TWO SUGGESTIONS OF HOW TO DEFINE A MULTISTATE COHERENT SYSTEM

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Abstract

One inherent weakness of traditional reliability theory is that the system and the components are always described just as functioning or failed. However, recent papers by Barlow and Wu (1978) and El-Neweihi, Proschan and Sethuraman (1978) have made significant contributions to start building up a theory for a multistate system of multistate components. Here the states represent successive levels of performance ranging from a perfect functioning level down to a complete failure level. In the present paper we will give two suggestions of how to define a multistate coherent system. The first one is more general than the one introduced in the latter paper, the results of which are, however, shown to be extendable. Furthermore, some new definitions and results are also given. The second one is similarly more general than the one introduced in Barlow and Wu (1978), the results of which are again shown to be extendable. In fact we do believe that most of the theory for the traditional binary coherent system can be extended to our second suggestion of a multistate coherent system.

MULTISTATE COHERENT SYSTEMS; COHERENT SYSTEMS; RELIABILITY; PERFORMANCE; REDUNDANCY.

1. Introduction

In reliability theory a key problem is to find out how the reliability of a complex system can be determined from knowledge of the reliabilities of its components. One inherent weakness of the traditional theory in this field is that the system and the components are always described just as functioning or failed. This approach represents an oversimplification in many real-life situations where the systems and their components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure.

Fortunately, some recent papers by Ross (1977), Barlow and Wu (1978) and especially El-Neweihi, Proschan and Sethuraman (1978) have made significant contributions to start building up a theory for a multistate system of multistate components. Consider a system with set of components $C = \{1,2,\ldots,n\}$. In these papers (and in some of their predecessors too) for each component and for the system itself the set of states is $S = \{0,1,\ldots,M\}$. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Let $(i=1,\ldots,n)$, x_i denote the state or performance level of the ith component and let $\underline{x} = (x_1,\ldots,x_n)$. It is assumed that the state φ of the system is a deterministic function of $\underline{x}$; i.e. $\varphi = \varphi(\underline{x})$. Here $\underline{x}$ takes values in S^n and φ takes values in S. The function φ is called the structure function of the system.

Before going into the specific restrictions Barlow and Wu (1978) and El-Neweihi, Proschan and Sethuraman (1978) claim on the function φ it is convenient at this stage to recall some basic definitions from the traditional binary theory; i.e. when $M=1$. This theory is nicely introduced in Barlow and Proschan (1975). The following notation is needed.

$$(\cdot_i,\underline{x}) = (x_1,\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n) .$$

Definition 1.1. A system is coherent iff

i) $\varphi(\underline{x})$ is nondecreasing in each argument

ii) Each component is relevant; i.e.

$$\forall i, \exists (\cdot_i, \underline{x}) \ni \varphi(1_i, \underline{x}) = 1 \text{ and } \varphi(0_i, \underline{x}) = 0 .$$

We often denote a coherent system by (C,φ) where C is defined above.

A component which is not relevant is said to be irrelevant. We note that an irrelevant component can never directly cause the failure of the system. As an example of such a component consider a condensor being in parallel with an electrical device in a large engine. The task of the condensor is to cut off high voltages which may have destroyed the electrical device. Hence although being irrelevant the condensor can be very important in increasing the life-time of the device and hence the life-time of the whole engine.

The limitation of Definition 1.1, claiming each component to be relevant, is inherited by the various definitions of a multistate coherent system discussed in the present paper.

Definition 1.2. A path set is a set of components whose functioning is sufficient for the system to function. A path set is minimal if it can not be reduced and still be a path set. A cut set is a set of components whose failure is sufficient to cause system failure. A cut set is minimal if it can not be reduced and still be a cut set.

We also need the following notation. Let $A \subseteq C$. Then

$$\coprod_{i\in A} x_i = 1 - \prod_{i\in A}(1-x_i) , \quad x_1 \amalg x_2 = 1-(1-x_1)(1-x_2)$$

$\underline{x}^A$ = vector with elements $x_i, i \in A$.

A^c = subset of C complementary to A.

Consider a coherent system φ with minimal path sets $P_1,\dots,P_p$ and mimimal cut sets $K_1,\dots,K_k$. Since the system is functioning iff for at least one minimal path set all the components are functioning, or alternatively, iff for all minimal cut sets at least one component is functioning, we have the two following representations for the structure function:

(1.1) $$\varphi(\underline{x}) = \coprod_{j=1}^{p} \prod_{i \in P_j} x_i = \max_{1 \leq j \leq p} \min_{i \in P_j} x_i$$

(1.2) $$\varphi(\underline{x}) = \prod_{j=1}^{k} \coprod_{i \in K_j} x_i = \min_{1 \leq j \leq k} \max_{i \in K_j} x_i$$

Definition 1.3. The coherent system (A,χ) is a module of the coherent system (C,φ) iff

$$\varphi(\underline{x}) = \psi[\chi(\underline{x}^A),\underline{x}^{A^c}] ,$$

where ψ is a coherent structure function and $A \subseteq C$.

Intuitively, a module is a coherent subsystem that acts as if it were just a component.

Definition 1.4. A modular decomposition of a coherent system (C,φ) is a set of disjoint modules $\{(A_k,\chi_k)\}_{k=1}^{r}$ together with an organizing coherent structure ψ ; i.e.

i) $$C = \bigcup_{i=1}^{r} A_i \quad \text{where} \quad A_i \cap A_j = \emptyset \quad i \neq j$$

ii) $$\varphi(\underline{x}) = \psi[\chi_1(\underline{x}^{A_1}),\ldots,\chi_r(\underline{x}^{A_r})] .$$

Definition 1.5. Given a coherent structure φ , its dual structure φ^D is given by

$$\varphi^D(\underline{x}) = 1 - \varphi(\underline{1}-\underline{x}) ,$$

where $\underline{1} - \underline{x} = (1-x_1,\ldots,1-x_n)$.

Definition 1.6. The random variables (r.v.'s) $T_1,\ldots,T_n$ are associated iff $\mathrm{Cov}[\Gamma(\underline{T}),\Delta(\underline{T})] \geq 0$ for all pairs of nondecreasing binary functions Γ,Δ .

We list some basic properties of associated r.v.'s:

P_1) Any subset of a set of associated r.v.'s is a set of associated r.v.'s.

P_2) The set consisting of a single r.v. is a set of associated r.v.'s.

P_3) Nondecreasing functions of associated r.v.'s are associated.

P_4) If two sets of associated r.v.'s are independent of each other, then their union is a set of associated r.v.'s.

We now return to the multinary theory and start by giving the structure function considered by Barlow and Wu (1978):

Definition 1.7. Let $P_1,\ldots,P_p$ be non-empty subsets of $C = \{1,\ldots,n\}$ such that $\bigcup_{i=1}^{p} P_i = C$ and $P_j \not\subset P_i \quad i \neq j$. Then

$$(1.3) \qquad \varphi(\underline{x}) = \max_{1\leq j\leq p} \min_{i\in P_j} x_i .$$

If the sets $\{P_1,\ldots,P_p\}$ are considered as minimal path sets, they uniquely determine a binary coherent system (C,φ_0), where φ_0 is defined by (1.1). On the other hand, starting out with a binary coherent system φ_0 its minimal path sets $\{P_1,\ldots,P_p\}$ are uniquely determined. Hence what Barlow and Wu (1978) essentially do when defining their structure function is just to extend the domain and range of (1.1) from $\{0,1\}$ to $\{0,1,\ldots,M\}$. It is hence a one-to-one correspondence between the binary structure function φ_0 and the multinary structure function φ. Furthermore, if $\{K_1,\ldots,K_k\}$ are the minimal cut sets of (C,φ_0), it follows from Theorem 3.5 (p. 12) of Barlow and Proschan (1975) that for $\varphi(\underline{x})$ of (1.3) we have

$$(1.4) \qquad \varphi(\underline{x}) = \min_{1\leq j\leq k} \max_{i\in K_j} x_i .$$

Specializing $p = 1$ in (1.3) and $k = 1$ in (1.4) we respectively get the multinary series and parallel structure functions.

El-Neweihi, Proschan and Sethuraman (1978) suggest the following definition of a multistate coherent system (MCS):

Definition 1.8. A system of n components is said to be an MCS iff its structure function φ satisfies:

i) $\varphi(\underline{x})$ is nondecreasing in each argument

ii) $\forall i, \forall j \in \{0,\ldots,M\}, \exists (\cdot_i, \underline{x}) \ni$

$\varphi(j_i, \underline{x}) = j$ and $\varphi(l_i, \underline{x}) \neq j$ for $l \neq j$

iii) $\forall j \in \{0,\ldots,M\}$ $\varphi(\underline{j}) = j$ $(\underline{j} = (j,j,\ldots,j))$

It is easy to see that the structure function of Definition 1.7 is just a special case of the one of Definition 1.8. Furthermore, note that i) and ii) above are generalizations of i) and ii) of Definition (1.1). In the binary case iii) is implied by i) and ii). This is not true in the multinary case.

In the present paper we will give two suggestions of how to define an MCS. These will respectively be called an MCS of type 1 and type 2 . In type 1 ii) of Definition 1.8 is replaced by a condition which is more general and we feel a more reasonable generalization of ii) of Definition 1.1. We will in Section 2 and 3 show that all results obtained by El-Neweihi, Proschan and Sethuraman (1978) for their MCS also hold for an MCS of type 1. Furthermore, some new definitions and some new theorems will be given.

The MCS of type 2 is a special case of the one of type 1. It is not a special case of the one suggested by El-Neweihi, Proschan and Sethuraman (1978) neither is it the other way round. However, the structure function of an MCS of type 2 is far more general than the one suggested by Barlow and Wu (1978). For instance considering systems of 3 components there are for all M just 9 different structure functions of the latter type whereas by choosing M = 7 we get 1665 structure functions in addition of the former type. We will in Section 5 show that all results obtained by Barlow and Wu (1978) extend to an MCS of type 2. In fact we do believe that most of the theory for a traditional binary coherent system can be extended to an MCS of type 2.

2. Deterministic properties of an MCS of type 1

Taking Definitions 1.1 and 1.8 into account it seems natural to claim the structure function φ of any MCS to be nondecreasing in each argument. This simply means that an improvement of the performance of a component can not have the opposite effect on the performance of the system. As shown by Ross (1977) it is possible to obtain interesting results without imposing further restrictions on φ.

We next question whether condition iii) of Definition 1.8 must enter when defining an MCS. The answer seems to be yes due to the following theorem which is easily proved (see Theorem 3.1 of El-Neweihi, Proschan and Sethuraman (1978)):

<u>Theorem 2.1</u>. Let $\varphi(\underline{x})$ be a (multinary) structure function which is nondecreasing in each argument. Then condition iii) of Definition 1.8. is equivalent with

$$(2.1) \qquad \min_{1 \leq i \leq n} x_i \leq \varphi(\underline{x}) \leq \max_{1 \leq i \leq n} x_i$$

Hence $\varphi(\underline{x})$ is bounded below (above) by the series (parallel) structure function. Now choose $j \in \{1,\ldots,M\}$ and let the states $\{0,\ldots,j-1\}$ correspond to the failure state if a binary approach had been applied. (2.1) says that for any j, i.e. for any way of distinguishing between the binary failure and functioning state, if all components are in the binary failure (functioning) state, the system itself is in the binary failure (functioning) state. This is consistent with the fact that iii) of Definition 1.8 holds in the binary case.

Following the binary approach above it seems natural, for any way of distinguishing between the failure and functioning state, to claim each component to be relevant. More precisely for any j and any component i, it should exist a vector $(\cdot_i,\underline{x})$ such that if the i-th component is in the binary failure (functioning) state, the system itself is in the binary failure (functioning) state. This motivates the following definition of an MCS of type 1:

<u>Definition 2.2</u>. A system of n components is said to be an MCS of type 1 iff its structure function φ satisfies:

i) $\varphi(\underline{x})$ is nondecreasing in each argument

ii) $\forall i, \forall j \in \{1,\ldots,M\}, \exists (\cdot_i,\underline{x}) \ni$

$$\varphi(j_i,\underline{x}) \geq j \quad \text{and} \quad \varphi((j-1)_i,\underline{x}) \leq j-1$$

iii) $\forall j \in \{0,1,\ldots,M\} \quad \varphi(\underline{j}) = j$

Note that i) and ii) of Definition 2.2 immediately imply:

$$\forall i, \forall j \in \{1,\ldots,M\}, \exists (\cdot_i, \underline{x}) \ni$$

$$\forall k \in \{j,\ldots,M\} \quad \varphi(k_i,\underline{x}) \geq j \quad \text{and} \quad \forall k \in \{0,\ldots,j-1\} \; \varphi(k_i,\underline{x}) \leq j-1 ,$$

which is just what we found natural to claim.

Comparing Definitions 1.8 and 2.2 we see that the MCS suggested by El-Neweihi, Proschan and Sethuraman (1978) is a special case of the MCS of type 1. Consider the MCS of 2 components having structure function tabulated in Table 1.

Component 2	2	1	1	2
	1	1	1	1
	0	0	1	1
		0	1	2
		Component 1		

Table 1

This is obviously an MCS of type 1, but not of the type suggested by El-Neweihi, Proschan and Sethuraman (1978). The MCS of 2 components having structure function tabulated in Table 2 is not of type 1.

Component 2	2	1	1	2
	1	0	1	1
	0	0	1	1
		0	1	2
		Component 1		

Table 2

If we had just claimed each component to be relevant, for <u>at least one</u> way of distinguishing between the binary failure and functioning state, the structure function of Table 2 would have satisfied the corresponding definition. This claim is, however, not strong enough to obtain for instance Theorems 2.4 and 2.5 below.

We start by noting that Lemma 3.1 of El-Neweihi, Proschan and Sethuraman (1978) is obviously valid for an MCS of type 1. The same is true for their definition of a dual structure, which naturally generalizes Definition 1.5:

Definition 2.3. Let φ be the structure function of an MCS of type 1. The dual structure function φ^D is given by:

$$\varphi^D(\underline{x}) = M - \varphi(M-x_1,\ldots,M-x_n) \tag{2.2}$$

This is the structure function of the dual system of an MCS of type 1. The following theorem is an almost trivial generalization of a result in the mentioned paper.

Theorem 2.4. The dual system of an MCS of type 1 is itself an MCS of type 1.

Proof. The conditions i) and iii) of Definition 2.2 are trivially satisfied for φ^D. Now applying ii) of this definition on φ we get

$$\forall i,\forall j \in \{1,\ldots,M\},\exists(\cdot_i,\underline{x}) \ni$$
$$\varphi^D(j_i,\underline{x}) = M - \varphi((M-j)_i,\underline{M}-\underline{x}) \geq M-(M-j) = j$$
$$\varphi^D((j-1)_i,\underline{x}) = M - \varphi((M+1)-j)_i,\underline{M}-\underline{x}) \leq M-(M+1-j) = j-1 ,$$

and the proof is completed.

The well-known principle that redundancy at the component level is preferable to redundancy at the system level (all other things being equal) also holds for an MCS of type 1. Again our theorem represents a generalized version of one given in El-Neweihi, Proschan and Sethuraman (1978). We need the following notation

$$x \vee y \overset{\text{def}}{=} \max(x,y)$$
$$\underline{x} \vee \underline{y} \overset{\text{def}}{=} (x_1 \vee y_1,\ldots,x_n \vee y_n)$$
$$x \wedge y \overset{\text{def}}{=} \min(x,y)$$
$$\underline{x} \wedge \underline{y} \overset{\text{def}}{=} (x_1 \wedge y_1,\ldots,x_n \wedge y_n) .$$

Theorem 2.5. Let φ be the structure function of an MCS of type 1. Then

i) $\varphi(\underline{x} \vee \underline{y}) \geq \varphi(\underline{x}) \vee \varphi(\underline{y})$

ii) $\varphi(\underline{x} \wedge \underline{y}) \leq \varphi(\underline{x}) \wedge \varphi(\underline{y})$

Equality holds in i) (ii)) for all $\underline{x}$ and $\underline{y}$ iff the structure function is parallel (series).

Proof. Here we will just show that equality in i) for all $\underline{x}$ and $\underline{y}$ implies the structure function to be parallel. The rest of the proof is identical to the one in the paper mentioned above.

Assume $\varphi(\underline{x} \vee \underline{y}) = \varphi(\underline{x}) \vee \varphi(\underline{y})$ for all $\underline{x}$ and $\underline{y}$. For all $i \in \{1,\dots,n\}$ and for all $j \in \{1,\dots,M\}$ there exists by Definition 2.2 $(\cdot_i,\underline{x})$ such that

$$\varphi(j_i,\underline{x}) \geq j \quad \text{and} \quad \varphi(0_i,\underline{x}) \leq j-1 \, .$$

Since $(j_i,\underline{x}) = (j_i,\underline{0}) \vee (0_i,\underline{x})$, we have

$$j \leq \varphi(j_i,\underline{x}) = \omega(j_i,\underline{0}) \vee \varphi(0_i,\underline{x}) = \varphi(j_i,\underline{0}) \leq \varphi(\underline{j}) = j \, .$$

Hence $\varphi(j_i,\underline{0}) = j$ for all $i \in \{1,\dots,n\}$ and all $j \in \{1,\dots,M\}$. By iii) of Definition 2.2 this is also true for $j = 0$. Now finally

$$\varphi(\underline{z}) = \varphi(z_1,0,\dots,0) \vee \varphi(0,z_2,0,\dots,0) \vee \dots \vee \varphi(0,\dots,z_n) =$$
$$= z_1 \vee z_2 \vee \dots \vee z_n = \max_{1 \leq i \leq n} z_i \, ,$$

and the proof is completed.

We will next give two new generalizations of each of the concepts "path set", "minimal path set", "cut set" and "minimal cut set" from binary theory. In the following $\underline{y} < \underline{x}$ means $y_i \leq x_i$ for $i = 1,\dots,n$ and $y_i < x_i$ for some i.

Definition 2.6. Let $j \in \{1,2,\dots,M\}$. A vector $\underline{x}$ is said to be a path vector to level j iff $\varphi(\underline{x}) \geq j$. The corresponding path sets to level j of type 1 and 2 are respectively given by

$$C_1^j(\underline{x}) = \{i \mid x_i \geq 1\} \quad \text{and} \quad C_2^j(\underline{x}) = \{i \mid x_i \geq j\}\ .$$

A path vector to level $j, \underline{x}$, is said to be minimal iff $\varphi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$. The corresponding path sets to level j of type 1 and 2 are also said to be minimal.

Definition 2.7. Let $j \in \{1,\dots,M\}$. A vector $\underline{x}$ is said to be a cut vector to level j iff $\varphi(\underline{x}) < j$. The corresponding cut sets to level j of type 1 and 2 are respectively given by

$$D_1^j(\underline{x}) = \{i \mid x_i < M\} \quad \text{and} \quad D_2^j(\underline{x}) = \{i \mid x_i < j\}\ .$$

A cut vector to level $j, \underline{x}$, is said to be minimal iff $\varphi(\underline{y}) \geq j$ for all $\underline{y} > \underline{x}$. The corresponding cut sets to level j of type 1 and 2 are also said to be minimal.

Note that following the binary approach mentioned earlier, for any way of distinguishing between the failure and functioning state, the definition of a path (cut) set to level j of type 2 above reduces to the corresponding one from binary theory. Note also that for $j \in \{1,,,.M\}$ the existence of a minimal path (cut) set to level j of both type 1 and 2 is guaranteed by Definition 2.2.

El-Neweihi, Proschan and Sethuraman (1978) give a lemma and a theorem for "upper critical connection vectors to level j". We conclude this section by giving corresponding results.

Lemma 2.8. For $j \in \{1,\dots,M\}$ the union of all minimal path sets to level j of type 1 (2) equals C. The same is true for the union of all minimal cut sets to level j of type 1 (2).

Proof. Any $i \in C$ is a member of the union since according to ii) of Definition 2.2 we can construct a minimal path (cut) vector to level j starting out with $(j_i, \underline{x})$ $((j-1)_i, \underline{x})$.

Note that Lemma 2.8 generalizes a well-known result from binary theory.

Theorem 2.9. Let φ be the structure function of an MCS of type 1. Furthermore, for $j \in \{1,\dots,M\}$ let $\underline{y}_r^j = (y_{1r}^j,\dots,y_{nr}^j)$ $r = 1,\dots,n_j$ $(\underline{z}_r^j = (z_{1r}^j,\dots,z_{nr}^j)$ $r = 1,\dots,m_j)$ be its minimal path (cut) vectors to level j and

$$C_1^j(\underline{y}_r^j) \quad r = 1,\dots,n_j \quad (D_1^j(\underline{z}_r^j) \quad r = 1,\dots,m_j)$$

the corresponding minimal path (cut) sets to level j of type 1. Then

i) $\varphi(\underline{x}) \geq j \Leftrightarrow \exists t \geq j$ and $1 \leq r \leq n_t \ni x_i \geq y_{ir}^t$ for $i \in C_1^t(\underline{y}_r^t)$

ii) $\varphi(\underline{x}) \geq j \Leftrightarrow \exists\, 1 \leq r \leq n_j \ni x_i \geq y_{ir}^j$ for $i \in C_1^j(\underline{y}_r^j)$

iii) $\varphi(\underline{x}) < j \Leftrightarrow \exists t \leq j$ and $1 \leq r \leq m_t \ni x_i \leq z_{ir}^t$ for $i \in D_1^t(\underline{z}_r^t)$

iv) $\varphi(\underline{x}) < j \Leftrightarrow \exists 1 \leq r \leq m_j \ni x_i \leq z_{ir}^j$ for $i \in D_1^j(\underline{z}_r^j)$

The proof is straightforward.

3. Stochastic performance of an MCS of type 1

In this section we concentrate on the relationship between the stochastic performance of the system and the stochastic performance of the components. Following El-Neweihi, Proschan and Sethuraman (1978) let X_i denote the random state of the i-th component and let $(i = 1,\dots,n; j=0,\dots,M)$

$$P(X_i=j) = p_{ij}$$

$$(3.1) \qquad P(X_i \leq j) = P_i(j)$$

$$\overline{P}(j) = 1-P_i(j)$$

P_i represents the performance distribution of the i-th component. Introduce the random vector $\underline{X} = (X_1,\dots,X_n)$. If now φ is a multinary structure function, $\varphi(\underline{X})$ is the corresponding random system state. Let $(j = 0,\dots,M)$

$$P[\varphi(\underline{X}) = j] = p_j$$

$$(3.2) \qquad P[\varphi(\underline{X}) \leq j] = P(j)$$

$$\overline{P}(j) = 1-P(j)$$

P represents the performance distribution of the system. We also introduce the performance function of the system, h ; defined by

(3.3) $$h = E\varphi(\underline{X}) .$$

In El-Neweihi, Proschan and Sethuraman (1978) $X_1,\ldots,X_n$ are assumed to be statistically independent. It is easily observed that all the theorems and lemmas of Section 4 of this paper also hold for an MCS of type 1. Especially it should be noted that their Lemma 4.1 and Theorem 4.3 also hold when $X_1,\ldots,X_n$ are statistically dependent. By repeated use of the generalized version of this lemma, or much more easily by a direct argument, one obtains

(3.4) $$h = \sum_{\underline{y}=(y_1,\ldots,y_n)\in S^n} P\left(\bigcap_{i=1}^{n}(X_i=y_i)\right)\varphi(\underline{y})$$

Assuming $X_1,\ldots,X_n$ to be independent, we get

(3.5) $$h = \sum_{\underline{y}=(y_1,\ldots,y_n)\in S^n} \prod_{i=1}^{n} p_{iy_i}\,\varphi(\underline{y})$$

The number of addends in (3.4) and (3.5) equals $(M+1)^n$, which easily gets far too large for any computer. Hence we have to find other ways of establishing h .

Section 4 of the mentioned paper is concluded by illustrating how the "upper critical connection vectors to level j " is used to establish bounds on the system performance distribution P and consequently on the system performance function h . The corresponding result for an MCS of type 1 is as follows.

<u>Theorem 3.1</u>. Let φ be the structure function of an MCS of type 1. Furthermore, for $j \in \{1,\ldots,M\}$ let $\underline{y}_r^j = (y_{1r}^j,\ldots,y_{nr}^j)$ $r=1,\ldots,n_j$ $(\underline{z}_r^j = (z_{1r}^j,\ldots,z_{nr}^j)$ $r=1,\ldots,m_j)$ be its minimal path (cut) vectors to level j and $C_1^j(\underline{y}_r^j)$ $r=1,\ldots,n_j$ $(D_1^j(\underline{z}_r^j)$ $r=1,\ldots,m_j)$ the corresponding minimal path (cut) sets to level j of type 1. Then for these values of j

(3.6) $$\overline{P}(j-1) = \sum_{k=1}^{n_j}(-1)^{k-1}S_k^j$$

(3.7) $$\overline{P}(j-1) = 1-\sum_{k=1}^{m_j}(-1)^{k-1}T_k^j ,$$

where

$$S_k^j = \sum_{1\le i_1<i_2<\dots<i_k\le n_j} P\Big[\bigcap_{i\in\bigcup_{s=1}^{k} C_1^j(\underline{y}^j_{i_s})} (X_i \ge \max_{1\le s\le k} y^j_{ii_s})\Big]$$

$$T_k^j = \sum_{1\le i_1<i_2<\dots<i_k\le m_j} P\Big[\bigcap_{i\in\bigcup_{s=1}^{k} D_1^j(\underline{z}^j_{i_s})} (X_i \le \min_{1\le s\le k} z^j_{ii_s})\Big]$$

Proof. For $j \in \{1,\dots,M\}$ we get by applying ii) of Theorem 2.9 and the general addition law of probability theory

$$\overline{P}(j-1) = P(\varphi(\underline{X})\ge j) = P\Big[\bigcup_{r=1}^{n_j} \bigcap_{i\in C_1^j(\underline{y}^j_r)} (X_i\ge y^j_{ir})\Big]$$

$$= \sum_{k=1}^{n_j}(-1)^{k-1} \sum_{1\le i_1<\dots<i_k\le n_j} P\Big[\bigcap_{s=1}^{k} \bigcap_{i\in C_1^j(\underline{y}^j_{i_s})} (X_i\ge y^j_{ii_s})\Big]$$

$$= \sum_{k=1}^{n_j}(-1)^{k-1} \sum_{1\le i_1<\dots<i_k\le n_j} P\Big[\bigcap_{i\in\bigcup_{s=1}^{k} C_1^j(\underline{y}^j_{i_s})} (X_i \ge \max_{1\le s\le k} y^j_{ii_s})\Big]$$

Hence (3.6) is proved. (3.7) is proved similarly by applying iv) of Theorem 2.9.

First note that it is sufficient to know the joint distribution of $X_1,\dots,X_n$ to obtain the expressions for $\overline{P}(j-1)$. If especially $X_1,\dots,X_n$ are independent, we have

$$S_k^j = \sum_{1\le i_1<\dots<i_k\le n_j} \prod_{i\in\bigcup_{s=1}^{k} C_1^j(\underline{y}^j_{i_s})} \overline{P}_i(\max_{1\le s\le k}(y^j_{ii_s})-1)$$

$$T_k^j = \sum_{1\le i_1<\dots<i_k\le m_j} \prod_{i\in\bigcup_{s=1}^{k} D_1^j(\underline{z}^j_{i_s})} P_i(\min_{1\le s\le k} z^j_{ii_s})$$

Furthermore, it is easy to see that the total number of addends in (3.6) and (3.7) is respectively equal to $2^{n_j}-1$ and $2^{m_j}-1$, which again may be too large for a computer. However, by the inclusion-exclusion principle of Feller (1968) (p. 98-101), we have

$$1 - T_1^j \leq \overline{P}(j-1) \leq S_1^j$$

$$S_1^j - S_2^j \leq \overline{P}(j-1) \leq 1 - (T_1^j - T_2^j)$$

$$1 - (T_1^j - T_2^j + T_3^j) \leq \overline{P}(j-1) \leq S_1^j - S_2^j + S_3^j$$

and so on, giving upper and lower bounds on $\overline{P}(j-1)$ for $j \in \{1,\ldots,M\}$. Since obviously

$$h = \sum_{j=1}^{M} \overline{P}(j-1)\,, \tag{3.8}$$

we automatically get upper and lower bounds on h too.

Note finally that the deductions above are based on ii) and iv) rather than i) and iii) of Theorem 2.9. First of all this simplifies Theorem 3.1 in a way that makes it easier to carry out the calculations to obtain exact expressions for the $\overline{P}(j-1)$'s. Secondly, when exact expressions can not be obtained, we feel that this makes the bounds better. This is at least true for the upper bound S_1^j and the lower bound $1 - T_1^j$. These two points seem to have been overlooked by El-Neweihi, Proschan and Sethuraman (1978).

For a binary coherent system a stochastic version of Theorem 2.5 is given in Barlow and Proschan (1975). In the following we will give a completely new generalization of this result, which is valid for an MCS of type 1.

Let for $i = 1,\ldots,n$ x_i be the state of a device, where x_i takes values in $S = \{0,\ldots,M\}$. Introduce the indicators $(j = 1,\ldots,M)$

$$I_j(x_i) = \begin{cases} 1 & \text{if} \quad x_i \geq j \\ 0 & \text{if} \quad x_i < j \end{cases} \tag{3.9}$$

and the indicator vector

$$\underline{I}_j(\underline{x}) = (I_j(x_1),\ldots,I_j(x_n))\,. \tag{3.10}$$

Now consider a multinary structure function $\varphi(\underline{x})$. If the states $\{0,1,\ldots,j-1\}$ correspond to the failure state when a binary approach is applied, then note that $\underline{I}_j(\underline{x})$ is the corresponding vector of binary component states and $I_j(\varphi(\underline{x}))$ the corresponding binary system state.

For x_i $(i = 1,\ldots,n)$ binary we have the following relations

$$\min_{1\le i\le n} x_i = \prod_{i=1}^{n} x_i \quad , \quad \max_{1\le i\le n} x_i = \coprod_{i=1}^{n} x_i$$

The corresponding relations in the multinary case are given in the following lemma which is easily proved.

Lemma 3.2. Let $\underline{x}$ be a vector taking values in S^n. Then

$$(3.11)\qquad \begin{aligned} \min_{1\le i\le n} x_i &= \sum_{j=1}^{M} \prod_{i=1}^{n} I_j(x_i) \\ \max_{1\le i\le n} x_i &= \sum_{j=1}^{M} \coprod_{i=1}^{n} I_j(x_i) \end{aligned}$$

Our new result is given in the following theorem.

Theorem 3.3. Let $\underline{X}$ and $\underline{X}'$ be statistically independent with $P(X_i=j) = p_{ij}$ and $P(X_i'=j) = p_{ij}'$ $(i = 1,\ldots,n;\ j = 0,\ldots,M)$. Furthermore, let φ be the structure function of an MCS of type 1. Then for all $\underline{p} = (p_{11},\ldots,p_{1M},p_{21},\ldots,p_{nM})$ and $\underline{p}' = (p_{11}',\ldots,p_{1M}',p_{21}',\ldots,p_{nM}')$ we have

i) $P[\varphi(\underline{X} \vee \underline{X}') \ge j] \ge P[(\varphi(\underline{X}) \vee \varphi(\underline{X}')) \ge j] \qquad j = 1,\ldots,M$

ii) $P[\varphi(\underline{X} \wedge \underline{X}') \ge j] \le P[(\varphi(\underline{X}) \wedge \varphi(\underline{X}')) \ge j] \qquad j = 1,\ldots,M$

iii) $E[\varphi(\underline{X} \vee \underline{X}')] \ge E[\varphi(\underline{X}) \vee \varphi(\underline{X}')] = \sum_{j=1}^{M} [E[I_j(\varphi(\underline{X}))] \amalg E[I_j(\varphi(\underline{X}'))]]$

iv) $E[\varphi(\underline{X} \wedge \underline{X}')] \le E[\varphi(\underline{X}) \wedge \varphi(\underline{X}')] = \sum_{j=1}^{M} [E[I_j(\varphi(\underline{X}))]\cdot E[I_j(\varphi(\underline{X}'))]]$

Equality holds in iii) (iv)) for all $\underline{p}$ and $\underline{p}'$ iff the structure function is parallel (series).

Proof.

$$\begin{aligned} &P[\varphi(\underline{X} \vee \underline{X}') \ge j] - P[(\varphi(\underline{X}) \vee \varphi(\underline{X}')) \ge j] \\ &= E[I_j(\varphi(\underline{X} \vee \underline{X}')) - I_j(\varphi(\underline{X}) \vee \varphi(\underline{X}'))] \\ &= \sum_{\underline{x}} \sum_{\underline{x}'} [I_j(\varphi(\underline{x} \vee \underline{x}')) - I_j(\varphi(\underline{x}) \vee \varphi(\underline{x}'))]\, P(\underline{X}=\underline{x})P(\underline{X}'=\underline{x}') \ge 0 \,, \end{aligned}$$

having applied the independence of $\underline{X}$ and $\underline{X}'$, i) of Theorem 2.5 and the fact that $I_j(x_i)$ is nondecreasing in x_i. Hence i) is established. The inequality of iii) follows from i) by applying (3.8). The equality of iii) follows most easily by applying Lemma 3.2. Similar arguments give ii) and iv).

Note that if $0 < p_{ij} < 1$, $0 < p'_{ij} < 1$ $(i = 1,\dots,n;\ j = 0,\dots,M)$, then $P(\underline{X}=\underline{x}) > 0$ for all $\underline{x}$ and $P(\underline{X}'=\underline{x}') > 0$ for all $\underline{x}'$. Hence equality in iii) for all $\underline{p}$ and $\underline{p}'$ is equivalent to

$$\varphi(\underline{x} \vee \underline{x}') = \varphi(\underline{x}) \vee \varphi(\underline{x}') \quad \text{for all } \underline{x} \text{ and } \underline{x}'.$$

Applying Theorem 2.5 this is again true iff the structure function is parallel. A similar argument establishes that equality holds in iv) for all $\underline{p}$ and $\underline{p}'$ iff the structure function is series. This completes the proof.

In Section 3 of Chapter 2 of Barlow and Proschan (1975) a series of bounds on h are given for a binary coherent system. Some of the results of this section will be generalized in the following.

Theorem 3.4. If $X_1,\dots,X_n$ are associated r.v.'s, we have respectively for the series and parallel structure functions $(j = 1,\dots,M)$

$$P[\min_{1\le i\le n} X_i \ge j] \ge \prod_{i=1}^{n} \overline{P}_i(j-1) \tag{3.12}$$

$$P[\max_{1\le i\le n} X_i \ge j] \le \coprod_{i=1}^{n} \overline{P}_i(j-1) \tag{3.13}$$

Proof. We obviously have

$$P[\min_{1\le i\le n} X_i \ge j] = P[\prod_{i=1}^{n} I_j(X_i) = 1]$$

$I_j(X_i)$ is nondecreasing in X_i, and so by property P_3 of associated r.v.'s, $I_j(X_1),\dots,I_j(X_n)$ are associated. (3.12) now follows from Theorem 3.1 (p. 32) of Barlow and Proschan (1975), this theorem being just Theorem 3.4 in the binary case. (3.13) is proved similarly.

Corollary 3.5. Let φ be the structure function of an MCS of type 1. Assume $X_1,\dots,X_n$ to be associated r.v.'s. Then

$$(3.14)\qquad \prod_{i=1}^{n} \overline{P}_i(j-1) \leq \overline{P}(j-1) \leq \coprod_{i=1}^{n} \overline{P}_i(j-1) \qquad j=1,\dots,M$$

$$(3.15)\qquad \sum_{j=1}^{M} \prod_{i=1}^{n} \overline{P}_i(j-1) \leq h \leq \sum_{j=1}^{M} \coprod_{i=1}^{n} \overline{P}_i(j-1)$$

Proof. (3.14) follows from Theorems 2.1 and 3.4. (3.15) follows from (3.14) by applying (3.8).

The corresponding result in the binary case is given by Theorem 3.3 (p.34) of Barlow and Proschan (1975). For the special case where $X_1,\dots,X_n$ are independent, this result is given by Theorem 4.4 of El-Neweihi, Proschan and Sethuraman (1978).

Theorem 3.6. Let φ be the structure function of an MCS of type 1 having associated components. Furthermore, for $j \in \{1,\dots,M\}$ let $\underline{y}_r^j = (y_{1r}^j,\dots,y_{nr}^j)$ $r=1,\dots,n_j$ $(\underline{z}_r^j = (z_{1r}^j,\dots,z_{nr}^j)$ $r=1,\dots,m_j)$ be its minimal path (cut) vectors to level j and $C_1^j(\underline{y}_r^j)$ $r=1,\dots,n_j$ $(D_1^j(\underline{z}_r^j)$ $r=1,\dots,m_j)$ the corresponding minimal path (cut) sets to level j of type 1. Then

$$(3.16)\qquad \prod_{r=1}^{m_j} [1-P(\bigcap_{i\in D_1^j(\underline{z}_r^j)} (X_i \leq z_{ir}^j))] \leq P[\varphi(\underline{X}) \geq j]$$

$$\leq \coprod_{r=1}^{n_j} P(\bigcap_{i\in C_1^j(\underline{y}_r^j)} (X_i \geq y_{ir}^j))$$

If furthermore $X_1,\dots,X_n$ are independent, with $P(X_i=j) = p_{ij}$ and $\underline{P} = (\overline{P}_1(0),\dots,\overline{P}_1(M-1),\overline{P}_2(0),\dots,\overline{P}_n(M-1))$, then

$$(3.17)\qquad l^j(\underline{P}) = \prod_{r=1}^{m_j} \coprod_{i\in D_1^j(\underline{z}_r^j)} \overline{P}_i(z_{ir}^j) \leq P[\varphi(\underline{X}) \geq j]$$

$$\leq \coprod_{r=1}^{n_j} \prod_{i\in C_1^j(\underline{y}_r^j)} \overline{P}_i(y_{ir}^j-1) = u^j(\underline{P})\ .$$

Proof. Introduce the following generalization of the minimal cut parallel structure from binary theory

$$I_{\underline{z}_r^j}(\underline{X}) = \begin{cases} 1 & \text{otherwise} \\ 0 & \text{if } X_i \leq z_{ir}^j \text{ for } i \in D_1^j(\underline{z}_r^j) \end{cases}$$

$I_{\underline{z}_r^j}(\underline{X})$ is obviously a nondecreasing function of $X_1,\ldots,X_n$. Hence by property P_3 of associated r.v.'s $I_{\underline{z}_1^j}(\underline{X}),\ldots,I_{\underline{z}_{m_j}^j}(\underline{X})$ are associated. By applying iv) of Theorem 2.9 and Theorem 3.1 (p.32) of Barlow and Proschan (1975), we get

$$P[\varphi(\underline{X}) \geq j] = P[\prod_{r=1}^{m_j} I_{\underline{z}_r^j}(\underline{X}) = 1]$$

$$\geq \prod_{r=1}^{m_j} [1-P(\bigcap_{i\in D_1^j(\underline{z}_r^j)} (X_i \leq z_{ir}^j))] ,$$

and the left inequality of (3.16) is proved. The right inequality is proved similarly by applying ii) of Theorem 2.9. (3.17) follows immediately from (3.16) since $X_1,\ldots,X_n$ now are independent.

The corresponding results in the binary case are given by Theorem 3.4 (p.34) and Corollary 3.5 (p.35) of Barlow and Proschan (1975).

Theorem 3.7. Let φ be the structure function of an MCS of type 1 having independent components, and let $l^j(\underline{P})$ and $u^j(\underline{P})$ be defined as in Theorem 3.6. Then

i) $l^j(\underline{P})$ and $u^j(\underline{P})$ are nondecreasing functions in each argument.

ii) $l^j(\underline{P}) < P[\varphi(\underline{X}) \geq j] < u^j(\underline{P})$ for $0 < p_{ij} < 1$ $(i = 1,\ldots,n; j \in \{0,M\})$ if at least two minimal path (cut) sets to level j of type 1 overlap.

Proof. The proof of i) is trivial. To prove ii) assume the two minimal cut sets to level j of type 1 $D_1^j(\underline{z}_1^j)$ and $D_1^j(\underline{z}_2^j)$ overlap. Introduce $I_{\underline{z}_r^j}(\underline{X})$ $r = 1,\ldots,m_j$ as in the proof of Theorem 3.6. Then since $0 < p_{ij} < 1$ $(i = 1,\ldots,n; j \in \{0,M\})$ none of the following two r.v.'s $I_{\underline{z}_1^j}(\underline{X})$ and $\prod_{r=2}^{m_j} I_{\underline{z}_r^j}(\underline{X})$ are identically equal to 0 or 1. Since $D_1^j(\underline{z}_1^j)$ and $D_1^j(\underline{z}_2^j)$ overlap it then also follows that the two r.v.'s mentioned are dependent. Furthermore, they are associated by property P_3 of associated r.v.'s. By applying Exercise 6 (p.31) of Barlow and Proschan (1975) we have

$$\mathrm{Cov}[I_{\underline{z}_1^j}(\underline{X}), \prod_{r=2}^{m_j} I_{\underline{z}_r^j}(\underline{X})] > 0 ,$$

which is equivalent to

$$P[\varphi(\underline{X}) \geq j] = E[\prod_{r=1}^{m_j} I_{\underline{z}_r^j}(\underline{X})] > E(I_{\underline{z}_1^j}(\underline{X}))E[\prod_{r=2}^{m_j} I_{\underline{z}_r^j}(\underline{X})]$$

By finally applying Theorem 3.1 (p.32) of Barlow and Proschan (1975) on the right hand side the left inequality of ii) follows. The right one is proved similarly.

The corresponding result in the binary case is given by Theorem 3.6 (p.35) of Barlow and Proschan (1975).

El-Neweihi, Proschan and Sethuraman (1978) conclude their paper by considering dynamic models; i.e. models in which the state of the system and of its components vary over time. At time 0, the system and each of its components are in state M. As time passes, the performance of each component and consequently of the system itself deteriorates to successively lower levels, until ultimately level 0 is attained. Theorem 5.1, due to Ross (1977), of the mentioned paper is immediately seen to hold for an MCS of type 1. Furthermore, by applying Theorem 2.5 their Theorem 5.2 is also shown to be valid for the latter system.

We conclude this section by generalizing Theorem 3.2 and Corollary 3.3 (p.33) of Barlow and Proschan (1975) where dynamic models are considered. Let $\{X_i(t), t \geq 0\}$ denote the stochastic process representing the state of component i as a function of time t, $i = 1,\ldots,n$. Introduce the r.v.'s

$$T_i^j = \inf\{t : X_i(t) \leq j\} \qquad i = 1,\ldots,n;\ j = 0,\ldots,M-1$$

representing the lifelength in the states $\{j+1,\ldots,M\}$ of component i.

Theorem 3.8. If $T_1^j,\dots,T_n^j$ are associated r.v.'s, then

(3.18) $$P[\bigcap_{i=1}^{n}(T_i^j > t_i)] \geqq \prod_{i=1}^{n} P(T_i^j > t_i)$$

(3.19) $$P[\bigcap_{i=1}^{n}(T_i^j \leqq t_i)] \geqq \prod_{i=1}^{n} P(T_i^j \leqq t_i)$$

Proof.

$$P[\bigcap_{i=1}^{n}(T_i^j > t_i)] = P[\bigcap_{i=1}^{n}(X_i(t_i) \geqq j+1)]$$

$$= P[\prod_{i=1}^{n} I_{j+1}(X_i(t_i)) = 1] .$$

Since $I_{j+1}(X_i(t_i))$ is nondecreasing in T_i^j, it follows by property P_3 of associated r.v.'s that $I_{j+1}(X_1(t_1)),\dots,I_{j+1}(X_n(t_n))$ are associated. (3.18) now follows from Theorem 3.1 (p.32) of Barlow and Proschan (1975). (3.19) is proved similarly.

Corollary 3.9. If $T_1^j,\dots,T_n^j$ are associated r.v.'s, then

(3.20) $$P[\min_{1\leqq i\leqq n} T_i^j > t] \geqq \prod_{i=1}^{n} P(T_i^j > t)$$

(3.21) $$P[\max_{1\leqq i\leqq n} T_i^j > t] \leqq \coprod_{i=1}^{n} P(T_i^j > t)$$

The proof is immediate from Theorem 3.8.

4. Deterministic properties of an MCS of type 2

We start by immediately giving the definition of an MCS of type 2.

Definition 4.1. A system of n components is said to be an MCS of type 2 iff there exist binary coherent structures φ_j $j=1,\dots,M$ such that its structure function φ satisfies

(4.1) $$\varphi(\underline{x}) \geq j \iff \varphi_j(\underline{I}_j(\underline{x})) = 1$$

for all $j \in \{1,\dots,M\}$ and all $\underline{x}$.

Choose $j \in \{1,\dots,M\}$ and let the states $\{0,\dots,j-1\}$ correspond to the failure state if a binary approach is applied. By the definition above φ_j will uniquely determine the system's binary state from the component's binary states.

The binary coherent structures φ_j $j = 1,\dots,M$ can not be chosen arbitrarily as is demonstrated in the following theorem.

Theorem 4.2. The binary coherent structures φ_j of an MCS of type 2 satisfy

(4.2) $$\varphi_j(\underline{z}) \geq \varphi_{j+1}(\underline{z})$$

for all $j \in \{1,\dots,M-1\}$ and all binary $\underline{z}$.

Proof. Choose $j \in \{1,\dots,M-1\}$. To prove (4.2), is equivalent to proving

i) $$\varphi_{j+1}(\underline{z}) = 1 \Rightarrow \varphi_j(\underline{z}) = 1 \qquad \forall \underline{z} \ni \varphi_{j+1}(\underline{z}) = 1$$

We will however show that i) is equivalent to

ii) $$\varphi(\underline{x}) \geq j+1 \Rightarrow \varphi(\underline{x}) \geq j \qquad \forall \underline{x} \ni \varphi(\underline{x}) \geq j+1 ,$$

which is trivially satisfied.

Assume that i) is true. Choose $\underline{x}$ arbitrarily such that $\varphi(\underline{x}) \geq j+1$. We then have

$$\varphi(\underline{x}) \geq j+1 \Rightarrow \varphi_{j+1}(\underline{I}_{j+1}(\underline{x})) = 1$$
$$\Rightarrow \varphi_j(\underline{I}_{j+1}(\underline{x})) = 1 \Rightarrow \varphi_j(\underline{I}_j(\underline{x})) = 1 \Rightarrow \varphi(\underline{x}) \geq j ,$$

and ii) is true. Finally assume ii) to be true. Choose $\underline{z}$ arbitrarily such that $\varphi_{j+1}(\underline{z}) = 1$. Introduce

$$x_i^j(z_i) = \begin{cases} j & \text{if } z_i = 1 \\ 0 & \text{if } z_i = 0 \end{cases}$$

and $\underline{x}^j(\underline{z}) = (x_1^j(z_1),\dots,x_n^j(z_n))$. We then have

$$\varphi_{j+1}(\underline{z}) = 1 \Rightarrow \varphi(\underline{x}^{j+1}(\underline{z})) \geq j+1$$

$$\Rightarrow \varphi(\underline{x}^{j+1}(\underline{z})) \geq j \Rightarrow \varphi(\underline{x}^{j}(\underline{z})) \geq j \Rightarrow \varphi_j(\underline{z}) = 1 ,$$

and i) is true.

Theorem 4.3. For an MCS of type 2 we have the following unique correspondence between the structure function φ and the binary coherent structures φ_j $j = 1,\dots,M$.

(4.3) $$\varphi(\underline{x}) = 0 \Leftrightarrow \varphi_1(\underline{I}_1(\underline{x})) = 0$$

(4.4) $$\varphi(\underline{x}) = j \Leftrightarrow \varphi_j(\underline{I}_j(\underline{x})) - \varphi_{j+1}(\underline{I}_{j+1}(\underline{x})) = 1 \quad j \in \{1,\dots,M-1\}$$

(4.5) $$\varphi(\underline{x}) = M \Leftrightarrow \varphi_M(\underline{I}_M(\underline{x})) = 1$$

Proof. The relations (4.3) - (4.5) follow immediately from (4.1). Note especially that by applying (4.3), (4.4) we obtain

$$\bigcup_{j=o}^{M-1} \{\varphi(\underline{x}) = j\} \Leftrightarrow \exists j \in \{1,\dots,M\} \ni \varphi_j(\underline{I}_j(\underline{x})) = 0$$

Since by applying (4.2)

$$\varphi_j(\underline{I}_j(\underline{x})) \geq \varphi_j(\underline{I}_M(\underline{x})) \geq \varphi_M(\underline{I}_M(\underline{x})) ,$$

it follows that

$$\bigcup_{j=o}^{M-1} \{\varphi(\underline{x}) = j\} \Leftrightarrow \varphi_M(\underline{I}_M(\underline{x})) = 0 ,$$

which is equivalent to (4.5). Hence the latter relation represents nothing new. Starting out with $\varphi_1,\dots,\varphi_M$ φ is uniquely determined by (4.3) and (4.4). On the other hand starting out with φ, φ_1 is uniquely determined by (4.3). Then $\varphi_2,\varphi_3,\dots,\varphi_M$ is uniquely determined by applying (4.4) for $j = 1,\dots,M-1$.

Theorem 4.4. An MCS of type 2 is also an MCS of type 1.

Proof. We have to show that the structure function of an MCS of type 2 satisfies the claims i) - iii) of Definition 2.2. The claim i) is seen to be satisfied from (4.1) since by i) of Definition 1.1 and (3.9), (3.10) $\varphi_j(\underline{I}_j(\underline{x}))$ is nondecreasing in each x_i for $j \in \{1,\ldots,M\}$. To show the claim ii) is equivalent to showing

$$\forall i, \forall j \in \{1,\ldots,M\}, \exists (\cdot_i, \underline{x}) \ni$$

$$\varphi_j(1_i, \underline{I}_j(\underline{x})) = 1 \quad \text{and} \quad \varphi_j(0_i, \underline{I}_j(\underline{x})) = 0 .$$

By applying ii) of Definition 1.1 on φ_j for $j = 1,\ldots,M$, the above statement is seen to be true. Finally to prove the claim iii), choose $k \in \{0,\ldots,M\}$. By applying Exercise 1 (p.8) of Barlow and Proschan (1975), we get for $j \in \{0,\ldots,M\}$

$$\varphi_j(\underline{I}_j(\underline{k})) = \begin{cases} 1 & \text{if} \quad j \leq k \\ 0 & \text{if} \quad j > k \end{cases}$$

This implies

$$\varphi(\underline{k}) \geqq k \quad \text{and} \quad \varphi(\underline{k}) < k+1 ,$$

and hence $\varphi(\underline{k}) = k$.

Consider the MCS of 2 components having structure function tabulated in Table 3:

Component 2	2	1	2	2
	1	1	1	2
	0	0	1	2
		0	1	2
		Component 1		

Table 3

This is obviously an MCS of type 1. However since $\varphi(1,2) = 2$ whereas $\varphi(0,2) = 1$, it is not an MCS of type 2.

As demonstrated earlier the MCS having structure function tabulated in Table 1 of Section 2 is not of the type suggested by El-Neweihi, Proschan and Sethuraman (1978). It is, however, an MCS of type 2 with φ_1 and φ_2 being respectively a parallel and series structure. The opposite case is demonstrated by the structure function tabulated in Table 4:

Component 2				
	2	1	2	2
	1	0	1	2
	0	0	0	1
		0	1	2
		Component 1		

Table 4

Theorem 4.5. Consider an MCS of type 2 having minimal path (cut) sets to level j of type 2 $C_2^j(\underline{y}_r^j)$ $r = 1,\dots,n_j$ $(D_2^j(\underline{z}_r^j)$ $r = 1,\dots,m_j)$ where $j \in \{1,\dots,M\}$. Then

$$(4.6) \qquad \varphi(\underline{x}) \geq j \Leftrightarrow \coprod_{r=1}^{n_j} \prod_{i \in C_2^j(\underline{y}_r^j)} I_j(x_i) = 1$$

$$(4.7) \qquad \varphi(\underline{x}) < j \Leftrightarrow \prod_{r=1}^{m_j} \coprod_{i \in D_2^j(\underline{z}_r^j)} I_j(x_i) = 0$$

Proof. The results follow immediately from Definitions 2.6, 2.7, 4.1 and (1.1), (1.2).

Note that the minimal path (cut) sets to level j of type 2 are for an MCS of type 2 identical to the minimal path (cut) sets of φ_j, $j = 1,\dots,M$. Having the theorem above it is then natural to believe that most of the theory for a binary coherent system can be extended to an MCS of type 2. This confidence is not weakened by the results given in the next section on stochastic performance of an MCS of type 2.

Theorem 4.6. The structure function of an MCS of type 2, where all the binary coherent structures φ_j are identical, reduces to the one suggested by Barlow and Wu (1978) given by Definition 1.7.

Proof. Denote the common binary coherent structure by φ_0 and its minimal path sets by $P_1,\dots,P_p$. Then for $j \in \{1,\dots,M\}$ it follows from (4.6) that

$$\varphi(\underline{x}) \geq j \Leftrightarrow \max_{1 \leq j \leq p} \min_{i \in P_j} I_j(x_i) = 1 \Leftrightarrow \max_{1 \leq j \leq p} \min_{i \in P_j} x_i \geq j$$

Hence (1.3) is satisfied and the proof is completed.

Pooling all relevant results we can give the following figure illustrating the relationships between the different MCS's considered in this paper.

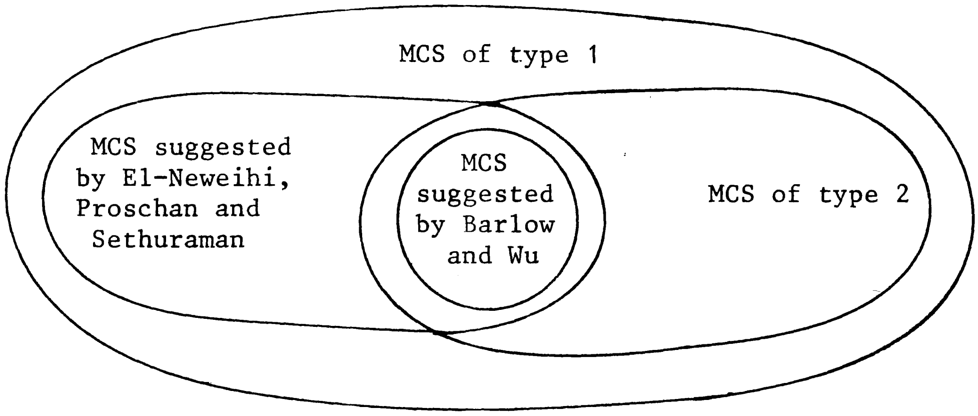

Figure 1

We will conclude this section by demonstrating that the structure function of an MCS of type 2 is far more general than the one suggested by Barlow and Wu (1978).

Theorem 4.7. Restricting to 2-component systems there are $M+1$ different MCS's of type 2, whereas just 2 of the type suggested by Barlow and Wu (1978). The corresponding numbers for 3-component systems are

$$9 + (M-1)30 + \binom{M-1}{2}46 + \binom{M-1}{3}33 + \binom{M-1}{4}9$$

and just 9 .

Proof. First consider the 2-component systems. The only possible binary coherent structures are then

$$\psi_1(z_1, z_2) = z_1 \amalg z_2 \qquad \psi_2(z_1, z_2) = z_1 \cdot z_2 ,$$

i.e. the parallel and series structure. Since

$$\psi_1(\underline{z}) \geq \psi_2(\underline{z}) \ \forall \underline{z} \quad \text{and} \quad \exists \underline{z} \ni \psi_1(\underline{z}) > \psi_2(\underline{z}) ,$$

which we abbreviate $\psi_1 > \psi_2$, it is according to Theorem 4.2 impossible to choose the binary coherent structures $\varphi_1,\dots,\varphi_M$ of an MCS of type 2 such that $\varphi_k = \psi_2$ and $\varphi_{k+1} = \psi_1$ $k=1,\dots,M-1$. By applying Theorems 4.3, 4.6 the results for the 2-component systems follow immediately.

Now consider the 3-component systems. From Barlow and Proschan (1975) (p.8) the only possible binary coherent structures are

$$\psi_1(\underline{z}) = z_1 \amalg z_2 \amalg z_3,\ \psi_2(\underline{z}) = z_1 \amalg (z_2 \cdot z_3),\ \psi_3(\underline{z}) = z_2 \amalg (z_1 \cdot z_3)$$

$$\psi_4(\underline{z}) = z_3 \amalg (z_1 \cdot z_2),\ \psi_5(\underline{z}) = (z_1 \cdot z_2) \amalg (z_1 \cdot z_3) \amalg (z_2 \cdot z_3),\ \psi_6(\underline{z}) = z_1 \cdot (z_2 \amalg z_3)$$

$$\psi_7(\underline{z}) = z_2 \cdot (z_1 \amalg z_3),\ \psi_8(\underline{z}) = z_3 \cdot (z_1 \amalg z_2),\ \psi_9(\underline{z}) = z_1 \cdot z_2 \cdot z_3$$

From Theorems 4.3, 4.6 there is just 9 different 3-component systems of the type suggested by Barlow and Wu (1978).

The structure functions above can be ordered in the following way

$$\psi_1 > \left\{ \begin{matrix} \psi_2 \\ \psi_3 \\ \psi_4 \end{matrix} \right\} > \psi_5 > \left\{ \begin{matrix} \psi_6 \\ \psi_7 \\ \psi_8 \end{matrix} \right\} > \psi_9$$

Among ψ_2, ψ_3, ψ_4 and among ψ_6, ψ_7, ψ_8 there is no ordering. This ordering divides the structure functions into 5 natural groups. Let now $(i = 1,\dots,5)$

a_i = the number of ways to choose i structure functions coming from different groups.

b_i = the number of ways to choose M elements from i groups such that all groups are represented.

By Theorems 4.2, 4.3 the number of different 3-component MCS's of type 2 equals $\sum_{i=1}^{5} a_i b_i$. It is not hard to see that

$$a_1 = \binom{9}{1} = 9$$

$$a_2 = \binom{3}{2} + \binom{2}{1}\binom{3}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{1} = 30$$

$$a_3 = \binom{3}{3} + \binom{3}{2}\binom{2}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{1}\binom{3}{1} = 46$$

$$a_4 = \binom{2}{1}\binom{3}{1} + \binom{3}{1}\binom{3}{1}\binom{3}{1} = 33$$

$$a_5 = \binom{3}{1}\binom{3}{1} = 9$$

If we can now show that

$$b_i = \binom{M-1}{i-1} \qquad i = 1,2,3,4,5 ,$$

our proof is completed. This, however, follows from Feller (1968) (p.38).

5. Stochastic performance of an MCS of type 2

In this section we will mainly demonstrate that all results obtained by Barlow and Wu (1978) extend to an MCS of type 2. We will, however, also try to indicate that most of the theory for a binary coherent system can be extended to an MCS of type 2.

We start by generalizing Theorem 2.1 of Barlow and Wu (1978).

Theorem 5.1. Consider an MCS of type 2 having binary coherent structure functions $\varphi_1,\ldots,\varphi_M$. Let h_j be the reliability of φ_j, i.e.

$$h_j = E\varphi_j(I_j(\underline{X})) \qquad j = 1,\ldots,M . \tag{5.1}$$

Then the performance distribution of the system is given by

$$p_0 = 1 - h_1$$

$$p_j = h_j - h_{j+1} \qquad j = 1,\ldots,M-1$$

$$p_M = h_M$$

$$\overline{P}(j-1) = h_j \qquad j = 1,\ldots,M$$

Furthermore, the performance function of the system, h, is given by

$$h = \sum_{j=1}^{M} h_j .$$

Proof. The results follow immediately from Theorem 4.3 and (3.8).

In Barlow and Wu (1978) $X_1,\ldots,X_n$ are assumed to be independent. Note that this is not assumed in the theorem above. Note also that in order to compute exact expressions for $p_0,\ldots,p_M$ and h , or to give upper and lower bounds for these quantities, we can just apply binary theory on $h_1,\dots,h_M$.

If the components are independent, or some simpler assumptions on the dependence are made, the binary version of Theorem 3.1 improved by Satyanarayana and Prabhakar (1978), can be applied to compute exact values of $h_1,\dots,h_M$. If not, upper and lower bounds for $h_1,\dots,h_M$ can be found from Natvig (1980) by choosing suitable modular decompositions for each φ_j , $j = 1,\dots,M$. Note especially that by assuming $X_1,\dots,X_n$ to be associated r.v.'s it follows by property P_3 of associated r.v.'s that $I_j(X_i)$ $i = 1,\dots,n$ are associated for fixed $j \in \{1,\dots,M\}$. This is often needed in Natvig (1980).

When components are repaired, the following definition seems natural.

Definition 5.2. Consider an MCS of type 2 having binary coherent structure functions $\varphi_1,\dots,\varphi_M$. Furthermore, consider a time interval $I = [t_A,t_B]$ and let $\tau(I) = \tau \cap I$, where generally the time $t \in \tau$. The availability, $h_{\varphi_j}^{(I)}$, and the unavailability, $g_{\varphi_j}^{(I)}$, to the level j in the time interval I for this system are given by $(j = 1,\dots,M)$:

$$h_{\varphi_j}^{(I)} = P[\varphi_j(I_j(\underline{X}(s))) = 1 \quad \forall s \in \tau(I)]$$

$$g_{\varphi_j}^{(I)} = P[\varphi_j(I_j(\underline{X}(s))) = 0 \quad \forall s \in \tau(I)]$$

By applying the theory of Natvig (1980) upper and lower bounds on $h_{\varphi_j}^{(I)}$ and $g_{\varphi_j}^{(I)}$ can be obtained in the case of maintained, interdependent components.

We now return to Barlow and Wu (1978). Their Proposition 2.2 is generalized by Theorem 4.2 of El-Neweihi, Proschan and Sethuraman (1978). The latter theorem is even valid for an MCS of type 1 as mentioned in Section 3. We next generalize Proposition 2.3 of the former paper. To do this we need some more notation.

If $X_1,\dots,X_n$ are independent, (5.1) is written in the form

(5.2) $\qquad h_j = h_j(\underline{\overline{P}}(j-1)) \qquad j = 1,\dots,M ,$

where

(5.3) $\quad \underline{\overline{P}}(j-1) = (\overline{P}_1(j-1),\ldots,\overline{P}_n(j-1))$.

$\overline{P}_i(j-1)$ is the <u>reliability</u> of the i th component to level j $(i = 1,\ldots,n;\ j = 1,\ldots,M)$. If especially $\overline{P}_i(j-1) = p_{0j}$ $i = 1,\ldots,n$, we write

(5.4) $\quad h_j = h_j(p_{0j}) \qquad j = 1,\ldots,M$.

<u>Theorem 5.2</u>. Consider an MCS of type 2 where $X_1,\ldots,X_n$ are independent. Let

$$(p_{i0},p_{i1},\ldots,p_{iM}) = (\alpha_0,\alpha_1,\ldots,\alpha_M) = \underline{\alpha} \qquad i = 1,\ldots,n .$$

Assume $h_j(p_{0j}) = p_{0j}$ for some $0 < p_{0j} < 1$, $j = 1,\ldots,M$. Then

(5.5) $$\sum_{r=j}^{M} \alpha_r \leq p_{0j} \quad j = 1,\ldots,M \Rightarrow h_j\Big(\sum_{r=j}^{M} \alpha_r\Big) \leq \sum_{r=j}^{M} \alpha_r \quad j = 1,\ldots,M$$

(5.6) $$\sum_{r=j}^{M} \alpha_r \geq p_{0j} \quad j = 1,\ldots,M \Rightarrow h_j\Big(\sum_{r=j}^{M} \alpha_r\Big) \geq \sum_{r=j}^{M} \alpha_r \quad j = 1,\ldots,M$$

<u>Proof</u>. The result follows immediately from Theorem 5.4 (the Moore-Shannon Theorem) (p.46) of Barlow and Proschan (1975).

The theorem allows us to compare the performance distribution of an arbitrary MCS of type 2 (with identical components) to the common performance distribution of its components. Note that if φ_j $(j = 1,\ldots,M)$ has no path sets or cut sets of size 1, then from the Moore-Shannon Theorem there exists $0 < p_{0j} < 1$ such that $h_j(p_{0j}) = p_{0j}$.

To generalize Proposition 2.4 of Barlow and Wu (1978) is straightforward and is left to the reader. We conclude this section by looking into their measure of component importance, which works equally well for an MCS of type 2.

Definition 5.2. Consider an MCS of type 2. Then component i is critical to the system at state j $(j = 0,\dots,M)$ iff

$$x_i = j \Leftrightarrow \varphi(\underline{x}) = j$$

The probability importance of component i with respect to system state j, I_{ij}, is defined by

$$I_{ij} = P[X_i = j \Leftrightarrow \varphi(\underline{X}) = j]$$

Theorem 5.3. Consider an MCS of type 2 where $X_1,\dots,X_n$ are independent. Then $(i = 1,\dots,n)$

$$(5.7) \qquad I_{i0} = h_1(1_i,\underline{\overline{P}}(0)) - h_1(0_i,\underline{\overline{P}}(0))$$

$$(5.8) \qquad I_{i1} = h_{j+1}(1_i,\underline{\overline{P}}(j)) - E[\varphi_j(0_i,\underline{I}_j(\underline{X}))\cdot\varphi_{j+1}(1_i,\underline{I}_{j+1}(\underline{X}))]$$
$$\leq h_{j+1}(1_i,\underline{\overline{P}}(j)) - h_{j+1}(0_i,\underline{\overline{P}}(j)) \qquad j = 1,\dots,M-1$$

$$(5.9) \qquad I_{iM} = h_M(1_i,\underline{\overline{P}}(M-1)) - h_M(0_i,\underline{\overline{P}}(M-1))$$

Proof. For $j = 1,\dots,M-1$ following the proof of Theorem 2.6 of Barlow and Wu (1978) we get:

$$\begin{aligned}
I_{ij} &= E[\varphi_j(1_i,\underline{I}_j(\underline{X}))\varphi_{j+1}(1_i,\underline{I}_{j+1}(\underline{X}))] \\
&- E[\varphi_j(1_i,\underline{I}_j(\underline{X}))\varphi_{j+1}(0_i,\underline{I}_{j+1}(\underline{X}))] \\
&- E[\varphi_j(0_i,\underline{I}_j(\underline{X}))\varphi_{j+1}(1_i,\underline{I}_{j+1}(\underline{X}))] \\
&+ E[\varphi_j(0_i,\underline{I}_j(\underline{X}))\varphi_{j+1}(0_i,\underline{I}_{j+1}(\underline{X}))] \\
&= E[\varphi_{j+1}(1_i,\underline{I}_{j+1}(\underline{X}))] - E[\varphi_{j+1}(0_i,\underline{I}_{j+1}(\underline{X}))] \\
&- E[\varphi_j(0_i,\underline{I}_j(\underline{X}))\varphi_{j+1}(1_i,\underline{I}_{j+1}(\underline{X}))] \\
&+ E[\varphi_{j+1}(0_i,\underline{I}_{j+1}(\underline{X}))] ,
\end{aligned}$$

having applied Theorem 4.2. Hence (5.8) follows. (5.7) and (5.9) are proved as in the mentioned Theorem 2.6.

We do not feel sure that Definition 5.2 is the most reasonable one. Let $\{0,1,\dots,j-1\}$ correspond to the failure state when a binary approach is applied. Then the following definition reduces to the binary one given by Birnbaum (1969).

Definition 5.4. Consider an MCS of type 2. Then component i is critical to the system to level j $(j = 1,\dots,M)$ iff

$$x_i \geqq j \Leftrightarrow \varphi(\underline{x}) \geqq j$$

The probability importance of component i to level j, I'_{ij}, is defined by

$$I'_{ij} = P[X_i \geqq j \Leftrightarrow \varphi(\underline{X}) \geqq j]$$

Theorem 5.5. Consider an MCS of type 2 where $X_1,\dots,X_n$ are independent. Then $(i = 1,\dots,n;\ j = 1,\dots,M)$

$$I'_{ij} = h_j(1_i,\underline{\overline{P}}(j-1)) - h_j(0_i,\underline{\overline{P}}(j-1)) .$$

Proof. The proof is identical to the one of (5.7) and (5.9).

6. Some final comments

When presenting a preliminary draft of the present paper at the "Reliability Days" at Chalmers Institute of Technology, Gothenburg, May 5.-6. 1980, it was objected against the assumption that both for each component and for the system itself the set of states is $S = \{0,\dots,M\}$. Some felt that the set of states for each component should be included in S. In particular a binary state space would be sufficient for several components.

To take this objection into account assume the set of states of the i th component to be S_i $(i = 1,\dots,n)$ where

$$\{0,M\} \subseteq S_i \subseteq S$$

Remembering Theorem 2.1, we give the following definition of a modified MCS of type 1 :

Definition 6.1. A system of n components is said to be a modified MCS of type 1 iff its structure function satisfies:

i) $\varphi(\underline{x})$ is nondecreasing in each argument

ii) $\forall i, \forall j \in \{1,\ldots,M\}, \exists(\cdot_i,\underline{x}) \ni$

$\varphi(k_i,\underline{x}) \geq j \quad \forall k \in S_i \cap \{j,j+1,\ldots,M\}$

$\varphi(k_i,\underline{x}) \leq j-1 \quad \forall k \in S_i \cap \{0,1,\ldots,j-1\}$

iii) $\min\limits_{1\leq i\leq n} x_i \leq \varphi(\underline{x}) \leq \max\limits_{1\leq i\leq n} x_i$

It can now be shown that almost all results given in Section 2 and 3 for an MCS of type 1 are also valid for a modified MCS of type 1. In fact the only results we are not able to generalize are that equality for all $\underline{x}$ and $\underline{y}$ in i) (ii)) of Theorem 2.5 implies the structure function to be parallel (series) and the corresponding part of Theorem 3.3.

It should be noted that the definition of an MCS of type 2 works equally well in the more general situation regarded here.

Concerning future research treating the multinary case we hope to return to measures of component importance in a later paper. For the time being strong efforts are made at this Department, mainly by Terje Aven, to build up a theory for optimal preventive maintenance. Two of the interesting questions are: To what levels should we allow the components to deteriorate before acting, and when acting, to what levels should the components be repaired ?

Acknowledgement

We are thankful to the Johan and Mimi Wesmann Foundation for supporting this research and to Terje Aven for helpful discussions.

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