FIRST- AND SECOND-ORDER HYDRODYNAMIC FORCES ON SUBMERGED, RESTRAINED CYLINDERS OF ARBITRARY SHAPE

by

Torgeir Vada

Dept. of Mech., University of Oslo,

Blindern, Oslo 3, Norway

Abstract.

The first- and second-order forces on a submerged, restrained cylinder is computed by an integral equation method. It is shown that the first-order and mean forces can be easily computed for cylinders of arbitrary shape. The solution is given for three specific contours. The second-order oscillatory force is computed only for a circular cylinder.

1. Introduction.

By the use of Green's theorem the diffraction problem for a submerged cylinder can be transformed to an integral equation. In a previous report in this series /1/ we solved this equation in the linear case for a circular cylinder by a boundary element method with cubic splines. A comparison with previous results given by Ogilvie /2/ and Mehlum /3/, obtained by different methods gave encouraging results. In this report we extend the use of this method in two different directions.

First we study the first-order problem for a smooth cylinder of arbitrary shape. It is shown that the first-order potential can be obtained by solving a very good conditioned set of linear equations. The first-order and mean forces can then be found by pressure integration. The numerical solution is given for three different cases.

We also have computed the oscillatory second-order force on a circular cylinder. By a simple application of Green's theorem, previously used by Søding /4/, this can be achieved without the full solution of the second-order problem. Because of a mathematical difficulty due to the presence of a reflected wave, the computation has not been extended to other contours.

2. Formulation of the problem.

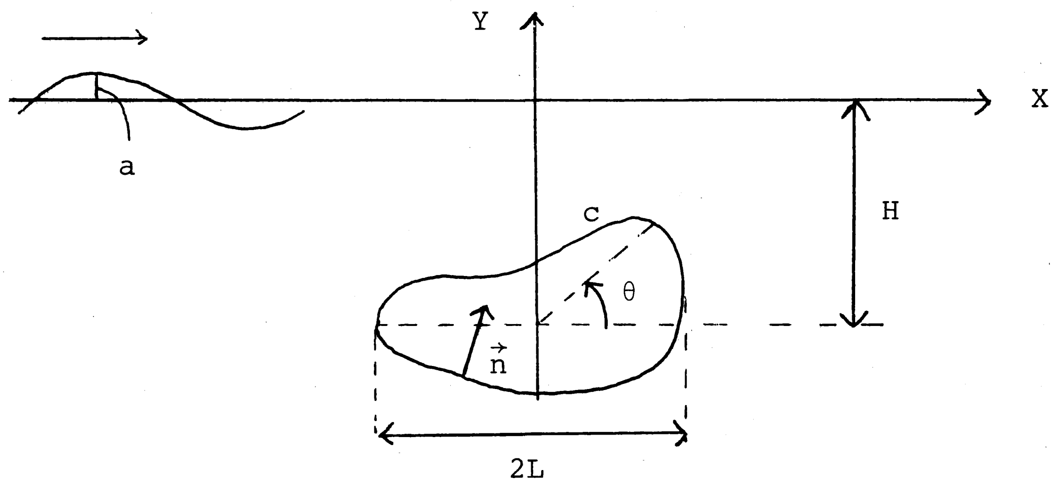


Figure 1.

A 2-dimensional periodic wave with amplitude a and frequency ω is disturbed by a submerged, restrained cylinder with contour C. The fluid is inviscid and incompressible and the motion is assumed to be irrotational. A velocity potential $\hat{\Phi}(X,Y,t)$ then exists.

We introduce dimensionless quantities

$$(2\text{-}1)\quad \begin{cases} x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad \varepsilon = \frac{a}{L}, \quad h = \frac{H}{L}, \quad k = \frac{L\omega^2}{g}, \quad \tau = \omega t, \\ \Phi(x,y,\tau) = \frac{1}{\omega L^2}\hat{\Phi}, \quad p(x,y,\tau) = \frac{1}{\rho g L}\hat{p}(X,Y,t) \\ \vec{F}(\tau) = \frac{1}{\rho g L^2}\hat{\vec{F}}(t) \end{cases}$$

where $\vec{F}$ is the force per unit length of the cylinder. We also assume that Φ can be expanded in a series

$$(2\text{-}2)\qquad \Phi = \sum_{n=1}^{\infty} \varepsilon^n \Phi^{(n)}$$

Since the wave is periodic we write

$$(2\text{-}3)\qquad \begin{aligned}\Phi^{(1)} &= \mathrm{Re}\{(\phi_0(x,y)+\phi_7(x,y))e^{-j\tau}\}\\ \Phi^{(2)} &= \mathrm{Re}\{\phi_{20}(x,y)+\phi_{22}(x,y)e^{-2j\tau}\}\end{aligned}$$

where j is the imaginary unit and

$$(2\text{-}4)\qquad \phi_0(x,y) = -\frac{1}{k}e^{ky}e^{jkx}$$

is the potential of the incoming wave. This potential is correct to second order since the depth is infinite. $\phi_7(x,y)$, $\phi_{20}(x,y)$ and $\phi_{22}(x,y)$ are the potentials for the disturbance of this wave due to the cylinder.

It follows from Bernoulli's equation that the time-independent part of the second-order potential only gives a third-order contribution to the force. We therefore neglect the potential ϕ_{20}.

Introduction of (2) in the exact boundary conditions and Taylor expansion around $Y = 0$ gives the boundary conditions for ϕ_7 and ϕ_{22}:

$$(2\text{-}5)\qquad \begin{cases}(\phi_7)_n = -(\phi_0)_n, & (x,y) \in C\\ (\phi_7)_y - k\phi_7 = 0, & y = 0\\ (\phi_7)_y = 0, & y = -\infty\end{cases}$$

$$(2\text{-}6)\qquad \begin{cases}(\phi_{22})_n = 0, & (x,y) \in C\\ (\phi_{22})_y - 4k\phi_{22} = f(x), & y = 0\\ (\phi_{22})_y = 0, & y = -\infty\end{cases}$$

where

$$(2\text{-}7)\qquad f(x) = \frac{kj}{2}\left[3k^2(\phi^{(1)})^2 + \phi^{(1)}_{xx}\,\phi^{(1)} + 2(\phi^{(1)}_x)^2\right]_{y=0}$$

and

$$\phi^{(1)} = \phi_0 + \phi_7 .$$

In addition ϕ_7 and ϕ_{22} must satisfy the two-dimensional Laplace equation and the radiation conditions

(2-9) $\quad (\phi_7)_x \mp jk\phi_7 = 0, \quad x = \pm\infty$

(2-10) $\quad (\phi_{22})_x \mp 4jk\phi_{22} = 0, \quad x = \pm\infty$

Applying Greens theorem with the Greens function

$$G(z,\xi,k) = \mathrm{Re}\{\ln(z-\xi)-\ln(z-\bar{\xi})+2\int_0^\infty \frac{e^{-i\nu(z-\bar{\xi})}}{k-\nu}d\nu-2\pi je^{-ik(z-\bar{\xi})}\}$$

we obtain (see for instance Potash /5/)

(2-11) $$\phi_7(z) = \frac{1}{2\pi}\int_C[\phi_7(\xi)\frac{\partial G(z,\xi,k)}{\partial n(\xi)} - \frac{\partial\phi_7(\xi)}{\partial n(\xi)}G(z,\xi,k)]ds(\xi)$$

where $z = x+iy$ is a point in the fluid. We now let z approach C. This limiting operation gives the integral eqation

(2-12) $$-\pi\phi_7(z)+\int_C\phi_7(\xi)\frac{\partial G(z,\xi,k)}{\partial n(\xi)}ds(\xi) = \int_C\frac{\partial\phi_7(\xi)}{\partial n(\xi)}G(z,\xi,k)ds(\xi).$$

This is an ordinary Fredholm equation of the second kind from which ϕ_7 can be found on C.

We describe the contour C by the parametric equations

$$x = x(\theta), \quad y = y(\theta), \quad \theta \in [0,2\pi]$$

These functions are assumed to be twice continuosly differentiable and may, if necessary, be approximated by a cubic spline. The continuity condition is imposed since, in the next section, we shall assume that the velocity potential satisfies the same condition.

Equation (12) may now be written

(2-13) $$-\pi\phi_7(\hat{\theta})+\int_0^{2\pi}\phi_7(\theta)\frac{\partial G(\hat{\theta},\theta,k)}{\partial n(\theta)}A(\theta)d\theta=\int_0^{2\pi}-\frac{\partial\phi_0(\theta)}{\partial n(\theta)}G(\hat{\theta},\theta,k)A(\theta)d\theta$$

where

$$A(\theta) = \sqrt{(x')^2+(y')^2}. \tag{2-14}$$

We have also used the boundary condition on C in (5).

3. The solution method.

We divide the interval $[0,2\pi]$ into N parts

$$0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{N-1} < \theta_N = 2\pi$$

and write the potential ϕ_7 as the spline

$$\phi_7(\theta) = \sum_{i=1}^{N} q_i B_i(\theta) \tag{3-1}$$

where B_i is the B-spline starting at θ_{i-2} (see appendix B), and we have

$$\theta_{i\pm N} = \theta_i \pm 2\pi. \tag{3-2}$$

When this is substituted in (2-13) we obtain by collocation at θ_i a set of linear equations from which the coefficients q_i can be found:

$$\begin{aligned} &-\pi(B_{i-1}(\theta_i)q_{i-1}+B_i(\theta_i)q_i+B_{i+1}(\theta_i)q_{i+1}) \\ &+\sum_{j=1}^{N} A_{ij}q_j = \sum_{j=1}^{N} B_{ij}, \quad i = 1,2,\cdots,N \end{aligned} \tag{3-3}$$

where

$$A_{ij} = \int_{\theta_{j-2}}^{\theta_{j+2}} B_j(\theta)\frac{\partial G(\theta_i,\theta,k)}{\partial n(\theta)}A(\theta)d\theta$$

$$B_{ij} = \int_{\theta_{j-1}}^{\theta_j} -\frac{\partial \phi_0(\theta)}{\partial n(\theta)}G(\theta_i,\theta,k)A(\theta)d\theta.$$

ϕ_0 is given from (2-4), and the integrals are computed by the 3-point Gauss formula.

This set of equations is very good conditioned, in fact it is almost diagonal-dominant and can therefore easily be solved by straight-forward Gaussian elimination.

4. The first-order and mean forces.

The dimensionless Bernoulli equation gives

$$(4\text{-}1) \qquad p(x,y,\tau) = -k(\Phi_\tau + \tfrac{1}{2}(\nabla\Phi)^2)$$

where the hydrostatic terms are neglected. The force per unit length of the cylinder is

$$\vec{F}(\tau) = \varepsilon\overrightarrow{F_1} + \varepsilon^2\overrightarrow{F_2} + O(\varepsilon^3)$$

where

$$\vec{F}_1 = \mathrm{Re}\{\overrightarrow{f_1}e^{-j\tau}\}$$

$$\vec{F}_2 = \mathrm{Re}\{\overrightarrow{f_{20}} + \overrightarrow{f_{22}}e^{-2j\tau}\}$$

Substitution of (2-2) and (2-3) into (1) then gives

$$(4\text{-}2) \qquad \overrightarrow{f_1} = \int_0^{2\pi} Kj\phi^{(1)}\vec{n}(\theta)A(\theta)d\theta$$

$$(4\text{-}3) \qquad \overrightarrow{f_{20}} = \int_0^{2\pi} -\tfrac{1}{4}K\phi_s^{(1)}\overline{\phi_s^{(1)}}\vec{n}(\theta)A(\theta)d\theta$$

$$(4\text{-}4) \qquad \overrightarrow{f_{22}} = \int_0^{2\pi} \{2Kj\phi_{22} - \tfrac{1}{4}K(\phi_s^{(1)})^2\}\vec{n}(\theta)A(\theta)d\theta$$

where

$$\phi^{(1)} = \phi_0 + \phi_7$$

and $A(\theta)$ is given from (2-14).

When the equations (3-3) are solved ϕ_7 is known from (3-1), and the integrals in (2) and (3) can be computed. Again the 3-point formula is used.

As numerical examples we have studied contours given by the equations

$$x(\theta) = \frac{\cos\theta - \alpha\cos 3\theta}{1-\alpha}$$

$$y(\theta) = b\,\frac{\sin\theta + \alpha\sin 3\theta}{1-\alpha} - h.$$

Contours of this type are often called Lewis-forms. We have made computations for the three cases shown in fig. 2.

The results are shown in fig. 3-11 and are as one should expect. We notice from (A-1) that the maximum values of the mean horisontal force corresponds to a reflection coefficient of 0.26 for the ellipse and 0.23 for the "square".

The computation time for one computation (i.e. one h and one k) ranged from 30 seconds for the most difficult case ("square", $h = 1.25$, $K<1$, 30 nodes) to 15 seconds for the easiest (circle, $K>1$, 20 nodes) when the nodes were distributed with some care. With equidistant nodes the computation time increases considerably.

The computations have been carried out by a set of ALGOL programs on a DEC-10 computer. With FORTRAN programs on a better computer the computation time should be reduced considerably.

5. The second-order oscillatory force.

We now introduce the radiation potentials for an oscillation with frequency 2ω. These potentials are the solutions of

$$(5\text{-}1)\quad \begin{cases} \nabla^2\phi_i = 0, & \text{in the fluid} \\ (\phi_i)_n = n_i, & (x,y) \in C \\ (\phi_i)_y - 4K\phi_i = 0, & y = 0 \quad i = 1,2 \\ (\phi_i)_y = 0, & y = -\infty \\ (\phi_i)_x \mp 4jK\phi_i = 0, & x = \pm\infty \end{cases}$$

and can therefore be found in exactly the same way as ϕ_7, with K replaced by 4K and $-(\phi_0)_n$ replaced by n_i in the integral equation (2-13).

Using Greens theorem, (2-6) and (1) one obtains

$$(5\text{-}2)\qquad \int_C \phi_{22} n_i ds = \int_{-\infty}^{\infty} \phi_i(x,0) f(x) dx$$

where $f(x)$ is given by (2-7).

When ϕ_7 is known on C it can be found anywhere else from (2-11). Differentiation of this equation also gives ϕ_x and ϕ_{xx}. Hence $f(x)$ can be computed when the equations (3-3) are solved. $\phi_i(x,0)$ are found in exactly the same way. This makes it possible to compute the first term in (4-4) and thereby the force component $\vec{f}_{22}$ without the solution of the second order problem.

If we write the potential of the reflected wave as

$$(5\text{-}3)\qquad \lim_{x\to-\infty} \phi_7(x,0) = \frac{R}{K} e^{-jKx}$$

we obtain from the radiation condition (2-9)

$$(5\text{-}4)\qquad \lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to-\infty} f(x) = -4KjR.$$

We therefore see that the right-hand side of (2) can be computed only when $R = 0$. This restricts the use of this method to a circular cylinder.

It may be of interest to estimate the range of parameter values for which the linear theory can give a fair approximation of the force on the cylinder. We are now able to do this.

One restriction on the linear theory is that the wave steepness must be small. We will assume that

$$(5\text{-}5) \qquad \frac{\varepsilon}{\lambda} < 0.1 .$$

Another restriction on the wave amplitude may be obtained by a comparison of $\vec{f}_1$ and $\vec{f}_{22}$, where the latter may be taken as a measure of the importance of non-linear effects. A possible criterion for the validity of linear theory is then

$$|\vec{f}_1| > 10\varepsilon|\vec{f}_{22}|$$

From fig. 12 we see that this implies

$$(5\text{-}6) \qquad \begin{cases} \varepsilon < 0.05 & \text{if} \quad h \approx 1.25 \\ \varepsilon < 0.1 & \text{if} \quad h \approx 1.5 \\ \varepsilon < 0.25 & \text{if} \quad h \approx 1.75 \end{cases}$$

for the most important wavelengths. We see from the figures that only wavelengths $\lambda > \pi$ $(K<2)$ are of interest here, and the most critical wavelengths are $\lambda \in [10,20]$. For all these wavelengths (5) is satisfied when ε is chosen as in (6). We also see that if we study waves with amplitudes of 1-10 m, linear theory can only be applied if $h>1.75$ unless L is very large.

The computation time needed in the second-order case was 5-6 minutes for $h>1.5$. It was necessary to use 30 nodes for $K<1$ and

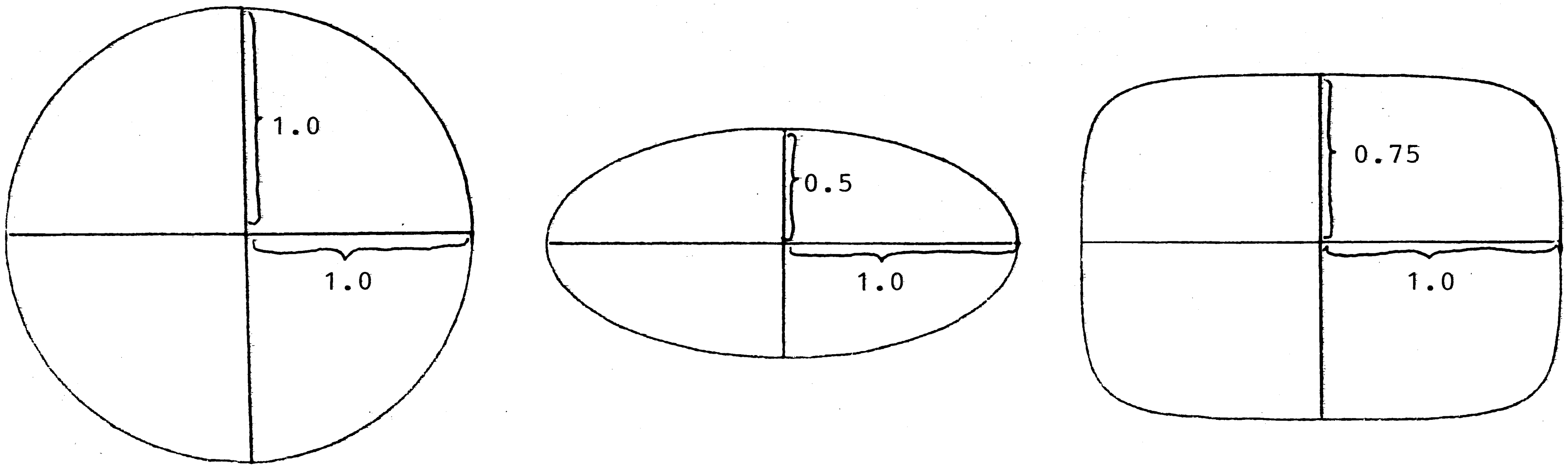
24 nodes for $k>1$, and the integrand in (2) was computed on the interval $[-7,7]$.

When $h<1.5$ the necessary computation time increases rapidly because the function $f(x)$ becomes very ill behaved. But as we have pointed out previously, such values should be treated with strongly nonlinear methods.

For the other two contours considered in the previous section we expect that the non-linear effects are at least as important as for the circle. We therefore only show results for cases where the distance from the surface to the top of the cylinder $(h-b)$ is at least 0.5

Acknowledgement.

This report has been written under surveillance of Prof. E. Palm and it is a pleasure to thank him for helpful advises.



a) b = 1, α = 0
(circle)

b) b = 0.5, α = 0
(ellipse)

c) b = 0.75, α = 0.1
("square")

Fig. 2: The contours we have studied

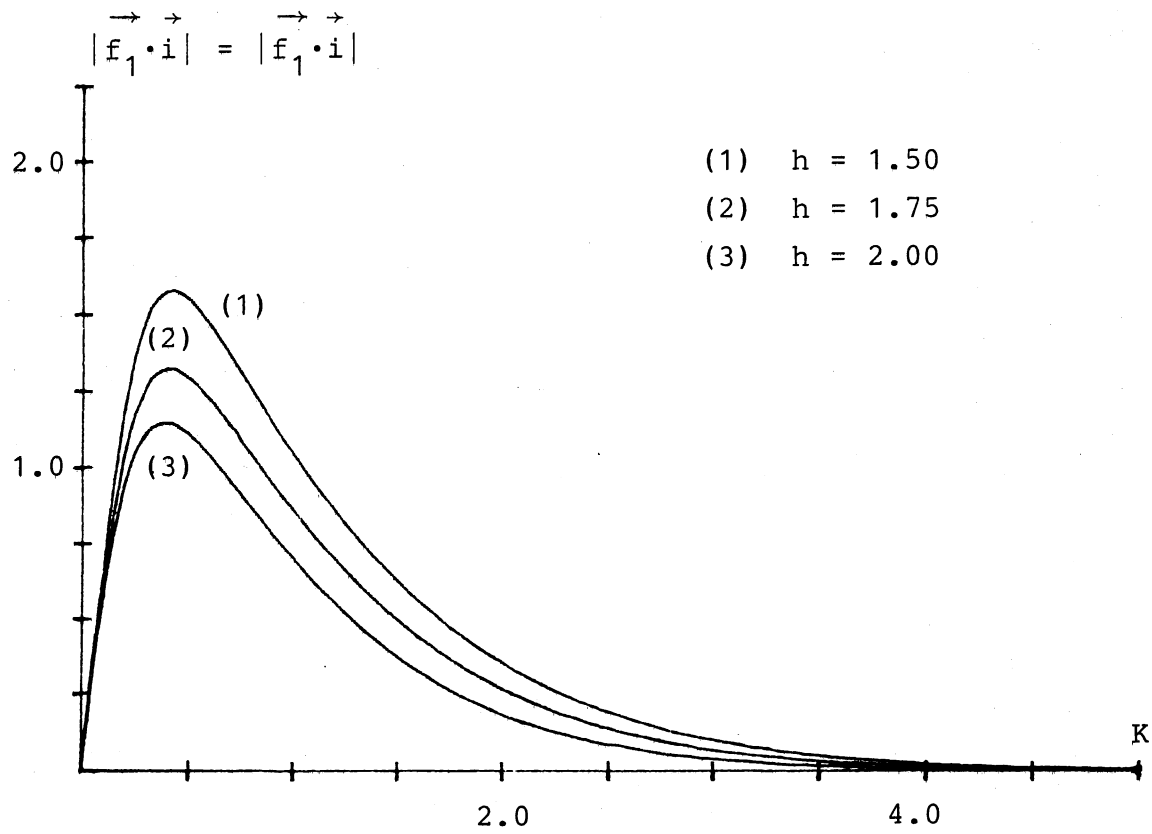


Fig. 3: Amplitude of first-order force for the circle

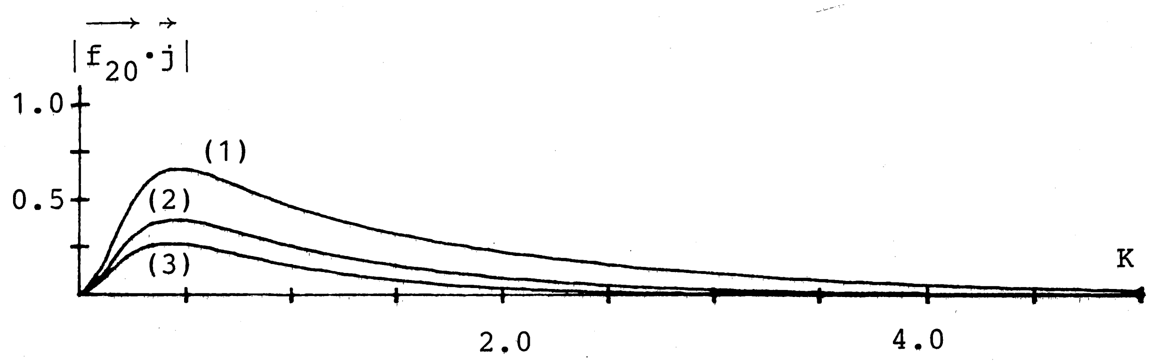


Fig. 4: Mean vertical force on the circle.

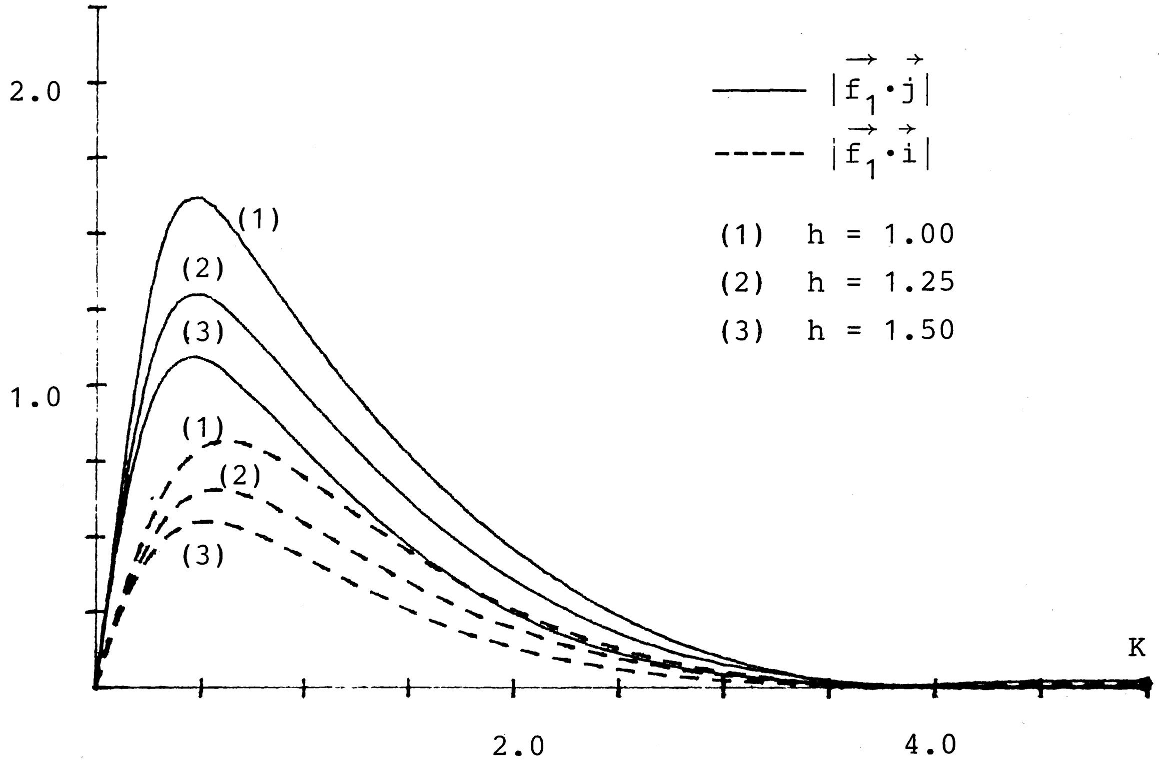


Fig. 5: Amplitude of the first-order force for the ellipse

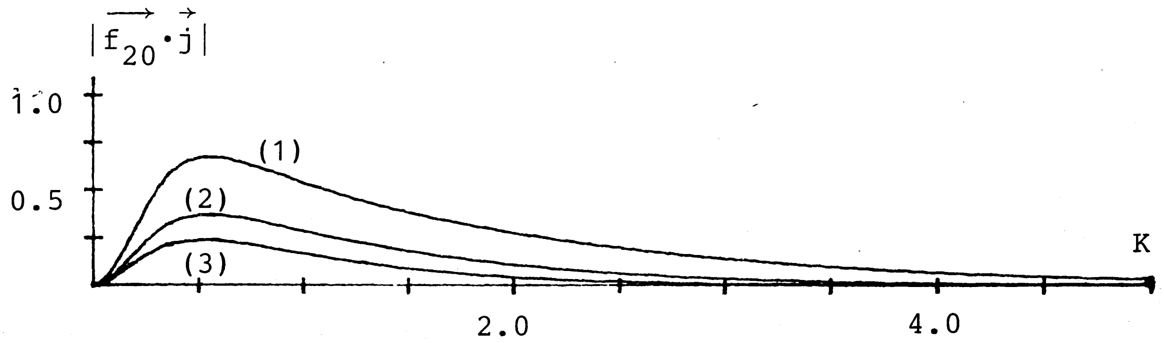


Fig. 6: Mean vertical force on the ellipse

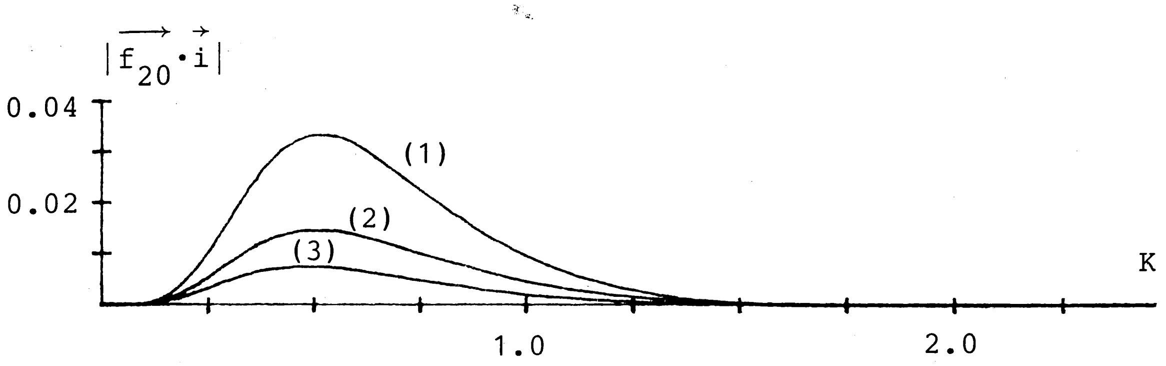


Fig. 7: Mean horisontal force on the ellipse

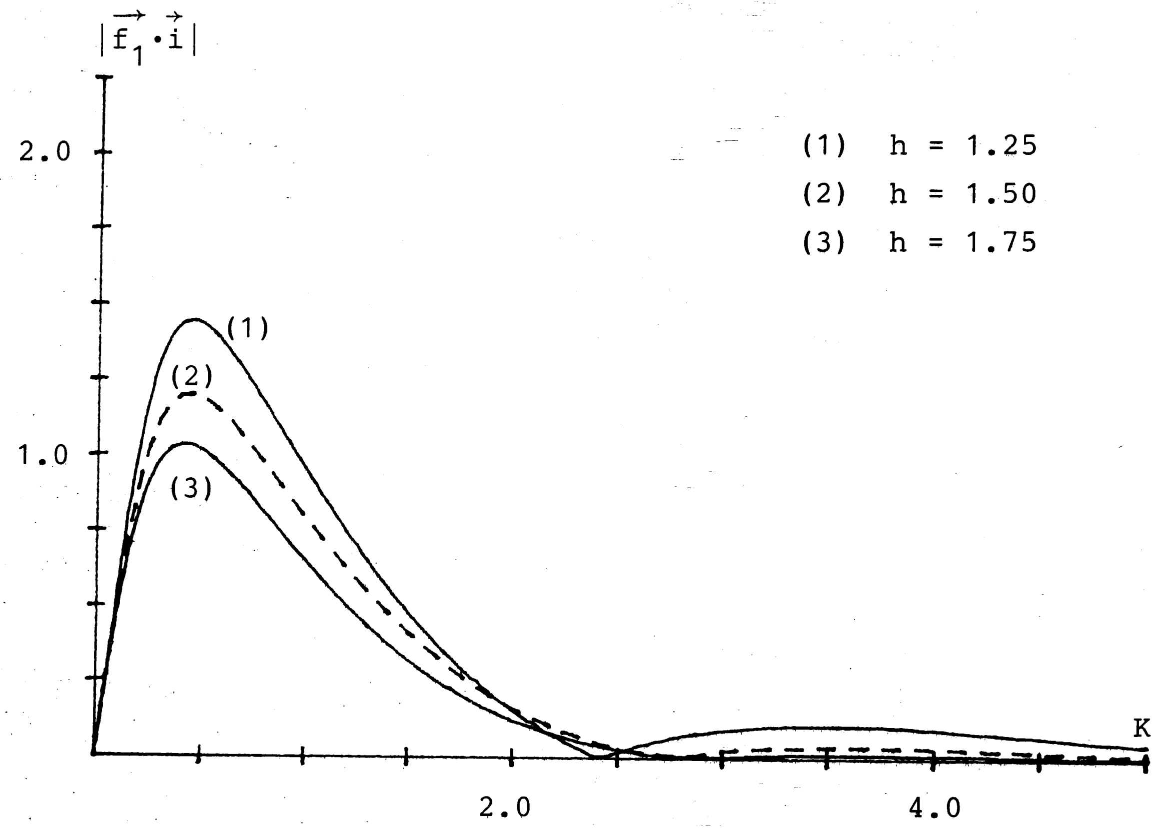


Fig. 8: Amplitude of horisontal first-order force on the "square"

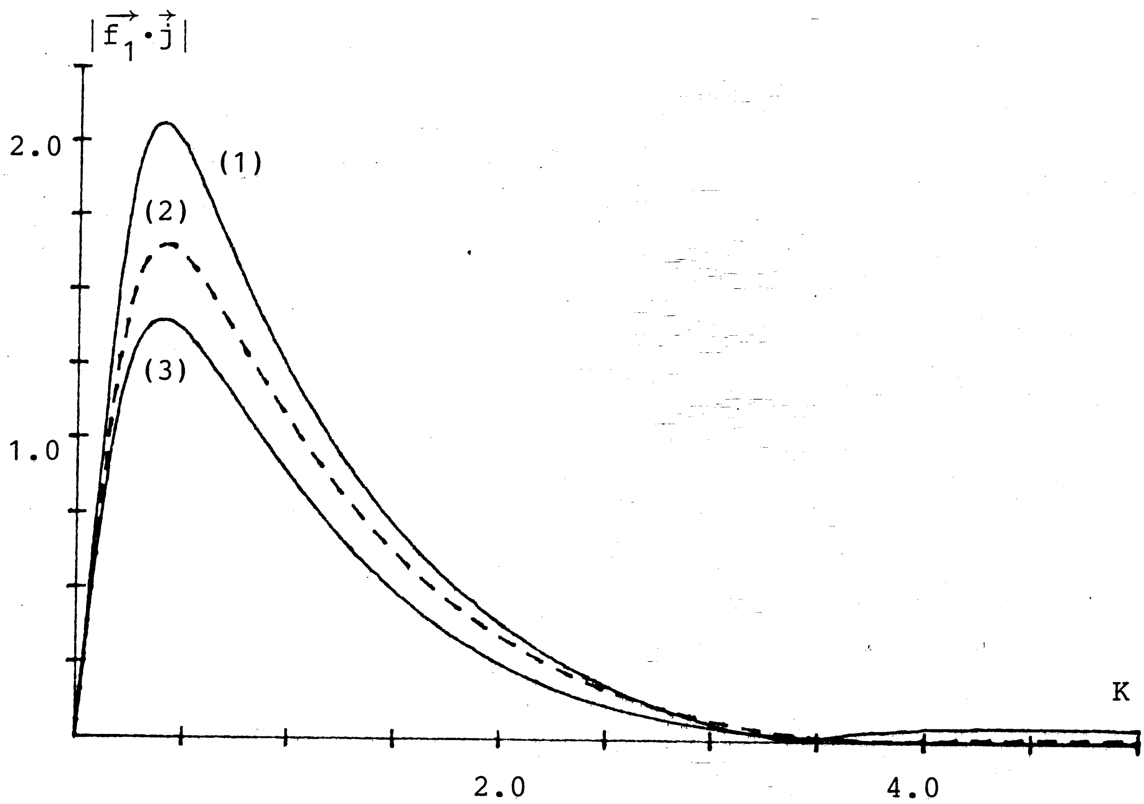


Fig. 9: Amplitude of vertical first-order force on the "square".

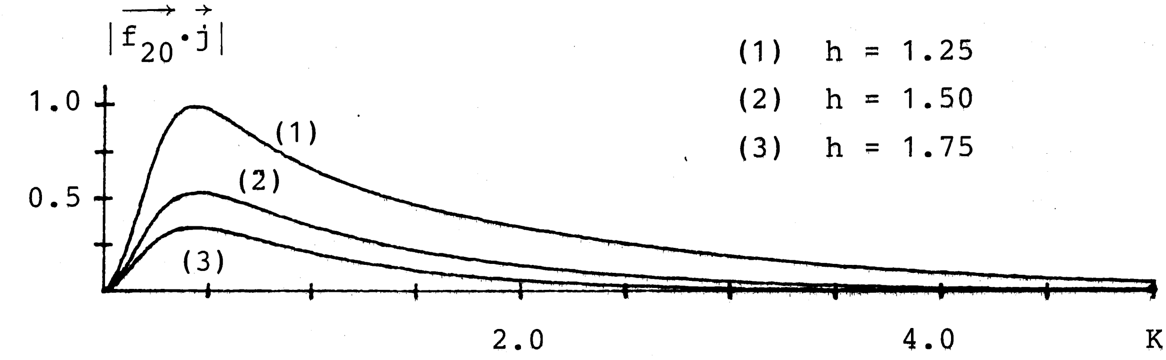


Fig. 10: Mean vertical force on the "square"

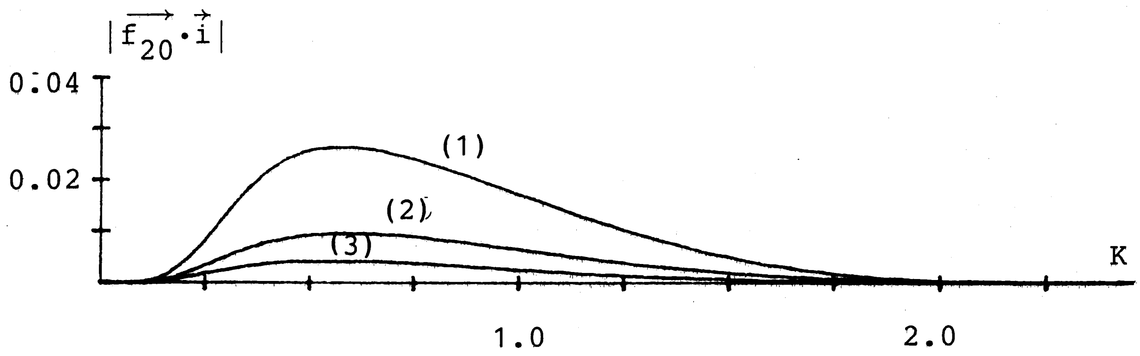


Fig. 11: Mean horisontal force on the "square"

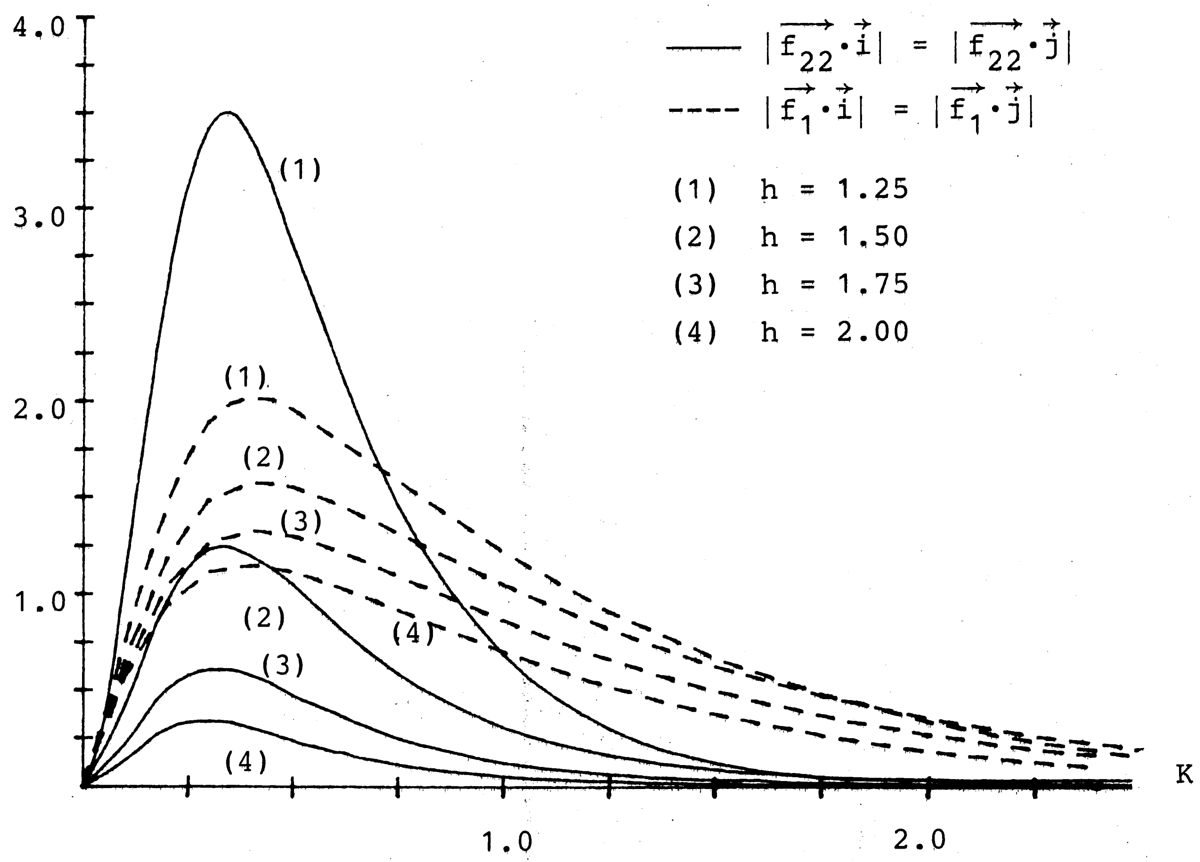


Fig. 12: Amplitude of first-and second-order oscillating forces on the circle

Appendix A. Numerical checks.

Apart from (5-4) there are three equations which can be used in order to check the computations.

From momentum conservation there follows a relation between the reflection coefficient, r, and the mean horisontal force (Longuet-Higgins /6/)

$$\text{(A-1)} \qquad \overrightarrow{f_{20}} \cdot \vec{i} = \tfrac{1}{2} r^2$$

where $r = |R|$ and R is defined in (5-3).

In Newman /7/ a relation between the damping and exciting forces is derived (p. 304). With the dimensionless quantities used in this report this relation is

$$\text{(A-2)} \qquad b_{11} = K(\vec{f_1} \cdot \vec{i})^2, \quad b_{22} = K(\vec{f_1} \cdot \vec{j})^2$$

where b_{11} and b_{22} are the dimensionless damping force for sway and heave respectively.

From energy conservation there also follows a relation between the damping force and the amplitude of the radiated wave, A_i, (/7/)

$$Kb_{ii} = |A_i|^2, \quad i = 1,2.$$

When the radiation problem is solved, A_i can be found in exactly the same way as R.

We have checked all these relations for some of the parameter values and have been satisfied with the numerical accuracy when they were all satisfied with an error of 1% or less.

Appendix B. Computation of the splines.

The B-splines are piecewise cubic polynomials which satisfies the following conditions

1. $B_i(\theta) = 0 \quad \theta<\theta_{i-2}$ or $\theta>\theta_{i+2}$
2. $B_i(\theta) > 0 \quad \theta_{i-2}<\theta<\theta_{i+2}$
3. $B_i'(\theta)$ and $B_i''(\theta)$ are continuous for all θ.
4. If $\theta \in [x_{i-1}, x_i]$ then $\sum_{j=i-2}^{i+1} B_i(\theta) = 1$.

An example is shown in fig. 13.

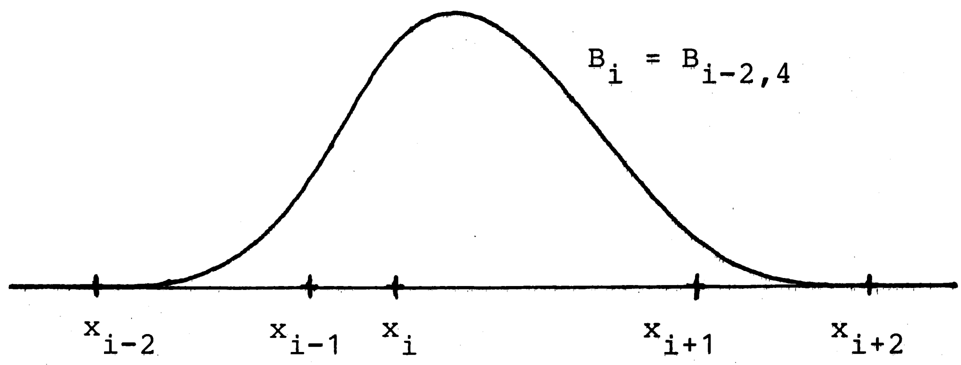


Fig. 13

The expression given in the appendix in /1/ is correct only when the nodes are equidistant, in the general case the expressions are more complicated, but the value can be easily computed from the recurrence relation

$$B_{j,1}(x) = \begin{cases} 1 & x_j<x<x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,k}(x) = \frac{x-x_i}{x_{i+k-1}-x_i} B_{i,k-1}(x) + \frac{x_{i+k}-x}{x_{i+k}-x_{i+1}} B_{i+1,k-1}, \quad k>1$$

We notice that this is a sum of positive quantities which can be evaluated without loss of numerical accuracy. The relation is taken from de Boor /8/, and the two-index notation is due to his work.

References.

1. Vada, T.: "The diffraction problem for a submerged circular cylinder solved by spline collocation". Inst. of Math., University of Oslo, Preprint Series, Jan. 1983.
2. Ogilvie, T. Francis: "First- and second-order forces on a cylinder submerged under a free surface". Journal of Fluid Mechanics 16 (1963).
3. Mehlum, E.: "A circular cylinder in water waves". Appl. Oc. Res. 1980.
4. Søding, H.: "Second-order forces on oscillating cylinders in waves". Schiffstechnik, bd. 23 - 1976.
5. Potash, R. L.: "Second-order theory of oscillating cylinders". Journal of Ship Research, dec. 1971.
6. Longuet-Higgins, M. S.: "The mean forces exerted by waves on floating and submerged bodies with applications to sand bars and wave power machines". Proc. Royal Soc., London, A.352 (1977).
7. Newman, J. N.: "Marine hydrodynamics". The MIT press.
8. de Boor, C.: "A practical guide to Splines". Springer-Verlag.

Several other references are given in /1/.