solution $\psi_{0,N}^{(H)}(\zeta)$ of $\psi_{0,N}(\zeta)$ can be constructed by superposition of terms $a_{m,n}\zeta^{m}e^{-n\zeta}$. Some details of the calculations determining $\{a_{m,n}\}$ are given in appendix A. The results of these calculations are,

$$
\begin{aligned}
\psi_{0,N}^{(H)}(\zeta) = {} & A_N e^{-\zeta} + B_N [1+(2N+1)\zeta e^{-\zeta} \\
& - \sum_{n=2}^{2N+1} \frac{(2N+1)2N(2N-1)\cdots(2N-n+2)}{(n!)^2(n-1)} e^{-n\zeta}] \\
& + C_N[-(6N+4)+\zeta+\frac{2N+1}{2}\zeta^2 e^{-\zeta} \\
& + \sum_{n=2}^{2N+1} (K_n - \frac{(2N+1)2N(2N-1)\cdots(2N-n+2)}{(n!)^2(n-1)}\zeta)e^{-n\zeta} \\
& + (2N+1)((2N)!)\sum_{k=2}^{\infty}(-1)^{k-1}\frac{(k-2)!}{((2N+k)!)^2(2N+k-1)}e^{-(2N+k)\zeta}]
\end{aligned}
\tag{17}
$$

where

$$K_2 = -\frac{(2N+1)(4N+1)}{4}$$

and for $n \geq 3$

$$
\begin{aligned}
K_n = {} & \frac{(2N-n+2)(n-2)}{n^3-n^2} K_{n-1} \\
& + \frac{(2N+1)(2N)(2N-1)\cdots(2N-n+3)}{((n-1)!)^2(n^3-n^2)}\left[\frac{2(N-n+2)}{n-2}\right. \\
& \left. - \frac{n(3n-2)(2N-n+2)}{n^2(n-1)}\right]
\end{aligned}
$$

According to equation (12) and (13)

$$\psi_{0,N}^{(H)} = \psi_{1,N}^{(H)} \tag{18}$$

when $N \geq 1$. For $N = 0$ we find,

(19) $$\psi_{1,0}^{(H)} = A_0\, e^{-\zeta} + B_0\, [1+\zeta e^{-\zeta}]$$

$$+ C_0[-4+\zeta + \tfrac{1}{2}\zeta^2 e^{-\zeta} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(k-2)!}{(k!)^2(k-1)} e^{-k\zeta}]$$

The particular solutions can now be found by variation of the parameters, but also by inspection matching the residual terms

(20) $$R_{m,n}(\zeta;N) = L_N(a_{m,n}\zeta^m e^{-n\zeta})$$

to the inhomogenity terms of (12) and (13) by choosing special values of $\{a_{m,n}\}$.

Applications

The results obtained above are now used to establish the fourth term $\psi_{0,3}$ of the Blasius series (9) subject to conditions (15). In this context we need to quote three first terms given by Riley (1965) equations (27), (28), (29) and (31), respectively, which in our notations can be written

(21) $$\psi_{0,0} = 1 - e^{-\zeta}$$

(22) $$\psi_{0,1} = \frac{1}{68}[12+(7+36\zeta)e^{-\zeta}-18e^{-2\zeta}-e^{-3\zeta}]$$

(23) $$\psi_{0,2} = f_2(\zeta) + \frac{3a_1^2}{a_2}\, g_2(\zeta)$$

where

(24) $$f_2(\zeta) = \frac{1}{18552}[1440+(7063+7200\zeta)e^{-\zeta}$$

$$- 7200e^{-2\zeta}-1200e^{-3\zeta}-1200^{-3\zeta}-100e^{-4\zeta}-3e^{-5\zeta}]$$

(25) $$g_2(\zeta) = \frac{1}{53615280}[99216-(5392720+477900\zeta+2504520\zeta^2)e^{-\zeta} + (5486940+5009040\zeta)e^{-2\zeta} - (140655-417420\zeta)e^{-3\zeta} - 53270e^{-4\zeta} + 489e^{-5\zeta}]$$

As usual, $\psi_{0,3}$ is written as a sum of functions f_3 , g_3 and h_3 which are independent of $\{a_N\}$, i.e.,

(26) $$\psi_{0,3} = f_3(\zeta) + \frac{a_1 a_2}{a_3} g_3(\zeta) + \frac{3a_1^3}{a_3} h_3(\zeta)$$

which give,

(27) $$\begin{cases} L_3 f_3(\zeta) = 0 \\ L_3 g_3(\zeta) = -5\psi''_{0,1} f_2 + 8\psi'_{0,1} f'_2 - 3\psi_{0,1} f''_2 \\ L_3 h_3(\zeta) = -5\psi''_{0,1} g_2 + 8\psi'_{0,1} g'_2 - 3\psi_{0,1} g''_2 \\ [f'_3(\zeta)]_{\zeta=0} = 1 , \quad f_3(0) = [f'_3(\zeta)]_{\zeta=\infty} = 0 \\ [g'_3(\zeta)]_{\zeta=0} = g_3(0) = [g'_3(\zeta)]_{\zeta=\infty} = 0 \\ [h'_3(\zeta)]_{\zeta=0} = h_3(0) = [h'_3(\zeta)]_{\zeta=\infty} = 0 \end{cases}$$

The solutions are,

(28) $$f_3(\zeta) = \frac{1}{3927660}\{151200 + (1957814+1058400\zeta)e^{-\zeta} - 1587600e^{-2\zeta} - 441000e^{-3\zeta} - 73500e^{-4\zeta} - 6615e^{-5\zeta} - 294^{-6\zeta} - 5e^{-7\zeta}\} ,$$

(29) $$g_3(\zeta) = \psi^{(g)}_{0,3}(\zeta) + \frac{1}{210256}\{7254 - 43200\zeta^2 e^{-\zeta} + (112611+129600\zeta)e^{-2\zeta} + (-7426+25200\zeta)e^{-3\zeta} + (-\frac{46315}{12}+2400\zeta)e^{-4\zeta} + (-\frac{30299}{80}+90\zeta)e^{-5\zeta} - \frac{2581}{200} e^{-6\zeta} + \frac{39}{560} e^{-7\zeta}\}$$

(30) $$h_3(\zeta) = \psi_{0,3}^{(h)}(\zeta) + \frac{1}{91145976}\{-\frac{4766634}{7} + (-8181\zeta^2+751356\zeta^3)e^{-\zeta}$$
$$-(\frac{19164177}{4} + 9430020\zeta+4508136\zeta^2)e^{-2\zeta}$$
$$+(\frac{23985321}{8} + \frac{833949}{2}\zeta - 563517\zeta^2)e^{-3\zeta}$$
$$+(\frac{1690601}{4} + 191772\zeta)e^{-4\zeta}$$
$$-(\frac{14325}{400} + \frac{4401}{2}\zeta)e^{-5\zeta}$$
$$+\frac{10074}{40}e^{-6\zeta} - \frac{2301}{112}e^{-7\zeta}\}$$

where,

$$\psi_{0,3}^{(g)}(\zeta) = \psi_{0,3}^{(H)}(\zeta;A_3;B_3;C_3)$$

with

$$A_3 \approx -0.704077008,\ B_3 \approx -0.96801791\times 10^{-7},\ C_3 = 0\ ;$$

and,

$$\psi_{0,3}^{(h)}(\zeta) = \psi_{0,3}^{(H)}(\zeta;A_3;B_3;C_3)$$

with

$$A_3 \approx 0.0880122017,\ B_3 \approx 0.334608072\times 10^{-7},\ C_3 = 0\ .$$

($\psi_{0,3}^{(H)}$ is given by equation 17.)

The first term of the second order approximation ψ_1 is treated in the same way, i.e.,

(31) $$\psi_{1,0}(\zeta) = p_0(\zeta) + \frac{1}{b_0}q_0(\zeta)$$

giving,

(32) $$\begin{cases} L_0 p_0(\zeta) = 0 \\ L_0 q_0(\zeta) = \zeta e^{-\zeta} \\ p_0(0) = [p_0'(\zeta)]_{\zeta=0} = 0 \\ [p_0'(\zeta)]_{\zeta=0} = 1 \\ q_0(0) = [q_0'(\zeta)]_{\zeta=0} = 0 \\ [q'(\zeta)]_{\zeta=\infty} = 0 \end{cases}$$

The solutions are,

(33) $$p_0(\zeta) = \psi_{1,0}^{(h)}(\zeta;A_0;B_0;C_0)$$

with

$$A_0 = \frac{5}{2} - \frac{1}{2}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{1}{k!(k-1)^2}$$

$$B_0 = \frac{3}{2} + \frac{1}{2}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{(k-2)!}{(k!)^2}$$

$$C_0 = 1 ,$$

and,

(34) $$q_0(\zeta) = \psi_{1,0}^{(H)}(\zeta;A_0;B_0;C_0) - 2 - \zeta$$

with

$$A_0 = 3 + \frac{1}{2}\sum_{k=2}^{\infty}(-1)^{k}\frac{(k-2)!(k+1)}{(k!)^2\,(k-1)}$$

$$B_0 = 3 + \frac{1}{2}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{(k-2)!}{(k!)^2}$$

$$C_0 = 1$$

where $\psi_{1,0}^{(H)}(\zeta;A_0;B_0;C_0)$ is given by equation (19).

Numerical examples

The slip velocity induced by the time averaged Reynolds stresses in the Stokes layer at a long circular sylinder placed orthogonal to a oscillatory flow field, generates a steady slip boundary layer outside the Stokes layer where (see Riley 1975 equation (9), note deviant notations),

$$\left[\frac{\partial\psi_0}{\partial\zeta}\right]_{\zeta=0} = \frac{3}{2}\sin 2\theta$$

which give,

$$c_0 = \sqrt{3} \ , \ a_0 = 1 \ , \ a_1 = -\frac{2}{3} \ , \ a_2 = \frac{2}{15} \ , \ a_3 = -\frac{8}{945} \ .$$

An approximate expression of the dimensionless momentumflux in the slip boundary layer is,

$$M(\theta) = c_0^2 \int_0^\infty \{ \sum_{N=0}^{3} [a_N \, \psi'_{0,N}(\zeta) \, \theta^{2N+1}] \}^2 d\zeta$$

which give,

$$M(\tfrac{\pi}{2}) \approx 0.975$$

This is unexpected close to a result given by Riley (1975, pp 807) based on numerical integration which gave

$$M(\tfrac{\pi}{2}) \approx 0.991$$

In figure 2 the dimensionless tangential velocity,

$$V = c_0^2 \sum_{N=0}^{N_1} a_N \frac{\partial \psi_{0,N}}{\partial \zeta} \, \theta^{2N+1}$$

for $N_1 = 2$ (Riley 1965) and $N_1 = 3$.

This figure indicate a three term Blasius series to give the tangential velocity with resonable accuracy for

$$|\theta| \leq \frac{\pi}{3}$$

while a four term series seems to be applicable for

$$|\theta| \leq \frac{5}{12}\pi \ .$$

Appendix A

The general residual term of a test solution $a_{m,n}\zeta^m e^{-n\zeta}$ is,

(A1) $$R_{m,n}(\zeta;N) = L_N\{a_{m,n}\zeta^m e^{-n\zeta}\}$$
$$= a_{m,n}\{[m(m-1)(m-2)\zeta^{m-3} + m(m-1)(1-3n)\zeta^{m-2}$$
$$+ mn(3n-2)\zeta^{m-1} + (n^2-n^3)\zeta^m]e^{-n\zeta}$$
$$+ [-m(m-1)\zeta^{m-2} + 2m(n-1-2N)\zeta^{m-1}$$
$$+ (-n+1+2N)(n-1)\zeta^m]e^{-(n+1)\zeta}\}$$
$$\equiv a_{m,n}\{P_{m,n}(\zeta;N)e^{-n\zeta} + Q_{m,n}(\zeta;N)e^{-(n+1)\zeta}\}$$

with the following properties,

(A2) $$R_{0,1}(\zeta;N) = 0$$

(A3) $$R_{0,2N+1}(\zeta;N) = -2a_{0,2N+1}N(2N+1)^2 e^{-(2N+1)\zeta}$$

(A4) $$R_{1,2N+1}(\zeta;N) = a_{1,2N+1}[(2N+1)(6N+1) - 2N(2N+1)^2\zeta]e^{-(2N+1)\zeta}$$

Equation (A2) indicates that $e^{-\zeta}$ is a homogeneous solution for every N. The construction of the other homogeneous solutions consists of choosing numerical values of $a_{m,n}$ such that

$$\sum_m [a_{m,n+1}P_{m,n+1}(\zeta;N) + a_{m,n}Q_{m,n}(\zeta;N)] = 0$$

for every n. The simplest expressions are obtained when the properties (A3) and (A4) are utilized.

List of references

Riley N. (1965) Mathematika 12, 161.

Riley N. (1975) J. Fluid Mech. 68, 801.

Stuart J.T. (1966) J. Fluid Mech. 24, 673.

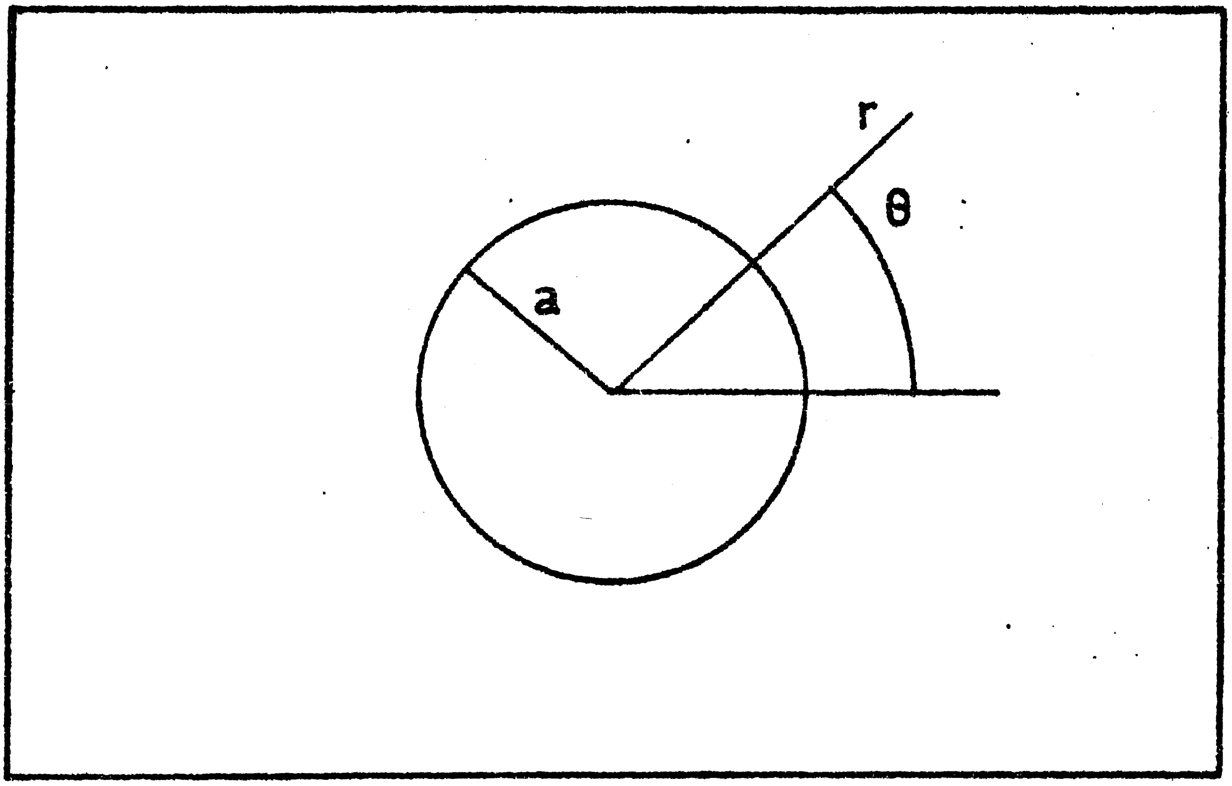

Figure 1. The polar coordinate system (r,θ) referred to in the paper.

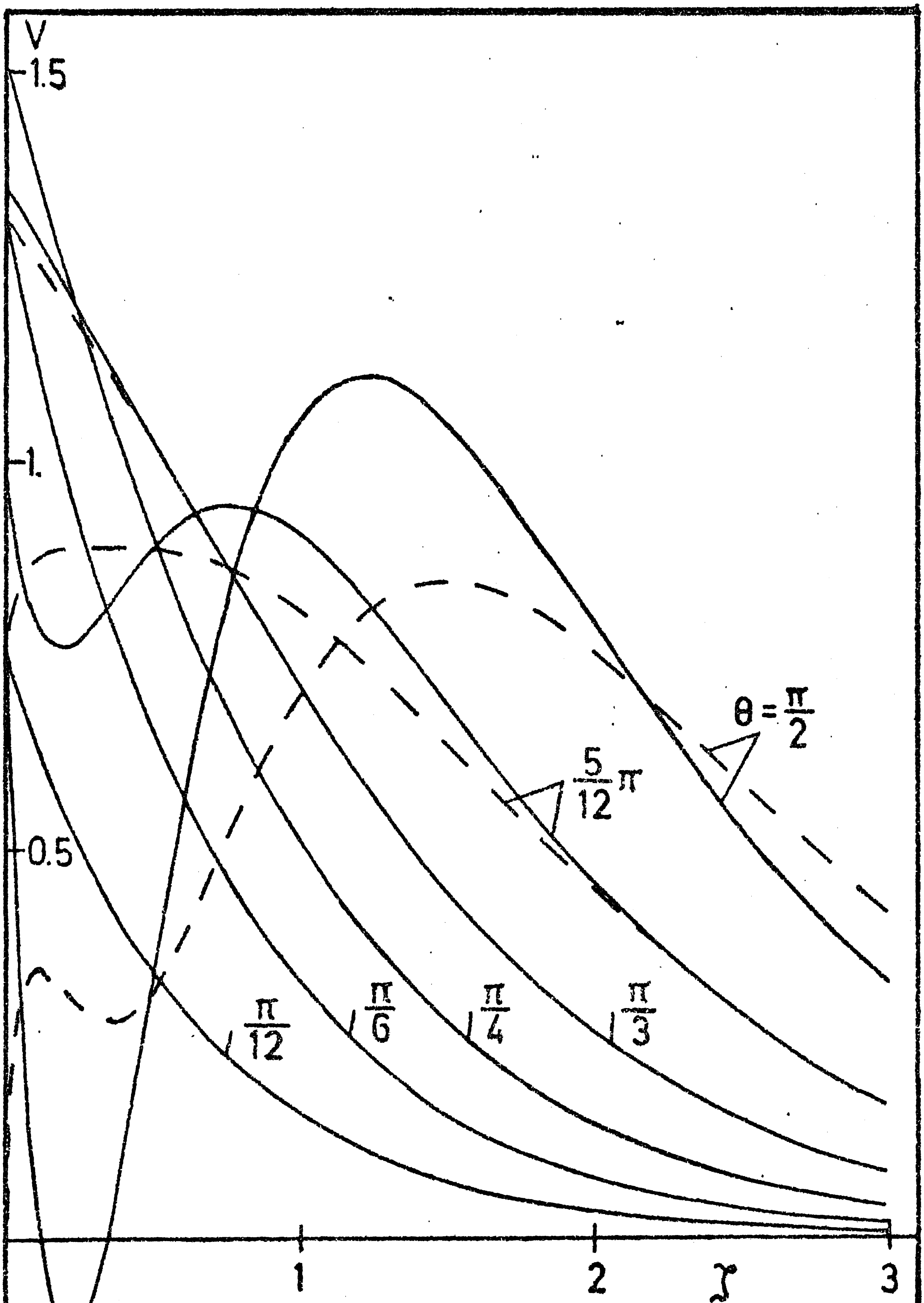

Figure 2. Dimensionless tangential velocity distribution at various angular positions. Full and dashed curves based on three and four terms of the Blasins series, respectively.