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UNIFORMLY MINIMUM VARIANCE UNBIASED (UMVU) ESTIMATORS BASED ON SAMPLES FROM RIGHT TRUNCATED AND RIGHT ACCUMULATED EXPONENTIAL DISTRIBUTIONS

by

Erik N. Torgersen

ABSTRACT

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Consider a sample; $X_1,\ldots,X_n$; of size n from the distribution F on $]0,1]$ given by:

$$F(]0,x[= 1-e^{-\lambda x} \; ; \; x \in]0,1[$$

and

$$F(\{1\}) = e^{-\lambda}$$

Here $\lambda > 0$ is an unknown parameter.

It is shown that this experiment does not admit a boundedly complete and sufficient statistic when $n \geqq 2$. We provide answers to the following problems:

Which functions of $(X_1,\ldots,X_n)$ are UMVU estimators of their expectations?

Which functions of λ has UMVU estimators?

Suppose a function of λ has an UMVU estimator. How do we find it?

How must n be chosen so that a given function of λ does have an UMVU estimator based on n obersvations?

Introduction

Consider a sample; $X_1,\ldots,X_n$; of size n from the distribution F on $]0,1]$ given by:

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Suppose a function of λ has an UMVU estimator. How do we find it?

How must n be chosen so that a given function of λ does have an UMVU estimator based on n observations?

The basic results on UMVU estimation in Lehmann and Scheffé [1950; Completeness, similar regions and unbiased estimation. Sankhyā 10, 305-340] and in Bahadur [1957; On unbiased estimates of uniformly minimum variance. Sankhyā 18, 211-224] are used constantly and without explicit references. Our use of Laplace transforms appears to be similar to the use of Laplace transforms in Linnik [1966, Statistical problems with nuisance parameters. Translations Monographs Volume 20, 1968, American mathematical society].

UMVU estimators based on samples from right truncated and right accumulated exponential distributions.

Suppose the probability of death within the infinitesimal time interval $(x, x + dx)$ is $\lambda e^{-\lambda x} dx$. Inference on λ based on, either the observed lifespans of n randomly chosen individuals, or on the time of occurence of the n-th death may - in both cases - be based on a complete and suffisient statistic. If, however, our experiment is obtained by observing the times of death within a given period for a fixed sample of at least two individuals, then, as we shall see, no complete and sufficient statistic is available*. UMVU estimation must then be based on first principles; and it is this analysis which is the subject of this section. We may, without loss of generality, assume that the period chosen is of unit length.

Our experiment, $\mathcal{E}^n$, is then obtained by making n independent observations $X_1,\ldots,X_n$ of a random variable X whose distribution is given by:

$$P\ (o < X < x) = 1-e^{-\lambda x} \quad ; \; x \in [o, 1[$$

and

$$P\ (X = 1) = e^{-\lambda}$$

We will assume that $\lambda > o$ is totally unknown. Most of the analysis, however, carries over to the case where the parameter set is a specified sub set of $]o, \infty[$ having at least one point of accumulation.

If $n = 1$, then our experiment is a sub experiment of a complete experiment and, consequently, is itself a complete experiment.

*) The problem of completeness was brought to the authors attention by E. Sverdrup.

In order to describe the joint distribution of $X_1, \ldots, X_n$ in a convenient way let us introduce the functions:

$$d(x) = 1 \text{ if } 0 \leqq x < 1$$
$$\quad = 0 \text{ if } x = 1$$
$$t(x) = x d(x) \; ; \; x \in [0, 1]$$
$$d_n(x_1, \ldots, x_n) = \sum_i d(x_i) \; ; \; x_1, \ldots, x_n \in [0, 1]$$

and

$$t_n(x_1, \ldots, x_n) = \sum_i t(x_i) \; ; \; x_1, \ldots, x_n \in [0, 1]$$

Let U and δ_1 denote, respectively, the rectangular distribution on $[0, 1]$ and the one point distribution in 1. Then

$$e^{-n\lambda} \lambda^{d_n(x)} e^{\lambda(d_n(x) - t_n(x))} \; ; \; x \in \,]0, 1]^n$$

is a version of $dP_\lambda^n / d(U + \delta_1)^n$.

It follows that $(d_n(X), t_n(X))$ constitutes a minimal sufficient statistic and that the family of distributions of this statistic is of the Darmois, Koopman type. The joint distribution of $D_n = d_n(X)$ and $T_n = t_n(X)$ is given by:

(i) $P_\lambda(D_n = d) = \binom{n}{d}(1-e^{-\lambda})^d e^{-(n-d)\lambda} \; ; \; d = 0, 1, \ldots, n$.

ie D_n is binomially distributed with sucsess parameter $1-e^{-\lambda}$.

(ii) The conditional distribution of T_n given $D_n = d$ has, for each $d = 0, 1, \ldots, n$, density:

$$\left[\frac{\lambda}{1-e^{-\lambda}}\right]^d e^{-\lambda t} \; ; \; t \in [0, d]$$

w.r.t U^{d*} where U^{d*} is the d fold convolution of U .

The density of T_n given $D_n = d$ w.r.t Lebesgue measure on $[0, d]$ is thus:

$$\left[\frac{\lambda}{1-e^{-\lambda}}\right]^d e^{-\lambda t} f_d(t) \; ; \; t \in [0, d]$$

where $f_d(t) = \frac{1}{(d-1)!} \quad [x^{d-1}-\binom{d}{1}(t-1)^{d-1}+\ldots+\binom{d}{[t]}(t-[t])^{d-1}]$; $t \in [o, d]$, is a version of the density of U^{d*} w.r.t.. Lebesgue measure an $[o, d]$.

Before proceeding let us note the fact that $\int |h(x)| U^{n*}(dx) < \infty$ if and only if $\int_o^n |h(x)| x^{n-1}(n-x)^{n-1} dx < \infty$. It follows that a statistic $\delta(D_n, T_n)$ is integrable if and only if:

$$\int_o^d |\delta(d, t) t^{d-1}(d-t)^{d-1} dt < \infty \; ; \; d = 1, \ldots, n$$

We shall use the fact that any integrable statistic may be represented by a sequence $\nu_{\delta,o}, \nu_{\delta,1}, \ldots, \nu_{\delta,n}$ of finite measures so that: $\nu_{\delta,i} \ll U^{i*}$; i=o, 1,..., n and

$$\left[\frac{d\nu_{\delta,i}}{dU^{i*}}\right]_t = \binom{n}{i} e^{-(n-i)} e^{-t} \delta(i, t) \; ; \; t \in [0, i]; \; i = 0, 1, \ldots, n$$

This representation is 1-1 onto and linear. The expectation of an integrable statistic may be expressed in terms of these measures as follows:

Proposition E. 1.

Let $\varkappa_o$ be the one point distribution in 0 and $\varkappa_i$, i = 1, 2,... the probability distribution on $[i, \infty[$ whose density w.r.t. Lebesgue measure on $[i, \infty[$ may be epecified as:

$$\Gamma(i)^{-1}(t-i)^{i-1} e^{-(t-i)} \; ; \; t \geqq i \; .$$

Then the expectation of an integrable statistic $\delta(D_n, T_n)$ may be written:

$$E\,\delta(D_n, T_n) = \lambda^n \int e^{(1-\lambda)t} [\sum_{d=o}^{n} \varkappa_{n-d} * \nu_{\delta,d}](dt) \; .$$

Proof:

$$E\,\delta(D\ ,\quad) = \sum_{d=o}^{n} \binom{n}{d}\lambda^{d} e^{-(n-d)\lambda} \int_{o}^{d} \delta(d,t)e^{-\lambda t}\, U^{d*}(dt)$$

$$= \sum_{d=o}^{n} e^{(1-\lambda)(n-d)}\lambda^{d} \int e^{(1-\lambda)t}\nu_{\delta,d}(dt)$$

$$= \lambda^{n} \sum_{d=o}^{n} \int e^{(1-\lambda)t}\varkappa_{n-d}(dt) \int e^{(1-\lambda)t}\nu_{\delta,d}(dt)$$

$$= \lambda^{n} \sum_{d=o}^{n} \int e^{(1-\lambda)t}[\varkappa_{n-d}*\nu_{\delta,d}](dt)$$

$$= \lambda^{n} \int e^{(1-\lambda)t}[\sum_{d=o}^{n} \varkappa_{n-d}*\nu_{\delta,d}](dt) \quad . \qquad \square$$

Corollary E. 2

$\delta(D_n, T_n)$ is an unbiased estimator of zero if and only if it is integrable and

(§) $\sum_{d=o}^{n} \varkappa_{n-d}*\nu_{\delta,d} = 0$

The corollary tells us - in principle - how to construct the most general unbiased estimator of zero. To see this rewrite (§) as:

(§§) $\nu_{\delta,n} = -\sum_{d=0}^{n-1} \varkappa_{n-d}*\nu_{\delta,d}$

The procedure is therefore: Choose $\delta(d,\cdot)$; d=o, 1,..., n-1 so that $\sum_{d=o}^{n-1} \varkappa_{n-d}*\nu_{\delta,d}$ has no mass on $[\ n, \infty[$. Finally $\nu_{\delta,n}$ and thus $\delta(n,\cdot)$ is obtained by (§§).

Proposition E. 3

Let $\delta(d,t)$; $t\epsilon[o,d]$: d=o,..., n-1 be given functions. Then there is a function $\delta(n,t)$ so that $\delta(D_n, T_n)$ is an unbiased estimator of zero if and only if:

(i) $\int_o^d |\delta(d,t)| t^{d-1}(d-t)^{d-1} dt < \infty$; $d=1,\ldots,n-1$

(ii) $\delta(o,o) = o$

and
$$\sum_{d=1}^{i} \frac{e^{n-d}}{\Gamma(n-d)} \binom{n-d-1}{n-i-1} \int_o^d e^x (d-x)^{i-d} \nu_{\delta,d}(dx) = o; \quad i=1,\ldots,n-1$$

Proof:

We must show that (ii) is - assuming (i) is satisfied - a necessary and sufficient condition. The measures $\varkappa_{n-d} * \nu_{\delta,d}$; $d=o,1,\ldots,n-1$ are absolutely continuous with, respectively, densities*:

$$\int_o^{d\wedge(t-n+d)} \Gamma(n-d)^{-1}(t-x-n+d)^{(n-d-1)} e^{-(t-x-n+d)} \nu_{\delta,d}(dx) \; ; \; t \geqq n-d$$

It follows that a version of the density of $\sum_{d=o}^{n-1} \varkappa_{n-d} * \nu_{\delta,d}$ is:

$$\Gamma(n)^{-1}(t-n)^{n-1} e^{-(t-n)} \nu_{\delta,o}(\{o\}) \, I_{[n,\infty[}$$
$$+ \sum_{d=1}^{n-1} \int_o^{d\wedge(t-n+d)} \Gamma(n-d)^{-1}(t-x-n+d)^{n-d-1} e^{-(t-x-n+d)} \nu_{\delta,d}(dx) \; ; \; t \geqq 1$$

On $[n, \infty[$ this reduces to

$$\Gamma(n)^{-1}(t-n)^{n-1} e^{-(t-n)} \nu_{\delta,d}(\{o\})$$
$$+ \sum_{d=1}^{n-1} \int_o^d \Gamma(n-d)^{-1}(t-x-n+d)^{n-d-1} e^{-(t-x-n+d)} \nu_{\delta,d}(dx)$$

It follows - by continuity - that $\sum_{d=o}^{n-1} \varkappa_{n-d} * \nu_{\delta,d}$ is concentrated on $[o, n]$ if and only if:

$$\Gamma(n)^{-1}(t-n)^{n-1} \nu_{\delta,d}(\{o\})$$
$$+ \sum_{d=1}^{n-1} \int_o^d \Gamma(n-d)^{-1}(t-x-n+d)^{n-d-1} e^{-(d-x)} \nu_{\delta,d}(dx) = o, \; t > n$$

i e: $\Gamma(n)^{-1} t^{n-1} \nu_{\delta,d}(\{o\})$
$$+ \sum_{d=1}^{n-1} \int_o^d \Gamma(n-d)^{-1}(t-x+d)^{n-d-1} e^{-(d-x)} \nu_{\delta,d}(dx) = o, \; t > 0$$

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*) If a and b are numbers then $a \wedge b \overset{\text{def}}{=} \min\{a, b\}$.

The left hand side of this identity is - in any case - a polynomical of degree at most n-1 . The proof is now completed by checking that the equations in (ii) just states that the cofficients of this polynomial are all zero. $\square$.

(ii) may be rewritten as:

(ii') $\delta(o,o) = o$

and

$$\frac{e^{n-j-1}}{\Gamma(n-j-1)} \int_o^{j+1} e^x \nu_{\delta,j+1}(dx) = -\sum_{d=1}^{j} \frac{e^{n-d}}{\Gamma(n-d)} \binom{n-d-1}{n-j-2} \int_o^d e^x (d-x)^{j+1-d} \nu_{\delta,d}(dx); \quad j=o,1,\dots,n-2$$

Suppose now that $\delta(d,\cdot)$ are constructed for $d=o,1,..,i \leqq n-2$ so that $\int_o^d |\delta(d,t)| t^{d-1}(d-t)^{d-t} < \infty$ and (ii) holds i.e:

$$\sum_{d=1}^{i'} \frac{e^{n-d}}{\Gamma(n-d)} \binom{n-d-2}{n-i'-1} \int_o^d e^x (d-x)^{i'-d} \nu_{\delta,d}(dx) = o; \; i'=1,2,\dots, i$$

and $\delta(o,o) = o$.

By proposition E.3 there are functions $\delta(d,\cdot)$; $i < d \leqq n$ so that $\delta(D_n, T_n)$ is an unbiased estimator of zero.

Consider now any UMVU estimator φ . By proposition E.3 there is no restriction on $\varphi(o,o)$. Let δ be an UMVU estimator of zero. By

(ii) $\int_o^1 e^x \nu_{\delta,1}(dx) = o$ ie: $\int_o^1 \delta(1,x)dx = o$. Hence

$\int_o^1 \varphi(1,x)\delta(x)dx = o$ for any square integrable function $\delta(x)$, $x\in[o,1]$ such that $\int_o^1 \delta(x)dx = o$. This imply that there is a constant b so that $\varphi(1,\cdot) = b$ a.e Lebesgue on $[o,1]$

We shall now demonstrate that $\varphi(d,\cdot) = b$ a.e Lebesgue on $[o,d]$ for $d = 1,2,\dots n-1$. Suppose we have shown that $\varphi(d,\cdot) = b$ a.e. Lebesgue on $[o,d]$ for $d = 1,2,\dots,j < n-1$.

Let δ be a square integrable unbiased estimator of zero.

Then $\varphi\delta$ is an unbiased estimator of zero so that:

$$\frac{e^{n-j-1}}{\Gamma(n-j-1)}\int_o^{j+1} e^x \nu_{\delta\varphi,j+1}(dx) = -\sum_{d=1}^{j}\frac{e^{n-d}}{\Gamma(n-d)}\binom{n-d-1}{n-j-2}\int_o^d e^x(d-x)^{j+1-d}\nu_{\delta\varphi,d}(dx)$$

Hence

$$\frac{e^{n-j-1}}{\Gamma(n-j-1)}\int_o^{j+1} e^x\varphi(j+1,x)\nu_{\delta,j+1}(dx) = -\sum_{d=1}^{j}\frac{e^{n-d}}{\Gamma(n-d)}\binom{n-d-1}{n-j-2}\int_o^d e^x(d-x)^{j+1-d}\varphi(j,x)\nu_{\delta,d}(dx)$$

$$= -b\sum_{d=1}^{j}\frac{e^{n-d}}{\Gamma(n-d)}\binom{n-d-1}{n-j-2}\int_o^d e^x(d-x)^{j+1-d}\nu_{\delta,d}(dx)$$

$$= b\frac{e^{n-j-1}}{\Gamma(n-j-1)}\int_o^{j+1} e^x\nu_{\delta,j+1}(dx)$$

It follows that:

$$\int_o^{j+1}\varphi(j+1,x)\delta(x)U^{(j+1)*}(dx) = b\int_o^{j+1}\delta(x)U^{(j+1)*}(dx) = o$$

for any bounded function δ on $[o,j+1]$ satisfying

$\int_o^{j+1}\delta(x)U^{(j+1)*}(dx) = o$. Hence $\varphi(j+1,\cdot) = b$ a.e $U^{(j+1)*}$ i e

$\varphi(j+1,\cdot) = b$ a.e Lebesgue. By induction: $\varphi(d,\cdot) = b$ a.e Lebesgue on $[o,d]$; $d=1,\dots, n-1$.

What can we say about $\varphi(n,\cdot)$?

Let δ be any unbiased estimator of zero. The density (strictly speaking; a version of) $\nu_{\delta,n}$ is $t \rightsquigarrow e^{-t}\delta(n,t)f_n(t)$ on $[1,n]$. By the identity: $\nu_{\delta,n} = -\sum_{d=o}^{n-1}\varkappa_{n-d}*\nu_{\delta,d}$, this density may also be written:

$$-\sum_{d=o}^{n-1}\int_o^{d\wedge(t-n+d)}\Gamma(n-d)^{-1}(t-x-n+d)^{n-d-1}e^{-(t-x-n+d)}\nu_{\delta,d}(dx)$$

It follows - provided $\delta(D_n, T_n)^2$ is integrable - that:

$$e^{-t}\delta(n,t)\varphi(n,t)f_n(t) = -\sum_{d=1}^{n-1}\int_o^{d\wedge(t-n+d)}\Gamma(n-d)^{-1}(t-x-n+d)^{n-d-1}e^{-(t-x-n+d)}b\nu_{\delta,d}(dx)$$

$$= b\, e^{-t}\delta(n,t)f_n(t) \text{ ; almost all } t\in[1,n]$$

Hence $[\varphi(n,t)-b]\delta(n,t)\ f_n(t) = o$; almost all $t\epsilon[1,n]$

Consider now the space of unbiased estimators δ of zero such that: $\delta(o,\cdot) = o,\ \delta(1,\cdot) = o, \ldots, \delta(n-2,\cdot) = o$

By proposition E.3 :
$$\int_o^{n-1} e^x \nu_{\delta,n-1}(dx) = o$$
i e:
$$\int_o^{n-1} \delta(n-1,x)\ f_{n-1}(x)dx = o\ ,$$ and this is the only requrement on $\delta(n-1,\cdot)$. By the density considerations above:
$$e^{-t}\delta(n,t)\ f_n(t) = -\int_o^{t-1} \Gamma(1)^{-1}(t-x-1)^o e^{-(t-x-n+n-1)}\nu_{\delta,n-1}(dx) =$$
$$= -ne^{-t}\int_o^{t-1} \delta(n-1,x)f_{n-1}(x)dx\ ;\ \text{almost all}\ \ t\epsilon[1,n]$$
It follows that
$$[\varphi(n,t)-b]\int_o^{t-1} \delta(n-1,x)f_{n-1}(x)dx = o,\ \text{on almost all}$$
$t\epsilon[1,n]$,provided $\delta(D_n, T_n)$ is square integrable.

It follows that
$$[\varphi(n,t)-b]\int_o^{t-1} \gamma(x)dx = o\ ;\ \text{for almost all}\ \ t\epsilon[1,n]\ .$$
provided
$$\int_o^{n-1} \gamma(x)f_{n-1}(x)dx\ = o,\ \int_o^{n-1} \gamma(x)^2 f_{n-1}(x)dx < \infty$$
and
$$\int_o^n [1/f_n(t)]\ [\int_o^{t-1} \gamma(x)f_{n-1}(x)dx]^2 dt < \infty\ .$$
These conditions are satisfied with $\gamma = \gamma_\epsilon$ where
$$\gamma_\epsilon(x) = \begin{cases} o & \text{when}\quad o < x < \epsilon \\ -1/f_{n-1}(x) & \text{"}\quad \epsilon \leqq x \leqq \frac{n-1}{2} \\ 1/f_{n-1}(x) & \text{"}\quad \frac{n-1}{2} < x \leqq n-1-\epsilon \\ o & \text{"}\quad x > n-1-\epsilon \end{cases}$$
and $\epsilon \in]o\ ,\ \frac{n-1}{2}[$ is a constant. Letting $\epsilon \to o$ we get:
$$[\varphi(n,t)-b]\int_o^{t-1} \Psi(x)dx = o\ ;\quad \text{for almost all}\ \ t\epsilon[1,n]$$
where $\Psi(x) = -1$ or $+1$ as $x < \frac{n-1}{2}$ or $x > \frac{n-1}{2}$

Hence

$$\varphi(n,\cdot) = b \text{ a.e. on } [1,n]$$

If δ is an unbiased estimator of zero then, by the identity: $\nu_{\delta,n} = -\sum_{d=1}^{n-1} \varkappa_{n-d} * \nu_{\delta,d}$, $\nu_{\delta,n}$ is consentrated on $[1,n]$. It follows that $\delta(n,\cdot) = o$ a. e on $[o,1]$. This unply that there is no restrictions on $\varphi(n,\cdot)$ on $[o,1]$, except for the condition of square integrability. We have proved the "only if" part of

Theorem E.4

$\varphi(D_n, T_n)$ is an UMVU estimator of its expectation if and only if:

(i) φ is a.e. a constant on

$\{(d,t): 1 \leqq d \leqq n-1\} \cup \{(n,t): 1 \leqq t \leqq n\}$

and

(ii) $\int_o^1 \varphi(n,t)^2 t^{n-1} dt < \infty$

Proof: Suppose φ satisfies (i) and (ii) and let δ be any square integrable estimator of zero. We must show that $\varphi\delta$ is an unbiased estimator of zero. (It is easily seen that φ is square integrable). Let b denote the constant in(i) . Then

$$\sum_{d=o}^{n-1} \varkappa_{n-d} * \nu_{\delta\varphi,d} = b \sum_{d=o}^{n-1} \varkappa_{n-d} * \nu_{\delta,d} = -b\nu_{\delta,n} = -\nu_{b\delta,n} = -\nu_{\varphi\delta,n} . \blacksquare .$$

Corollary E.5

$\varphi(D_n, T_n)$ is an UMVU estimator of its expectation if and only if it is square integrable and measurable w.r.t. the σ-algebra $\mathfrak{B}$ of sets generated by the sets:

$[D_n = P_n = o]$,

$[1 \leqq D_n \leqq n-1] \cup [D_n = n, T_n \geqq 1]$

and

$[D_n = n] \wedge [T_n \leqq t]$; $t \in [o,1]$.

Let $\varphi(D_n, T_n)$ be $\mathcal{S}$ measurable and integrable. Put

$a = \varphi(o,o)$,

$b = \varphi(D_n, T_n)$; a.s. on the set $[1 \leq D_n \leq n-1] \cup [D_n=n, T_n \geq 1]$

and write $c(t) = \varphi(n,t)$ when $t \in [o,1]$. Then

$$E\varphi(D_n,T_n) = aP(D_n = T_n = o)$$

$$+ b\,[1-P(D_n = T_n = o) - P(D_n = n, P_n < 1)]$$

$$+ P(D_n = n) \int_o^1 c(t)\, P(T_n \in dt \mid D_n = n)$$

$$= a\, e^{-n\lambda} + b(1-e^{-n\lambda} - \lambda^n \int_o^1 e^{-\lambda t} \frac{t^{n-1}}{(n-1)!} dt)$$

$$+ \lambda^n \int_o^1 c(t) e^{-\lambda t} \frac{t^{n-1}}{(n-1)!}\, dt$$

$$= A + Be^{-n\lambda} + \lambda^n \int_o^1 C(t) e^{-\lambda t} dt$$

where $A = b$, $B = a-b$ and $C(t) = \frac{t^{n-1}}{(n-1)!}\,(c(t) - b)$

P-integrability of φ is equivalent with Lebesgue integrability of C. We have almost proved:

<u>Proposition E.6</u>

$\mathcal{S}$ is complete and a function g of λ has a $\mathcal{S}$ measurable unbiased estimator if and only if g is of the form:

$$(\S)\quad g(\lambda) = A + Be^{-n\lambda} + \lambda^n \int_o^1 C(t) e^{-\lambda t} dt; \quad \lambda > o$$

where A and B are constants and C is Lebesgue integrable on $[o,1]$. If g is of this form then its $\mathcal{S}$ measurable unbiased estimator φ is given by:

$$\varphi(o,o) = A + B$$

$$\varphi(d,t) = A \,;\; 1 < d < n \text{ or } (d = n \text{ and } t > 1)$$

and

$$\varphi(n,t) = A + \frac{(n-1)!}{t^{n-1}}\, C(t) \;;\; t \in [o,1]$$

<u>Completion of the proof the proposition</u>:

Consider any function

$$g(\lambda) = A + Be^{-n\lambda} + \lambda^n \int_o^1 C(t) d^{-\lambda t} dt; \quad \lambda > o$$

where A and B are constants and C is integrable (Lebesgue) on $[o,1]$. Define φ as in the proposition. By the calculations imediately before the proposition, φ is an unbiased estimator of g. Let φ be a measurable unbiased estimator of zero. Write $a = \varphi(o,o)$, $b = \varphi(d,t)$ when $1 \leqq d < n$ or ($d = n$ and $t \geqq 1$) and $C(t) = \varphi(n,t)$; $t \in [o,1]$. Then - by the same calculations:

$$g(\lambda) \overset{\text{def}}{=} A + Be^{-n\lambda} + \lambda^n \int_o^1 C(t)e^{-\lambda t}dt \equiv o$$

where $A = b$, $B = a-b$ and $C(t) = \frac{t^{n-1}}{(n-1)!}(C(t)-b)$.

We may - without loss of generality - assume $n \geqq 2$. Then

$o = \lim_{\lambda \to o} g(\lambda) = A+B$ and $o = \lim_{\lambda \to o} g'(\lambda) = -nB$ so that

$A = B = 0$. Hence $C = 0$ so that $\varphi = o$. ▨.

<u>Corollary E.7</u>

<u>A function g of λ has an UMVU estimator if and only if it is of the form:</u>

<u>$$g(\lambda) = A+Be^{n\lambda}+\lambda^n \int_o^1 C(t)e^{-\lambda t}dt; \ \lambda > o$$</u>

<u>where A and B are constants and C is a function on $[o,1]$ such that $\int_o^1 \frac{C(t)^2}{t^{n-1}} dt < \infty$.</u>

<u>If g is of this form then its UMVU estimator is the function φ defined in proposition E.6.</u>

Let F denote the probability distribution on $[o,\infty[$ whose density (w.r.t. Lebesgue Measure) is $x \curvearrowright e^{-x}$

Then $\int e^{(1-\lambda)t}F(dt) = \lambda^{-1}$; $\lambda > o$

Let g be of the form (§). Then:

$$g(\lambda)\lambda^{-n} = \int_o^\infty e^{(1-\lambda)t}[AF^{n*}+Be^{-n}F^{n*}*\delta_n + \varkappa(C)](dt); \ \lambda > o$$

where δ_n is the one point distribution in n and $\varkappa$ is the distribution on $[o,1]$ whose density w.r.t. Lebesque measure is $t \curvearrowright e^{-t}C(t)$.

It follows that a function $g(\lambda)$; $\lambda > o$ has a $\mathfrak{S}_n$measurable* unbiased estimator if and only if it is of the form:

(§§) $g(\lambda) = \lambda^n \int_o^\infty e^{(1-\lambda)t}[AF^{n*}+Be^{-n}F^{n*}*\delta_n+\varkappa](dt)$; $\lambda > o$

where A and B are constants and $\varkappa$ is an absolutely continuous finite measure on $[o,1]$. This representation is unique (ie A,B and $\varkappa$ are determined by g) and a g of the form (§§) has an UMVU estimator if and only if

$$\int_o^1 [(\frac{d\varkappa}{dt})^2 / t^{n-1}]dt < \infty$$

or equivalently that:

$$\int_o^1 \frac{C(t)^2}{t^{n-1}} dt < \infty$$

where $\varkappa(C) = \varkappa$.

We hall use this representation to study the set of positive integers n such that a given g has an UMVU estimator based on n observations.

<u>Proposition E.8</u>

Let $m < n$ be positive integers, A_m, B_m, A_n and B_n constants; and C_m and C_n Lebesgue integrable functions on $[o,1]$. Consider first the case $m > 1$. Then

$$A_m+B_m\, e^{-m\lambda}+\lambda^m \int_o^1 e^{-\lambda t}C_m(t)dt \underset{\lambda}{\equiv}$$
$$A_n+B_n e^{-n\lambda}+\lambda^n \int_o^1 e^{-\lambda t}C_n(t)dt$$

if and only if:

(i) $A_m = A_n$

(ii) $B_m = B_n = o$ and

(iii) $\varkappa(C_n) = \varkappa(C_m)*F^{(n-m)*}$

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*) From here on we write $\mathfrak{S}_n$ instead of $\mathfrak{S}$.

Let $n > 1$. Then

$$A_1+B_1e^{-\lambda}+\lambda^1 \int_o^1 e^{-\lambda t}C_1(t)dt \underset{\lambda}{\equiv}$$

$$A_n+B_ne^{-n\lambda}+\lambda^n \int_o^1 e^{-\lambda t}C_n(t)dt \quad \text{if and only if}$$

(i') $A_1 + B_1 = A_n$

(ii') $B_n = o$

and

(iii') $\varkappa(C_n) = \varkappa(C_1)*F^{(n-1)*}+B_1F^{n*}[e^{-1}\delta_1-\delta_o]$

Proof:

The identity may - in both cases - be written:

$$\lambda^m \int_o^\infty e^{(i-\lambda)t}[A_mF^{m*}+B_me^{-m}F^{m*}*\delta_m+\varkappa(C_m)]dt$$

$$\underset{\lambda}{\equiv}\lambda^n\int_o^\infty e^{(1-\lambda)t}[A_nF^{n*}+B_ne^{-n}F^{n*}*\delta+\varkappa(C_n)]dt$$

Dividing by λ^n on both sides and using the identity $\int e^{(1-\lambda)t} F(dt) = \lambda^{-1}$; $\lambda > o$ we may write this:

$$A_mF^{n*}+B_me^{-m}F^{n*}*\delta_m+\varkappa(C_m)*F^{(n-m)*}$$

$$(\dagger)$$

$$= A_nF^{n*}+B_ne^{-n}F^{n*}*\delta_n+\varkappa(C_n) \quad .$$

Further more - by letting $\lambda \to o$ in the identity as it is written in the proposition - we obtain $A_m+B_m = A_n+B_n$.

On $[m,n]$ $(\dagger)$ may be written:

$$A_mF^{n*}+B_me^{-m}F^{n*}*\delta_m+\varkappa(C_m)*F^{(n-m)*}$$

$$= A_nF^{n*} \quad .$$

Or - equivalently:

$$(A_n-A_m)F^{n*}-B_me^{-m}F^{n*}*\delta_m = \varkappa(C_m)*F^{(n-m)*}$$

Writing out densities we get:

$$(A_n-A_m) \frac{x^{n-1}}{\Gamma(n)} e^{-x}-B_me^{-m} \frac{(x-m)^{n-1}}{\Gamma(n)} e^{-(x-m)}$$

$$= \int_o^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} e^{-(x-s)}e^{-s}C_m(s)ds$$

i e:

$$(A_n - A_m)\frac{x^{n-1}}{\Gamma(n)} - B_m \frac{(x-m)^{n-1}}{\Gamma(n)} = \int_0^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} C_m(s)ds$$

for almost (Lebesgue) all x on $[m,n]$. Hence - since both sides of this equalion are polynomials - this extends to all real numbers x . The right hand side is of degree $\leqq n-m-1 \leqq n-2$. It follows that the coefficient of x^{n-1} on the left hand side must vanish ie:

$$A_n - A_m = B_m \text{ . Hence}$$

$$B_n = A_n + B_n - A_n = A_m + B_m - A_n = o$$

Let us now distinguish the two cases.

1^o. $m > 1$. Then the coefficient of x^{n-2} on the left is zero i e:

$$B_m = o$$

It follows that $A_m = A_n$ and that $B_m = B_n = o$ (iii) follows now byinserting this in (†). By (†) again (i), (ii) and (iii) are sufficient.

2^o. $m = 1$. We have shown that the identity imply (i') and (ii') . It follows that (†) may be written as (iii'): □

Using proposition E.8 we will show that the set of positive integers n such that a given g has an UMVU estimator based on n observations is an interval.

Proposition E.9

Suppose g has an unbiased $\mathcal{S}_n$ measurable estimator for $n = n_1$ and $n = n_2$. Then g has a $\mathcal{S}_m$ measurable unbiased estimator for any integer m between n_1 and n_2 . If, moreover, g has an UMVU estimator based on n ovservations for $n = n_1$ and $n = n_2$ then g has an UMVU estimator based on m observations for any m between n_1 and n_2 .

Proof:

If suffices to prove the following statement:

Statement: Let $n > m+1$. Suppose g has an unbiased $\mathcal{B}_m$ measurable estimator as well as an unbiased $\mathcal{B}_n$ measurable estimator. Then g has an unbiased $\mathcal{B}_{m+1}$ measurable estimator. If, moreover, g has an UMVU estimator based on m observations and an UMVU estimator based on n observations then g has an UMVU estimator based on $m+1$ observations.

proof of the statement:

1°. $m > 1$. Suppose g has an unbiased $\mathcal{B}_m$ measurable estimator and an unbiased $\mathcal{B}_n$ measurable estimator. Then we may - by propositions E.6 and E.8 - write:

$$g(\lambda) = A+\lambda^{-m} \int_0^1 e^{-\lambda t} C_m(t)dt$$

$$= A+\lambda^{-n} \int_0^1 e^{-\lambda t} C_n(t)dt$$

We must show that there exists a C_{m+1} so that

$$g(\lambda) = A+\lambda^{-m-1} \int_0^1 e^{-\lambda t} C_{m+1}(t)dt$$

Or equivalently that:

$$\varkappa(C_{m+1}) = \varkappa(C_m)*F$$

It suffices therefor to show that $\varkappa(C_m)*F$ is supported by $[o,1]$.

The density of $\varkappa(C_m)*F$ on $[1,\infty[$ is:

$$x \wedge\!\!\longrightarrow \int_0^1 e^{-(x-s)} e^{-s} C_m(s)ds = e^{-x}\int_0^1 C_m(s)ds$$

Hence we will be through if we can show that $\int_0^1 C_m(s)ds = o$.

By proposition E.8 $\varkappa(C_m)*F^{(n-m)*}$ is consentrated on $[o,1]$ i e $\int_0^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} C_m(s)ds = 0$ a.e. and hence - by continuity - everywhere on $[1,\infty]$. The left hand side is a polynomial in x with $\int_0^1 \Gamma(n-m)^{-1}C_m(s)ds$ as coefficient of x^{n-m-1} .

It follows that $\int_0^1 C_m(s)ds = o$. Suppose now that g has an

UMVU estimator based on m observations, i e:
$\int_0^1 \frac{C_m(t)^2}{t^{m-1}}\, dt < \infty$. It follows from the equation $\varkappa(C_{m+1}) = \varkappa(C_m)*F$ that $e^{-x}C_{m+1}(x) = \int_0^x e^{-(x-s)}e^{-s}C_m(s)ds$ for almost all x in $[o,1]$.

We may as well assume that:

$$C_{m+1}(x) = \int_0^x C_m(s)ds \ ; \ x \in [o,1]$$

We get sucsesively:

$$\int_0^1 \frac{C_{m+1}(x)^2}{x^m}\, dx = \int_0^1 \frac{(x\int_0^x C_m(s)\frac{ds}{x})^2}{x^m}\, dx \leqq$$

$$\leqq \int_0^1 \frac{x^2\int_0^x C_m(s)^2\frac{ds}{x}}{x^m}\, dx = \int_0^1 \frac{\int_0^x C_m(s)^2 ds}{x^{m-1}}\, dx =$$

$$= \int_0^1 \int_0^1 \frac{C_m(s)^2}{x^{m-1}}\, I_{[o,x]}(s)ds\, dx$$

$$= \int_0^1 C_m(s)^2 \left[\int_s^1 \frac{1}{x^{m-1}}\, dx\right] ds$$

If $m \geqq 3$ then this may be written:

$$\int_0^1 C_m(s)^2 \frac{1}{m-2}\left(\frac{1}{s^{m-2}}-1\right)ds$$

$$\leqq \int_0^1 \frac{sC_m(s)^2}{s^{m-1}}ds \leqq \int_0^1 \frac{C_m(s)^2}{s^{m-1}}\, ds < \infty .$$

If $m = 2$ then we get:

$$\int_0^1 C_m(s)^2 \left[\int_s^1 \frac{1}{x^{m-1}}\, dx\right]ds = \int_0^1 \frac{C_2(s)^2}{s}\, [-s \log s]\, ds < \infty \ .$$

2^o. $m = 1$. Suppose g has an unbiased $\mathfrak{B}_1$ measurable estimator and an unbiased $\mathfrak{B}_n$ measurable estimator. Then we may - by propositions E.6 and E.8 - write:

$$g(\lambda) = A_1+B_1e^{-\lambda}+\lambda^1\int_0^1 e^{-\lambda t}C_1(t)dt$$

$$= A_1+B_1+\lambda^n \int_0^1 e^{-\lambda t}C_n(t)dt$$

where

$$\varkappa(C_n) = \varkappa(C_1)F^{(n-1)*}+B_1F^{n*}*(e^{-1}\delta_1-\delta_o)$$

We must try to find a C_2 so that

$$g(\lambda) = A_1+B_1+\lambda^2 \int_o^1 e^{-\lambda t}C_2(t)dt$$

i e C_2 must satisfy:

$$\varkappa(C_2) = \varkappa(C_1)*F +B_1F^{2*}*(e^{-1}\delta_1-\delta_o)$$

It suffices therefor to show that

$\varkappa(C_1)*F+B_1F^{2*}*(e^{-1}\delta_1-\delta_o)$ is concentrated on $[o,1]$. The density on $[1,\infty[$ may be written:

$$x \curvearrowright \int_o^1 e^{-(x-s)}e^{-s}C_1(s)ds+e^{-1}B_1(x-1)e^{-(x-1)}-B_1xe^{-x}$$

It follows that we will be through if we could show that:

$\int_o^1 C_1(s)ds +B_1(x-1)-B_1x \equiv o$ i e that:

$$\int_o^1 C_1(s)ds = B_1$$

By proposition E.8:

$\varkappa(C_1)*F^{(n-1)*}+B_1F^{n*}*[e^{-1}\delta_1-\delta_o]$ is concentrated on $[o,1]$, i e

$$\int_o^1 \frac{(x-s)^{n-2}}{\Gamma(n-1)} e^{-(x-s)}e^{-s}C_1(s)ds+B_1e^{-1} \frac{(x-1)^{n-1}}{\Gamma(n)} e^{-(x-1)}-B_1 \frac{x^{n-1}}{\Gamma(n)}e^{-x} \underset{x}{\equiv} o$$

or equivalently that:

$$\int_o^1 \frac{(x-s)^{n-2}}{\Gamma(n-1)} C_1(s)ds+B_1 \frac{(x-1)^{n-1}}{\Gamma(n)} -B_1 \frac{x^{n-1}}{\Gamma(n)} \underset{x}{\equiv} o$$

The coefficient of x^{n-2} on the left hand side is:

$\int_o^1 \frac{1}{\Gamma(n-1)} C_1(s)ds-B_1 \frac{(n-1)}{\Gamma(n)}$. It follows that $\int_o^1 C_1(s)ds = B_1$. Suppose finally that g has an UMVU estimator based on 1 observation, i e that $\int_o^1 C_1(s)^2ds < \infty$. The density of $\varkappa(C_2) = \varkappa(C_1)*F+B_1F^{2*}*[e^{-1}\delta_1-\delta_o]$ on $[o,1]$ is (almost everywhere)

$$e^{-x}C_2(x) = \int_o^x e^{-(x-s)}e^{-s}C_1(s)ds$$
$$+ e^{-1}B_1(x-1)e^{-(x-1)}-B_1xe^{-x}.$$

We may therefore as well assume that:

$$C_2(x) = \int_0^x C_1(s)ds - B_1 = \int_x^1 C_1(s)ds = -\int_0^x C_1(s)ds$$

Hence $\displaystyle\int_0^1 \frac{C_2(x)^2}{x}dx = \int_0^1 \frac{[\int_0^x C_1(s)ds]^2}{x}dx$

$$= \int_0^1 \frac{x^2[\int_0^x C_1(s)\frac{ds}{x}]^2}{x}dx \leqq \int_0^1 \frac{x^2 \int_0^x C_1(s)^2 \frac{ds}{x}}{x}dx$$

$$= \int_0^1 [\int_0^x C_1(s)^2 ds]dx = \int_0^1 C_1(s)^2(1-s)ds < \infty . \qquad \square .$$

<u>Proposition E.9</u>

Consider an estimand g of the form:
$g(\lambda) = A_m + B_m e^{-m\lambda} + \lambda^m \int_0^1 e^{-\lambda t} C_m(s)ds$; $\lambda > 0$ where $B_m = 0$ when $m > 1$.

Then g has a $\mathcal{S}_n$ measurable unbiased estimator if and only if

$$\int_0^1 (C_m(s) - B_m)s^i \, ds = 0 \; ; \; i = 0,1,\ldots n-m-1$$

It follows that the set of integers n such that a given non constant g has a $\mathcal{S}_n$ measurable unbiased estimator is a finite interval. In particular the set of integers n so that a given non constant g has an UMVU estimator based on n observations is a finite interval.

<u>Proof:</u>

Suppose $m > 1$. By proposition E.8 g has a $\mathcal{S}_n$ measurable unbiased estimator if and only if $\kappa(C_m) * F^{(n-m)*}$ is consentrated on $[0,1]$ i.e. if and only if:

$$\int_0^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} e^{-(x-s)} e^{-s} C_m(s)ds \underset{x}{\equiv} 0$$

or equivalently that:

$$\int_0^1 s^i C_m(s)ds = 0 \; ; \; i = 0,1,\ldots n-m-1 .$$

If this holds for arbitrarily large n then $C_m = o$ a.e. i.e. $g = A$. This proves the last two statements of the proposition.

It remains to consider the oase $m = 1$.

Then $g(\lambda) = A_1 + B_1 e^{-\lambda} + \lambda^1 \int_o^1 e^{-\lambda t} C_1(s)ds$

By proposition E.8 g has an unbiased estimator based on n observations if and only if

$$\varkappa(C_1) * F^{(n-1)*} + B_1 F^{n*} * (e^{-1}\delta_1 - \delta_0)$$

is concentrated on $[o,1]$ i.e. if and only if:

$$\int_o^1 \frac{(x-s)^{n-2}}{\Gamma(n-1)} C_1(s)ds + B_1 \frac{(x-1)^{n-1}}{\Gamma(n)} - B_1 \frac{x^{n-1}}{\Gamma(n)} \underset{x}{\equiv} o$$

i.e. if and only if

$$\int_o^1 (C_1(s) - B_1)s^k ds = o \; ; \; k = o, 1 \ldots, n-2$$

∎