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Erik N. Torgersen

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Abstract

The weak compactness lemma is a useful tool for proving the existence of optimal procedures. Occasionally this lemma is formulated and proved for separable sample spaces only. One reason might be that the effort spent in getting rid of the separability condition might seem to outweigh the dividind of more general statements. The purpose of this note is to present a simple minded proof, using elementary properties of conditional expectations only.

The separability criterion in the weak compactness lemma.

<u>How to get rid of separability by conditional expectations.</u>

Let $\delta_1, \delta_2, \ldots$ be a uniformly integrable sequence of real random variables on a probability space $(\chi, \mathcal{A}, P)$. The weak compactness lemma asserts the existence of a $\delta \in L_1(P)$ and a sub sequence $\delta_{n_1}, \delta_{n_2}, \ldots$ so that

$$\lim_k E\delta_{n_k} h = E\delta h \ ; \ h \in L_\infty(P) \ .$$

For separable $\mathcal{A}$ the proof is often based on Cantor's diagonal argument and the Radon Nikodym theorem.

We include - for the sake of completeness - the proof here. The arguments run as follows:

Let $\mathcal{B}$ be a countable basis for $\mathcal{A}$. We may - without loss of generality - assume that $\mathcal{B}$ is an algebra and that $\delta_n \geq 0$, $n = 1,2,\ldots$. By Cantor's diagonal method there is - since $E\delta_1, E\delta_2, \ldots$ is bounded - a sub sequence $\delta_{n_1}, \delta_{n_2}, \ldots$ so that $\varphi(B) \underset{\text{definition}}{=} \lim_k EI_B\delta_{n_k}$ exists for all $B \in \mathcal{B}$. (For any set A in $\mathcal{A}$; I_A is the indicator function of A i.e. $I_A(x) = 0$ or 1 as $x \notin A$ or $x \in A$.)

To any $\varepsilon > 0$ there is - by uniform absolute continuity - a $\delta_\varepsilon > 0$ so that $P(B) < \delta_\varepsilon \Rightarrow \varphi(B) < \varepsilon$ and $\sup_n EI_B\delta_n < \varepsilon$. It follows that φ is σ-additive and bounded, and consequently has a unique extension to a finite measure φ on $\mathcal{A}$. Furthermore there is - since $\mathcal{B}$ is an algebra which generates $\mathcal{A}$ - to any $\varepsilon > 0$ a $B_\varepsilon \in \mathcal{B}$ so that $P(A \,\Delta\, B_\varepsilon) < \delta_\varepsilon$. This implies that $\lim_k EI_A\delta_{n_k} = \varphi(A)$ for any $A \in \mathcal{A}$. In particular $\varphi(A) = 0$ when $P(A) = 0$.

By the Radon Nikodym theorem there is a δ in $L_1(P)$ so that $EI_A\delta = \varphi(A)$; $A \in \mathcal{A}$. By linearity $\lim_k E\delta_{n_k}h = E\delta h$ for any measurable simple function h, - and by approximation this extends to any $h \in L_\infty(P)$.

The proof might have been slightly simplified by appealing to the fact that $L_1^* = L_\infty$.

If $\delta_1, \delta_2, \ldots$ had not been uniformly integrable, then it might be shown (The Vitali Hahn Saks theorem), that there is a sub sequence posessing no weakly convergent sub sequence.

Assuming the weak compactness lemma in the separable case, the general case may be deduced as follows:

Let $\mathcal{C}$ be the smallest σ-algebra on χ making $\delta_1, \delta_2, \ldots$ measurable. By assumption - since $\mathcal{C}$ is separable - there is a sub sequence $\delta_{n_1}, \delta_{n_2}, \ldots$ and a measurable δ in $L_1(P)$ so that $\lim_k E\delta_{n_k} h = E\delta h$ for any $\mathcal{C}$ measurable h in $L_\infty(P)$.

For any $h \in L_\infty(P)$, however, $E^{\mathcal{C}} h$ is in $L_\infty(P)$. Hence

$E\delta_{n_k} h = E\delta_{n_k} E^{\mathcal{C}} h \to E\delta E^{\mathcal{C}} h = E\delta h.$

In the particular case of a uniformly L_∞ bounded sequence the extension to $h \in L_1(P)$ and P σ-finite may be carried out by approximating h with its truncations and using the fact that any σ-finite measure is equivalent with a probability measure.

The uniform integrability criterion for weak conditional compactness of sub sets of $L_1(P)$ in the weak topology induced by $L_\infty(P)$, is just as efficient as the weak compactness lemma in proving existence of optimal procedures. Moreover this criterion, together with Eberlein's theorem on the equivalence of sequence compactness and net compactness, yields another proof of the weak compactness lemma.