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TESTS MAXIMIZING MINIMUM POWER

by

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ABSTRACT

In a multiple hypotheses testing problem involving q different alternative hypotheses if the null hypothesis is rejected, the form of the tests maximizing minimum power over certain alternatives is derived. The result is used on the slippage problem for means and variances of normal populations, test for a change in a parameter occuring at an unknown time point, the three-decision problem, and two slippage problems for discrete distribution. In the latter case, attention is restricted to unbiased tests. In the case of the slippage problems the regularity assumptions which seem to have been imposed in earlier works on this subject, are not required. For example in the slippage problem for the means of normal populations, it is not required that the number of observations from each population should be equal. The form of the tests is rather complicated.

1. Introduction

Paulson [12] was the first to prove an optimality property of a test for a slippage problem involving means of normal populations. The optimality property was maximizing the minimum power over certain alternatives. Paulson's technique was later used to find optimal tests for other slippage problems, see e.g. [4] for references. The results, however, were not completely general. In the problem with normal means, for example, it seemed to be necessary to have an equal number of observations from each population to be able to prove the optimality property. Recently, Hall and Kudô [6] and Hall, Kudô and Yeh [7] used an other criterion, symmetry in power, but also their results depend upon the same kind of symmetry as Paulson's.

Pfanzagl [13] assumed that the various alternatives had certain known probabilities, and using that he found tests which maximized the average power over various alternatives with respect to the given probabilities. Pfanzagl's results did not depend upon the kind of regularity assumption as used in [12].

In the present paper we will find tests which are optimal in the sense of Paulson but without requiring the regularity assumptions of the earlier papers. The results apply, however, not only to slippage problems, the general setting (see (2.1)) is any problem where one has to choose between a finite number of disjoint alternative hypotheses when the null hypothesis is rejected.

2. Statement of the problem

Let X be a random variable with distribution function P_Θ where P_Θ belongs to a class $\{P_\Theta: \Theta\varepsilon\Omega\}$ of distribution functions. Consider the hypothesis testing problem

(2.1) $H: \Theta\varepsilon\Omega_0$ against $K_1: \Theta\varepsilon\Omega_1$ or $K_2: \Theta\varepsilon\Omega_2$ or ... or $K_q: \Theta\varepsilon\Omega_q$,

where $\Omega_0, \Omega_1, \ldots, \Omega_q$ are disjoint subsets of Ω. We define a test of (2.1) to consist of q elements $(\Psi_1(x), \ldots, \Psi_q(x))$ where the $\Psi_i(x)$, $i=1,\ldots,q$, are ordinary test functions and

(2.2) $\sum_{i=1}^{q} \Psi_i(x) \leq 1$

If x is observed, we reject H with probability $\sum_{i=1}^{q}\Psi_i(x)$, and accept the alternative K_i with probability $\Psi_i(x)$, $i=1,\ldots,q$. Only one of the alternatives $K_1,\ldots,K_q$ is accepted. If H is not rejected, our conclusion is $\Theta\varepsilon\Omega$, not $\Theta\varepsilon\Omega_0$. A test is called a level α test if

$$(2.3)\qquad \sup_{\Theta\varepsilon\Omega_0} \sum_{i=1}^{q} E_\Theta \Psi_i(X) \leq \alpha$$

Let

$$(2.4)\qquad \beta(\Theta,\Psi_i) = E_\Theta \Psi_i(X) \qquad\qquad i=1,\ldots,q$$

We define the power function of a test to be the vector $(\beta(\Theta,\Psi_1),\ldots,\beta(\Theta,\Psi_q))$. We shall say that a test $(\phi_1,\ldots,\phi_q)$ is more powerful than a test $(\Psi_1,\ldots,\Psi_q)$ if $\beta(\Theta,\phi_i) \geq \beta(\Theta,\Psi_i)$, $\Theta\varepsilon\Omega_i$, $i=1,\ldots,q$. We would like to find a level α test such that $\beta(\Theta,\Psi_i)$ is large when $\Theta\varepsilon\Omega_i$, $i=1,\ldots,q$.

3. Tests that maximize minimum average power and minimum power

If we try to find a test which, subject to (2.3), maximizes $E_\Theta \Psi_i(X)$, $\Theta\varepsilon\Omega_i$, for a particular i, it would generally lead to small values of $E_\Theta \Psi_j(X)$, $\Theta\varepsilon\Omega_j$ when $j\neq i$.

We will therefore try to find tests which maximize the average power over the q alternatives, or maximize minimum power over the q alternatives. Denote the class of tests satisfying (2.2) and (2.3) by $S(\alpha)$. Let ω_i be a subset of Ω_i, $i=1,\ldots,q$.

A test $\phi\varepsilon S(\alpha)$ satisfying $\min\left(\inf_{\Theta\varepsilon\omega_1} E_\Theta\phi_1(X),\ldots,\inf_{\Theta\varepsilon\omega_q} E_\Theta\phi_q(X)\right)=\sup_{\Psi\varepsilon S(\alpha)} \min\left(\inf_{\Theta\varepsilon\omega_1} E_\Theta\Psi_1(X),\ldots,\inf_{\Theta\varepsilon\omega_q} E_\Theta\Psi_q(X)\right)$, we call a <u>test maximizing the minimum power over $\omega_1,\ldots,\omega_q$</u>. A test $\phi\varepsilon S(\alpha)$ satisfying

$$\inf_{\Theta_1\varepsilon\omega_1,\ldots,\Theta_q\varepsilon\omega_q} \sum_{i=1}^{q} E_{\Theta_i}\phi_i(X) = \sup_{\Psi\varepsilon S(\alpha)} \inf_{\Theta_1\varepsilon\omega_1,\ldots,\Theta_q\varepsilon\omega_q} \sum_{i=1}^{q} E_{\Theta_i}\Psi_i(X)$$ we

call a test maximizing the minimum average power over $w_1,\ldots,w_q$.

In the following $f_0,f_1,\ldots,f_q$ will be q+1 real-valued functions, integrable with respect to a σ-finite measure μ on a Euclidean space.

The following theorem will be helpful when determining tests that maximize minimum power.

Theorem 1. Consider the problem to maximize

(3.1) $\min(\int\Psi_1 f_1 d\mu,\ldots,\int\Psi_q f_q d\mu)$,

where $\Psi_1,\ldots,\Psi_q$ are test functions satisfying (2.2) and

(3.2) $\int(\sum_{i=1}^q\Psi_i)f_0 d\mu = c$

Suppose that there exist constants $k_1,\ldots,k_q$ with $\sum_{i=1}^q k_i > 0$ and tests $\phi_1,\ldots,\phi_q$ such that

(3.3)
$$\phi_i(x) = \begin{cases} 1 \text{ when } k_i f_i(x)>f_0(x) \text{ and } k_i f_i(x)>\max_{j\neq i} k_j f_j(x) \\ 0 \text{ when } k_i f_i(x)<f_0(x) \end{cases}$$
$$\sum_{\{i:k_i f_i(x)=\max_j k_j f_j(x)\}} \phi_i(x) = 1 \text{ when } \max_j k_j f_j(x)>f_0(x)$$

and

(3.4) $\int\phi_1 f_1 d\mu = \ldots = \int\phi_q f_q d\mu$

Then $\phi_1,\ldots,\phi_q$ maximize (3.1) subject to (2.2) and (3.2).

Proof. That there exists a set of test functions maximizing (3.1) is easily seen by using the weak compactness theorem for test functions (see [9] p. 354).

We will first show that there also exists a test satisfying (3.4) which maximizes (3.1). Let $(\Psi_1,\ldots,\Psi_q)$ maximize (3.1) and suppose that

(3.4) is not satisfied. Then let β_1 and β_2 be defined by

$$\beta_1 = \min_i \int \Psi_i f_i d\mu < \max_i \int \Psi_i f_i d\mu = \beta_2$$

Let $I_1 = \{i : \int \Psi_i f_i d\mu = \beta_1\}$ and $I_2 = \{i : \int \Psi_i f_i d\mu = \beta_2\}$. Let $\beta_1 < \delta < \beta_2$ and let n be the number of elements in I_1. Define new tests by

$$(3.5)\quad \begin{aligned} \Psi_i^* &= \Psi_i + n^{-1}(1-\delta/\beta_2) \textstyle\sum_{i\varepsilon I_2} \Psi_i && i\varepsilon I_1 \\ \Psi_i^* &= (\delta/\beta_2)\Psi_i && i\varepsilon I_2 \\ \Psi_i^* &= \Psi_i && i\notin I_1 \cup I_2 \end{aligned}$$

We have $\sum_{i=1}^q \Psi_i = \sum_{i=1}^q \Psi_i^*$, hence $(\Psi_1^*,\ldots,\Psi_q^*)$ also satisfies (2.2) and (3.2). Furthermore,

$$(3.6)\quad \begin{aligned} \int \Psi_i^* f_i d\mu &= \beta_1 + n^{-1}(1-\delta/\beta_2)\int(\textstyle\sum_{j\varepsilon I_2} \Psi_j)f_i d\mu \geq \beta_1 && i\varepsilon I_1 \\ \int \Psi_i^* f_i d\mu &= (\delta/\beta_2)\int \Psi_i f_i d\mu = \delta > \beta_1 && i\varepsilon I_2 \\ \int \Psi_i^* f_i d\mu &= \int \Psi_i f_i d\mu > \beta_1 && i\varepsilon I_1 \cup I_2 \end{aligned}$$

Since $(\Psi_1,\ldots,\Psi_q)$ maximizes (3.1) we must have equality sign for at least one index i, i_0 say, in the first equation of (3.6). Hence

$$(3.7)\quad \int(\textstyle\sum_{j\varepsilon I_2} \Psi_j)f_{i_0} d\mu = 0$$

Define new tests by

$$\begin{aligned} \Psi_{i_0}^{**} &= \Psi_{i_0} + (1-\beta_1/\beta_2)\textstyle\sum_{j\varepsilon I_2} \Psi_j && \\ \Psi_i^{**} &= (\beta_1/\beta_2)\Psi_i && i\varepsilon I_2 \\ \Psi_i^{**} &= \Psi_i && \text{otherwise.} \end{aligned}$$

It is easily seen that $(\Psi_1^{**},\ldots,\Psi_q^{**})$ satisfies (2.2) and (3.2). By (3.7) it is also found that $\int \Psi_i^{**} f_i d\mu = \beta_1$, $i\varepsilon I_1 \cup I_2$. If $(\Psi_1^{**},\ldots,\Psi_q^{**})$ does not satisfy (3.4), we may proceed as above using $I_1 \cup I_2$ as I_1, and so on until we end up with tests satisfying (3.4) with all the integrals equal to β_1.

Now return to the proof of the theorem. Let $(\Psi_1,\ldots,\Psi_q)$ be a test maximizing (3.1) and satisfying (2.2), (3.2) and (3.4). (As shown above this is no restriction.) Since both $(\phi_1,\ldots,\phi_q)$ and $(\Psi_1,\ldots,\Psi_q)$ satisfy (3.4) we have

$$(\textstyle\sum_{i=1}^{q} k_i)\int \phi_1 f_1 d\mu = \int(\sum_{i=1}^{q} k_i \phi_i f_i)d\mu$$

and

$$(\textstyle\sum_{i=1}^{q} k_i)\int \Psi_1 f_1 d\mu = \int(\sum_{i=1}^{q} k_i \Psi_i f_i)d\mu$$

Hence

$$(3.8)\qquad (\textstyle\sum_{i=1}^{q} k_i)(\int \phi_1 f_1 d\mu - \int \Psi_1 f_1 d\mu) =$$
$$= \int\big(\textstyle\sum_{i=1}^{q} (\phi_i - \Psi_i)(k_i f_i - f_0)\big)d\mu.$$

Look at the integrand

$$(3.9)\qquad \textstyle\sum_{i=1}^{q} \big(\phi_i(x) - \Psi_i(x)\big)\big(k_i f_i(x) - f_0(x)\big)$$

If $\max_i k_i f_i(x) > f_0(x)$ the integrand is equal to
$k_t f_t(x) - f_0(x) - \sum_{i=1}^{q} \Psi_i(x)\big(k_i f_i(x) - f_0(x)\big)$
$\geq k_t f_t(x) - f_0(x) - \sum_{i=1}^{q} \Psi_i(x)\big(k_t f_t(x) - f_0(x)\big) = \big(k_t f_t - f_0(x)\big)\big(1 - \sum_{i=1}^{q} \Psi_i(x)\big) \geq 0$,
where t is an index such that $k_t f_t(x) = \max_i k_i f_i(x)$. If $\max_i k_i f_i(x) < f_0(x)$, then the integrand is $-\sum_{i=1}^{q} \Psi_i(x)\big(k_i f_i(x) - f_0(x)\big) \geq 0$, since then $k_i f_i(x) - f_0(x) < 0$ for all i. If $\max_i k_i f_i(x) = f_0(x)$, then the integrand is $-\sum_{\{i : k_i f_i(x) < f_0(x)\}} \Psi_i(x)\big(k_i f_i(x) - f_0(x)\big) \geq 0$. Hence the integrand is

always non-negative, and (3.8) is greater or equal to 0. Since $\sum_{i=1}^{q} k_i > 0$ and $\int \phi_i(x) f_i(x) d\mu(x)$ does not depend upon i, the theorem is proved.

In the examples to follow it will not always be obvious that there exist tests satisfying (3.2), (3.3) and (3.4). The next theorem gives conditions for existence of such tests.

Theorem 2. In addition to the assumptions of Theorem 1, let $f_i \geq 0$, $i=0,\dots,q$, $0 < c < \int f_0 d\mu$ and

$$(3.10) \qquad \int_A f_i d\mu = 0 \Rightarrow \int_A f_j d\mu = 0 \qquad j \neq i$$

If a test maximizes (3.1) subject to (2.2) and (3.2), then it is of the form (3.3) with $\sum_{i=1}^{q} k_i > 0$ and satisfies (3.4).

Proof. Let N be the set of all points $(\int \Psi_1 f_1 d\mu, \dots, \int \Psi_q f_q d\mu, \int (\Psi_1 + \dots + \Psi_q) f_0 d\mu)$. N is closed and convex. (Compare [9] p. 83.) Let $(u_1, \dots, u_{q+1})$ denote a general point in N. For fixed $u_{q+1} = c$, there exists a point $(a_1, \dots, a_q, c) \varepsilon N$ such that min $(a_1, \dots, a_q)$ is equal to

$$\sup_{(u_1, \dots, u_q, c) \varepsilon N} \min (u_1, \dots, u_q).$$

Because of the condition (3.10), we must have $a_1 = \dots = a_q$. Furthermore $(a_1, \dots, a_q, c)$ is a boundary point of N. Let

$$\sum_{i=1}^{q+1} k_i u_i = \sum_{i=1}^{q} k_i a_i + k_{q+1} c$$

be a hyperplane through this point such that all points in N are on the same side of the hyperplane.

Let M be the set of all points $(\int \Psi_1 f_1 d\mu, \dots, \int \Psi_q f_q d\mu)$ where $(\Psi_1, \dots, \Psi_q)$ varies over all test functions satisfying (2.2). M is closed and convex, and using the fact $0 < c < \int f_0 d\mu$ we see that $(a_1,\dots,a_q)$ is inner point of M.

Let a^* and a^{**} be the minimum and maximum last coordinate, respectively, of points in N for fixed first q coordinates $(a_1,\dots,a_q)$. We must have $a^* = c$, since $a^* < c$ would imply that $\min(a_1,\dots,a_q) < \sup_{(u_1,\dots,u_q,c)\varepsilon N} \min(u_1,\dots,u_q)$.

Suppose first $c < a^{**}$. Then $(a_1,\dots,a_q, (c+a^{**})/2)$ is an inner point of N. It then follows that $k_{q+1} \neq 0$ in the equation of the hyperplane, since $k_{q+1} = 0$ would imply that $(a_1,\dots,a_q,(c+a^{**})/2)$ is on the hyperplane. Taking $k_{q+1} = -1$ the equation of the hyperplane is
$\sum_{i=1}^{q} k_i u_i - u_{q+1} = \sum_{i=1}^{q} k_i a_i - c$, and

$$(3.11) \qquad \sum_{i=1}^{q} k_i u_i - u_{q+1} \leq \sum_{i=1}^{q} k_i a_i - c$$

when $(u_1,\dots,u_{q+1})\varepsilon N$. Hence for all test functions $(\Psi_1,\dots,\Psi_q)$ we have

$$(3.12) \qquad \int\sum_{i=1}^{q} \Psi_i(k_i f_i - f_0)d\mu \leq \int\sum_{i=1}^{q} \Psi_i^*(k_i f_i - f_0)d\mu$$

where $(\psi_1^*,\dots,\Psi_q^*)$ is a test function giving the point $(a_1,\dots,a_q,c)$. Define $(\phi_1,\dots,\phi_q)$ as in (3.3). Then as in the argument after (3.9)

$$\sum_{i=1}^{q} (\phi_i - \Psi_i \ )(k_i f_i - f_0) \geq 0$$

But by (3.12) with $\phi_1,\dots,\phi_q$ is $\Psi_1,\dots,\Psi_q$

$$\int\sum_{i=1}^{q} (\phi_i - \Psi_i^*)(k_i f_i - f_0)d\mu \leq 0$$

Hence

$$\sum_{i=1}^{q} (\phi_i - \Psi_i^*)(k_i f_i - f_0) = 0 \quad \text{a.e.}\mu.$$

It then follows that $(\Psi_1^*,\dots,\Psi_q^*)$ is defined in the same way as

$(\phi_1,\dots,\phi_q)$ a.e.μ.

If $a^* = c = a^{**}$ we find by an argument similar to |9| p. 86, that N is on the hyperplane

$$u_{q+1} = \sum_{i=1}^{q} k_i u_i .$$

Hence $\int(\sum_{i=1}^{q}\Psi_i)f_0 d\mu = \sum_{i=1}^{q}\int k_i \Psi_i f_i d\mu$,

or $\sum_{i=1}^{q}\int\Psi(k_i f_i - f_0)d\mu = 0$

for all $\Psi_1,\dots,\Psi_q$. That implies $k_i f_i = f_0$ a.e.μ, $i=1,\dots,q$. Hence all tests are trivially of the form (3.3) a.e.μ.

Clearly, we must have all $c_i > 0$, otherwise the corresponding test would have power 0. This completes the proof.

In Pfanzagl [13] p. 39 is given the form of the tests which maximize

$$\sum_{i=1}^{q}\int\Psi_i f_i d\mu$$

among tests satisfying (3.2). They are of the form (3.3) with $k_1=\dots=k_q=k$ and where k is determined so that (3.2) is satisfied.

The following corollary to Theorems 1 and 2 gives a condition under which the test maximizing $\sum_{i=1}^{q}\int\Psi_i f_i d\mu$ and min $(\int\Psi_1 f_1 d\mu,\dots,\int\Psi_q f_q d\mu)$ coincide.

<u>Corollary.</u> <u>Let</u> $(\phi_1,\dots,\phi_q)$ <u>be of the form (3.3) with</u> $k_1=\dots=k_q>0$, <u>and hence maximizes</u> $\sum_{i=1}^{q}\int\Psi_i f_i d\mu$ <u>subject to (2.2) and (3.2).</u> <u>If</u> $\int\phi_1 f_1 d\mu=\dots=\int\phi_q f_q d\mu$, <u>then</u> $(\phi_1,\dots,\phi_q)$ <u>also maximizes</u> <u>min</u> $(\int\Psi_1 f_1 d\mu,\dots,\int\Psi_q f_q d\mu)$ <u>subject to</u> (2.2) <u>and</u> (3.2).

<u>Proof.</u> Follows trivially from Theorem 1 since $(\phi_1,\dots,\phi_q)$ is of the form (3.3) and satisfies (3.4).

The following lemma, the proof of which is obvious, will be used when we determine tests maximizing minimum power.

Lemma. Suppose that there exist a test $\phi = (\phi_1,\ldots,\phi_q) \in S(\alpha)$ such that (I) there exist points $\Theta_1^*,\ldots,\Theta_q^*$, where $\Theta_i^* \in \omega_i$, $i=1,\ldots,q$, such that ϕ maximizes $\min\,(E_{\Theta_1^*}\Psi_1(X),\ldots,E_{\Theta_q^*}\Psi_q(X))$, (II) $\inf_{\Theta\in\omega_i} E_\Theta\phi_i(X) = E_{\Theta_i^*}\phi_i(X)$, $i=1,\ldots,q$. Then ϕ maximizes $\min\,(\inf_{\Theta\in\omega_1} E_\Theta\Psi_1(X),\ldots,\inf_{\Theta\in\omega_q} E_\Theta\Psi_q(X))$ among tests $\Psi\in S(\alpha)$.

4. Application to some simple problems without nuisance parameters.

A. The slippage problem for normal means.

Let X_{ij} be independent $N(\mu_i,1)$, $i=1,\ldots,q$, $j=1,\ldots,n_i$. Consider the problem

$$H:\mu_1=\ldots=\mu_q=0 \text{ against } K_i:\mu_1=\ldots=\mu_{i-1}=\mu_i-\Delta=\mu_{i+1}=\ldots=\mu_q=0 \quad i=1,\ldots,q,$$

where $\Delta > 0$. This problem seems to be due to Mosteller [10]. Paulson [12] found the test maximizing the minimum power over alternatives $\Delta \geq \Delta_1$, in the case when $n_1=\ldots=n_q$. Pfanzagl [13] found the test maximizing the average power over the same alternatives for general $n_1,\ldots,n_q$. We will now in the general case derive the test maximizing the minimum power over the alternatives ω_i defined by $\mu_j=0$, $j\neq i$, $\mu_i\geq\Delta_i$, $i=1,\ldots,q$.

Let f_0 be the density of the observations under H, and let f_i be the density under K_i with $\Delta=\Delta_i$. The ratio f_i/f_0 is then

$$(4.1) \qquad \exp(\Delta_i n_i x_{i\cdot} - \tfrac{1}{2} n_i \Delta_i^2)$$

where $x_{i\cdot}= \sum_{j=1}^{n_i} x_{ij}/n_i$. Multiplying (4.1) with k_i and taking the logarithm we get

(4.2) $\Delta_i n_i x_{i.} - \frac{1}{2} n_i \Delta_i^2 + c_i$

where $c_i = \log k_i$. Denote (4.2) by V_i. According to Theorems 1 and 2 we shall have

$$(4.3)\quad \phi_i(x) = \begin{cases} 1 & \text{when } V_i > 0 \text{ and } V_i > V_j \quad i \neq j \\ 0 & \text{otherwise} \end{cases}$$

where $c_1, \ldots, c_q$ are determined so that

$$(4.4)\quad P_0(\max_i V_i > 0) = \alpha,$$

and

$$(4.5)\quad P_i(V_i > 0 \text{ and } V_i > \max_{j \neq i} V_j)$$

is independent of i, where P_i denotes that the probabilities are calculated with respect to the density f_i, $i=0,\ldots,q$. This will give the test maximizing the minimum power over alternatives with $\Delta = \Delta_i$ when we consider the alternative K_i. It is easily seen that the elements of the power function is strictly increasing in Δ, hence by the Lemma the test maximizes the minimum power over the alternatives $\omega_1, \ldots, \omega_q$.

In general, it is very difficult to determine the constants $c_1, \ldots, c_q$ such that (4.4) and (4.5) are satisfied. In some special cases, however, it is only one constant c to determine. If $n_1 = \cdots = n_q$ and $\Delta_1 = \cdots = \Delta_q$, then $c_1 = \cdots = c_q$. If the $\{n_i\}$ are not all equal it might be natural to consider alternatives of the form $\Delta_i = \gamma n_i^{-\frac{1}{2}}$, since our "best" estimate of Δ_i is $x_{i.}$ with variance $n_i^{-\frac{1}{2}}$. In that case we will also find that we have $c_1 = \cdots = c_q$, so there is only one constant c to determine.

B. Test for a change in a parameter occurring at an unknown time point.

Let $X_1,\dots,X_q$ be independent $N(\mu_i,1)$. Consider the hypothesis

(4.6) $H:\mu_1=\dots=\mu_q=0$ against $K_i:\mu_1=\dots=\mu_{q-i}=0,\ \mu_{q-i+1}=\dots=\mu_q>0\ \ i=1,\dots,q.$

For the origin of this problem see Page [11]. Tests of H against the alternative "H is not true" have been proposed in [1], [2], [8]. Pfanzagl |13| found the test of (4.6) maximizing average power over alternatives with $\mu_{q-i+1}=\dots=\mu_q\geq\Delta$ $i=1,\dots,q.$

Consider now the alternatives ω_i defined by $\mu_{q-i+1}=\dots=\mu_q\geq\Delta_i$. Let f_0 be the density when H is true and let f_i be the density under K_i when $\mu_{q-i+1}=\dots=\mu_q = \Delta_i$.

The expression corresponding to (4.2) is in this case

(4.7) $\Delta_i S_i - \frac{1}{2} i \Delta_i^2 + c_i$

where $S_i = \sum_{j=q-i+1}^{q} X_i$. From this we can find a test similar to (4.3) with conditions as in (4.4) and (4.5). It will be q constants $c_1,\dots,c_q$ to determine.

Arguing as in A, we might consider alternatives with $\Delta_i = i^{-\frac{1}{2}}\gamma$. Then (4.7) is proportional to

(4.8) $i^{-\frac{1}{2}}S_i - c_i'$

where $c_i' = -\frac{1}{2} + c_i/\gamma$. Even in this case there will be q constants to determine to find the test maximizing minimum power. To find the test maximizing average power we put $c_1' = \quad = c_q' = c'$. Then it is only one constant to determine, and if we reject H we accept the alternatives with the largest $i^{-\frac{1}{2}}S_i$. This is contrary to traditional cumulative sum tests (see e.g. [3])where one accepts the alternatives with the largest S_i. The quantity $i^{-\frac{1}{2}}S_i$ is more stable than S_i since Var $(i^{-\frac{1}{2}}S_i) = 1$ while Var $S_i = i$.

5. Use of invariance.

It is easy to see that if a hypothesis testing problem of the form (2.1) is invariant under a group G of transformations, and G satisfies the conditions of the Hunt-Stein theorem (see [9] p. 336), then there exists an invariant test maximizing minimum power.

A. The slippage problem for normal means. Let X_{ij} be independent $N(\mu_i, \sigma^2)$, $i=1,\dots,q$, $j=1,\dots,n_i$. Consider the problem

$H: \mu_1 = \dots = \mu_q$ against $K_i: \mu_1 = \dots = \mu_{i-1} = \mu_i - \Delta_i\sigma = \mu_{i+1} = \dots = \mu_q$ $i=1,\dots,q$

where $\Delta_i > 0$. This problem is invariant under translations and change of scale, a maximal invariant being $(T_1,\dots,T_{q-1})$ where

$$T_i = \frac{X_{i\cdot} - \bar{X}}{S} \frac{n_i}{(n-q)^{\frac{1}{2}}} \qquad i=1,\dots,q$$

where $X_{i\cdot}$ and n is defined as in Section 4, $\bar{X} = \sum_{i=1}^{q} n_i X_i/n$ and $S^2 = \sum_{i=1}^{q}\sum_{j=1}^{n_i}(X_{ij}-X_{i\cdot})^2/(n-q)$. We have $\sum_{i=1}^{q} T_i = 0$. The joint density of the T_i-s under the alternative K_i with $\Delta = \Delta_i$ is (see [13] p 26).

$$(5.1) \quad f_i = C(1+\sum_{j=1}^{q} t_j^2/n_j)^{-(n-1)/2}\exp(-\tfrac{1}{2}n_i(1-n_i/n)\Delta_i^2)$$

$$I(t_i\Delta_i/(1+\sum_{j=1}^{q} t_j^2/n_j)^{\frac{1}{2}}) \qquad i=1,\dots,q$$

where C is a constant and

$$I(\tau) = \int_0^\infty \exp(\tau x^{\frac{1}{2}} - x/2)\, x^{(n-3)/2} dx \;.$$

The density f_0 under H is obtained from (5.1) by putting $\Delta_i = 0$. Hence the ratio f_i/f_0 is

$$(5.2) \; f_i/f_0 = \exp(-\tfrac{1}{2}n_i(1-n_i/n)\Delta_i^2) I(t_i\Delta_i/(1+\sum_{j=1}^{q} t_j^2/n_j)^{\frac{1}{2}}) I(0)^{-1}.$$

It is seen that it will in general be very complicated to determine the test maximizing minimum power. A simplification occurs if we argue as in

Section 4 A, and choose $\Delta_i = \gamma/(n_i(1-n_i/n))^{\frac{1}{2}}$ since the best estimate of $\Delta_i\sigma$ is $X_{i\cdot} - \sum_{j\neq i} n_j X_{j\cdot}/(n-n_j)$ with variance $(n_i(1-n_i/n))^{-1}$. Using this Δ_i in (5.2) we get

$$f_i/f_0 = \exp(-\tfrac{1}{2}\gamma)I(t_i\gamma/((1+\textstyle\sum_{j=1}^q t_j^2/n_j)(n_i(1-n_i/n))^{\frac{1}{2}})I(0)^{-1}$$

Hence $k_i f_i > f_0$ is equivalent to

$$(5.3) \quad \frac{t_i}{(n_i(1-n_i/n))^{\frac{1}{2}}(1+\sum_{j=1}^q t_j^2/n_j)^{\frac{1}{2}}} - c_i > 0$$

where c_i is a new constant. Furthermore $k_i f_i > k_\nu f_\nu$ is equivalent to

$$(5.4) \quad \frac{t_i}{(n_i(1-n_i))^{\frac{1}{2}}(1+\sum_{j=1}^q t_j^2/n_j)^{\frac{1}{2}}} - \frac{t_\nu}{(n_\nu(1-n_\nu/n))^{\frac{1}{2}}(1+\sum_{j=1}^q t_j^2/n_j)^{\frac{1}{2}}} - c_i + c_\nu > 0$$

To determine $c_1,\ldots,c_q$ so that the test defined by (3.3) (and now obtained from (5.3) and (5.4)) satisfies (3.4) is, of course, numerically very difficult.

B. <u>The slippage problem for normal variances.</u>

Let X_{ij} be independent $N(\mu_i,\sigma_i^2)$, $i=1,\ldots,q$, $j=1,\ldots,n_i$, and consider the problem

$$H:\sigma_1^2=\ldots=\sigma_q^2 \text{ against } K_i:\sigma_1^2=\ldots=\sigma_{i-1}^2=\sigma_i^2/\Delta_i=\sigma_{i+1}^2=\ldots=\sigma_q^2 \quad i=1,\ldots,q$$

where $\Delta_i > 1$.

The test maximizing minimum power over alternatives $\Delta > 1$ in the case $n_1=\ldots=n_q$ was found by Truax [14]. Pfanzagl [13] found the test maximizing average power over the same alternatives in the general case.

Using invariance under translations and change of scale, we find that a maximal invariant is $(V_1,\ldots,V_{q-1})$ where

$$V_i = \frac{n_i-1}{n-q}\frac{S_i^2}{S^2} \qquad i=1,\ldots,q,$$

and where $S_i^2 = \sum_{j=1}^{n_i}(X_{ij}-X_{i.})^2/(n_i-1)$

and $S^2 = \sum_{i=1}^{q}(n_i-1)S_i^2/(n-q)$. We have $\sum_{i=1}^{q}V_i=1$. Let f_i be the density of the maximal invariant under K_i with $\Delta = \Delta_i$ and f_0 the density under H. Then using Pfanzagl's results ([13] pp.30-31) we find that $k_i f_i/f_0$ is equivalent to

$$v_i(1-\Delta_i^{-1}) - c_i > 0$$

and $k_i f_i > k_\nu f_\nu$ is equivalent to

$$v_i(1-\Delta_i^{-1}) - v_\nu(1-\Delta_\nu^{-1}) - c_i + c_\nu > 0,$$

where $c_1,\ldots,c_q$ are new constants.

If we have $n_1=\ldots=n_q$ and choose $\Delta_1=\ldots=\Delta_q$, it turns out that $c_1=\ldots=c_q$ and we are back to Truax's [14] result.

6. The three-decisions problem.

Consider the one-parameter exponential family

$$C(\Theta)\, e^{\Theta T(x)} h(x),$$

and let the problem be

$$H:\Theta=\Theta_0 \quad \text{against } K_1:\Theta<\Theta_0 \text{ or } K_2:\Theta>\Theta_0$$

Let $\omega_1 = \{\Theta:\Theta\leq\Theta_1<\Theta_0\}$ and $\omega_2 = \{\Theta:\Theta\geq\Theta_2>\Theta_0\}$ where Θ_1 and Θ_2 are given values of Θ, and let us find the test maximizing minimum power over ω_1 and ω_2. Choosing the density when $\Theta = \Theta_i$ as f_i, $i=0,1,2$, we find that the test maximizing minimum power over ω_1 and ω_2 is as follows:

$$\phi_1(x) = \begin{cases} 1 & \text{when } T(x)<c_1 \\ \gamma_1 & \text{when } T(x)=c_1 \\ 0 & \text{when } T(x)>c_1 \end{cases}$$

and

$$\phi_2(x) \quad \begin{matrix} 1 & \text{when} & T(x)>c_2 \\ \gamma_2 & \text{when} & T(x)=c_2 \\ 0 & \text{when} & T(x)<c_2 \end{matrix}$$

Here c_1, c_2, γ_1, γ_2 are determined so that

$$P_{\Theta_0}(T(X)<c_1) + P_{\Theta_0}(T(X)>c_2) + \sum_{i=1}^{2}\gamma_i P_{\Theta_0}(T(X)=c_i) = \alpha$$

and

$$P_{\Theta_1}(T(X)<c_1) + \gamma_1 P_{\Theta_1}(T(X)=c_1) = P_{\Theta_2}(T(X)>c_2) + \gamma_2 P_{\Theta_2}(T(X) = c_2).$$

<u>An example.</u> Let $X_1,\ldots,X_n$ be independent $N(0,\sigma^2)$ and consider the problem

$$H:\sigma=\sigma_0 \quad \text{against} \quad K_1:\sigma<\sigma_0 \quad \text{or} \quad K_2:\sigma>\sigma_0 \; .$$

We get

$$\phi_1(x) = 1 \quad \text{when} \quad \sum_{i=1}^{n} x_i^2 < k_1$$

and

$$\phi_2(x) = 1 \quad \text{when} \quad \sum_{i=1}^{n} x_i^2 > k_2$$

The constants k_1 and k_2 are determined by

$$F_n(k_1/\sigma_0^2) + 1 - F_n(k_2/\sigma_0^2) = \alpha$$

and

$$F_n(k_1/\sigma_1^2) = 1 - F_n(k_2/\sigma_2^2),$$

where F_n is the cumulative chi-square distribution with n degrees of freedom. This test is different both from the unbiased test of H and the test maximizing minimum power in the traditional sense (see Lehmann [9] p. 332).

7. Use of unbiasedness.

We will call a test $(\Psi_1,\ldots,\Psi_q)$ of (2.1) unbiased if

$$\sup_{\Omega_0} E_\Theta \Psi_i(X) \le \inf_{\Omega_i} E_\Theta \Psi_i(X) \qquad i=1,\ldots,q.$$

Let Ω_{i0} be the set of common accumulation points of Ω_0 and Ω_i. If the power function of any test is continuous in Θ, then unbiasedness implies

$$(7.1) \qquad E_\Theta \Psi_i(X) = \alpha_i,\ \Theta\varepsilon\Omega_{i0}, \qquad i=1,\ldots,q,$$

where α_i is some constant. Furthermore, if T is a complete and sufficient statistics relative to $\Omega_{00} = \bigcap_{i=1}^{q}\Omega_{i0}$, then (7.1) is equivalent to

$$E_\Theta(\Psi_i(X)|t) = \alpha_i \quad \text{a.e.},\ \Theta\varepsilon\Omega_{00} \qquad i=1,\ldots,q.$$

Let $\Theta_i\varepsilon\Omega_i$, $i=1,\ldots,q$, and let $(\phi_1,\ldots,\phi_q)$ be the test which maximizes $\sum_{i=1}^{q} E_{\Theta i}(\Psi_i(X)|t)$ among tests satisfying (2.2) and $\sum_{i=1}^{q} E_\Theta(\Psi_i(X)|t) = \alpha$, $\Theta\varepsilon\Omega_{00}$. We can find this $(\phi_1,\ldots,\phi_q)$ by the methods of Section 3. It is easily seen that if $(\phi_1,\ldots,\phi_q)$ is unbiased, then it maximizes the average power over the alternatives $\Theta_1,\ldots,\Theta_q$ among unbiased tests.

If, in addition, it turns out that $E_{\Theta_1}\phi_1(X)=\ldots=E_{\Theta_q}\phi_q(X)$, the test also maximizes the minimum power over $\Theta_1,\ldots,\Theta_q$ among unbiased tests.

A. <u>The slippage problem for the Poisson distribution.</u>

Let $X_1,\ldots,X_q$ be independent with Poisson distributions

$$P[X_j=x_j] = \frac{(\mu_j)^{x_j}}{x_j!} e^{-\mu_j} \qquad j=1,\ldots,q$$

Consider the problem

$$H: \mu_j = p_j\mu \quad \text{against} \quad K_i: \mu_i=\gamma_i p_i \mu \quad \mu_j = \frac{1-\gamma_i p_i}{1-p_i} p_j \mu \quad j\neq i$$

where $p_1,\ldots,p_q$ are known constants with $\sum_{j=1}^{q} p_j=1$, and and $\gamma_i > 1$ are unknown parameters. (See Doornbos and Prince [5].) The joint distribution under H is

$$\frac{\Pi_{j=1}^{q} p_j^{x_j}}{\Pi_{j=1}^{q} x_j!} \mu^{\sum_{j=1}^{q} x_j} e^{-\mu} ,$$

Hence $T = \sum_{j=1}^{q} X_j$ is sufficient and complete. The conditional distributions given T under H and K_i are, respectively,

$$\frac{\Pi_{j=1}^{q} p_j^{x_j}}{\Pi_{j=1}^{q} x_j!} t!$$

and

$$\frac{\Pi_{j=1}^{q} p_j^{x_j}}{\Pi_{j=1}^{q} x_j!} \left(\frac{1-\gamma_i p_i}{1-p_i}\right)^t \left(\frac{\gamma_i-\gamma_i p_i}{1-\gamma_i p_i}\right)^{x_i} t! \quad .$$

The test maximizing the average power over alternatives with $\gamma = \gamma_i^*$ is of the form, accept K_i if

$$x_i \log((\gamma_i^*-\gamma_i^* p_i)/1-\gamma_i^* p_i)) - t \log((1-\gamma_i^* p_i)/(1-p_i)) > k_t$$

and

$$x_i \log((\gamma_i^*-\gamma_i^* p_i)/(1-\gamma_i^* p_i)) - t \log((1-\gamma_i^* p_i)/(1-p_i))$$

$$> x_j \log((\gamma_j^*-\gamma_j^* p_j)/(1-\gamma_j^* p_j)) - t \log((1-\gamma_j^* p_j)/(1-p_j)), \quad j \neq i.$$

Here K_t is determined so that the conditional probability of rejecting H is α.

If $p_1 = \ldots = p_q = 1/q$ and $\gamma_1^* = \ldots = \gamma_q^*$, the test is, accept K_i if $x_i > k_t$ and $x_i > x_j$, $j \neq i$. Because of the symmetry of the situation the powers are equal at the alternatives K_i with $\gamma_i = \gamma^*$ and $\mu_i = \mu^*$ for any γ^* and μ^*. It is easily seen by an argument similar to Lehmann [9] p 142 that the power function is increasing in γ_i and μ_i. Hence the test maximizes both the minimum and average power over alternatives $\omega_i = \{(\mu_i,\gamma_i): \ \mu_i \geq \mu^*, \ \gamma_i \geq \gamma^*\}$ for any μ^* and γ^*.

B. <u>The slippage problem for the binomial distribution.</u>

Let $X_1,\ldots,X_q$ be independently distributed with binomial distributions

$$P[X_i=x_i] = \binom{n_i}{x_i} p_i^{x_i}(1-p_i)^{n_i-x_i} \qquad i=1,\ldots,q.$$

Let $\Theta_i = p_i/(1-p_i)$, $i=1,\ldots,q$, and consider the problem

$$H:\Theta_1=\ldots=\Theta_q \quad \text{against} \quad K_i:\Theta_1=\ldots=\Theta_{i-1}=\gamma_i\Theta_i=\Theta_{i+1}=\ldots=\Theta_q \qquad i=1,\ldots,q$$

where $\gamma < 1$. The joint distribution of $X_1,\ldots,X_q$ under K_i

$$\left(\Pi_{j=1}^{q}\binom{n_j}{x_j}\right) \Theta^{\sum_{j=1}^{q}x_j}(1-\Theta)^{-N}\gamma_i^{-x_i}\left\{(1-\Theta)/(1-\Theta/\gamma_i)\right\}^{n_i}$$

where $N = \sum_{j=1}^{q}n_j$. The conditional distribution given $T = \sum_{j=1}^{q}X_j$, which is sufficient and complete under H, is of the form

$$\left(\Pi_{j=1}^{q}\binom{n_j}{x_j}\right)C(t,n_i,\Theta\gamma_i)\gamma_i^{-x_i}$$

Hence the test maximizing the average power over alternatives $\gamma_i = \gamma_i^*$, $i=1,\ldots,q$, $\Theta = \Theta^*$, will be of the form $\phi_i(x) = 1$ when $x_i >$ constant and $C(t,n_i,\Theta^*,\gamma_i^*)\gamma_i^{*-x_i} = \max_j \ C(t,n_j,\Theta^*,\gamma_j^*)\gamma_j^{*-x_j}$. In the case $n_1=\ldots=n_q$ and $\gamma_1^*=\ldots=\gamma_q^*$ this reduces to $x_i >$ constant and $x_i = \max_j x_j$. The constant is determined so that the conditional probability of rejecting H given T is equal to α. The power of the test depends upon Θ and γ. It is a decreasing function of γ for fixed Θ, hence it maximizes the minimum power and minimum average power over alternatives $\omega_i = \{(\Theta,\gamma):\Theta=\Theta^*, \ \gamma\leq\gamma^*\}$ $i=1,\ldots,q$, for any Θ^* and γ^*.

Remark 7.1. In Doornbos [4] is shown an optimum property of some slippage tests for discrete distributions where optimality is defined relative to the conditional distribution given the sufficient and complete statistics. In this paper optimality for the conditional distribution is used as an intermediate step to derive optimality for the unconditional distribution. As we have seen the fact that a test maximizes average power in the conditional distribution also carries over to the unconditional distribution. An interesting question is whether this is, in general, the case for the test maximizing minimum power. My conjecture is that this is not always so.

Remark 7.2. One might think that for example for the problem in A it would be a stronger statement to state that the test maximizes the minimum over all alternatives with $\gamma \geq 1$. The minimum power over $\gamma \geq 1$ is, however, $= \alpha$ for any unbiased test. (See Lehmann [9]).
Hence the result in A is stronger.

Remark 7.3. In [6] and [7] is discussed a class of tests called symmetric in power for slippage problems, and most powerful tests are derived in the case when the probability distribution has certain symmetric properties. In the cases studied in this paper where the probability distribution satisfies these symmetric properties, the test maximizing minimum power will be the same as the most powerful test which is symmetric in power.

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