

PREFACE

The authors have often been annoyed by the insufficient documentation of numerical methods used in applied fluid mechanics. In traditional papers on such subjects there is no room for a proper description of the numerical methods which are used. The present preprint is completely devoted to the documentation of a particular numerical method. We believe it will be helpful for the understanding of later applications. The motivation for developing this method stems from interest in two specific problems: Diffraction and refraction of swells in shallow regions and propagation of long waves generated by slides or avalanches.

NUMERICAL SOLUTION OF THE THREE DIMENSIONAL
BOUSSINESQ EQUATIONS FOR DISPERSIVE SURFACE WAVES
PART 1

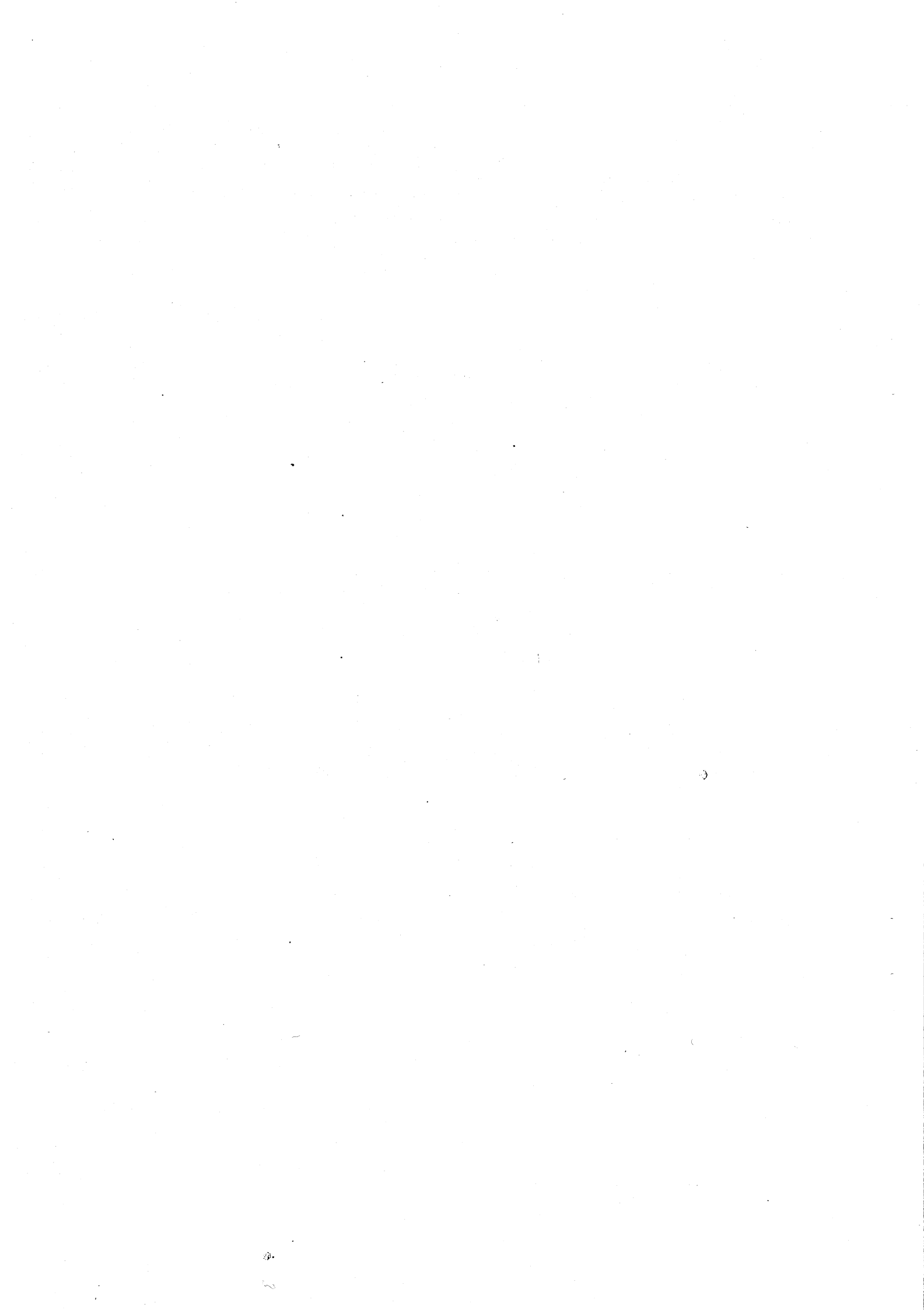
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ABSTRACT

A finite difference method for integrating the three-dimensional Boussinesq equations has been developed. Dispersion relations and the numerical stability criterion are derived. The advancement from one time step to next involves solving a large coupled set of equations for the accelerations. The solution is achieved by a line by line iterative technique. Different initial values are tested by simple simulations. Correction terms to the basic numerical scheme are found. These terms are of similar mathematical form as the dispersion terms in the Boussinesq equations, and are thus easily included in the difference equations.

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1. INTRODUCTION

1.1. Introductory remarks

In models of wave propagation in shallow water the hydrostatic pressure approximation is commonly used. The hydrostatic equations may often be effectively solved by very simple numerical schemes even if effects of variable depth or non-linearity are included. If the ratio between typical wavelengths and depths is less than say 10, deviations from hydrostatic pressure become important. These deviations may be approximately accounted for by adding correction terms, usually called dispersion terms, to the equations. The resulting Boussinesq equations are substantially more troublesome to handle numerically than the hydrostatic shallow water equations. To avoid instability the finite difference representations of the dispersion terms have to be implicit and large coupled sets of ordinary equations have to be solved at each time step. In three dimensional cases this set of equations have to be solved by time consuming iterative techniques.

A review of commonly used numerical methods for solving standard wave equations is given by Meisinger and Arakawa (1976). Methods for numerical integration of the two dimensional Boussinesq, or related, equations are reported by many authors, among which we mention Peregrine (1967), Madsen & Mey (1969), Miles (1979) and Pedersen & Gjevik (1983). Numerical integration of the three dimensional Boussinesq equations are to our knowledge hardly reported at all.

The following text is the first part of the documentation on a finite difference technique for solving these equations. Inclusion of non-linearity and tests involving complicated geometry are postponed to the next part. We use a staggered grid which allow implementation of boundaries of rather complex shapes. At every time step implicit equations for the accelerations are solved by a line by line iteration procedure. In the last section we introduce correction terms which improve the accuracy of the numerical scheme. These correction terms will be less useful in the presence of complicated boundaries. The numerical dispersion relations and tests of the iteration procedure indicate that the numerical method is sufficiently effective to solve complicated problems.

1.2. Basic equations

A cartesian coordinate system with horizontal axes Ox and Oy and vertical axis Oz is introduced. A typical wavelength λ_c , and a typical depth h_c , are used as horizontal and vertical length scales respectively. By introducing $\lambda_c (gh_c)^{-1/2}$ as time scale, the linearized non-dimensional Boussinesq equations may be written in the form: (Peregrine 1972).

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (h\vec{v}) \quad (1.1)$$

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \eta + \varepsilon \left\{ \frac{1}{2} h \nabla \nabla \cdot (h \frac{\partial \vec{v}}{\partial t}) - \frac{1}{6} h^2 \nabla^2 \frac{\partial \vec{v}}{\partial t} \right\} + O(\varepsilon^2) \quad (1.2)$$

where η is the surface displacement, h the equilibrium depth, $\vec{v} = u\vec{i} + v\vec{j}$ the mean horizontal velocity, $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y}$ the horizontal gradient operator, and $\varepsilon = (h_c/\lambda_c)^2$ a small parameter. The mean velocity \vec{v} with components u and v is defined by

$$\vec{v}(x, y, t) = u\vec{i} + v\vec{j} = \frac{1}{h} \int_{-h}^0 \{ \tilde{u}(x, y, z, t) \vec{i} + \tilde{v}(x, y, z, t) \vec{j} \} dz \quad (1.3)$$

where \tilde{u} and \tilde{v} are the actual horizontal velocity components. From (1.3) it follows immediately that \vec{v} is irrational only to order ε^0 if the depth is non-constant. We may thus write:

$$\nabla x \vec{v} = O(\varepsilon) \frac{\partial h}{\partial x} + O(\varepsilon) \frac{\partial h}{\partial y} \quad (1.4)$$

Rewriting the right hand side of (1.2) by using (1.4) leads to:

$$\begin{aligned} \dot{u} = & -\frac{\partial \eta}{\partial x} + \frac{1}{2} \varepsilon h \left[\frac{\partial^2}{\partial x^2} (h\dot{u}) + \frac{\partial}{\partial y} (h \frac{\partial}{\partial y} \dot{u}) + \frac{\partial}{\partial y} (\frac{\partial h}{\partial x} \dot{v}) \right] \\ & - \frac{1}{6} \varepsilon h^2 \nabla^2 \dot{u} + O(\varepsilon^2) \end{aligned} \quad (1.5)$$

$$\begin{aligned} \dot{v} = & -\frac{\partial \eta}{\partial y} + \frac{1}{2} \varepsilon h \left[\frac{\partial^2}{\partial y^2} (h\dot{v}) + \frac{\partial}{\partial x} (h \frac{\partial}{\partial x} \dot{u}) + \frac{\partial}{\partial x} (\frac{\partial h}{\partial y} \dot{v}) \right] \\ & - \frac{1}{6} \varepsilon h^2 \nabla^2 \dot{v} + O(\varepsilon^2) \end{aligned} \quad (1.6)$$

where we have introduced the notations $\frac{\partial u}{\partial t} = \dot{u}$ and $\frac{\partial v}{\partial t} = \dot{v}$. (1.2) and (1.5-1.6) are identical in the case of constant depth, $h = h_0$, for which we get:

$$\frac{\partial \eta}{\partial t} = -h_0 \nabla \cdot \vec{v} \quad (1.7)$$

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \eta + \frac{\epsilon}{3} h_0^2 \nabla^2 \frac{\partial \vec{v}}{\partial t} \quad (1.8)$$

For two dimensional problems the two formulations of the equation of motion are again identical.

2. THE NUMERICAL SCHEME

2.1. Grid and notations

The numerical approximation to a quantity F at the gridpoint with coordinates $(\alpha \Delta x, \beta \Delta y, \gamma \Delta t)$ is denoted by $F_{\alpha, \beta}^{\gamma}$. Δx , Δy and Δt are the grid increments. The space discretization is done on a standard staggered grid which is illustrated in figure 1. Approximations to η , \dot{u} , \dot{v} and u , v are calculated at different values of the time t . The quantities to be calculated, are thus:

$$\eta_{j+\frac{1}{2}, p+\frac{1}{2}}^n, \dot{u}_{j, p+\frac{1}{2}}^n, \dot{v}_{j+\frac{1}{2}, p}^n, u_{j, p+\frac{1}{2}}^{n+\frac{1}{2}}, v_{j+\frac{1}{2}, p}^{n+\frac{1}{2}}$$

where j , k and n are integers. We define a difference operator δ_x by:

$$\delta_x F_{\alpha, \beta}^{\gamma} = \frac{1}{\Delta x} (F_{\alpha+\frac{1}{2}, \beta}^{\gamma} - F_{\alpha-\frac{1}{2}, \beta}^{\gamma}) \quad (2.1)$$

and correspondingly δ_y and δ_t . These operators will be combined to give higher order differences. An important example is:

$$\begin{aligned} \delta_x^2 F_{\alpha, \beta}^{\gamma} &= \delta_x (\delta_x F)_{\alpha, \beta}^{\gamma} = \frac{1}{\Delta x} (\delta_x F_{\alpha+\frac{1}{2}, \beta}^{\gamma} - \delta_x F_{\alpha-\frac{1}{2}, \beta}^{\gamma}) \\ &= \frac{1}{\Delta x^2} (F_{\alpha+1, \beta}^{\gamma} - 2F_{\alpha, \beta}^{\gamma} + F_{\alpha-1, \beta}^{\gamma}) \end{aligned} \quad (2.2)$$

The right hand side is easily recognized as the standard symmetric three point difference for $\partial^2 F / \partial x^2$. If F is a simple harmonic function of x , say $F = \hat{F} e^{ikx}$, we get from the definition of δ_x :

$$\delta_x F_{\alpha, \beta}^Y = iK F_{\alpha, \beta}^Y$$

where $K = \frac{2}{\Delta x} \sin \frac{k\Delta x}{2}$. Because K has the same relation to the operator δ_x as k has to $\partial/\partial x$ we will denote it as discrete wavenumber. Likewise we define $L = \frac{2}{\Delta y} \sin \frac{l\Delta y}{2}$ and the discrete frequency $\Omega = \frac{2}{\Delta t} \sin \frac{\tilde{\omega}\Delta t}{2}$ where $\tilde{\omega}$ is the wave frequency obtained from the numerical scheme which we will refer to as the numerical frequency. Analogously to the difference operators we introduce average operators by:

$$(\bar{F}^X)_{\alpha, \beta}^Y = \frac{1}{2}(F_{\alpha+\frac{1}{2}, \beta}^Y + F_{\alpha-\frac{1}{2}, \beta}^Y) \text{ etc.} \quad (2.3)$$

By use of (2.1) and (2.3) it is possible to state a difference analogue to the product rule of derivation:

$$\delta_x (FG)_{\alpha, \beta}^Y = (\bar{F}^X)_{\alpha, \beta}^Y \delta_x G_{\alpha, \beta}^Y + (\bar{G}^X)_{\alpha, \beta}^Y \delta_x F_{\alpha, \beta}^Y \quad (2.4)$$

By use of notations like (2.1) most of the terms in the difference equations will get identical indexes. To abbreviate the expressions we often collect terms of equal indexes within square brackets leaving the indexes outside the bracket. As an example we abbreviate the difference equation:

$$\frac{1}{\Delta t} (u_{j, p+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j, p+\frac{1}{2}}^{n-\frac{1}{2}}) = - \frac{1}{\Delta x} (\eta_{j+\frac{1}{2}, p+\frac{1}{2}}^n - \eta_{j-\frac{1}{2}, p+\frac{1}{2}}^n)$$

to

$$\delta_t u_{j, p+\frac{1}{2}}^n = -\delta_x \eta_{j, p+\frac{1}{2}}^n$$

by introduction of the difference operators, and by use of square brackets to:

$$[\delta_t u = -\delta_x \eta]_{j, p+\frac{1}{2}}^n$$

For the standard five point representation of the Laplacian we will use the notation \square . The formal definition of \square reads:

$$\square = \delta_x^2 + \delta_y^2 \quad (2.5)$$

2.2. The numerical scheme

The equilibrium depth h may be represented by an analytical function or by discrete values in an array. We assume that values for h are available at the points in space at which u and v are calculated.

The difference form of the equation of continuity (1.1) reads:

$$[\delta_t \eta = -\delta_x(hu) - \delta_y(hv)]_{j+\frac{1}{2}, p+\frac{1}{2}}^{n-\frac{1}{2}} \quad (2.6)$$

For the x component of the equation of motion (1.5) we get:

$$\begin{aligned} 0 = & [\dot{u} + \delta_x \eta^{-\frac{1}{2}} \epsilon h \{ \delta_x^2(h\dot{u}) + \delta_y(\bar{h}^y \delta_y \dot{u}) \} \\ & + \frac{1}{\sigma} \epsilon h^2 \square \dot{u} - \frac{1}{2} \epsilon h \delta_y(\bar{v}^x \delta_x h)]_{j, p+\frac{1}{2}}^n \end{aligned} \quad (2.7)$$

and correspondingly for the y -component

$$\begin{aligned} 0 = & [\dot{v} + \delta_y \eta^{-\frac{1}{2}} \epsilon h \{ \delta_y^2(h\dot{v}) + \delta_x(\bar{h}^x \delta_x \dot{v}) \} \\ & + \frac{1}{\sigma} \epsilon h^2 \square \dot{v} - \frac{1}{2} \epsilon h \delta_x(\bar{u}^y \delta_y h)]_{j+\frac{1}{2}, p}^n \end{aligned} \quad (2.8)$$

The relation between the accelerations \dot{u} , \dot{v} and the velocities u , v are implemented by:

$$[\delta_t u = \dot{u}]_{j, p+\frac{1}{2}}^n, \quad [\delta_t v = \dot{v}]_{j+\frac{1}{2}, p}^n \quad (2.9)$$

The implicitness of equations (2.7) and (2.8) are essential for the stability of the scheme. If all quantities for $t < (n-\frac{1}{2})\Delta t$ are calculated, the advancement to $t = (n+\frac{1}{2})\Delta t$ may be sketched:

- (i) Calculate η^n by (2.6).
- (ii) Solve the implicit equations (2.7) and (2.8) for \dot{u}^n, \dot{v}^n by an iterative method. (Chapter 3.)
- (iii) Calculate $u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}$ by (2.9).

For constant depth, $h = h_0$, the difference equations (2.6-8) simplify considerably:

$$[\delta_t \eta = -h_0 (\delta_x u + \delta_y v)]_{j+\frac{1}{2}, p+\frac{1}{2}}^{n+\frac{1}{2}} \quad (2.10)$$

$$0 = [\dot{u} + \delta_x \eta - \frac{1}{3} \epsilon h_0^2 \square \dot{u}]_{j, p+\frac{1}{2}}^n \quad (2.11)$$

$$0 = [\dot{v} + \delta_y \eta - \frac{1}{3} \epsilon h_0^2 \square \dot{v}]_{j+\frac{1}{2}, p}^n \quad (2.12)$$

2.3 Boundary conditions

At a rigid wall we have the boundary condition $\vec{v} \cdot \vec{n} = 0$ where \vec{n} is a normal to the boundary. In the hydrostatic case, $\epsilon = 0$, there is no need for additional conditions as long as the boundary consists of line segments parallel to the axes, and are passing through the grid-points. The approximation to a boundary of general shape by such a series of segments, is illustrated in figure 1b. The non-hydrostatic case is somewhat more complicated. If the boundary is located at, say $y = 0$ (the fluid at $y > 0$), the quantities $\dot{u}_{j, -\frac{1}{2}}^n$ appear in the dispersion terms in equation (2.7). We have to assign values to these fictive quantities outside the domain by use of the boundary conditions and the governing equations. Because the fictive quantities appear only in the $O(\epsilon)$ terms, we may implement

boundary conditions valid only to $O(\varepsilon^0)$. At the boundary $y = 0$ described above, we may thus use the irrotational condition (1.4) which gives: $\dot{u}_{j, -\frac{1}{2}}^n = \dot{u}_{j, \frac{1}{2}}^n$. This technique will work also for the piecewise straight boundaries described earlier.

2.4. Dispersion relations, stability analysis

The stability analysis is limited to the case of uniform depth and to solutions in form of simple harmonics:

$$(\eta, u, v) = (\hat{\eta}, \hat{u}, \hat{v}) e^{i(kx + ly + \omega t)} \quad (2.13)$$

From (1.7-1.8) we get the dispersion relation:

$$\omega^2 = \frac{h_0 a^2}{1 + \frac{\varepsilon}{3} h_0^2 a^2} \quad (2.14)$$

where $a = (k^2 + l^2)^{\frac{1}{2}}$ is the absolute value of the wave number, and the amplitude relations

$$(\hat{u}, \hat{v}) = \frac{-\hat{\eta}}{\omega(1 + \frac{\varepsilon}{3} h_0^2 a^2)} (k, l) \quad (2.15)$$

If a solution of the form:

$$\eta_{j+\frac{1}{2}, p+\frac{1}{2}}^{n+\frac{1}{2}} = \tilde{\eta} e^{i(k(j+\frac{1}{2})\Delta x + l(p+\frac{1}{2})\Delta y + \tilde{\omega}(n+\frac{1}{2})\Delta t)} \quad \text{etc.}$$

is substituted into (2.9)-(2.12) we immediately obtain the relations corresponding to (2.14-2.15) simply by replacing k, l and ω by the discrete analogues K, L and Ω :

$$\Omega^2 = \frac{h_0 A^2}{1 + \frac{\varepsilon}{3} h_0^2 A^2} \quad (2-16)$$

$$(\tilde{u}, \tilde{v}) = \frac{-\tilde{\eta}}{\Omega(1 + \frac{\varepsilon}{3} h_0^2 A^2)} (K, L) \quad (2.17)$$

$$(\tilde{u}, \tilde{v}) = i\Omega(\tilde{u}, \tilde{v}) \quad (2.18)$$

where $A^2 = K^2 + L^2$. By requiring $\tilde{\omega}$ real we obtain from (2.16) the stability criterion:

$$h_0 \Delta t^2 < \frac{1}{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}} + \frac{4}{3} \varepsilon h_0^2 \quad (2.19)$$

If $\Delta t < \sqrt{\frac{4}{3\varepsilon}} h_0$ the scheme is unconditionally stable independent of the value of Δx and Δy . This is due to the implicitness in the dispersion terms. If $\varepsilon = 0$ (2.19) reduces to a standard Courant condition. An expansion of the right hand side of (2.14) in terms of a gives:

$$\omega^2 = h_0 (a^2 - \frac{\varepsilon}{3} h_0^2 a^4 + O(a^6)) \quad (2.20)$$

If the numerical frequency $\tilde{\omega}$ is found from (2.16), and expanded in the same manner, we get:

$$\begin{aligned} \tilde{\omega}^2 = h_0 (a^2 - \frac{\varepsilon}{3} h_0^2 a^4 + \frac{1}{12} (h_0 \Delta t^2 a^4 - \Delta x^2 k^4 - \Delta y^2 l^4) \\ + O(a^6 (\varepsilon \Delta t^2, \Delta t^4) \text{etc})) \end{aligned} \quad (2.21)$$

In figure 2 we have compared ω and $\tilde{\omega}$ for various values of Δx , Δy and Δt . Both the figure and (2.21) illustrate the improvement of the numerical scheme obtained if Δt^2 is close to $(\Delta x^2 k^4 + \Delta y^2 l^4) / (h_0 a^4)$.

3. SOLUTION OF THE IMPLICIT EQUATIONS FOR THE ACCELERATIONS

3.1. Iteration procedure

The equations (2.7) and (2.8) are solved by a line by line iterative technique. Values obtained from the previous time step

are used as initial values for the iteration. If approximations to $\dot{u}_{j,p+\frac{1}{2}}^n$ and $\dot{v}_{j+\frac{1}{2},p}^n$ are denoted by $\dot{u}_{j,p+\frac{1}{2}}^{\circ}$ and $\dot{v}_{j+\frac{1}{2},p}^{\circ}$ respectively, improved values $\dot{u}_{j,p+\frac{1}{2}}^*$ and $\dot{v}_{j+\frac{1}{2},p}^*$ are obtained through the following four steps:

i) Solve the sets of tridiagonal equations for the intermediate quantities $\dot{u}_{j,p+\frac{1}{2}}^+$ defined by:

$$0 = [\dot{u}^+ + r_x (\dot{u}^+ - \dot{u}) + \delta_x \eta^n + \frac{1}{2} \epsilon h \delta_x^2 (h \dot{u}^+) + \frac{1}{6} \epsilon h^2 \delta_x^2 \dot{u}^+ - \frac{1}{2} \epsilon h \delta_y (\bar{h}^y \delta_y \dot{u}) + \frac{1}{6} \epsilon h^2 \delta_y^2 (\dot{u}) - \frac{1}{2} \epsilon h \delta_y (\bar{v}^x \delta_x h)]_{j,p+\frac{1}{2}} \quad (3.1)$$

ii) Solve the equations for u^* :

$$0 = [\dot{u}^* + r_y (\dot{u}^* - \dot{u}^+) + \delta_x \eta^n - \frac{1}{2} \epsilon h \delta_x^2 (h \dot{u}^+) + \frac{1}{6} \epsilon h^2 \delta_x^2 \dot{u}^+ - \frac{1}{2} \epsilon h \delta_y (\bar{h}^y \delta_y \dot{u}^*) + \frac{1}{6} \epsilon h^2 \delta_y^2 (\dot{u}^*) - \frac{1}{2} \epsilon h \delta_y (\bar{v}^x \delta_x h)]_{j,p+\frac{1}{2}} \quad (3.2)$$

iii) & iv) Apply the analogous procedures to equation (2.8) using \dot{u}^* instead of \dot{u} .

The coefficients r_x and r_y serve as relaxation factors and this will be discussed in the next sections.

3.2. The iteration scheme in the case of constant depth

For $h = h_0 = \text{constant}$ the equations (3.1) and (3.2) simplify

$$0 = [\dot{u}^+ + r_x (\dot{u}^+ - \dot{u}) + \delta_x \eta^n - \frac{1}{3} \epsilon h_0^2 (\delta_x^2 \dot{u}^+ + \delta_y^2 \dot{u})]_{j,p+\frac{1}{2}} \quad (3.3)$$

$$0 = [\dot{u}^* + r_x (\dot{u}^* - \dot{u}^+) + \delta_x \eta^n - \frac{1}{3} \epsilon h_0^2 (\delta_x^2 \dot{u}^+ + \delta_y^2 \dot{u}^*)]_{j,p+\frac{1}{2}} \quad (3.4)$$

We note that the set of equations for \dot{u} and \dot{v} are not coupled

in this case. The iteration procedure defined by (3.3), (3.4) may be regarded as a quasi time integration of the equation:

$$\frac{\partial \phi}{\partial t_*} = -\phi + \frac{1}{3}\epsilon h_0^2 \nabla^2 \phi + f(x, y) \quad (3.5)$$

where ϕ corresponds to \dot{u} , f to $-\delta\eta/\partial x$ and t_* is the quasi time. By defining $\Delta t_* = 1/r_x + 1/r_y$, $\alpha = 1/(r_x \Delta t_*)$, $\phi^s = \dot{u}$, $\phi^{s+\alpha} = \dot{u}^+$ and $\phi^{s+1} = \dot{u}^*$ (3.3)-(3.4) can be written in the form:

$$\left[\frac{1}{\alpha \Delta t_*} (\phi^{s+\alpha} - \phi^s) = -\phi^{s+\alpha} + \frac{1}{2}\epsilon h_0^2 \delta_x^2 \phi^{s+\alpha} + \frac{1}{3}\epsilon h_0^2 \delta_y^2 \phi^s + f \right]_{j, p+\frac{1}{2}} \quad (3.6)$$

$$\left[\frac{1}{(1-\alpha)\Delta t_*} (\phi^{s+1} - \phi^{s+\alpha}) = -\phi^{s+1} + \frac{1}{2}\epsilon h_0^2 \delta_x^2 \phi^{s+\alpha} + \frac{1}{3}\epsilon h_0^2 \delta_y^2 \phi^{s+1} + f \right]_{j, p+\frac{1}{2}} \quad (3.7)$$

which is a standard ADI (Alternating direction implicit) scheme applied to equation (3.5). The relaxation factors r_x and r_y have thus interpretation as the reciprocals of the quasi time half steps in the ADI scheme.

Convergence to the stationary solution of (3.5) is obtained when the transients vanishes. The transients are solutions of the homogeneous counterpart of (3.5). For a transient in form of a single harmonic component we may write:

$$[\phi^{s+1} = \beta_y \phi^{s+\alpha} = \beta_y \beta_x \phi^s \equiv \beta \phi^s]_{j, p+\frac{1}{2}} = \beta^{s+1} e^{i(kj\Delta x + l(p+\frac{1}{2})\Delta y)} \quad (3.8)$$

where β_x , β_y and $\beta = \beta_x \beta_y$ are damping factors. Substitution of this component in (3.6) and (3.7) gives

$$\beta_x = \frac{r_x - \frac{1}{3}\epsilon h_0^2 L^2}{r_x + 1 + \frac{1}{3}\epsilon h_0^2 K^2}, \quad \beta_y = \frac{r_y - \frac{1}{3}\epsilon h_0^2 K^2}{r_y + 1 + \frac{1}{3}\epsilon h_0^2 L^2} \quad (3.9)$$

where K and L are as defined in paragraph (2.1). The absolute value of the product $\beta = \beta_x \beta_y$ is obviously less than 1 for all positive r_x and r_y , and the iteration scheme is thus unconditionally stable. The difference between \dot{u} etc at two adjacent time steps (real time) is generally dominated by changes in phases rather than energy distributions on the wavenumbers. If we are able to predict the dominant wavelengths the rate of convergence can thus be increased considerably by making the appropriate choices of r_x and r_y . In the case of non-constant depth it may be advantageous to vary r_x and r_y from gridpoint to gridpoint. This may correspond to quasi time integration of an equation of form:

$$\frac{\partial \phi}{\partial t^*} = q(x, y) \left(-\phi + \frac{1}{3} \epsilon h_0 \nabla^2 \phi + f \right) \quad (3.10)$$

3.3. Initiation of the iteration scheme

In this paragraph we will investigate different choices for the initial values of the iteration scheme described in paragraph (3.1). The initial values are denoted by $(\dot{u}^0)^n_{j, p+\frac{1}{2}}$ and $(\dot{v}^0)^n_{j+\frac{1}{2}, p}$. We will discuss three different possibilities:

i) The values from the previous time step:

$$(\dot{u}^0)^n = \dot{u}^{n-1} ; (\dot{v}^0)^n = \dot{v}^{n-1} \quad (3.11)$$

ii) Values obtained from (2.7) and (2.8) by ignoring the dispersion terms:

$$[(\dot{u}^0) = -\delta_x \eta]^n_{j, p+\frac{1}{2}} , [(\dot{v}^0) = -\delta_y \eta]^n_{j+\frac{1}{2}, p} \quad (3.12)$$

These values must be expected to be poor for relatively short waves.

iii) Values extrapolated from the previous time step accounting for changes in the η -term but not for changes in the dispersion terms:

$$\begin{aligned} [(\dot{u}^0)^n - \dot{u}^{n-1} = -\delta_x \eta^n + \delta_x \eta^{n-1}]_{j, p+\frac{1}{2}} \\ [(\dot{v}^0)^n - \dot{v}^{n-1} = -\delta_y \eta^n + \delta_y \eta^{n-1}]_{j+\frac{1}{2}, p} \end{aligned} \quad (3.13)$$

(3.13) corresponds to an integration of the equation of motion with a backward representation of the dispersion terms

$$\begin{aligned} [(\dot{u}^0)^n = -\delta_x \eta^n + \{\text{"dispersion terms"}\}^{n-1}]_{j, p+\frac{1}{2}} \\ [(\dot{v}^0)^n = -\delta_y \eta^n + \{\text{"dispersion terms"}\}^{n-1}]_{j+\frac{1}{2}, p} \end{aligned} \quad (3.14)$$

If the values calculated by (3.14) are used as final values for \dot{u}^n, \dot{v}^n we would get an unstable scheme.

As a test example we choose eigenoscillations in a closed rectangular basin. For constant depth the oscillations are simple harmonics and both the analytical equations (1.7), (1.8) and the numerical counterparts (2.10-12) are easily solved exact. In figure 3 we have compared time series of η at a fixed point obtained by using initiation (i), (ii) or (iii) and iteration to those obtained from the exact solution of (2.10-12). The point is located approximately midway between the left wall of the basin and the nearest crest. For all series depicted in the figure we have: $h_0 = 1$, $k = 4\pi/5$, $\lambda = 2\pi/25$, $\Delta x = 0.25$, $\Delta y = 1.25$ and $\Delta t = 0.25$. This corresponds to a wave length about two and a half times the depth. The other parameters are irrelevant in this context.

By use of only one iteration at each time step considerable deviations are observed for all choices of initial values. Alternative (i) seems to be the best one if only the magnitude of the error is considered. on the other hand, the slow growth in amplitude calls for a numericists disapproval. The time story related to alternative (iii) is subjected to a considerable damping while the corresponding for (ii) exhibits a much larger frequency than the exact discrete solution. The latter may be understood by the recognition of (3.12) as equations of motion in the hydrostatic case which has a dispersion relation different from the one possessed by the Boussinesq equations. If two iterations are used the deviations are scarcely visible (relative errors 1-2%) for any of the alternatives. The corresponding graphs are thus omitted in the figure. Although two iterations are sufficient for this problem we must expect that the necessary number of iterations and the sort of initiation that really should be used are probably very dependent on specific problem. As a conclusion we may say that the investigations in this section indicate that a good choice for r_x and r_y is more likely to reduce the necessary number of iterations than good initiation is.

4. CORRECTION TERMS TO THE BASIC NUMERICAL SCHEME

4.1. A preliminary derivation

For waves parallel to the y-axis we get from (2.21):

$$\tilde{\omega} = h_0 \left(k^2 - \left\{ \frac{\varepsilon}{3} h_0^2 + \frac{1}{12} (\Delta x^2 - h_0 \Delta t^2) \right\} k^4 + O(k^6 (\varepsilon \Delta t^2, \Delta t^4 \text{ etc})) \right) \quad (4.1)$$

or by calculating the square-root for each side:

$$\tilde{\omega} = h_0^{\frac{1}{2}} k \left\{ 1 - \left(\frac{\varepsilon h_0^2}{6} + \frac{1}{24} (\Delta x^2 - h_0 \Delta t^2) \right) k^2 + O(k^4 (\varepsilon \Delta t^2, \Delta t^4 \text{ etc})) \right\} \quad (4.2)$$

We note that the leading error term has the same mathematical form as the leading dispersion term in (4.2). The leading error term arise from discretization error in the $O(\varepsilon^0)$ terms in (1.7) and (1.8). This error term vanishes for $\frac{\Delta x}{\Delta t} = h_0^{\frac{1}{2}}$. In this case the numerical dispersion relation will be exact for $\varepsilon = 0$ and very accurate for $\varepsilon \neq 0$ even for large values of Δx . The error term may also be removed by changing the value of ε , or in other words, adding an artificial dispersion term to the equations. Including the artificial dispersion term the equation of motion will read:

$$\dot{u} = - \frac{\partial \eta}{\partial x} + \frac{\varepsilon h_0^2}{3} \frac{\partial^2 \dot{u}}{\partial x^2} + \frac{1}{12} (h_0 \Delta t^2 - \Delta x^2) \frac{\partial^2 \dot{u}}{\partial x^2} \quad (4.3)$$

which by discretization gives:

$$[\dot{u} = -\partial_x \eta + \left\{ \frac{\varepsilon h_0^2}{3} + \frac{1}{12} (h_0 \Delta t^2 - \Delta x^2) \right\} \partial_x^2 \dot{u}]_j^n \quad (4.4)$$

The addition of the extra term corresponds to changing the value of ε by the amount

$$\varepsilon' = \frac{1}{4} \left(\frac{\Delta t^2}{h_0} - \left(\frac{\Delta x}{h_0} \right)^2 \right) \quad (4.5)$$

For $\varepsilon = 0$ we get from (4.2):

$$\begin{aligned} \tilde{\omega} &= h_0^{\frac{1}{2}} k \left\{ 1 + O(k^4 (\varepsilon' \Delta t^2, \Delta t^4 \text{ etc})) \right\} \\ &= h_0^{\frac{1}{2}} k \left\{ 1 + O(k^4 \Delta t^4 \text{ etc}) \right\} \end{aligned} \quad (4.6)$$

In this case the scheme defined by (4.4) becomes a fourth order scheme. For $\varepsilon \neq 0$ the scheme will not be of fourth order but the numerical dispersion relation will be correct to the same order of k as the Boussinesq equations themselves. From (4.6) it is clear

that the additional term in (4.4) has interpretation as a correction term for the basic numerical scheme applied to the hydrostatic equations. Such correction terms may be expected to be found in form of artificial dispersion terms also in the case of variable depth.

4.2. The general derivation of the correction terms

In the hydrostatic case, elimination of \vec{v} from (1.1) and (1.2) gives:

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla \cdot (h \nabla \eta) = 0 \quad (4.7)$$

The corresponding elimination applied to (2.6)-(2.9) gives:

$$\langle \rangle \eta_{j+\frac{1}{2}, p+\frac{1}{2}}^{n+\frac{1}{2}} \equiv [\delta_t^2 \eta - \delta_x (h \delta_x \eta) - \delta_y (h \delta_y \eta)]_{j+\frac{1}{2}, p+\frac{1}{2}}^{n+\frac{1}{2}} = 0 \quad (4.8)$$

where we have introduced the notation $\langle \rangle$ for the difference form of the wave operator. (4.8) is a standard mid-point discretization of (4.7). We denote the exact analytical solution of (4.7) by ζ . By use of Taylor series we get:

$$\begin{aligned} \langle \rangle \zeta_{j+\frac{1}{2}, p+\frac{1}{2}}^{n+\frac{1}{2}} &= \frac{1}{12} \Delta t^2 \frac{\partial^4 \zeta}{\partial t^4} - \frac{\Delta x^2}{24} \left\{ \frac{\partial^3}{\partial x^3} (h \frac{\partial \zeta}{\partial x}) + \frac{\partial}{\partial x} (h \frac{\partial^3 \zeta}{\partial x^3}) \right\} \\ &\quad - \frac{\Delta y^2}{24} \left\{ \frac{\partial^3}{\partial y^3} (h \frac{\partial \zeta}{\partial y}) + \frac{\partial}{\partial y} (h \frac{\partial^3 \zeta}{\partial y^3}) \right\} \\ &\quad + O(\Delta t^4, \Delta x^4, \Delta y^4) \end{aligned} \quad (4.9)$$

where the right hand side is to be calculated at the node-point. A fourth order scheme for (4.7) may be constructed by introducing second order differences for the derivatives or their equivalents, in (4.9).

We assume, according to the ideas of the previous paragraph, that a fourth order scheme for the hydrostatic equations can be constructed by adding artificial dispersion terms to the discretized equations of motion. Since no mixed space derivatives appear at the right hand side of (4.9), the corrected difference equations are expected to be of the form:

$$[\dot{u} = -\delta_x \eta + \alpha_x \delta_t^2 \delta_x \eta + \beta_x \delta_x^2 \dot{u} + \gamma_x h^{-1} \delta_x^2 (h\dot{u})]_{j,p+\frac{1}{2}}^n \quad (4.10)$$

$$[\dot{v} = -\delta_y \eta + \alpha_y \delta_t^2 \delta_y \eta + \beta_y \delta_y^2 \dot{v} + \gamma_y h^{-1} \delta_y^2 (h\dot{v})]_{j+\frac{1}{2},p}^n \quad (4.11)$$

The same manipulation which leads to (4.8) applied to the set (2.6), (4.10) and (4.11) gives an equation of the form (4.9). By use of $\dot{u} = -\delta_x \eta + O(\Delta x^2, \Delta t^2)$ etc we find that the dispersion terms give rise to a second order representation of the derivatives on the right hand side of (4.9) if:

$$\alpha_x = \alpha_y = -\frac{1}{12}\Delta t^2, \quad \beta_x = \gamma_x = -\frac{1}{24}\Delta x^2, \quad \beta_y = \gamma_y = -\frac{1}{24}\Delta y^2 \quad (4.12)$$

The generalized version of (4.4) is obtained by replacing the terms

$$\delta_t^2 \delta_x \eta \quad \text{and} \quad \delta_t^2 \delta_y \eta \quad \text{in (4.10) and (4.11) by their equivalents}$$

$$-\{\delta_x^2 (h\dot{u}) + \delta_y (\bar{h}^y \delta_y \dot{u}) + \delta_y (\bar{v}^x \delta_x h)\} \quad \text{and} \quad -\{\delta_y^2 (h\dot{v}) + \delta_x (\bar{h}^x \delta_x \dot{v}) + \delta_x (\bar{u}^y \delta_y h)\}.$$

(See paragraph 1.2.)

$$0 = [\dot{u} + \delta_x \eta - \{\frac{1}{2}\epsilon h + \frac{1}{12}(\frac{1}{2}h^{-1}\Delta x^2 - \Delta t^2)\} \delta_x^2 (h\dot{u})$$

$$- (\frac{1}{2}\epsilon h - \frac{1}{12}\Delta t^2) \{\delta_y (\bar{h}^y \delta_y \dot{u}) + \delta_y (\bar{v}^x \delta_x h)\} \quad (4.13)$$

$$+ (\frac{1}{6}\epsilon h^2 - \frac{1}{24}\Delta x^2) \delta_x^2 \dot{u} + \frac{1}{6}\epsilon h^2 \delta_y^2 \dot{u}]_{j,p+\frac{1}{2}}^n$$

$$\begin{aligned}
 0 = & \left[\dot{v} + \delta_y \eta - \left\{ \frac{1}{2} \epsilon h + \frac{1}{12} \left(\frac{1}{2} h^{-1} \Delta y^2 - \Delta t^2 \right) \right\} \delta_y^2 (h \dot{v}) \right. \\
 & - \left. \left(\frac{1}{2} \epsilon h - \frac{1}{12} \Delta t^2 \right) \left\{ \delta_x (\bar{h}^x \delta_x \dot{v}) + \delta_y (\bar{u}^y \delta_y h) \right\} \right. \\
 & \left. + \left(\frac{1}{6} \epsilon h^2 - \frac{1}{24} \Delta x^2 \right) \delta_x^2 \dot{u} + \frac{1}{6} \epsilon h^2 \delta_y^2 \dot{u} \right]_{j+\frac{1}{2}, p}^n
 \end{aligned} \tag{4.14}$$

Because no new terms have appeared the replacement of (2.7) and (2.8) by (4.13) and (4.14) will not be accompanied by a significant increase in neither computer time nor programming effort. For constant depth $h = h_0$ (2.6), (4.13) and (4.14) give the dispersion relation:

$$\begin{aligned}
 \Omega^2 = & \frac{K^2}{1 + \left(\frac{1}{3} \epsilon h_0^2 - \frac{1}{12} h_0 \Delta t^2 \right) A^2 + \frac{\Delta y^2}{12} K^2} \\
 & + \frac{L^2}{1 + \left(\frac{1}{3} \epsilon h_0^2 - \frac{1}{12} h_0 \Delta t^2 \right) A^2 + \frac{\Delta y^2}{12} L^2}
 \end{aligned} \tag{4.15}$$

In figure 4 (4.15) is compared to the dispersion relation of the uncorrected scheme, (2.16). In some cases the improvements are substensial. Corresponding corrections to conditions at complex boundaries are hard to construct. In cases where reflexions from such boundaries are essential the correction terms in (4.13) and (4.14) are of less interest.

CONCLUSION

A numerical method for solving the Boussinesq equations has been described. In the case of uniform depth the scheme is proved to reproduce the analytical dispersion relation well. The dispersion relation of the scheme is further improved by inclusion of

the correction terms derived in section 4. In the simple tests in section 3 convergence for the accelerations were achieved after two iterations. Use of the proposed relaxation factors will in many cases accelerate the convergence. The tests and discussions in this preprint should provide a good basis for examining the application of the method to problems involving nonlinearity, variations in depth and complicated boundaries. These will be the topics of the next part of the documentation.

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FIGURE CAPTIONS

Figure 1: The grid.

Figure 2: Dispersion relations. The phase velocities c are plotted as functions of the wavelength $\lambda = 2\pi/a$. Since there is no adequate characteristic wave length we choose $\lambda_c = h_c \cdot \lambda$ is thus the ratio between the real wave length and h_c . The full drawn lines correspond to analytical dispersion relations. ϕ denotes the angle between the wave number vector and the x-axis.

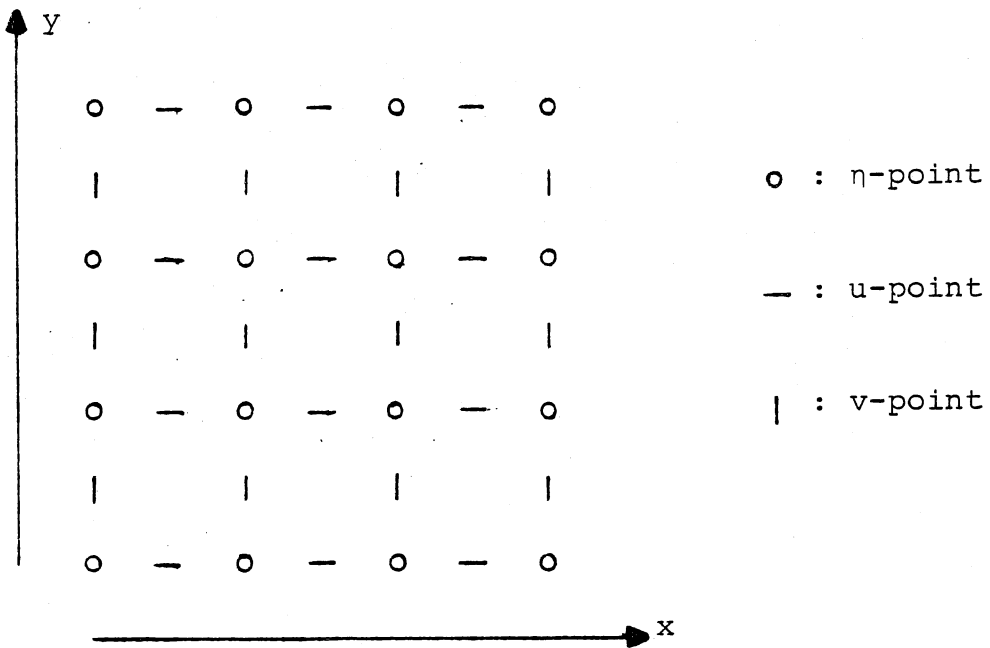
- a) (i) The fully inviscid set $c = (\tanh(a)/a)^{1/2}$
(ii) The Boussinesq equations $c = a/(1 + \frac{1}{3}a^2)^{1/2}$
(iii) The KDV equation $c = a - \frac{1}{2}a^2$
- b) — : analytical dispersion relation for the Boussinesq equations (same as a) (ii))
+++ : $\Delta t = \Delta x = 1.5$, $\phi = 0$
--- : $2\Delta t = \Delta x = 1.5$, $\phi = 0$
- c) — : same as in b)
+++ : $\Delta t = \Delta x = 0.25$, $\phi = 0$
--- : $2\Delta t = \Delta x = 2.5$, $\phi = 0$
- d) — : same as in b)
+++ : same as in b)
--- : $\Delta x = \Delta t = 1.5$, $\phi = \pi/4$

Figure 3: Time-series of η . All important data are given in paragraph 3.3. The series obtained from (2.10) through (2.12) and the iteration procedure are depicted by fully drawn line and dotted points respectively. The numbers i, ii and iii correspond to those in the paragraph.

Figure 4: The dispersion relation (4.15) for the corrected scheme compared to the dispersion relation for the uncorrected scheme (4.16) and the analytical dispersion relation of the Boussinesq equation.

- : analytical dispersion relation
- : uncorrected dispersion relation with $2\Delta t = \Delta x = 1.5$,
 $\phi = 0$ (same as in figure 2b)
- +++ : corrected dispersion relation with $2\Delta t = \Delta x = 1.5$,
 $\phi = 0$.

a)



b)

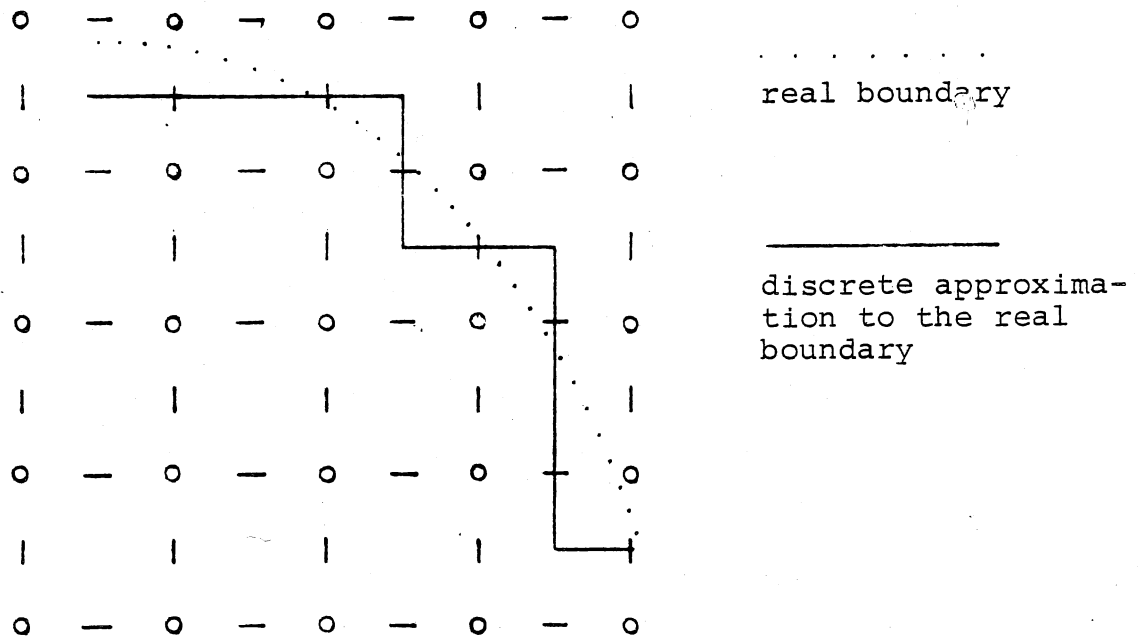


Figure 1.

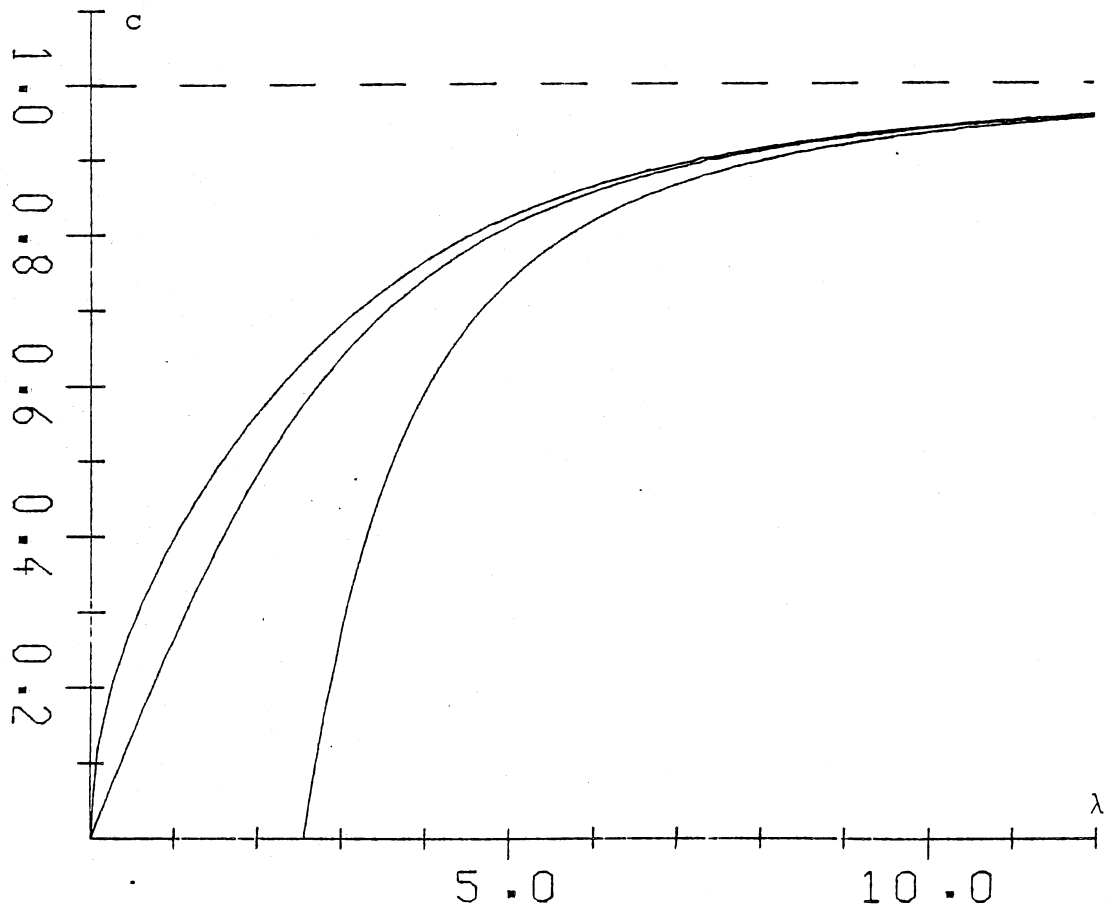


Figure 2a.

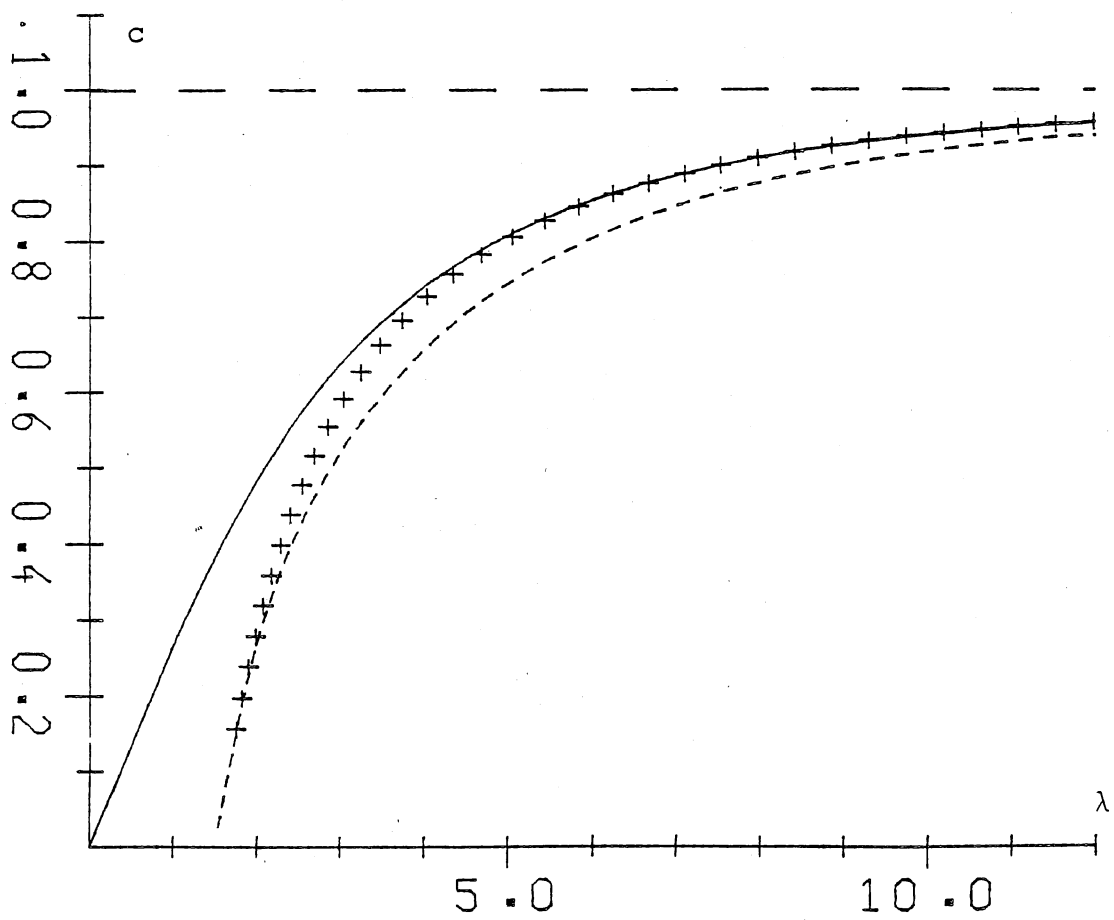


Figure 2b.

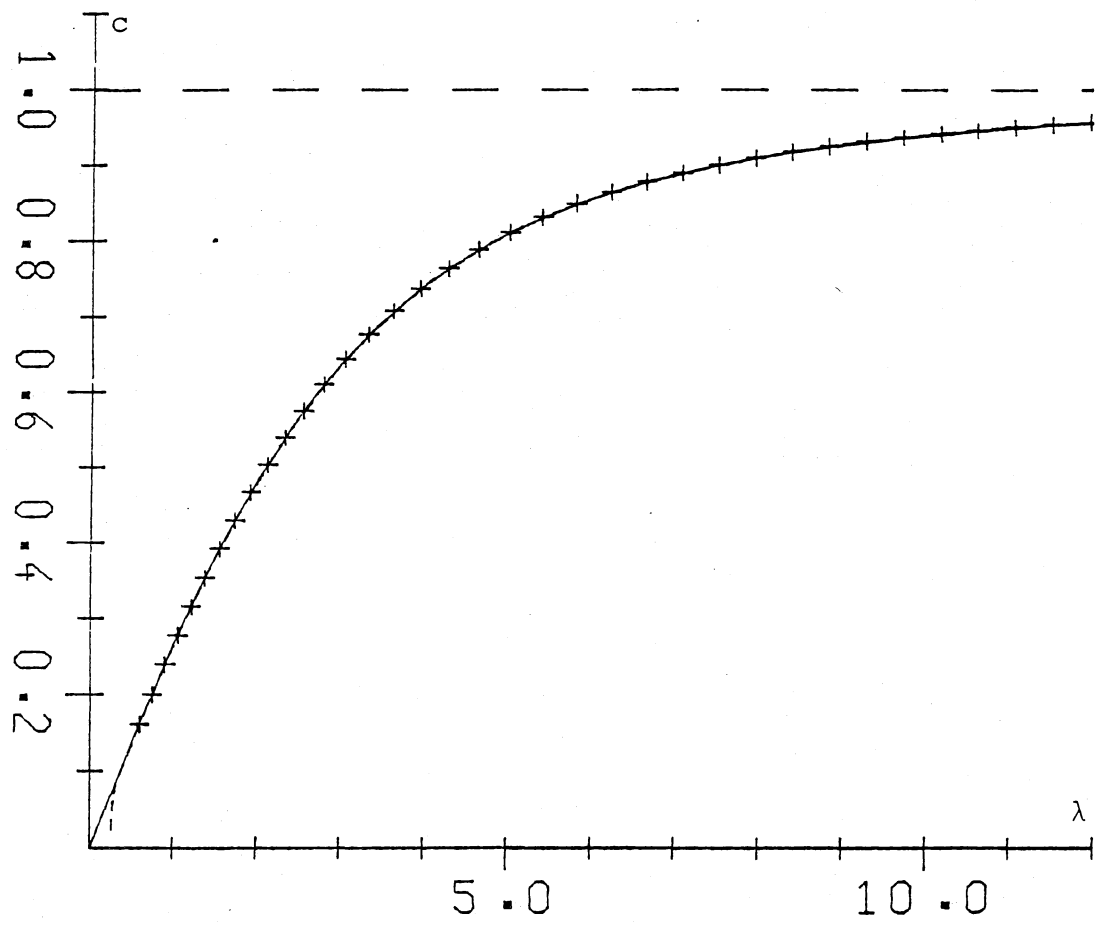


Figure 2c.

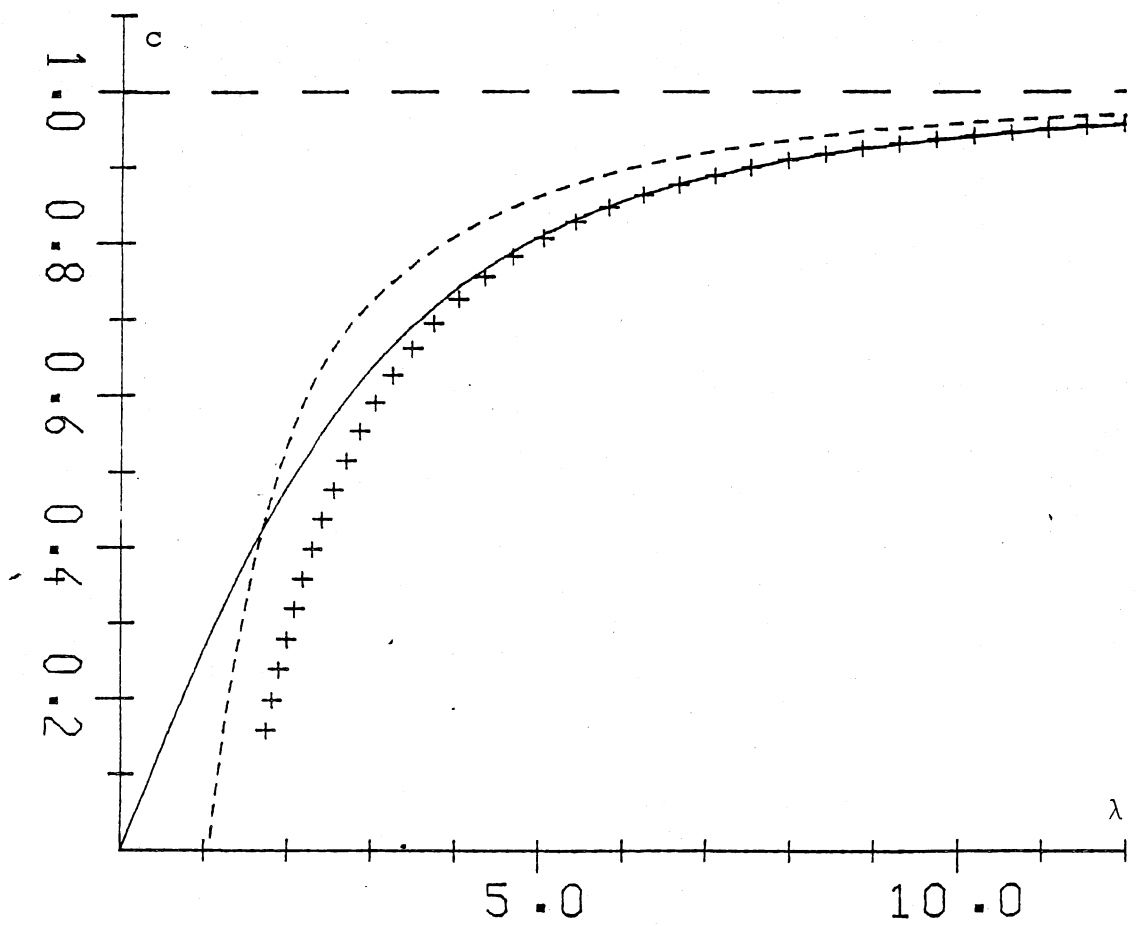
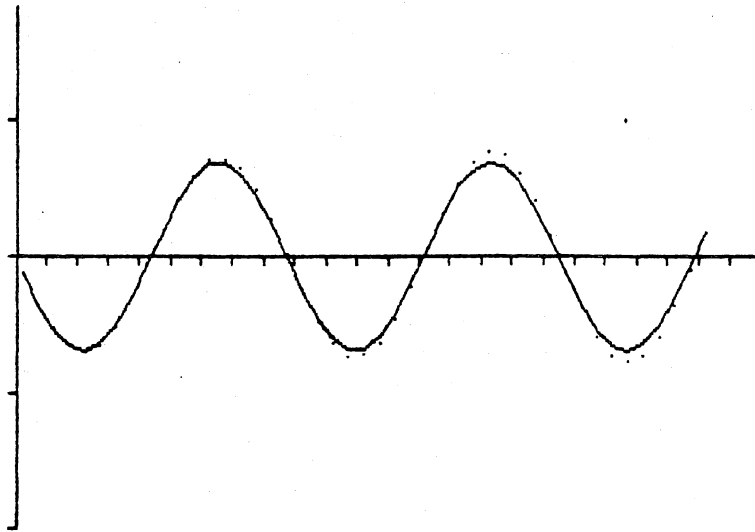
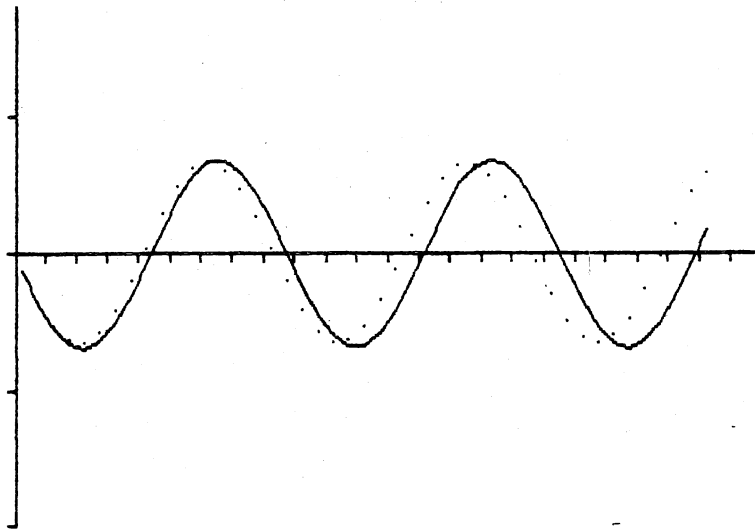


Figure 2d.

(i)



(ii)



(iii)

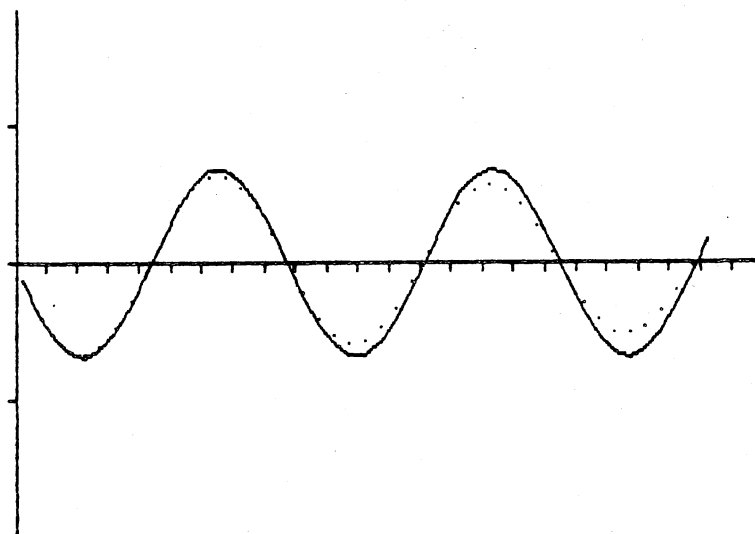


Figure 3.

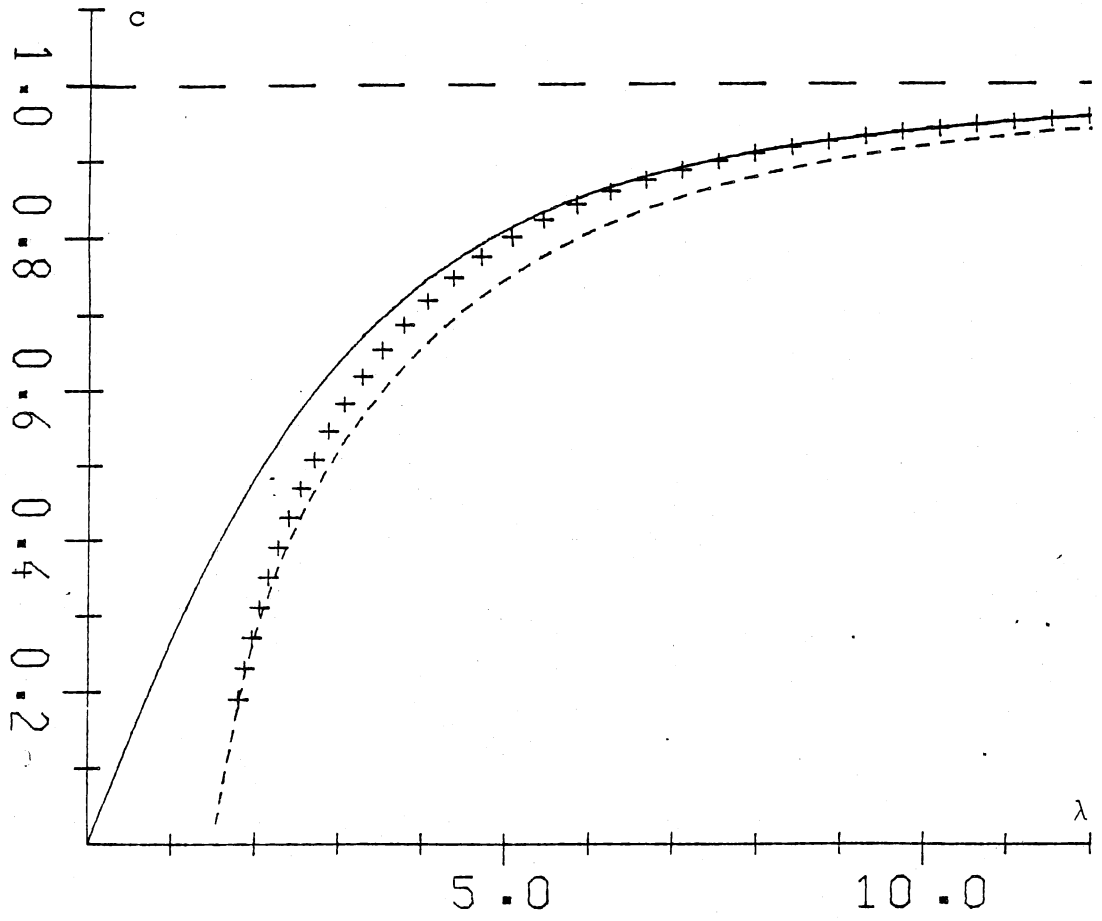


Figure 4.

