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TESTS OF SIGNIFICANCE IN PERIODOGRAM ANALYSIS

by

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SUMMARY

The paper treats the problem of detection of hidden periodicities in periodogram analysis. The problem is stated as a problem of selection. A competitor to Fisher's classical test is proposed and analyzed. The distribution of the test criterion is derived. It is established that in most alternatives the new method is more powerful than Fisher's.

CONTENTS

1. INTRODUCTION

Let $X_1,\dots,X_T$ be an observed time series described by the model

$$X_t = \xi_t + U_t \quad , \quad t=1,\dots,T \tag{1.1}$$

where $U_1,\dots,U_T$ are pure white noise, i.e. independent and identically distributed random variables. We shall assume throughout that their common distribution is normal with unknown variance σ^2. From the observations of $X_1,\dots,X_T$ we want to identify periodical components of the trend ξ_t. Although the periods are unknown and may be any among a large number, it is known that only a few of them are present. In mathematical terms the problem may be stated as follows. Assume that ξ_t can be written

$$\xi_t = \alpha_0 + \sum_{\nu=1}^{p} A_\nu \cos(2\pi\lambda_\nu t - \varphi_\nu) \tag{1.2}$$

where $A_1,\dots,A_p > 0$. All the parameters in (1.2) (including p) are unknown. It is assumed that p is small compared to m. We are to find out if $p = 0$ or > 0, and in the latter case we want to determine $\lambda_1,\dots,\lambda_p$. The decision problem will be restated in chapter 2 in a more convenient form.

The statistical problem described above is a classical one. (For early references see [10] and [13]). The standard method found in textbooks is due to Fisher [5]. It is described and studied in chapter 3. The main object of the paper is to propose and analyze a new procedure which is introduced in chapter 4. Fisher's method is seen to be included as a special case. The distribution of the test criterion is derived exactly and approximately. Critical values are computed and tabulated. It is established that the new method represents a considerable improvement over Fisher's.

Various other approaches to the problem, some considered in the past and some new ones are briefly discussed in chapter 5.

The paper is based on a part of the author's Cand.Real. thesis at the University of Oslo. The presentation has on several points been extended. Theorem 4.1 is stated in a new form with a new proof.

2. FORMULATION OF THE DECISION PROBLEM

$\xi_1,\ldots,\xi_T$ can be expanded uniquely in an ortogonal trigonometric series, i.e. for t=1,...,T

$$(2.1) \qquad \xi_t = \alpha_0 + \sum_{j=1}^{[\frac{T-1}{2}]} \alpha_j \cos(2\pi \tfrac{j}{T} t) + \sum_{j=1}^{[\frac{T-1}{2}]} \beta_j \sin(2\pi \tfrac{j}{T} t) + \alpha_{T/2}(-1)^t \; (*)$$

the last term being excluded when T is odd. (2.1) is equivalent to

$$(2.2) \qquad \xi_t = \alpha_0 + \sum_{j=1}^{[\frac{T-1}{2}]} \rho_j \cos(2\pi \tfrac{j}{T} t - \theta_j) + \alpha_{T/2}(-1)^t$$

where

$$\rho_j = \sqrt{\alpha_j^2 + \beta_j^2}$$
$$\theta_j = \arctan\left(\frac{\beta_j}{\alpha_j}\right) .$$

The last term of (2.2) represents an inessential, but unwanted complication, and we shall throughout assume that $T = 2m+1$ in which case (2.2) can be rewritten

(*) For a real number y , the symbol $[y]$ here and later denotes the largest integer less than or equal to y .

(2.3) $$\xi_t = \alpha_0 + \sum_{j=1}^{m} \rho_j \cos(2\pi \tfrac{j}{T} t - \theta_j) , \quad t=1,\ldots,T .$$

We shall refer to $\rho_1,\ldots,\rho_m$ as the amplitudes of (2.3).

Comparing with (1.2), it is clear that $\rho_1,\ldots,\rho_m$ are functions of $\lambda_1,\ldots,\lambda_p$. Suppose

(2.4) $$\lambda_\nu = \frac{j_\nu}{T} , \quad \nu=1,\ldots,p$$

where $j_1,\ldots,j_\nu$ are integers less than $\frac{T}{2}$. Then $\rho_{j_\nu} = A_\nu$, $\nu=1,\ldots,p$, and $\rho_j = 0$ otherwise. (2.4) means that the periods are integral divisors of the series length. This is sometimes a reasonable a priori assumption, for instance in connection with monthly, seasonly or annual data. But even if there is no such knowledge, as is usually the case, there still are among the quantities $\rho_1,\ldots,\rho_m$ a few dominating ones corresponding to values of $\frac{j}{T}$ close to one of the periods $\lambda_1,\ldots,\lambda_p$. As an example, suppose that $p=1$ in (1.2). Then

(2.5) $$\xi_t = \alpha_0 + \rho_1 \cos(2\pi\lambda_1 t - \varphi_1) .$$

In [1] it is proved that if $\lambda_1 \in (\frac{k}{T}, \frac{k+1}{T})$, then ρ_k and ρ_{k+1} will be the two largest of $\rho_1,\ldots,\rho_m$, and the two of them will account for at least 81% of the sum $\sum_{j=1}^{m} \rho_j^2$, the minimum value being attained when $\lambda = \frac{k+\frac{1}{2}}{T}$. Thus, under (2.5) most of the quantities $\rho_1,\ldots,\rho_m$ are small with one or two dominating the others, the indices approximately determining λ_1. The situation is analogous when $p > 1$.

It is clear from the reasoning above that the problem of identifying the periods $\lambda_1,\ldots,\lambda_p$ may (approximately) be regarded as a problem of selection of the large ones among $\rho_1,\dots,\rho_m$.

Estimates of $\rho_1,\dots,\rho_m$ can be constructed from the least-squares estimates of $\alpha_0,\alpha_1,\dots,\alpha_m$ and $\beta_1,\dots,\beta_m$, i.e.

$$\hat{\alpha}_0 = \bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$$

$$\hat{\alpha}_j = \frac{2}{T}\sum_{t=1}^{T} X_t \cos(2\pi \tfrac{j}{T}t)$$

$$\hat{\beta}_j = \frac{2}{T}\sum_{t=1}^{T} X_t \sin(2\pi \tfrac{j}{T}t)$$

$$\hat{\rho}_j^2 = \hat{\alpha}_j^2 + \hat{\beta}_j^2 .$$

It is easily seen that $\hat{\rho}_1^2,\dots,\hat{\rho}_m^2$ are independent, the distribution of $\frac{\hat{\rho}_j^2}{\sigma^2}$ being non-central χ_2^2 with eccentricity $\frac{\rho_j^2}{\sigma^2}$. The selection rule will be based on $\hat{\rho}_1^2,\dots,\hat{\rho}_m^2$. This is hardly any restriction as it can easily be proved that $\hat{\rho}_1^2,\dots,\hat{\rho}_m^2$ together with $\hat{\alpha}_0$ is a sufficient set of statistics.

The decision problem will be interpreted as a choice, for each $j=1,\dots,m$, between the statements "state $\rho_j > 0$" and "state nothing". A decision rule will be represented by a vector-valued function $\psi = (\psi_1,\dots,\psi_m)$ where $\psi_j = 1$ and 0 respectively for the two statements above. We shall require the methods to have level ϵ , i.e. that the probability of stating that ξ_t has periodical components when indeed there are none $(\rho_1=\dots\rho_m=0)$ is at most ϵ . For the decision rules of chapter 3 and 4 this will also control the probability of making false statements for arbitrary values of $\rho_1,\dots,\rho_m$.

3. FISHER's TEST

The standard method for the problem formulated in chapter 2 was proposed by R.A. Fisher [5] :

$$(3.1) \qquad \psi_k^{(o)} = \begin{cases} 1 & , \text{ if } \hat{\rho}_k^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c \\ 0 & , \text{ otherwise .} \end{cases}$$

The constant c is to be determined from

$$(3.2) \qquad \Pr[\max_k \hat{\rho}_k^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c \,|\, \rho_1 = \ldots = \rho_m = 0] = \varepsilon \;.$$

To derive an expression for the left hand side, note that

$$\Pr[\max_k \hat{\rho}_k^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c] = \sum_{k=1}^{m} (-1)^{k-1} \sum_{p_1 < \ldots < p_k} \Pr[\min_{1 \leq j \leq k} \hat{\rho}_{p_j}^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c] \;.$$

When $\rho_1 = \ldots = \rho_m = 0$, this yields by symmetry

$$(3.3) \qquad \Pr[\max_k \hat{\rho}_k^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c] = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \Pr[\min_{1 \leq j \leq k} \hat{\rho}_j^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c].$$

It is well known that the two statistics, $k \circ \min_{1 \leq j \leq k} \hat{\rho}_j^2$ and $\sum_{j=1}^{k} \hat{\rho}_j^2 - k \min_{1 \leq j \leq k} \hat{\rho}_j^2$ are independently distributed as χ_2^2 and $\chi_{2(k-1)}^2$ respectively.
It easily follows that

$$\Pr[\min_{1 \leq j \leq k} \hat{\rho}_j^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c] = \{\max(1-kc, 0)\}^{m-1}.$$

Combining this with (3.3), we obtain

$$(3.4) \qquad \Pr[\max_k \hat{\rho}_k^2 / \sum_{j=1}^{m} \hat{\rho}_j^2 > c \,|\, \rho_1 = \dots = \rho_m = 0] = \sum_{k=1}^{[c^{-1}]} (-1)^{k-1} \binom{m}{k} (1-kc)^{m-1} .$$

Other proofs of (3.4) are given in [1], [4] and [14]. The derivation above, which is new, at least to the author, is of a more elementary character and also simpler than the earlier proofs.

It is easily shown that the probability of stating falsely that $\rho_k > 0$ for some k for which $\rho_k = 0$, attains its maximum when $\rho_1 = \dots = \rho_m = 0$.

Let N_ρ be the number of non-zero amplitudes, i.e. $N_\rho = \#\{j \,|\, \rho_j > 0\}$. It can be proved that the test has optimum properties when $N_\rho = 1$. To study the power in more general alternatives, note that the test statistic in (3.1) is an increasing function of $\hat{\rho}_k^2 / \sum_{j \neq k} \hat{\rho}_j^2$. In this expression the numerator and the denominator may, in the context, be interpreted as estimates of ρ_k^2 and σ^2 respectively. This means that when there are non-zero amplitudes other than ρ_k^2, σ^2 will be over-estimated, and the test may become unsensitive. In the extreme case when $\rho_l = \infty$ for an $l \neq k$, the power is zero.

We now proceed to study this effect quantitatively. Denote the power function by

$$\beta_k^{(c)} = E(\psi_k^{(c)} \,|\, \rho_1, \dots, \rho_m, \sigma) .$$

In general the distribution of the test statistic (3.1) will be complicated as the ratio of two non-central χ^2-distributed variables. For our purpose it will be sufficiently accurate to apply a result due to Patnaik (see Appendix) which approximates a non-central χ^2-distribution with a central one. Accordingly, replace on the right of (3.1) $\hat{\rho}_k^2$ and $\sum_{j \neq k} \rho_j^2$ with $s_k Y_{\nu_k}$ and $r_k W_{\gamma_k}$ respectively, where Y_{ν_k} and W_{γ_k} are independent, $Y_{\nu_k} \sim \chi^2_{\nu_k}$, $W_{\gamma_k} \sim \chi^2_{\gamma_k}$

and

$$s_k = 1 + \frac{\rho_k^2}{2\sigma^2+\rho_k^2}$$

$$\nu_k = 2 + \frac{\rho_k^4}{2\sigma^4+2\sigma^2\rho_k^2}$$

$$r_k = 1 + \frac{\sum_{j\neq k}\rho_j^2}{2(m-1)\sigma^2+\sum_{j\neq k}\rho_j^2}$$

$$\gamma_k = 2(m-1) + \frac{(\sum_{j\neq k}\rho_j^2)^2}{2(m-1)\sigma^4+2\sigma^2\sum_{j\neq k}\rho_j^2} \quad .$$

We hereby easily obtain

$$\beta_k^{(o)} \approx 1 - B\left[\frac{r_k c}{r_k \cdot c+s_k(1-c)} \; ; \frac{\nu_k}{2} \, , \; \frac{\gamma_k}{2}\right] \tag{3.5}$$

where $B(\circ \; ; \frac{\nu_k}{2} , \frac{\gamma_k}{2})$ is the central Beta-distribution with parameters $\frac{\nu_k}{2}$ and $\frac{\gamma_k}{2}$. Nummerical results are shown in Figure 3.1 and 3.2.

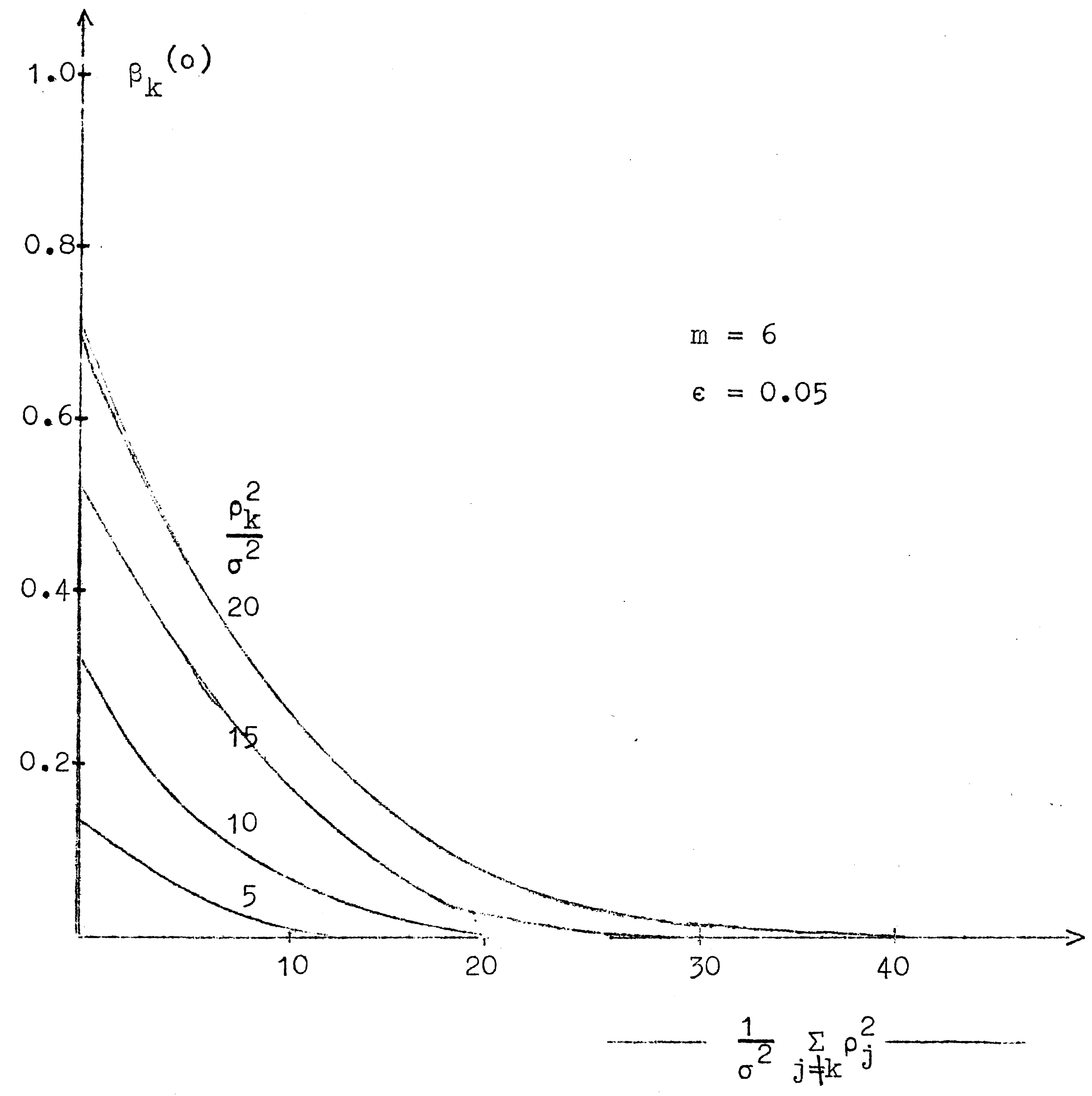

Figure 3.1. The power $\beta_k^{(\cdot)}$ of Fisher's method as a function of $\frac{1}{\sigma^2} \sum_{j \neq k} \rho_j^2$ for several values of $\frac{\rho_k^2}{\sigma^2}$. m = 6 , ε = 0.05 .

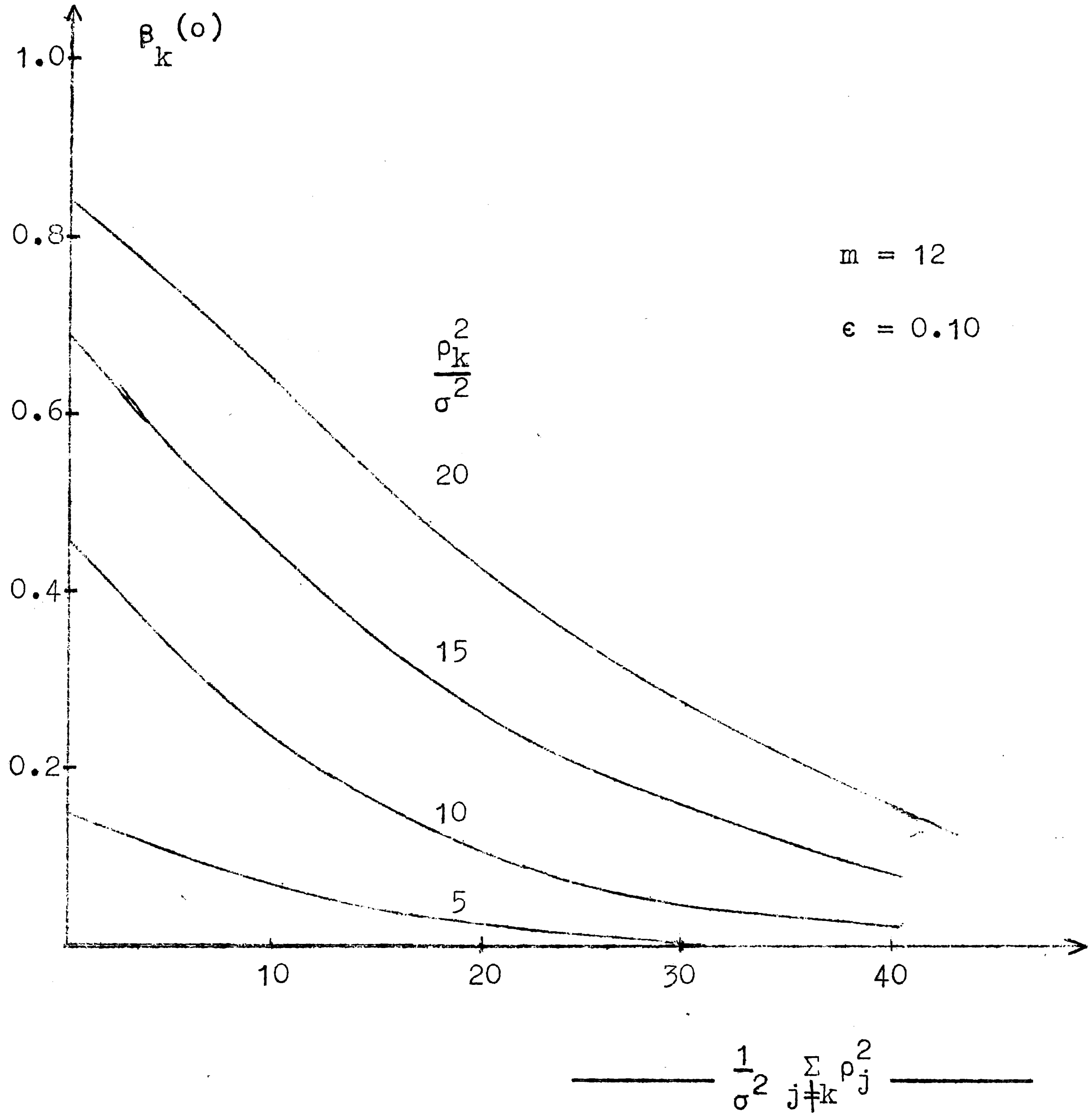

Figure 3.2. The power $\beta_k^{(o)}$ of Fisher's method as a function of $\frac{1}{\sigma^2} \sum_{j \neq k} \rho_j^2$ for several values of $\frac{\rho_k^2}{\sigma^2}$. $m = 12$, $\epsilon = 0.10$.

Summing up, the optimum properties of the test for $N_\rho = 1$, are in my opinion of little importance in view of the fact that the power rather drastically decreases when $N_\rho > 1$. As an example let $m = 12$, $\epsilon = 0.10$. The power in alternatives where $N_\rho = 2$, $\rho_k^2 = \rho_j^2 = \Delta > 0$, is roughly 50% of the value attained when

$N_\rho = 1$, $\rho_k = \Delta$. For the case $m = 6$, $\epsilon = 0.05$ the corresponding number is about 10-20% .

As another illustration assume that the model is described by (2.5). Let ρ_k^2 be the largest of $\rho_1^2,\dots,\rho_m^2$. Then the value of $\sum_{j \neq k} \rho_j^2$ may be as much as 50% larger than ρ_k^2 . Thus the sensivity of the test may clearly be poor. As an example let $m = 12$, $\epsilon = 0.10$. The probability of stating $\rho_k > 0$ which is 0.85 when $\rho_k^2 = 20\,\sigma^2$ and $\sum_{j \neq k} \rho_j^2 = 0$, decreases to 0.28 when the value of $\sum_{j \neq k} \rho_j^2$ increases to $30\,\sigma^2$.

4. A NEW METHOD

The reasoning of the preceding chapter indicates that Fisher's test can be improved by replacing the denominator in (3.1) by a "robust" estimate of σ^2 , "robust" in the sense that it is reasonably unsensitive to the number of large amplitudes. In theory there are many possibilities. This author has chosen a trimming procedure. For a specified integer $a \geqq 0$ leave out the a largest of $\hat{\rho}_1^2,\dots,\hat{\rho}_m^2$ and use an estimate based on the average of the remaining. Introducing $\hat{\rho}_{(1)}^2 \leqq \hat{\rho}_{(2)}^2 \leqq \dots \leqq \hat{\rho}_{(m)}^2$ as the order statistics of $\hat{\rho}_1^2,\dots,\hat{\rho}_m^2$ we are lead to the test $\psi^{(a)}$ defined by

$$\psi_k^{(a)} = \begin{cases} 1 , & \text{if } \hat{\rho}_k^2 / \sum_{j=1}^{m-a} \hat{\rho}_{(j)}^2 > c \\ 0 , & \text{otherwise.} \end{cases} \tag{4.1}$$

Obviously (4.1) reduces to (3.1) when $a = 0$.

It is easy to see that the probability of stating falsely that $\rho_k > 0$ for some $\rho_k = 0$ attains its maximum when $\rho_1 = \dots = \rho_m = 0$.

Thus to control the error, we have to calculate the quantity $\Pr[\hat\rho^2_{(m)}/\sum_{j=1}^{m-a}\hat\rho^2_{(j)} > c]$ under this condition. More generally, as it is only a trifle more difficult, we shall, for an arbitrary integer $r \geqq 0$, derive the distribution of

$$F_r^{(a)} = \hat\rho^2_{(m-r)}/\sum_{j=1}^{m-a}\hat\rho^2_{(j)} \tag{4.2}$$

i.e. we shall find an expression for

$$H_r^{(a)}(c) = \Pr(F_r^{(a)} > c \mid \rho_1 = \ldots = \rho_m = 0). \tag{4.3}$$

The derivation is an extension of the method used in establishing (3.4). As a first step we shall in the next section prove a result which may be taken as a generalisation of equation (3.3).

4.1. A lemma.

For variables $Y_1,\ldots,Y_m$ and integers q and p, $q < p$, let $Y_{(j;q,p)}$, $j=1,\ldots,p-q+1$ denote the order statistics of $Y_q,\ldots,Y_p$. Introduce the events

$$A_k = \begin{cases} \{Y_{(1;1,k)}/\sum_{j=1}^{m-a} Y_{(j;k+1,m)} > c\} & , \quad k \leqq a-1 \\ \{Y_{(1;1,k)}/\left[\sum_{j=a+1}^{k} Y_{(j;1,k)} + \sum_{j=k+1}^{m} Y_j\right] > c\} & , \quad k \geqq a \end{cases} \tag{4.4}$$

and define

$$T_k = \binom{m}{k}\binom{k-1}{k-r-1} \Pr(A_k) \quad , \quad k=r+1,\ldots,m\ . \tag{4.5}$$

Then the following lemma holds.

Lemma 4.1.

Let the distribution of $Y_1,\dots,Y_m$ be invariant with respect to permutations (*). Then, for non-negative integers r and a

$$(4.6)\qquad \Pr[Y_{(m-r)}/\sum_{j=1}^{m-a} Y_{(j)} > c] = \sum_{k=r+1}^{m} (-1)^{k-r-1} T_k \,.$$

Furthermore, for an integer $p > 0$,

$$(4.7)\qquad \sum_{k=r+1}^{r+2p} (-1)^{k-r-1} T_k \leqq \Pr[Y_{(m-r)}/\sum_{j=1}^{m-a} Y_{(j)} > c] \leqq \sum_{k=r+1}^{r+2p-1} (-1)^{k-r-1} T_k \,.$$

Proof. For arbitrary $k \geqq r+1$, let

$$(4.8)\qquad N_k = \binom{m}{r+1} \prod_{j=r+2}^{k} (m-j+1)\,.$$

$$(4.9)\qquad e_k = N_k \Pr[A_k \cap \{Y_{(1;1,r+1)} < Y_{r+2} < Y_{r+3} < \dots < Y_k < Y_{(m-k;k+1,m)}\}].$$

We shall prove that for arbitrary $p > 0$

$$(4.10)\qquad \Pr[Y_{(m-r)}/\sum_{j=1}^{m-a} Y_{(j)} > c] = \sum_{k=r+1}^{r+p} (-1)^{k-r-1} T_k + (-1)^p e_{r+p}\,.$$

The lemma is an immediate consequence of this result.

By symmetry

$$e_k = N_k(m-k) \Pr[A_k \cap \{Y_{(1;1,r+1)} < Y_{r+2} < Y_{r+3} < \dots < Y_{k+1},$$
$$Y_{k+1} > Y_{(m-k-1;k+2,m)}\}].$$

Recalling the definition of A_k , it follows that

(*) Without this assumption a more complicated version of the lemma still holds. The method of proof is essentially the same as the one employed here.

$$e_k = N_{k+1} \Pr[A_{k+1} \cap \{Y_{(1;1,r+1)} < Y_{r+2} < Y_{r+3} < \ldots < Y_{k+1},$$
$$Y_{k+1} > Y_{(m-k-1;k+2,m)}\}].$$

From the elementary formula $\Pr(A \cap B) = \Pr(A) - \Pr(A \cap B^c)$ we easily deduce

$$(4.11) \qquad e_k = N_{k+1} \Pr[A_{k+1} \cap \{Y_{(1:1,r+1)} < Y_{r+2} < Y_{r+3} < \ldots < Y_{k+1}\}] - e_{k+1} .$$

By symmetry it easily follows that

$$\Pr[A_{k+1} \cap \{Y_{(1;1,r+1)} < Y_{r+2} < Y_{r+3} < \ldots < Y_{k+1}\}]$$
$$= (r+1) \Pr[A_{k+1} \cap \{Y_1 < Y_{(1;2,r+1)},\ Y_1 < Y_{r+2} < Y_{r+3} < \ldots < Y_{k+1}\}]$$
$$= (r+1) \left[\prod_{j=k-r+1}^{k} j\right] \Pr[A_{k+1} \cap \{Y_1 < Y_2 < Y_3 < \ldots < Y_{k+1}\}]$$
$$= (r+1)\left[\prod_{j=k-r+1}^{k} j\right] \frac{1}{(k+1)!} \Pr[A_{k+1}] = \frac{r+1}{(k+1)(k-r)!} \Pr[A_{k+1}] .$$

Combining with (4.11) and (4.8) this yields

$$e_k = \binom{m}{k+1}\binom{k}{k-r} \Pr(A_{k+1}) - e_{k+1} \qquad \text{or}$$

$$(4.12) \qquad e_k = T_{k+1} - e_{k+1} .$$

Now, from an easy symmetry argument, recalling the definitions of A_r and A_{r+1}

$$\Pr[Y_{(m-r)} / \sum_{j=1}^{m-a} Y_{(j)} > c] = \binom{m}{r+1} \Pr[A_r \cap \{Y_{(1;1,r+1)} < Y_{(m-r-1;r+2,m)}\}]$$
$$= \binom{m}{r+1} \Pr(A_{r+1}) - e_{r+1} = T_{r+1} - e_{r+1} .$$

This is (4.10) with $p = 1$. For arbitrary p, (4.10) follows from (4.12) by induction.

Note that when $r = a = 0$, (4.6) with $Y_j = Z_j$, $j=1,\dots,m$, reduces to (3.3). Furthermore, inequality (4.7) is then an easy application of the Bonferoni inequality. In this case (4.6) is an immediate consequence of the well known expansion $\Pr(\bigcup_i B_i) = \sum_i \Pr(B_i) - \sum_{i<j} \Pr(B_i \cap B_j) + \dots$. In general, however, (4.6) can not be established in this manner.

It will be shown in the next section, that when $Y_1,\dots,Y_m$ are independently χ_2^2-distributed, it is possible to derive simple analytical expressions for T_k . Although this may not be possible in other cases, it is believed that the lemma is of interest beyond the problem considered here. In general the first terms on the right of (4.6) are dominating. Indeed, $\Pr(A_k) = 0$ if $k > [\frac{1}{c}] + a$. Typically the very first term provides a reasonable approximation. We shall return to this subject in section 4.3.

4.2. The exact distribution of the test criterion when $\rho_1=\dots=\rho_m = 0$.

Introduce $Z_j = \hat{\rho}_j^2 / \sigma^2$, $j=1,\dots,m$. Assuming $\rho_1=\dots=\rho_m = 0$, $Z_1,\dots,Z_m$ are independently χ_2^2-distributed. We are to calculate the distribution of $F_r^{(a)} = Z_{(m-r)} / \sum_{j=1}^{m-a} Z_{(j)}$. Applying the results of the preceding section, let from now on in (4.4) and (4.6) $Y_j = Z_j$, $j=1,\dots,m$. An expression for $\Pr(A_k)$ and hence for T_k will be derived from three lemmas. The first one enables us to write the statistic defining A_k as the ratio of a χ_2^2-distributed variable and a linear sum of independent χ_2^2-variables.

Lemma 4.2.

Let $Z_{(1)}<\dots< Z_{(m)}$ be an ordered sample from the χ_2^2 - distribution. Put $U_1 = mZ_{(1)}$, $U_j = (m-j+1)\ (Z_{(j)}-Z_{(j-1)})$, $j=2,\dots,m$. Then $U_1,\dots,U_m$ are independent and χ_2^2-distributed.

This is a well known result. The proof is elementary and is omitted.

The next two lemmas are believed to be new.

Lemma 4.3

Let Z and Y be independent. Suppose that $Z \sim \chi_2^2$ and that $\Pr(Y\geq 0) = 1$. Put $V = Z - Y$. Then the conditional distribution of V given $V > 0$ is χ_2^2 .

Remark: Consider two independent streams of events A and B. Suppose A are Poisson events with intensity $\frac{1}{2}$. Interprete Z and Y as the time of first occurrence of A and B respectively. Then the lemma states the well known fact that the Poisson process has "no memory".

Proof: Assume for simplicity that Y has a density $g(y)$. It is easy to see that the density of V can be written

$$q(v) = \tfrac{1}{2}\, e^{-\frac{1}{2}v} \int_{\max(o,-v)}^{\infty} g(y) e^{-\frac{1}{2}y} dy$$

from which the lemma is an immediate consequence.

Lemma 4.4

Let $U_0,U_1,\dots,U_n$ be an independent sample from the χ_2^2- distribution, and let $l_1,\dots,l_n$ be non-negative numbers. Then

$$(4.13) \qquad \Pr[U_0/\sum_{j=1}^{n} l_j U_j > c] = \prod_{j=1}^{n} (1+l_j c)^{-1} .$$

<u>Proof:</u> For $k=1,2,\ldots,n$, let

$$B_k = \{U_o / \sum_{j=1}^{k} l_j U_j > c\} .$$

Since l_k is non-negative, $B_k \subset B_{k-1}$ and hence

$$\Pr(B_k) = \Pr(B_k | B_{k-1}) \Pr(B_{k-1}) .$$

Introducing $W = U_o - c \sum_{j=1}^{k-1} l_j U_j$, we may write

$$\Pr(B_k | B_{k-1}) = \Pr(W/U_k > l_k c | W > 0) .$$

As U_k and W are independent, it follows from Lemma 4.3 that conditioned on $W > 0$ the ratio W/U_k is F-distributed with 2 degrees of freedom in numerator and denominator. This yields

$$\Pr(B_k | B_{k-1}) = (1+l_k c)^{-1} .$$

Thus

$$\Pr(B_k) = (1+l_k c)^{-1} \Pr(B_{k-1})$$

implying

$$\Pr(B_k) = \sum_{j=1}^{k} (1+l_j c)^{-1}$$

which was to be proved.

We now turn to the calculation of $\Pr(A_k)$. It is necessary to distinguish between two cases.

<u>Case 1. $k \leqq a-1$.</u> Let

$$U_j = \begin{cases} k\, Z_{(1;1,k)} & , j=0 \\ (m-k) Z_{(1;k+1,m)} & , j=1 \\ (m-k-j+1)(Z_{(j;k+1,m)} - Z_{(j-1;k+1,m)}) & , j=2,\ldots,m-k \end{cases}$$

($Z_{(j,1,k)}$, $j=1,\ldots,k$ and $Z_{(j;k+1,\ldots,m)}$, $j=1,\ldots,m-k$ are the order statistics of $Z_1,\ldots,Z_k$ and $Z_{k+1},\ldots,Z_m$ respectively.) From Lemma 4.2 it is easily deduced that $U_o,U_1,\ldots,U_{m-k}$ are independent and χ_2^2-distributed. In terms of these variables A_k may be written

$$A_k = \{U_o / \sum_{j=1}^{m-a} \frac{j}{a-k+j} U_{m-a-j+1} > k\ c\} .$$

Applying Lemma 4.4, it follows that

$$\text{(4.14)} \qquad \Pr(A_k) = \prod_{j=1}^{m-a} (1 + \frac{jk}{a-k+j} c)^{-1} .$$

<u>Case 2. $k \geqq a$.</u> In this case we use the transformation

$$U_j = \begin{cases} k\ Z_{(1;1,k)} & , \quad j=1 \\ (k-j+1)\ (Z_{(j;1,k)} - Z_{(j-1;1,k)}) & , \quad j=2,\ldots,k \\ Z_j & , \quad j=k+1,\ldots,m \end{cases}$$

As in case 1, $U_1,\ldots,U_m$ are independently χ_2^2-distributed. It is easily seen that

$$A_k = \left\{ \frac{U_1}{\sum_{j=2}^{a} \frac{k-a}{k-j+1} U_j + \sum_{j=a+1}^{m} U_j} > \frac{k\ c}{1-(k-a)c} \right\} , \text{ if } \quad (k-a)c < 1$$

(If $(k-a)c \geqq 1$, then A_k is empty.) Another application of Lemma 4.4 yields

$$\text{(4.15)} \qquad \Pr(A_k) = \frac{[\max(1-(k-a)c,0)]^{m-1}}{(1+ac)^{m-a} \prod_{j=1}^{a-1} (1+c \frac{(k-a)j}{k-j})} .$$

Finally by inserting (4.14) and (4.15) into (4.5) and (4.6), the distribution of $F_r^{(a)}$ can be written down. We state the result as a theorem.

Theorem 4.1.

Let $Z_1,\dots,Z_m$ be an independent sample from the χ_2^2-distribution. Put $F_r^{(a)} = Z_{(m-r)} / \sum_{j=1}^{m-a} Z_{(j)}$. Then

$$(4.16)\qquad H_r^{(a)}(c) = \Pr(F_r^{(a)} > c) = \sum_{k=r+1}^{[c^{-1}]+a} (-1)^{k-r-1} \binom{m}{k} \binom{k-1}{k-r-1} Q_k(c)$$

where

$$(4.17)\qquad Q_k(c) = \begin{cases} \prod_{j=1}^{m-a} \left(1+ \frac{j\,k}{a-k+j}\, c\right)^{-1} & , \quad k \leqq a-1 \\ \dfrac{(1-(k-a)c)^{m-1}}{(1+ac)^{m-a} \prod_{j=1}^{a-1} \left(1+ \frac{(k-a)j}{k-j}\, c\right)} & , \quad a \leqq k \leqq [c^{-1}]+a \end{cases}$$

Furthermore, for arbitrary integer $p > 0$,

$$(4.18)\qquad \sum_{k=r+1}^{r+2p} (-1)^{k-r-1} \binom{m}{k} \binom{k-1}{k-r-1} Q_k(c) \leqq H_r^{(a)}(c) \leqq \sum_{k=r+1}^{r+2p-1} (-1)^{k-r-1} \binom{m}{k} \binom{k-1}{k-r-1} Q_k(c).$$

If $a = 0$, (4.16) reduces to

$$(4.19)\qquad H_r^{(o)}(c) = \Pr(F_r^{(o)} > c) = \sum_{k=r+1}^{[c^{-1}]} (-1)^{k-r-1} \binom{m}{k} \binom{k-1}{k-r-1} (1-kc)^{m-1}$$

which was proved in [1], [6] and [11] by other, more complicated methods. Note that when $r = 0$, (4.19) is the same expression as (3.4)

4.3. Approximations.

We only consider the most important case $r = 0$. As indicated in section 4.1, the sum of the lowest terms on the right of (4.16) provide good approximations to $H^{(a)}(c) = H_0^{(a)}(c)$. Using p terms, the approximation can be written

$$(4.20) \qquad \tilde{H}_p^{(a)}(c) = \sum_{k=1}^{p} (-1)^{k-1} \binom{m}{k} Q_k(c).$$

In particular

$$(4.21) \qquad \tilde{H}_1^{(a)}(c) = m \prod_{j=1}^{m-a} \left(1 + \frac{j}{j+a-1}\, c\right)^{-1} .$$

It is clear that $\tilde{H}_p^{(a)}(c)$ is larger or smaller than $H^{(a)}(c)$ according to p being even or odd. An indication of the accuracy is given in Table 4.1, where the difference $| \tilde{H}_p^{(a)}(c) - H^{(a)}(c) |$ is recorded for several values of a, m, c and p .

		a=1	a=2	a=3	a=4
$H^{(a)}(c)$=0.05	m=12	0.0000 0.0000	0.0021 0.0000	0.0036 0.0002	0.0052 0.0005
	m=20	0.0001 0.0000	0.0016 0.0000	0.0023 0.0000	0.0029 0.0001
$H^{(a)}(c)$=0.10	m=12	0.0001 0.0000	0.0064 0.0000	0.0104 0.0006	0.0146 0.0017
	m=20	0.0006 0.0000	0.0053 0.0000	0.0076 0.0004	0.0094 0.0007

Table 4.1. The difference $|\tilde{H}_p^{(a)}(c) - H^{(a)}(c)|$, $p=1,2$ is recorded for values of c satisfying $H^{(a)}(c) = 0.05$, 0.10 . For each m the upper line of the table corresponds to $p = 1$, the lower line to $p = 2$.

As was to be expected the approximation is better the smaller the value of a and the larger the value of m. It is clear that $\tilde{H}_p^{(a)}(c)$ as p increases, very rapidly approaches $H^{(a)}(c)$. In applications the accuracy will almost always be better the larger the value of p. However, this is not true in general.

Applying the approximation $\tilde{H}_1^{(a)}(c)$ is for many practical purposes sufficiently accurate. Suppose c is determined from

$$\tilde{H}_1^{(a)}(c) = \epsilon \; . \tag{4.22}$$

Then $H^{(a)}(c)$ is less than ϵ, and as is clear from Table 4.1 not much less. Asymptotically, as $m \to \infty$, it is possible to derive a lower bound. From the Bonferoni inequality, it is easy to see that

$$H^{(a)}(c) \geqq m \Pr(V_1 > c) - \binom{m}{2} \Pr(V_1 > c, V_2 > c) \tag{4.23}$$

where $V_1 = Z_1/(\text{sum of the } m-a \text{ smallest of } Z_2, \dots, Z_m)$

$V_2 = Z_2/(\text{sum of the } m-a \text{ smallest of } Z_1, Z_3, \dots, Z_m)$.

Asymptotically V_1 and V_2 are independent. Hence (4.23) as $m \to \infty$ can be rewritten

$$H^{(a)}(c) \geqq m \Pr(V_1 > c) - \binom{m}{2} \Pr(V_1 > c) \Pr(V_2 > c) \; .$$

Inserting $\Pr(V_1 > c) = \frac{1}{m} \tilde{H}_1^{(a)}(c) = \frac{\epsilon}{m}$, we obtain

$$H^{(a)}(c) \geqq \epsilon - \frac{\epsilon^2}{2} \; , \quad \text{as } m \to \infty \; . \tag{4.24}$$

It is clear from Table 4.1 that this inequality does not hold in general for finite m, unless $a = 0, 1$. In the latter cases

(4.24) can indeed be proved, using essentially the same method as above (see [3]).

The approximations should be particularly useful in the following context. Let

$$F_k = \frac{\hat{\rho}_k^2}{\sum_{j=1}^{m-a} \rho_{(j)}^2} \quad , \quad k=1,\ldots,m \ .$$

Instead of determining c from $H_0^{(a)}(c) = \epsilon$, one might in practice prefer to calculate the quantities $H^{(a)}(F_k)$, $k=1,\ldots,m$, and state $\rho_k > 0$ if $H^{(a)}(F_k) \leqq \epsilon$. A quick and simple method, with level slightly less than ϵ, is to state $\rho_k > 0$ if

$$\tilde{H}_1^{(a)}(F_k) = m \prod_{j=1}^{m-a} \left(1+ \frac{1}{j+a-1} F_k\right)^{-1} \leqq \epsilon \ .$$

4.4. Performance

The object of this section is to study the performance of procedure (4.1) for different values of a and compare with Fisher's test $(a=0)$ in particular. In this respect two questions arise:

1) How much is lost by using $a > 0$ in the case of only one non-zero amplitude (when Fisher's test is optimal) ?

2) How much can be gained in other cases ?

Let the power function of the test $\psi_k^{(a)}$ be denoted

$$\beta_k^{(a)} = E(\psi_k^{(a)} | \rho_1,\ldots,\rho_m,\sigma) \quad , \quad k=1,\ldots,m$$

Since the distribution of the test statistics for general values of $\rho_1, \ldots, \rho_m$ is extremely complicated, the simplest way to go about seems to be by Monte Carlo technique. The results below are obtained from 1000 simulations on the computer of the University of Oslo. A well-tested random number generator was used. The accuracy of the results is indicated in Table 4.2. In the simplest case of $a=0$ the power, as derived in chapter 3 from an approximate analytical expression, is compared to returns from the Monte Carlo simulations. It is seen that the absolute error is at most 0.01-0.015 , which is good enough for our needs.

	ρ_k^2/σ^2	5.0	10.0	15.0	20.0
$\sum_{j \neq k} \rho_j = 0$	Analyt.	0.181	0.453	0.697	0.856
	Monte C.	0.177	0.464	0.692	0.841
$\sum_{j \neq k} \rho_j = 10\ \sigma^2$	Analyt.	0.068	0.231	0.446	0.648
	Monte C.	0.069	0.232	0.454	0.660

Table 4.2. The power $\beta_k^{(o)}$ of Fisher's test in the case $m=12$, $\epsilon=0.10$. Comparision between results obtained analytically and from Monte Carlo simulations.

As above let N_ρ be the total number of non-zero amplitudes. Results in the case $N_\rho = 1$ are shown in Table 4.3. It is clear that the loss in power by using $a > 0$ is slight. This is true even for small values of m . For instance, when $m=6$, the absolute loss is for $a=2$ at most 0.06-0.08 , (admittedly the discrepancy is 2-3 times greater for $a=3$, but this seems to be a very large value of a for such a small m .)

	$a \backslash \rho_k^2/\sigma^2$	5.0	10.0	15.0	20.0
m=6	0	0.12	0.32	0.53	0.69
	1	0.12	0.32	0.53	0.69
ε=0.05	2	0.12	0.28	0.47	0.62
	3	0.11	0.24	0.38	0.50
	0	0.18	0.46	0.69	0.84
m=12	1	0.18	0.46	0.69	0.84
	2	0.17	0.45	0.69	0.84
ε=0.10	3	0.17	0.44	0.68	0.82

Table 4.3. The power $\beta_k^{(a)}$ in alternatives with only one non-zero amplitude. Cases considered are $m=6$, $\epsilon=0.05$ and $m=12$, $\epsilon=0.10$.

To answer the second question above, consider alternatives where $\rho_k > 0$ and the non-zero amplitudes other than ρ_k have the same magnitude ρ_A . In Figure 4.1 and 4.2 the power $\beta_k^{(a)}$ in the case $m=12$, $\epsilon=0.10$ is plotted as a function of ρ_A^2/σ^2 for $N_\rho=2$ and $N_\rho=3$ respectively. The test $a=1$ is not included as its performance differed little from $a=0$. Also, when $N_\rho=2$, the power functions of methods $a=2$ and $a=3$ were close, and the latter is omitted.

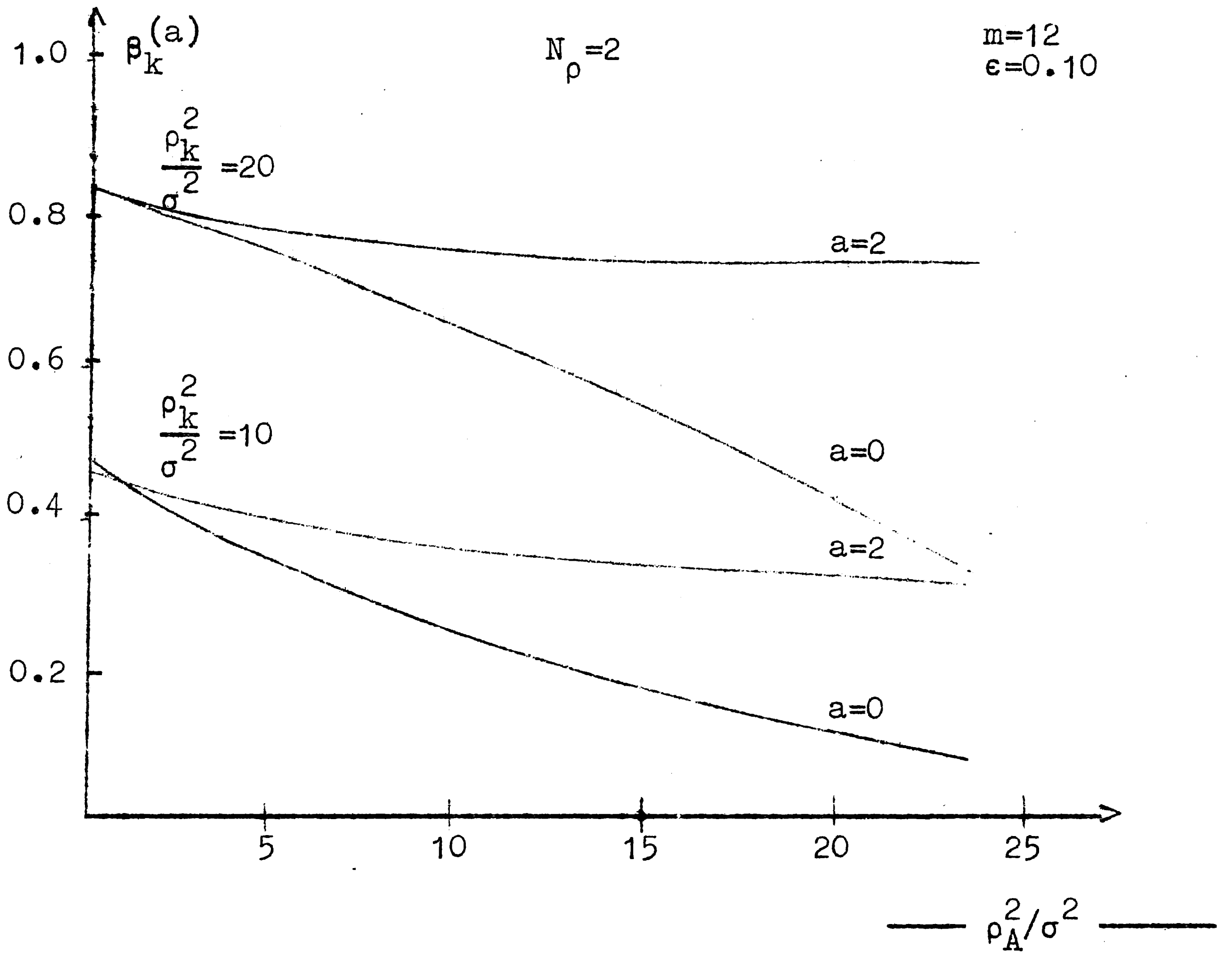

Figure 4.1. The power $\beta_k^{(a)}$ as a function of ρ_A^2/σ^2 when there is outside ρ_k exactly one non-zero amplitude ρ_A . m=12 , ϵ=0.10 .

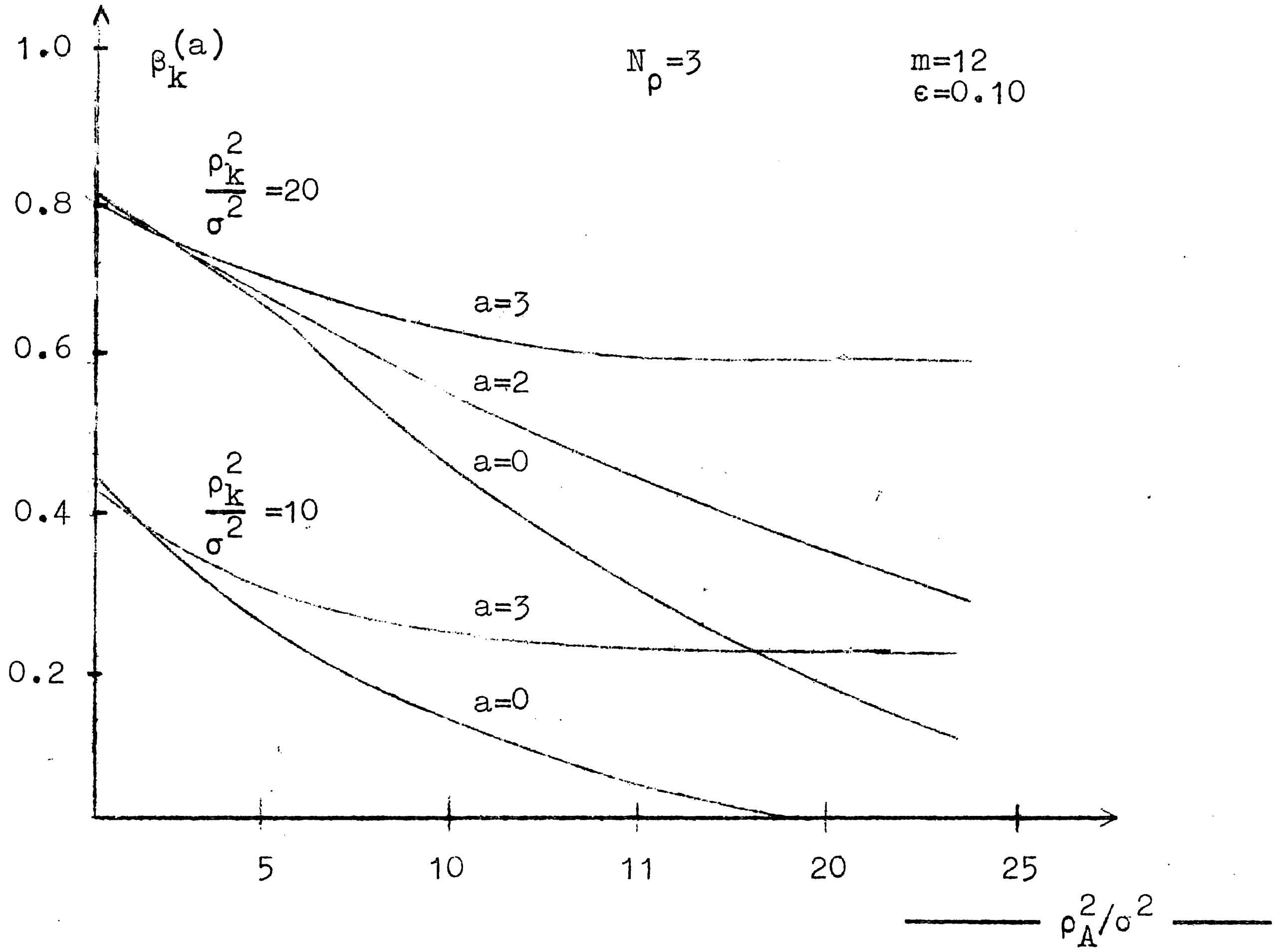

Figure 4.2. The power $\beta_k^{(a)}$ as a function of ρ_A^2/σ^2 when there outside ρ_k are exactly two non-zero amplitudes of equal magnitude ρ_A . m=12 , ε=0.10 .

It is clear that when $N_\rho > 1$, the Fisher method very rapidly becomes inferior to the other methods. For example, if $N_\rho=2$, $\rho_k^2 > \rho_j^2 \geqq 0$, then $\beta_k^{(2)}$ and $\beta_k^{(o)}$ become equal when ρ_j^2 is about 10-15 % of ρ_k^2 . On the other hand the gain in using $a \geqq 2$ instead of $a=0$ may be substantial. As an example, suppose $\rho_A = \rho_k$. In Figure 4.1 where $N_\rho=2$, the power is increased by 60-80 %. When $N_\rho=3$ (Figure 4.2) the power is doubled or threedoubled by using $a=3$ instead of $a=0$.

It was chosen to present results for the case m=12. Smaller values of m will tend to increase the discrepancy between the methods. Larger values have the opposite effect.

The results above also give insight into how a should be selected. It is clear that the performance may be poor if a is chosen too small. If the number of large ρ_k's is believed to be bounded by, say s, it may be reasonable to put $a=s$, for instance if the model is described by (2.5), our choice could be $a=2$. However, if m is not too small, it is clear that there is not much to lose by choosing a for safety somewhat larger than s. This would be all the more appropriate in cases where some of $\rho_1,\dots,\rho_m$, although uninteresting concerning the periodical nature of ξ_t, may not be exactly zero (cf. the discussion in chapter 2.)

5. OTHER METHODS

In this chapter other methods suggested in the past are briefly discussed along with a couple of new propositions.

Anderson [1] derives Bayes procedures assuming that the non-zero amplitudes have the same magnitude(*). These procedures, however, are rather complicated and are, perhaps, not so interesting from a practical point of view. Also, no effort is made in [1] to calculate the relevant constants (which would be far from easy).

From the same starting point another method can be derived by applying a technique suggested by Doornbos for a related problem (see [3]). Let us for a moment assume that the number of potential amplitudes is exactly k (or zero). Then a reasonable decision

(*) In addition Anderson assumes that the number of non-zero amplitudes is at most two, but his results can easily be extended on this point.

rule, optimal when the non-zero amplitudes have the same magnitude, would be to state (*) $\rho_{[m]},\ldots,\rho_{[m-j+1]} > 0$ if the statistic

$$(5.1) \qquad V_k = \frac{\sum_{j=1}^{k} \hat{\rho}^2_{(m-j+1)}}{\sum_{k=1}^{m} \hat{\rho}^2_k}$$

is sufficiently large. To define the test in the general case, let $v_k^{(o)}$ be the observed value of V_k and introduce for $k=1,2,\ldots,a$ (a is an integer fixed in advance)

$$(5.2) \qquad P_k = \Pr(V_k > v_k^{(o)} \mid \rho_1=\ldots=\rho_m=0) .$$

Define the integer $\overset{*}{k}$ by

$$(5.3) \qquad P_{\overset{*}{k}} = \min_{1\leq k\leq a} P_k .$$

If $P_{\overset{*}{k}} \leq \frac{\epsilon}{a}$, then $\rho_{[m]},\ldots,\rho_{[m-\overset{*}{k}+1]}$ are stated positive. Otherwise no statement is made.

It is immediately recognized that the probability of stating incorrectly that some $\rho_k > 0$ is bounded by ϵ when $\rho_1=\ldots=\rho_m=0$. However, it is not clear whether this is the case for general values of $\rho_1,\ldots,\rho_m$.

To transform the test proposition into a working method we must have a way of calculating P_k as given by (5.2). It does not seem that the proof of Theorem 4.1 easily can be extended to

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(*) For $k=1,2,\ldots,m$ $\rho_{[k]}$ is the amplitude corresponding to $\hat{\rho}_{(k)}$.

to cover this case. However, in [2] the present author arrived at a solution by another method. By Lemma 3.2 the numerator and the denominator of (5.1) may be written as linear sums of the same independently distributed χ_2^2-variables. Thus, their joint characteristic functions are not difficult to obtain, and their joint density can be found by the inversion formula for characteristic functions. Finally an expression for P_k can be derived by straightforward integration. Omitting the details (which are given in [2]) we are here content to state the result:

$$(5.4) \qquad P_k = 1 - \sum_{j=\left[\frac{k}{v_k^{(o)}}\right]+1}^{m} (-1)^{m+j}\left(\frac{k}{j-k}\right)^{k-1}\binom{m}{j}\binom{j-1}{k-1}\left(\frac{j}{k}v_k^{(o)}-1\right)^{m-1}$$

$k=1,2,\ldots,a$.

It can easily be verified that when $k=1$, (5.4) coincides with (3.4), as it should.

A reasonable approximation is provided by the first term of the Bonferoni inequality, i.e.

$$P_k \leqq \binom{m}{k} \Pr\left[\frac{\sum_{j=1}^{k} \hat{\rho}_j^2}{\sum_{j=1}^{m} \hat{\rho}_j^2} > v_j^{(o)} \,|\, \rho_1=\ldots=\rho_m=0 \right] .$$

Introducing $B(x;\nu,\mu)$ as the cumulative Beta-distribution, this yields

$$(5.5) \qquad P_k \leqq \binom{m}{k}\left(1 - B\left(v_k^{(o)};\, \frac{k}{2},\frac{m-k}{2}\right)\right) .$$

The difference between the two sides of (5.5) is typically small.

Whittle (see for example [1]) has suggested to make the inference stepwise. As the first step state $\rho_{[m]} > 0$ if

$$(5.6) \qquad \frac{\hat{\rho}^2_{(m)}}{\sum_{j=1}^{m} \hat{\rho}_j^2} > w_m .$$

Otherwise no statement is made and the process is terminated. The constant w_m is to be determined from

$$\Pr\left(\frac{\hat{\rho}^2_{(m)}}{\sum_{j=1}^{m} \hat{\rho}_j^2} > w_m \,|\, \rho_1 = \ldots = \rho_m = 0\right) = \epsilon$$

which can be done by (3.4). As the k-th step, assuming $\rho_{[m]}, \ldots, \rho_{[m-k+2]}$ to have been stated positive, state $\rho_{[m-k+1]} > 0$ if

$$(5.7) \qquad \frac{\hat{\rho}^2_{(m-k+1)}}{\sum_{j=1}^{m-k+1} \hat{\rho}^2_{(j)}} > w_{m-k+1} .$$

Otherwise terminate the process and make no further statement. How the constant w_{m-k+1} should be determined, is discussed in [1].

A serious objection to this procedure is that it may not at all get started if several of $\rho_1, \ldots, \rho_m$ are large, especially if their magnitudes are not far apart. As was demonstrated in chapter 3, the probability that (5.6) is satisfied, may then be quite low.

To overcome this difficulty one might consider to reverse the process. For a specified interger a, state $\rho_{[m-a+1]} > 0$, and in addition $\rho_{[m-a+2]}, \ldots, \rho_{[m]} > 0$, if

$$\frac{\hat{\rho}^2_{(m-a+1)}}{\sum_{j=1}^{m-a} \hat{\rho}^2_{(j)}} > l_{m-a+1} \quad .$$

Otherwise it is concluded that there is no basis for stating $\rho_{[m-a+1]} > 0$, and $\hat{\rho}^2_{(m-a+1)}$ is to be included in the estimate of σ^2. As the next step, state $\rho_{[m-a+2]} > 0$, as well as $\rho_{[m-a+3]},\ldots,\rho_{[m]} > 0$ if

$$\frac{\hat{\rho}^2_{(m-a+2)}}{\sum_{j=1}^{m-a+1} \hat{\rho}^2_{(j)}} > l_{m-a+2} \quad .$$

Otherwise continue the process as above until we can either state $\rho_{[m-a+k]},\ldots,\rho_{[m]} > 0$ for an index $k \leqq a$ or it is concluded that no amplitude can significantly be judged positive.

When $\rho_1=\ldots=\rho_m = 0$, the probability of making a false statement is bounded by the quantity

$$\sum_{k=1}^{a} \Pr\left[\frac{\hat{\rho}^2_{(m-k+1)}}{\sum_{j=1}^{m-k} \hat{\rho}^2_{(j)}} > l_{m-k+1}\right]$$

which by Theorem 4.1 is equal to $\sum_{k=1}^{a} \epsilon_k$ if l_{m-k+1} is determined from

(5.8) $$H^{(k)}_{k-1}(l_{m-k+1}) =$$

$$\sum_{j=k}^{[l^{-1}_{m-k+1}]+k} (-1)^{j-k}\binom{m}{j}\binom{j-1}{j-k} \frac{(1-(j-k)l_{m-k+1})^{m-1}}{(1+kl_{m-k+1})^{m-k} \prod_{i=1}^{k-1} (1+ \frac{(j-k)i}{j-i}\, l_{m-k+1})} = \epsilon$$

This only gives an upper bound for the error. The actual value is likely to be much smaller. Also it is an open question whether the error for arbitrary values of $\rho_1,\ldots,\rho_m$ is controlled by the quantity $\sum_{k=1}^{a} \epsilon_k$.

The question arises how to choose $\epsilon_1,\ldots,\epsilon_a$. Conceivably it would be a mistake to make the level to low for the first steps, as this could result in low-powered tests at the start leading to over-estimation of σ^2 later on. Anyway, from a practical viewpoint, I judge this procedure for a number of reasons to be clearly inferior to the one described in chapter 4.

6. TABLES OF CRITICAL VALUES

CRITICAL VALUES FOR METHOD (4.1)

LEVEL = 0.01

M	A = 1	A = 2	A = 3	A = 4	A = 5	A = 6	A = 7	A = 8
4	6.3681							
5	3.7287	10.6323						
6	2.5944	5.7780	15.0409					
7	1.9798	3.8100	7.8358	19.6278				
8	1.5985	2.7912	5.0025	9.9437	24.3818			
9	1.3403	2.1826	3.5725	6.2087	12.1071	29.2869		
10	1.1544	1.7831	2.7362	4.3544	7.4365	14.3240	34.3282	
11	1.0144	1.5031	2.1975	3.2854	5.1447	8.6873	16.5912	39.4929
12	.9051	1.2970	1.8258	2.6053	3.8372	5.9459	9.9610	18.9049
13	.8176	1.1394	1.5561	2.1414	3.0130	4.3941	6.7585	11.2565
14	.7459	1.0155	1.3526	1.8081	2.4555	3.4230	4.9572	7.5827
15	.6860	.9155	1.1942	1.5589	2.0579	2.7702	3.8362	5.5267
16	.6353	.8333	1.0678	1.3664	1.7626	2.3075	3.0867	4.2532
17	.5919	.7646	.9649	1.2140	1.5360	1.9656	2.5578	3.4053
18	.5541	.7063	.8795	1.0907	1.3575	1.7046	2.1688	2.8094
19	.5211	.6564	.8077	.9890	1.2139	1.4999	1.8730	2.3727
20	.4919	.6130	.7465	.9040	1.0961	1.3358	1.6419	2.0417
21	.4659	.5751	.6937	.8319	.9979	1.2018	1.4573	1.7839
22	.4427	.5416	.6479	.7702	.9151	1.0906	1.3070	1.5786
23	.4217	.5118	.6076	.7167	.8444	.9971	1.1826	1.4119
24	.4027	.4852	.5721	.6699	.7834	.9174	1.0783	1.2743
25	.3854	.4613	.5404	.6287	.7302	.8489	.9897	1.1592
26	.3696	.4396	.5120	.5922	.6836	.7894	.9137	1.3616
27	.3551	.4199	.4865	.5597	.6423	.7373	.8478	.9781
28	.3417	.4020	.4634	.5304	.6056	.6913	.7903	.9059
29	.3294	.3855	.4424	.5041	.5728	.6506	.7397	.8429
30	.3179	.3704	.4232	.4802	.5432	.6141	.6948	.7876
31	.3073	.3564	.4057	.4584	.5165	.5814	.6549	.7387
32	.2974	.3435	.3895	.4385	.4922	.5519	.6190	.6952
33	.2881	.3315	.3746	.4203	.4701	.5251	.5867	.6562
34	.2794	.3204	.3608	.4035	.4498	.5008	.5575	.6212
35	.2713	.3099	.3480	.3880	.4312	.4785	.5309	.5895
36	.2636	.3002	.3360	.3736	.4140	.4581	.5067	.5607
37	.2564	.2911	.3249	.3602	.3981	.4393	.4845	.5345
38	.2495	.2825	.3145	.3478	.3834	.4219	.4641	.5106
39	.2431	.2744	.3048	.3362	.3697	.4059	.4453	.4886
40	.2370	.2668	.2956	.3254	.3570	.3910	.4279	.4683
41	.2312	.2596	.2870	.3152	.3451	.3771	.4118	.4496
42	.2256	.2528	.2789	.3057	.3340	.3642	.3968	.4323
43	.2204	.2464	.2712	.2967	.3235	.3521	.3829	.4162
44	.2154	.2403	.2640	.2883	.3137	.3408	.3698	.4012
45	.2107	.2345	.2571	.2803	.3045	.3302	.3576	.3873
46	.2061	.2290	.2506	.2727	.2958	.3202	.3462	.3742
47	.2018	.2237	.2445	.2656	.2875	.3108	.3355	.3620
48	.1976	.2187	.2386	.2588	.2798	.3019	.3254	.3505
49	.1936	.2139	.2330	.2523	.2724	.2935	.3159	.3397
50	.1898	.2093	.2277	.2462	.2654	.2856	.3069	.3296

CRITICAL VALUES FOR METHOD (4.1)

LEVEL = 0.025

M	A = 1	A = 2	A = 3	A = 4	A = 5	A = 6	A = 7	A = 8
4	4.4288							
5	2.7606	7.3803						
6	1.9926	4.2699	10.4330					
7	1.5578	2.9210	5.7859	13.6115				
8	1.2797	2.1926	3.8321	7.3400	16.9079			
9	1.0871	1.7445	2.8041	4.7544	8.9363	20.3110		
10	.9459	1.4440	2.1854	3.4166	5.6940	10.5731	23.8103	
11	.8380	1.2298	1.7784	2.6233	4.0363	6.6520	12.2481	27.3969
12	.7529	1.0700	1.4929	2.1079	3.0636	4.6651	7.6281	13.9583
13	.6840	.9465	1.2831	1.7506	2.4376	3.5084	5.3033	8.6215
14	.6270	.8484	1.1230	1.4906	2.0072	2.7694	3.9584	5.9509
15	.5792	.7687	.9973	1.2940	1.6964	2.2647	3.1041	4.4139
16	.5384	.7027	.8962	1.1410	1.4631	1.9023	2.5237	3.4421
17	.5032	.6472	.8133	1.0188	1.2826	1.6318	2.1090	2.7848
18	.4726	.5999	.7442	.9193	1.1393	1.4235	1.8008	2.3169
19	.4456	.5591	.6857	.8368	1.0232	1.2589	1.5644	1.9704
20	.4216	.5236	.6357	.7675	.9274	1.1261	1.3783	1.7056
21	.4003	.4924	.5924	.7085	.8473	1.0170	1.2287	1.4978
22	.3810	.4648	.5546	.6576	.7793	.9261	1.1062	1.3312
23	.3637	.4401	.5213	.6135	.7211	.8492	1.0044	1.1952
24	.3479	.4180	.4918	.5747	.6706	.7835	.9186	1.0824
25	.3335	.3981	.4655	.5406	.6265	.7268	.8454	.9876
26	.3203	.3800	.4418	.5101	.5877	.6774	.7824	.9070
27	.3081	.3636	.4205	.4829	.5533	.6340	.7277	.8377
28	.2969	.3485	.4011	.4585	.5226	.5956	.6797	.7776
29	.2866	.3347	.3835	.4363	.4951	.5615	.6374	.7251
30	.2769	.3219	.3674	.4162	.4702	.5309	.5998	.6788
31	.2679	.3102	.3525	.3979	.4477	.5034	.5662	.6378
32	.2595	.2993	.3389	.3811	.4273	.4785	.5360	.6012
33	.2517	.2891	.3263	.3657	.4085	.4559	.5088	.5683
34	.2443	.2796	.3146	.3514	.3914	.4353	.4841	.5387
35	.2374	.2708	.3037	.3383	.3756	.4164	.4616	.5119
36	.2309	.2625	.2936	.3261	.3610	.3991	.4410	.4876
37	.2247	.2548	.2841	.3147	.3475	.3831	.4222	.4653
38	.2189	.2475	.2753	.3042	.3350	.3684	.4048	.4449
39	.2134	.2406	.2669	.2943	.3233	.3547	.3888	.4262
40	.2082	.2341	.2591	.2850	.3125	.3420	.3740	.4090
41	.2032	.2279	.2518	.2763	.3023	.3301	.3602	.3930
42	.1985	.2221	.2448	.2682	.2928	.3191	.3474	.3782
43	.1940	.2166	.2383	.2605	.2838	.3087	.3355	.3644
44	.1897	.2114	.2321	.2532	.2754	.2990	.3243	.3516
45	.1857	.2064	.2262	.2464	.2675	.2899	.3138	.3396
46	.1817	.2016	.2206	.2399	.2600	.2813	.3040	.3284
47	.1780	.1971	.2153	.2337	.2530	.2732	.2948	.3179
48	.1744	.1928	.2103	.2279	.2463	.2656	.2861	.3081
49	.1710	.1887	.2054	.2224	.2399	.2584	.2779	.2988
50	.1677	.1848	.2009	.2171	.2339	.2515	.2702	.2900

CRITICAL VALUES FOR METHOD (4.1)

LEVEL = 0.05

M	A = 1	A = 2	A = 3	A = 4	A = 5	A = 6	A = 7	A = 8
4	3.3089							
5	2.1623	5.4982						
6	1.6052	3.3350	7.7645					
7	1.2787	2.3467	4.5141	10.1263				
8	1.0648	1.7951	3.0753	5.7240	12.5775			
9	.9139	1.4478	2.2933	3.8136	6.9677	15.1096		
10	.8016	1.2109	1.8120	2.7930	4.5664	8.2440	17.7146	
11	.7149	1.0398	1.4899	2.1740	3.2990	5.3346	9.5507	20.3857
12	.6458	.9107	1.2612	1.7653	2.5386	3.8128	6.1179	10.8857
13	.5894	.8101	1.0912	1.4783	2.0411	2.9071	4.3347	6.9155
14	.5425	.7296	.9604	1.2673	1.6949	2.3190	3.2803	4.8647
15	.5029	.6637	.8570	1.1064	1.4422	1.9123	2.5996	3.6583
16	.4689	.6089	.7733	.9802	1.2509	1.6172	2.1312	2.8830
17	.4395	.5625	.7043	.8788	1.1018	1.3951	1.7931	2.3520
18	.4137	.5229	.6464	.7959	.9827	1.2229	1.5398	1.9702
19	.3909	.4885	.5973	.7268	.8858	1.0860	1.3441	1.6851
20	.3706	.4585	.5551	.6684	.8054	.9749	1.1891	1.4656
21	.3525	.4321	.5185	.6186	.7379	.8833	1.0639	1.2924
22	.3361	.4086	.4864	.5755	.6804	.8066	.9609	1.1528
23	.3213	.3876	.4581	.5380	.6310	.7415	.8749	1.0384
24	.3078	.3687	.4329	.5050	.5881	.6857	.8022	.9430
25	.2954	.3516	.4104	.4757	.5505	.6374	.7400	.8626
26	.2841	.3361	.3901	.4497	.5173	.5952	.6863	.7940
27	.2736	.3220	.3718	.4264	.4878	.5580	.6394	.7349
28	.2639	.3090	.3551	.4053	.4614	.5251	.5983	.6835
29	.2550	.2971	.3399	.3862	.4377	.4957	.5620	.6384
30	.2466	.2861	.3260	.3689	.4163	.4694	.5296	.5986
31	.2389	.2759	.3132	.3530	.3968	.4456	.5007	.5632
32	.2316	.2664	.3013	.3385	.3791	.4241	.4746	.5316
33	.2248	.2576	.2904	.3251	.3629	.4045	.4510	.5032
34	.2183	.2494	.2802	.3128	.3480	.3867	.4296	.4776
35	.2123	.2417	.2708	.3013	.3343	.3703	.4101	.4544
36	.2066	.2345	.2620	.2907	.3216	.3552	.3922	.4332
37	.2013	.2277	.2537	.2808	.3098	.3413	.3758	.4139
38	.1962	.2213	.2460	.2716	.2989	.3284	.3607	.3961
39	.1914	.2153	.2387	.2630	.2887	.3165	.3467	.3798
40	.1868	.2096	.2319	.2549	.2792	.3054	.3337	.3647
41	.1824	.2043	.2255	.2473	.2703	.2950	.3217	.3507
42	.1783	.1992	.2194	.2401	.2620	.2853	.3105	.3378
43	.1744	.1943	.2136	.2334	.2541	.2763	.3000	.3257
44	.1706	.1897	.2082	.2270	.2468	.2677	.2902	.3145
45	.1670	.1854	.2030	.2210	.2398	.2597	.2811	.3040
46	.1636	.1812	.1981	.2153	.2332	.2522	.2724	.2942
47	.1603	.1772	.1934	.2099	.2270	.2451	.2643	.2849
48	.1571	.1734	.1890	.2048	.2211	.2384	.2567	.2763
49	.1541	.1698	.1848	.1999	.2155	.2320	.2495	.2681
50	.1512	.1663	.1807	.1952	.2102	.2260	.2426	.2604

CRITICAL VALUES FOR METHOD (4.1)

LEVEL = 0.10

M	A = 1	A = 2	A = 3	A = 4	A = 5	A = 6	A = 7	A = 8
4	2.4200							
5	1.6591	3.9984						
6	1.2679	2.5446	5.6358					
7	1.0301	1.8434	3.4372	7.3446				
8	.8701	1.4383	2.4109	4.3547	9.1200			
9	.7550	1.1769	1.8339	2.9870	5.2989	10.9556		
10	.6681	.9953	1.4702	2.2315	3.5753	6.2689	12.8454	
11	.6000	.8622	1.2225	1.7626	2.6348	4.1762	7.2628	14.7844
12	.5452	.7606	1.0441	1.4474	2.0573	3.0448	4.7895	8.2790
13	.5001	.6807	.9100	1.2230	1.6730	2.3558	3.4617	5.4144
14	.4622	.6162	.8059	1.0561	1.4017	1.9005	2.6582	3.8854
15	.4301	.5631	.7228	.9277	1.2015	1.5813	2.1305	2.9649
16	.4023	.5185	.6552	.8263	1.0486	1.3473	1.7625	2.3631
17	.3781	.4807	.5991	.7442	.9285	1.1695	1.4939	1.9453
18	.3569	.4482	.5518	.6766	.8320	1.0305	1.2908	1.6416
19	.3380	.4199	.5115	.6201	.7529	.9194	1.1327	1.4128
20	.3212	.3951	.4767	.5721	.6870	.8287	1.0067	1.2353
21	.3061	.3732	.4464	.5309	.6314	.7535	.9043	1.0943
22	.2924	.3536	.4197	.4952	.5839	.6902	.8197	.9801
23	.2799	.3361	.3961	.4640	.5429	.6363	.7487	.8859
24	.2686	.3203	.3751	.4365	.5071	.5899	.6884	.8071
25	.2582	.3060	.3562	.4120	.4757	.5497	.6367	.7404
26	.2486	.2929	.3392	.3902	.4479	.5144	.5918	.6833
27	.2398	.2810	.3237	.3705	.4231	.4832	.5527	.6339
28	.2316	.2700	.3097	.3528	.4009	.4555	.5182	.5908
29	.2239	.2599	.2968	.3367	.3809	.4308	.4876	.5529
30	.2168	.2506	.2850	.3220	.3628	.4085	.4603	.5194
31	.2102	.2419	.2741	.3086	.3464	.3884	.4358	.4895
32	.2040	.2339	.2641	.2962	.3313	.3702	.4137	.4628
33	.1982	.2264	.2548	.2848	.3175	.3536	.3937	.4387
34	.1927	.2194	.2461	.2743	.3049	.3384	.3755	.4170
35	.1875	.2128	.2380	.2646	.2931	.3244	.3588	.3972
36	.1826	.2066	.2305	.2555	.2823	.3115	.3436	.3791
37	.1780	.2008	.2234	.2470	.2723	.2996	.3296	.3626
38	.1736	.1954	.2168	.2391	.2629	.2886	.3166	.3474
39	.1695	.1902	.2106	.2317	.2542	.2784	.3047	.3335
40	.1656	.1853	.2047	.2248	.2460	.2688	.2936	.3205
41	.1618	.1807	.1992	.2182	.2384	.2599	.2832	.3086
42	.1582	.1763	.1939	.2121	.2312	.2516	.2736	.2974
43	.1548	.1721	.1890	.2063	.2245	.2438	.2646	.2871
44	.1516	.1682	.1843	.2008	.2181	.2365	.2562	.2774
45	.1485	.1644	.1798	.1956	.2121	.2296	.2482	.2683
46	.1455	.1608	.1756	.1907	.2064	.2231	.2408	.2598
47	.1426	.1574	.1716	.1860	.2010	.2169	.2338	.2519
48	.1399	.1541	.1677	.1816	.1959	.2111	.2272	.2444
49	.1373	.1509	.1640	.1773	.1911	.2056	.2209	.2373
50	.1347	.1479	.1605	.1733	.1865	.2004	.2150	.2306

Appendix.

An approximation to the non-central χ^2-distribution.

Let $Z \sim \chi_n^2(\lambda)$. Patnaik [8] has suggested approximating Z with γY where $Y \sim \chi_\nu^2$ and

$$\gamma = 1 + \frac{\lambda}{\lambda+n}$$

$$\nu = n + \frac{\lambda^2}{2\lambda+n} \quad .$$

Table A.1, computed in [8], indicates the accuracy of the approximation.

In chapter 3 the approximation is used in the following way: Let V and W be independent and non-central χ^2-distributed and let V^* and W^* be Patnaik's approximations.

If we compare $R = \frac{V}{V+W}$ with $R^* = \frac{V^*}{V^*+W^*}$, it is easily shown that

$$|\Pr(R^*>r)-\Pr(R>r)| \leqq \sup_w |\Pr(W^*>w)-\Pr(W>w)| + \sup_w |\Pr(V^*>v)-\Pr(V>v)| \; .$$

The actual error is no doubt much less.

n	λ	z	Approx.	Exactly
4	4	1.765	0.0399	0.0500
	4	10.000	0.7191	0.7118
	4	17.309	0.9492	0.9500
	4	24.000	0.9913	0.9925
	10	10.000	0.3178	0.3148
7	1	4.000	0.1621	0.1628
	1	10.004	0.9499	0.9500
	16	10.257	0.0430	0.0500
	16	24.000	0.5947	0.5898
	16	38.970	0.9482	0.9500
12	6	24.000	0.8187	0.8174
	18	24.000	0.2936	0.2901
16	8	30.000	0.7895	0.7880
	8	40.000	0.9626	0.9632
	32	30.000	0.0590	0.0609
	32	60.000	0.8329	0.8316
24	24	36.000	0.1556	0.1567
	24	48.000	0.5333	0.5296
	24	72.000	0.9656	0.9667

Table A.1. The accuracy of Patnaik's approximation. Approximate and exact values of the cumulative distribution function of the non-central χ^2-distribution are recorded.

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