STATISTICAL RESEARCH REPORT No 10
Institute of Mathematics 1971
University of Oslo

A SIMPLE TECHNIQUE FOR PROVING UNBIASEDNESS OF TESTS AND CONFIDENCE REGIONS

by

Emil Spjøtvoll

ABSTRACT

In situations without nuisance parameters it is known that under certain assumptions that confidence sets based upon the likelihood function are unbiased. It is shown how this result can be used to prove the unbiasedness of certain modified likelihood ratio tests and likelihood ratio confidence sets.

1. INTRODUCTION

Let X be a random variable with probability density $f(x,\theta)$. The parameter θ belongs to a set Ω. In a recent paper the author (1971) has studied confidence sets based upon the likelihood function. The sets are of the form

$$S(x) = \{\theta : \frac{f(x,\theta)}{\sup_{\theta} f(x,\theta)} \geq c\},$$

where c is the largest c such that

$$P_{\theta}\{\theta \in S(X)\} \geq 1-\alpha . \qquad (1)$$

From the confidence sets we can also easily derive a test of the hypothesis

$$H : \theta = \theta_o \text{ against } \theta \neq \theta_o ;$$

the test rejects if $\theta_o \notin S(x)$. Then the probability of a false rejection is

$$P_{\theta_o}\{\theta_o \notin S(X)\} \leq \alpha .$$

The author (1971) has shown that under certain assumptions the confidence sets (1) are unbiased. Then the test which rejects H when $\theta_o \notin S(x)$ is also unbiased.

In this paper we shall consider the situation where the distribution of X depends upon two parameters θ and η, where η is a nuisance parameter. Suppose that there exists a statistics $Y(X)$ with density $g(y,\theta)$ (with respect to a measure μ) which depends only upon θ and not upon η. Then, from the likelihood function $g(y,\theta)$, we can construct confidence sets of the form

$$T(y) = \{\theta : \frac{g(y,\theta)}{\sup_{\theta} g(y,\theta)} \geq c\} .$$

Under the following assumptions A 1-3 (see Spjøtvoll (1971)) the confidence sets $T(y)$ are unbiased.

A 1. Ω is a separable topological space.

A 2. $g(y,\theta)$ is continuous in θ for all y .

A 3. The family of densities $\{g(y,\theta) : \theta \in \Omega\}$ is invariant under a group G of measurable transformations, of the sample space and μ is absolutely continuous with respect to μg^{-1} for all $g \in G$." Furthermore, the induced group $\bar{G}$ of transformations of Ω is transitive over Ω , and the transformations $\bar{g} \in \bar{G}$ are continuous.

2. CONFIDENCE SETS FOR UNKNOWN VARIANCES

Let the variables X_{ij} , $i = 1,\ldots,r$, $j = 1,\ldots,N_i$ be independent with normal distributions, where $EX_{ij} = \mu_i$ and $\mathrm{Var}\, X_{ij} = \sigma_i^2$. The parameters of interest are $\sigma_1^2,\ldots,\sigma_r^2$. To find a confidence set for $\sigma_1^2,\ldots,\sigma_r^2$ consider the variables

$$S_i = \frac{1}{n_i} \sum_{j=1}^{N_i} (X_{ij} - \bar{X}_i)^2$$

where

$$n_i = N_i - 1$$

$$\bar{X}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij} .$$

The joint distribution of $S_1,\ldots,S_r$ is proportional to

$$\prod_{i=1}^{r} \{ s_i^{\frac{1}{2}n_i - 1} \sigma_i^{-n_i} \exp(-\tfrac{1}{2} n_i s_i / \sigma_i^2) \} . \qquad (2)$$

A 1-3 are satisfied with G a group of scale changes for each variable S_i . The maximum of (2) w. r. t. $\sigma_1,\ldots,\sigma_r$ is attained for the values

$$\hat{\sigma}_i^2 = s_i \qquad i = 1,\ldots,r .$$

The value of the maximum is proportional to

$$\prod_{i=1}^{r} s_i^{-1} . \tag{3}$$

The ratio of (2) to (3) is

$$\prod_{i=1}^{r} \{(s_i/\sigma_i^2)^{\frac{1}{2}n_i} \exp(-\tfrac{1}{2}n_i s_i/\sigma_i^2)\} .$$

It follows from the results of Section 1 that the confidence region

$$\{(\sigma_1,\ldots,\sigma_r) : \prod_{i=1}^{r} \{ (s_i/\sigma_i^2)^{\frac{1}{2}n_i} \exp(-\tfrac{1}{2}n_i s_i/\sigma_i^2)\} \geq \text{constant}\}$$

is unbiased.

Next, consider a confidence region for ratios of the σ_i - s . Introduce the variables

$$T_i = \frac{S_i}{S_r} \qquad i = 1,\ldots,r-1 .$$

The joint density of $T_1,\ldots,T_{r-1}$ is found to be proportional to

$$(\prod_{i=1}^{r-1} t_i^{\frac{1}{2}n_i-1} \gamma_i^{-\frac{1}{2}n_i})(\sum_{i=1}^{r-1} n_i t_i/\gamma_i + n_r)^{-\frac{1}{2}\Sigma_{i=1}^{r} n_i} \tag{4}$$

where

$$\gamma_i = \frac{\sigma_i^2}{\sigma_r^2} \qquad i = 1,\ldots,r-1 .$$

A 1-3 are again satisfied with G a group of scale changes for each variable T_i . The maximum of (4) w. r. t. $\gamma_1,\ldots,\gamma_{r-1}$ takes place for the values $\hat{\gamma}_i = t_i$, $i=1,\ldots,r-1$, and the maximum is proportional to

$$(\prod_{i=1}^{r-1} t_i^{-1}) \tag{5}$$

The ratio of (4) to (5) is

$$L(t_1,\ldots,t_{r-1},\gamma_1,\ldots,\gamma_{r-1}) = \{\prod_{i=1}^{r-1} (t_i/\gamma_i)^{\frac{1}{2}n_i}\} (\sum_{i=1}^{r-1} n_i t_i/\gamma_i + n_r)^{-\frac{1}{2}\Sigma_{i=1}^{r} n_i} .$$

The confidence region

$$\{(\gamma_1,\ldots,\gamma_{r-1}) : L(t_1,\ldots,t_{r-1},\gamma_1,\ldots,\gamma_{r-1}) \geq \text{constant}\}$$

is unbiased.

The test which rejects the hypothesis $\gamma_1=\ldots=\gamma_{r-1} = 1$ when $L(t_1,\ldots,t_{r-1},1,\ldots,1) < \text{constant}$ is also unbiased. We have

$$L(t_1,\ldots,t_{r-1},1,\ldots,1)=\left(\prod_{i=1}^{r-1} t_i^{\frac{1}{2}n_i}\right)\left(\sum_{i=1}^{r-1} n_i t_i+n_r\right)^{-\frac{1}{2}\Sigma_{i=1}^{r} n_i},$$

which also can be written in the form

$$\left(\prod_{i=1}^{r} s_i^{\frac{1}{2}n_i}\right)\left(\sum_{i=1}^{r} n_i s_i\right)^{-\frac{1}{2}\Sigma_{i=1}^{r} n_i}.$$

Hence the test obtained is Bartlett's (1937) test for testing equality of variances. The result that Bartlett's test is unbiased is not new, it was proved by Pitman (1939) using a different method.

3. CONFIDENCE SET FOR AN UNKNOWN COVARIANCE MATRIX

Let $p\times 1$ vectors $X_1,\ldots,X_N$, $(N > p)$, be a random sample from a multivariate normal distribution $N(\mu,\Sigma)$ where both μ and Σ are unknown. Consider the problem to find tests and confidence sets for Σ. It is natural to start with the density of

$$S = \sum_{i=1}^{N} (X_i-\overline{X})(X_i-\overline{X})'$$

where $\overline{X} = N^{-1} \Sigma_{i=1}^{N} X_i$.

The distribution of S is a Wishart distribution with $n = N-1$ degrees of freedom and covariance matrix Σ, hence the density is proportional to

$$|\Sigma|^{-\frac{1}{2}n} \, |S|^{\frac{1}{2}(n-p-1)} \exp\{-\tfrac{1}{2}\mathrm{tr}(\Sigma^{-1}S)\} \tag{6}$$

The maximum of (6) w. r. t. occurs for $\hat{\Sigma} = n^{-1}S$ (see, e.g., Anderson (1958) pp 46-47), and the maximum is proportional to

$$|S|^{-\frac{1}{2}(p+1)} \tag{7}$$

The ratio of (6) to (7) is

$$L(S,\Sigma) = |S\Sigma^{-1}|^{\frac{1}{2}n} \exp\{-\tfrac{1}{2}\operatorname{tr}(\Sigma^{-1}S)\} .$$

A 1-3 are satisfied with G the group of all transformations CSC' of S where C is a nonsingular matrix. An unbiased confidence region for Σ is given by

$$\{\Sigma : L(S,\Sigma) \geq \text{constant}\} ,$$

and an unbiased test of the hypothesis

$$H : \Sigma = \Sigma_0 \quad \text{against} \quad \Sigma \neq \Sigma_0$$

is given by rejecting H when

$$L(S,\Sigma_0) < \text{constant} \tag{8}$$

Sugiura and Nagao (1968, pp 1686-88) proved the unbiasedness of the test (8) by a different method using a generalization of Pitman's (1939) technique.

4. JOINT CONFIDENCE SET FOR THE MEAN VECTOR AND THE COVARIANCE MATRIX

If we want joint tests and confidence regions for both μ and Σ, we can start with the joint density of $X_1,\ldots,X_N$ which is proportional to

$$|\Sigma|^{-\frac{1}{2}N} \exp\{-\tfrac{1}{2}(S\Sigma^{-1}+N(\overline{X}-\mu)(\overline{X}-\mu)')\} . \tag{9}$$

The maximum of (9) w. r. t. and Σ occur for $\hat{\mu} = \overline{X}$ and $\hat{\Sigma} = N^{-1}S$.

The maximum of (9) is then proportional to

$$|S|^{-\frac{1}{2}N} . \tag{10}$$

The ratio of (9) to (10) is

$$L(\overline{X},S,\Sigma,\mu) = |S\Sigma^{-1}|^{\frac{1}{2}N} \exp\{-\tfrac{1}{2}\operatorname{tr}(S\Sigma^{-1}+N(\overline{X}-\mu)(\overline{X}-\mu)'\} .$$

A 1-3 are satisfied with G a group of translations of $\overline{X}$ and transformations CSC' of S. The unbiased confidence region for μ and Σ, and the unbiased test for the hypothesis $\mu = \mu_0$ and

$\Sigma = \Sigma_0$ is found in the usual way. The results agree with those of Sugiura and Nagao (1968, pp 1691-92) where a different method of proof is used.

5. THE TEST FOR SPHERICITY

Consider again the example of Section 3. We shall derive a confidence region for the ratios

$$\gamma_{ij} = \frac{\sigma_{ij}}{\sigma_{11}} .$$

Let S_{ij} be the (i,j)-th element of S, and introduce the variables

$$T_{ij} = \frac{S_{ij}}{S_{11}} .$$

Let T be the pxp matrix with elements T_{ij}. The joint density of the T_{ij}-s and S_{11} is proportional to

$$|\Sigma|^{-\frac{1}{2}n} |T|^{\frac{1}{2}(n-p-1)} s_{11}{}^{\frac{1}{2}(np)-1} \exp\{-\tfrac{1}{2} s_{11} \mathrm{tr}(T\Sigma^{-1})\} .$$

Integrating over s_{11} the density of T is found to be proportional to

$$|\Sigma|^{-\frac{1}{2}n} |T|^{\frac{1}{2}(n-p-1)} \{\mathrm{tr}(T\Sigma^{-1})\}^{-\frac{1}{2}(np)} \tag{11}$$

Let Δ be the matrix with elements γ_{ij}. Then (11) can be written

$$|\Delta|^{-\frac{1}{2}n} |T|^{\frac{1}{2}(n-p-1)} \{\mathrm{tr}(T\Delta^{-1})\}^{-\frac{1}{2}(np)} . \tag{12}$$

Using methods analogous to Anderson (1957, pp 46-47) it is found that the maximum of (12) takes place for $\hat{\Delta} = T$. The maximum is then proportional to

$$|T|^{-\frac{1}{2}(p+1)} . \tag{13}$$

The ratio of (12) to (13) is

$$L(T,\Delta) = |T\Delta^{-1}|^{\frac{1}{2}n} \{\mathrm{tr}(T\Delta^{-1})\}^{-\frac{1}{2}(np)} .$$

A 1-3 are satisfied with the group of transformation that G in Section 3 induces on the space of matrices T.

The confidence region Δ based upon $L(T,\Delta)$ is therefore unbiased.

The hypothesis of sphericity

$$H : \Sigma = \sigma^2 I \text{ for some } \sigma^2 \text{, against } \Sigma \neq \sigma^2 I$$

is equivalent to

$$H' : \Delta = I \text{ against } \Delta \neq I .$$

An unbiased test of H' is given by rejecting when

$$|T|^{\frac{1}{2}n} (\text{tr } T)^{-\frac{1}{2}(np)} < \text{constant.} \tag{14}$$

Using the relationship between T and S (14) can be written

$$|S|^{\frac{1}{2}n} (\text{tr } S)^{-\frac{1}{2}(np)} < \text{constant .} \tag{15}$$

The unbiasedness of the test (15) was first proved by Gleser (1966), and later by Sugiura and Nagao (1968, pp 1681-1691). The technique used in this paper is different from both the earlier ones.

REFERENCES

ANDERSON, T.W. (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.

BARTLETT, M.S. (1937). Properties of sufficiency and statistical tests. Proc. Roy. Soc. London, series A, 160, 268-282.

GLESER, L.J. (1966). A note on the sphericity test. Ann. Math. Statist. 37, 464-467.

PITMAN, E.J.G. (1939). Tests of hypothesis concerning location and scale parameters. Biometrika, 31, 200-215.

SPJØTVOLL, E. (1971). Unbiasedness of likelihood ratio confidence sets in cases without nuisance parameters. To appear in J. R. Statist. Soc. B.

SUGIURA, N. and NAGAO, H. (1968). Unbiasedness of some test criteria for the equality of one or two covariance matrices. Ann. Math. Statist., 39, 1686-1692.