MULTISTATE RELIABILITY THEORY - A CASE STUDY

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1. Introduction to multistate reliability theory

Fortunately the traditional reliability theory, where the system and the components are always described simply as functioning or failed, is now being replaced by a theory for multistate systems of multistate components. Recent reviews of this development are given in Natvig (1985 a,b). However, there is a need for several convincing case studies demonstrating the practicability of the generalizations introduced. One such study could be of an offshore pipeline system where one lets the system state be the amount of oil running through a crucial point.

In this paper we will study an electrical power generation system for two nearby oilrigs. The amount of power, that may possibly be supplied to the two oilrigs, are considered as system states. Before proceeding to this study we give a short introduction to some main concepts in multistate reliability theory.

Let $S = \{0,1,\ldots,M\}$ be the set of states of the system; the $M+1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Let furthermore, $C = \{1,\ldots,n\}$ be the set of components and

$S_i \{i=1,\ldots,n)$ the set of states of the ith component. We claim $\{0,M\} \subseteq S_i \subseteq S$. Hence the states 0 and M are chosen to represent the endpoints of a performance scale that might be used for both the system and its components.

Let $x_i (i=1,\ldots,n)$ denote the state or performance level of the ith component and $\underline{x} = (x_1,\ldots,x_n)$. It is assumed that the state, ϕ, of the system is given by the structure function $\phi = \phi(\underline{x})$. In this paper we consider the following type of multistate systems for which a series of results can be derived:

Definition 1.1.

A system is a multistate monotone system (MMS) iff its structure ϕ satisfies

i) $\phi(\underline{x})$ is nondecreasing in each argument

ii) $\phi(\underline{0}) = 0$ and $\phi(\underline{M})=M$ $(\underline{0}=(0,\ldots,0)$, $\underline{M} = (M,\ldots,M))$.

The first assumption roughly says that improving one of the components cannot harm the system, whereas the second says that if all components are in the complete failure (perfect functioning) state, then the system is in the complete failure (perfect functioning) state.

In the following $\underline{y} < \underline{x}$ means $y_i \leq x_i$ for $i=1,\ldots,n$, and $y_i < x_i$ for some i.

Definition 1.2.

Let ϕ be the structure function of an MMS and let $j \in \{1,\ldots,M\}$. A vector $\underline{x}$ is said to be a minimal path (cut) vector to level j iff $\phi(\underline{x}) \geq j$ and $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$ $(\phi(\underline{x}) < j$ and $\phi(\underline{y}) \geq j$ for all $\underline{y} > \underline{x})$.

Definition 1.3.

The *performance process of the ith component* $(i=1,\ldots,n)$ is a stochastic process $\{X_i(t),\ t\in[0,\infty)\}$, where for each fixed $t\in[0,\infty), X_i(t)$ is a random variable (r.v.) which takes values in S_i. The *joint performance process for the components* $\{\underline{X}(t),\ t\in[0,\infty)\} = \{(X_1(t),\ldots,X_n(t)),\ t\in[0,\infty)\}$ is the corresponding vector stochastic process. The *performance process of an MMS* with structure function ϕ is a stochastic process $\{\phi(\underline{X}(t)),\ t\in[0,\infty)\}$, where for each fixed $t\in[0,\infty)$, $\phi(\underline{X}(t))$ is a r.v. which takes values in S.

Definition 1.4.

Let $j \in \{1,\ldots,M\}$. The *availability*, $h_\phi^{j(I)}$, and the *unavailability*, $g_\phi^{j(I)}$, *to level j in the time interval I for an MMS* with structure function ϕ are given by

$$h_\phi^{j(I)} = P[\phi(\underline{X}(s)) \geq j\ \forall s\in I],\quad g_\phi^{j(I)} = P[\phi(\underline{X}(s))<j\ \forall s\in I].$$

Note that $h_\phi^{j(I)} + g_\phi^{j(I)} \leq 1$, with equality for the case $I = [t,t]$. In Funnemark and Natvig (1985) bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$ are arrived at, based on corresponding information on the multistate components, generalizing earlier work by the first present author for the case M=1. The components are assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system noting that exact expressions are obtainable just for trivial systems. It is the aim of this paper to give such bounds for our power generation system.

2. An offshore electrical power generation system

In Figure 1 an outline of an offshore electrical power generation system is given.

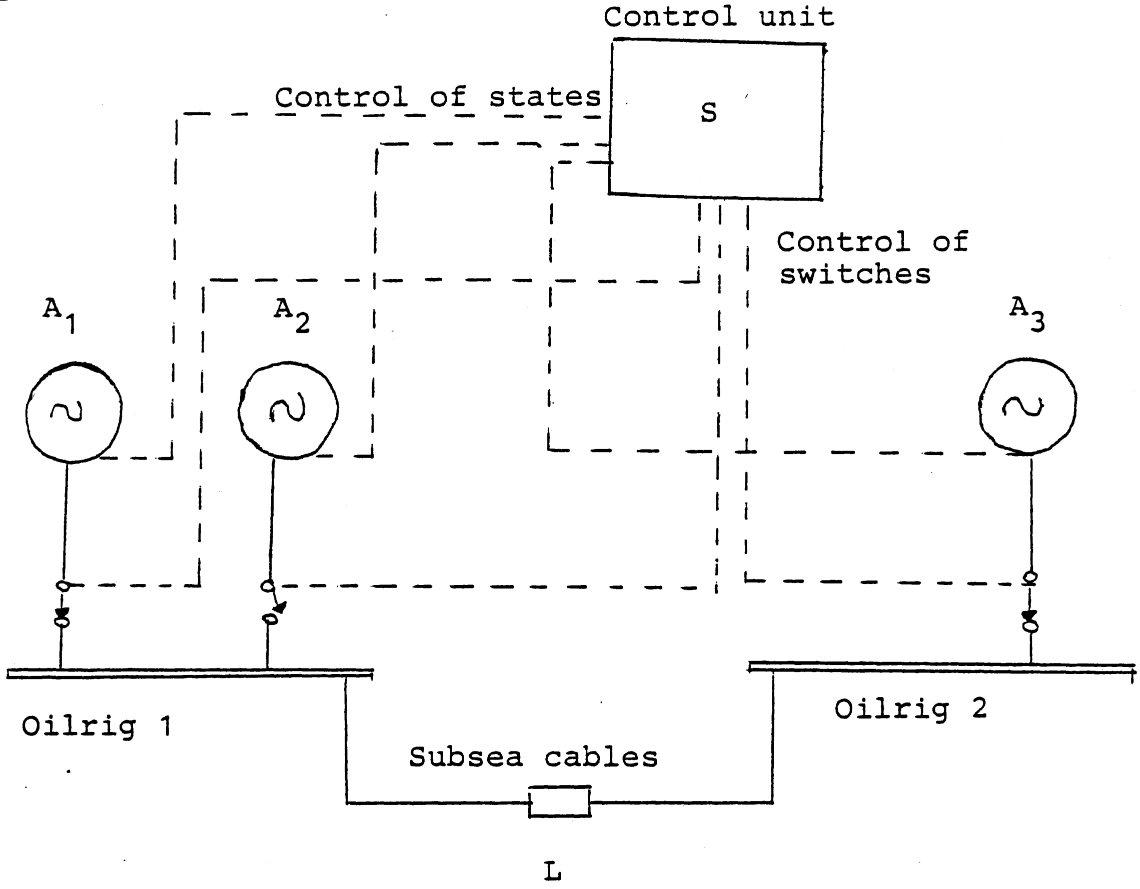

Figure 1. Outline of an offshore electrical power generation system

The purpose of this system is to supply two nearby oilrigs with electrical power. Both oilrigs have their own main generation, represented by equivalent generators A_1 and A_3, each having a capacity of 50MW. In addition oilrig 1 has a standby generator A_2, that is switched on the network in case of outage of A_1 or A_3 or may be used in extreme load situations in either of the two oilrigs. The latter situation is for simplicity not treated here. A_2 is in cold standby which means that a short startup time is needed before switched on the network. This time is neglected in the following model. Also A_2 has a capacity of 50MW. The control unit, S, is continously supervising the supply from each of the generators with automatic control of the switches. If for instance the supply from A_3 to oilrig 2 is not sufficient, whereas the supply from A_1 to oilrig 1 is sufficient, S can activate A_2 to supply oilrig 2 with electrical power through the subsea cables L.

The components to be considered in the following are A_1, A_2, A_3, S and L. We will let the perfect functioning level M equal 4 and let the set of states of all components be $\{0,2,4\}$. For A_1, A_2 and A_3 these states are interpreted as

0: The generator cannot supply any power

2: The generator can supply maximum 25MW

4: The generator can supply maximum 50MW

Note that as an approximation we have for these generators chosen to describe their supply capacity on a discrete scale of three points. The supply capacity is not a measure of the actual amount of power delivered at a fixed point of time. There is a continuous power-frequency control to match the generation to actual load, keeping electrical frequency within prescribed limits.

The control unit S has the states

0: S will by mistake switch the main generators A_1 and A_3 off without switching A_2 on

2: S will not switch A_2 on when needed

4: S is functioning perfectly.

The subsea cables L are actually assumed to be constructed as a double cable transferring one half of the power through each simple cable. This leads to the following states of L

0: No power is transferred

2: 50% of the power is transferred

4: 100% of the power is transferred

Let us now for simplicity assume that the mechanism that distributes the power from A_2 to platform 1 or 2 is working pefectly. Furthermore, as a start, assume that this mechanism is a simple one either transferring no power from A_2 to platform 2, if A_2 is needed at platform 1, or transforming all power from A_2 needed at platform 2.

Let now

$\phi_1(S,A_1,A_2)$= The amount of power that can be supplied to platform 1

$\phi_2(S,A_1,L,A_2,A_3)$ = The amount of power that can be supplied to platform 2

ϕ_1 will now just take the same states as the generators whereas ϕ_2 in addition can take the following states.

1: The amount of power that can be supplied is maximum 12.5 MW

3: The amount of power that can be supplied is maximum 37.5 MW

Let for an arbitrary event E the indicator function $I(E)$ be given by

$$I(E) = \begin{cases} 1 & \text{if } E \text{ is occuring} \\ 0 & \text{if } E \text{ is not occuring} \end{cases}$$

Then it is not too hard to be convinced that ϕ_1 and ϕ_2 are given by respectively

$$\phi_1(S,A_1,A_2) = I(S>0)\min(A_1+A_2 I(S=4),4)$$

$$\phi_2(S,A_1,L,A_2,A_3) = I(S>0)\min(A_3+A_2 I(S=4)I(A_1=4)L/4,4).$$

Hence it is obvious that both are structure functions of an MMS.

Let us still assume that the mechanism that distributes the power from A_2 to platform 1 or 2 is working perfectly. However, let it now be more advanced transferring excess power from A_2 to platform 2 if platform 1 is ensured a delivery corresponding to state 4. Of course in a more refined model this mechanism should be treated as a component. The structure functions are now given by

$$\phi_1^*(S,A_1,A_2) = \phi_1(S,A_1,A_2)$$

$$\phi_2^*(S,A_1,L,A_2,A_3) = I(S>0)\min(A_3+\max(A_1+A_2I(S=4)-4,0)L/4,4),$$

noting that $\max(A_1+A_2I(S=4)-4,0)$ is just the excess power from A_2 which one tries to transfer to platform 2.

Note that the above analysis can easily be more refined. Let $M=4n^2$ and with obvious interpretations let the states of A_1,A_2,A_3 and L be $\{0,2n,4n,\ldots,4n^2\}$ and the ones of S be $\{0,2n^2,4n^2\}$. Then

$$\phi_1(S,A_1,A_2) = \phi_1^*(S,A_1,A_2) = I(S>0)\min(A_1+A_2I(S=M),M)$$

with states $\{0,2n,4n,\ldots,4n^2\}$.

Furthermore,

$$\phi_2(S,A_1,L,A_2,A_3) = I(S>0)\min(A_3+A_2I(S=M)I(A_1=M)L/M,M)$$

$$\phi_2^*(S,A_1,L,A_2,A_3) = I(S>0)\min(A_3+\max(A_1+A_2I(S=M)-M,0)L/M,M)$$

with states $\{0,1,2,\ldots,4n^2\}$. Note that the structure functions are still of the MMS type.

Returning to the case where $M=4$ we list in Tables 1,2,3,4 the minimal path and cut vectors to the various levels of ϕ_1 and ϕ_2. Note that the same vector may be a minimal path vector to more than one level. The same is true for a minimal cut vector.

Levels	S	A_1	A_2
2	2	2	0
2	4	0	2
4	2	4	0
4	4	0	4
4	4	2	2

Table 1. Minimal path vectors of ϕ_1.

Levels	S	A_1	A_2
2	4	0	0
2	2	0	4
2,4	0	4	4
4	4	2	0
4	2	2	4
4	4	0	2

Table 2. Minimal cut vectors of ϕ_1.

Levels	S	A_1	L	A_2	A_3
1	4	4	2	2	0
1,2	2	0	0	0	2
2	4	4	2	4	0
2	4	4	4	2	0
3	4	4	2	2	2
3,4	2	0	0	0	4
4	4	4	2	4	2
4	4	4	4	2	2
3,4	4	4	4	4	0

Table 3. Minimal path vectors of ϕ_2.

Levels	S	A_1	L	A_2	A_3
1,2	4	4	4	0	0
1,2,3,4	0	4	4	4	4
1,2	4	4	0	4	0
1,2	4	2	4	4	0
1,2	2	4	4	4	0
2	4	4	2	2	0
3,4	4	4	4	0	2
3,4	4	4	0	4	2
3,4	4	2	4	4	2
3,4	2	4	4	4	2
3,4	4	4	2	4	0
3,4	4	4	4	2	0
4	4	4	2	2	2

Table 4. Minimal cut vectors of ϕ_2.

As examples of how to arrive at these tables note that

$$\phi_2(4,4,4,4,0) = 4, \text{ whereas}$$
$$\phi_2(2,4,4,4,0) = \phi_2(4,2,4,4,0) = 0$$
$$\phi_2(4,4,2,4,0) = \phi_2(4,4,4,2,0) = 2$$

Hence $(4,4,4,4,0)$ is a minimal path vector both to level 3 and 4. Similarly

$$\phi_2(4,4,4,2,0) = 2, \text{ whereas}$$
$$\phi_2(4,4,4,4,0) = \phi_2(4,4,4,2,2) = 4 .$$

Hence $(4,4,4,2,0)$ is a minimal cut vector both to level 3 and 4.

3. Availabilities and unavailabilities of the components

As mentioned at the end of Section 1 bounds for the availabilities and unavailabilities in a fixed time interval for an MMS are based on corresponding information on the multistate components. Hence, with obvious definitions, denote the availability and unavailability to level j in the time interval I for the ith component of an MMS by $p_i^{j(I)}$ and $q_i^{j(I)}$ respectively, $i=1,\ldots,n$; $j\in\{1,\ldots,M\}$. In this section we will establish these quantities for the components A_1,A_2,A_3,S and L of the preceding section.

Assume that the performance processes of the components are Markovian and introduce in the general case

$$S_{i\,j}^0 = S_i \cap \{0,1,\ldots,j-1\}$$

$$S_{i\,j}^1 = S_i \cap \{j,\ldots,M\}$$

$$p_i^{(k,\ell)}(t_1,t_2) = P(X_i(t_2)=\ell \mid X_i(t_1)=k).$$

Furthermore, denote the corresponding transition probabilities when E is a set of absorbing states by $p_i^{(k,\ell)E}(t_1,t_2)$. Finally assume that at time $t=0$ all components are in the perfect functioning state M; i.e. $\underline{X}(0) = \underline{M}$. Then for $I=[t_1,t_2]$

$$p_i^{j(I)} = \sum_{k\in S_{i\,j}^1} p_i^{(M,k)}(0,t_1)\Big[1-\sum_{\ell\in S_{i\,j}^0} p_i^{(k,\ell)S_{ij}^0}(t_1,t_2)\Big] \tag{3.1}$$

$$q_i^{j(I)} = \sum_{k\in S_{i\,j}^0} p_i^{(M,k)}(0,t_1)\Big[1-\sum_{\ell\in S_{i\,j}^1} p_i^{(k,\ell)S_{ij}^1}(t_1,t_2)\Big] \tag{3.2}$$

Note that we get $q_i^{j(I)}$ from $p_i^{j(I)}$ by replacing S_{ij}^1 by the "dual" set S_{ij}^0.

Let now

$$\mu_i^{(k,\ell)}(s) = \lim_{h\to 0} p_i^{(k,\ell)}(s,s+h)/h \qquad k\neq\ell,$$

be the transition intensities of $\{X_i(t), t\in[0,\infty)\}$. For simplicity we will assume that the performance processes of the components are time-homogeneous; i.e

$$p_i^{(k,\ell)}(t_1,t_2) = p_i^{(k,\ell)}(t_2-t_1)$$

$$\mu_i^{(k,\ell)}(s) = \mu_i^{(k,\ell)} \quad \text{for all} \quad s\in[0,\infty),\ k\neq\ell.$$

Hence, all what is needed to arrive at expressions for $p_i^{j(I)}$ and $q_i^{j(I)}$, and hence bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$, are these time independent transition intensities.

Returning to the components of the preceding section, with set of states $\{0,2,4\}$, introduce the matrices

$$\underline{P}_i(t) = \{p_i^{(k,\ell)}(t)\}_{\substack{k=0,2,4\\ \ell=0,2,4}} .$$

Furthermore, assume that $\mu_i^{(0,2)} = 0$, i.e. we will always repair a completely failed component to the perfect functioning level. Finally, assume that the performance processes of the components are conservative, implying that the corresponding intensity matrices are given by:

$$\underline{A}_i = \begin{bmatrix} -\mu_{04} & 0 & \mu_{04} \\ \mu_{20} & -(\mu_{20}+\mu_{24}) & \mu_{24} \\ \mu_{40} & \mu_{42} & -(\mu_{40}+\mu_{42}) \end{bmatrix}$$

By applying standard theory for finite state continuous time Markov processes, see Karlin and Taylor (1975), we have

$$\underline{P}_i(t) = e^{\underline{A}_i t} = \underline{I} + \sum_{n=1}^{\infty} \frac{\underline{A}_i^n t^n}{n!}$$

where $\underline{I}$ is the identity matrix and the initial condition is $\underline{P}_i(0) = \underline{I}$. Now by introducing the Lagrange interpolation coefficients (k=1,2,3)

$$\underline{L}_k(\underline{A}_i) = \prod_{\substack{j=1 \\ j \neq k}}^{3} \frac{\underline{A}_i - \gamma_j \underline{I}}{\gamma_k - \gamma_j} ,$$

where $\gamma_1, \gamma_2, \gamma_3$ are the eigenvalues of $\underline{A}_i$, we get from Apostol (1969)

$$\underline{P}_i(t) = \sum_{k=1}^{3} e^{\gamma_k t} \underline{L}_k(\underline{A}_i).$$

By solving the equation

$$\det|\underline{A}_i - \gamma \underline{I}| = 0,$$

we find

$$\gamma_1 = 0, \quad \gamma_2 = -(B+\sqrt{B^2-4C})/2, \quad \gamma_3 = -(B-\sqrt{B^2-4C})/2,$$

where

$$B = \mu_{40} + \mu_{42} + \mu_{24} + \mu_{04} + \mu_{20}$$

$$C = \mu_{40}\mu_{20} + \mu_{24}\mu_{40} + \mu_{20}\mu_{42} + \mu_{04}\mu_{42} + \mu_{04}\mu_{20} + \mu_{04}\mu_{24}$$

Hence, the choice of just three possible states of the components has the advantage of leading to a second order equation for these eigenvalues. Straightforward algebra now gives:

$$p_i^{(4,0)}(t) = [\mu_{40}(\mu_{24}+\mu_{20}) + \mu_{20}\mu_{42}]/\gamma_2\gamma_3$$

$$+ [(\mu_{20}\mu_{42} - \mu_{40}(\mu_{04}+\mu_{40}+\mu_{42}) - \gamma_3\mu_{40})/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+ [(\mu_{20}\mu_{42} - \mu_{40}(\mu_{04}+\mu_{40}+\mu_{42}) - \gamma_2\mu_{40})/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

$$p_i^{(4,2)}(t) = \mu_{42}\mu_{04}/\gamma_2\gamma_3 + [\mu_{42}(\mu_{04}+\gamma_2)/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+ [(\mu_{42}(\mu_{04}+\gamma_3)/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

$$p_i^{(4,4)}(t) = (\mu_{20}+\mu_{24})\mu_{04}\ /\gamma_2\gamma_3$$

$$+[(\mu_{04}\mu_{40}+\mu_{24}\mu_{42}+(\mu_{40}+\mu_{42})^2+\gamma_3(\mu_{40}+\mu_{42}))/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+[(\mu_{04}\mu_{42}+\mu_{24}\mu_{42}+(\mu_{40}+\mu_{42})^2+\gamma_2(\mu_{40}+\mu_{42})/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

$$p_i^{(2,0)}(t) = (\mu_{40}(\mu_{24}+\mu_{20})+\mu_{20}\mu_{42})/\gamma_2\gamma_3$$

$$+[(\mu_{24}\mu_{40}-\mu_{20}(\mu_{04}+\mu_{20}+\mu_{24})-\gamma_3\mu_{20})/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+[(\mu_{24}\mu_{40}-\mu_{20}(\mu_{04}+\mu_{20}+\mu_{24})-\gamma_2\mu_{20})/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

$$p_i^{(2,4)}(t) = (\mu_{20}+\mu_{24})\mu_{04}/\gamma_2\gamma_3$$

$$+[(\mu_{20}\mu_{04}-\mu_{24}(\mu_{20}+\mu_{24}+\mu_{40}+\mu_{42})-\gamma_3\mu_{24})/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+[(\mu_{20}\mu_{04}-\mu_{24}(\mu_{20}+\mu_{24}+\mu_{40}+\mu_{42})-\gamma_2\mu_{24})/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

$$p_i^{(0,4)}(t) = (\mu_{20}+\mu_{24})\mu_{04}/\gamma_2\gamma_3$$

$$+[(\mu_{04}(\mu_{20}+\mu_{24}+\gamma_2)/\gamma_2(\gamma_2-\gamma_3)]\exp(\gamma_2 t)$$

$$+[(\mu_{04}(\mu_{20}+\mu_{24}+\gamma_3)/\gamma_3(\gamma_3-\gamma_2)]\exp(\gamma_3 t)$$

By specializing $\mu_i^{(0,4)}$ in $p_i^{(4,0)}(t)$ and $p_i^{(2,0)}(t)$, we get $p_i^{(4,0)\{0\}}(t)$ and $p_i^{(2,0)\{0\}}(t)$. Similarly by specializing $\mu_i^{(4,2)} = \mu_i^{(4,0)} = 0$ in $p_i^{(2,4)}(t)$ and $p_i^{(0,4)}(t)$, we get $p_i^{(2,4)\{4\}}(t)$ and $p_i^{(0,4)\{4\}}(t)$.

From (3.1) and (3.2) we see that we have calculated all that is needed:

$$p_i^{4(I)} = p_i^{(4,4)}(t_1)\exp(-(\mu_{40}+\mu_{42})(t_2-t_1))$$

$$p_i^{2(I)} = p_i^{(4,4)}(t_1)[1-p_i^{(4,0)\{0\}}(t_2-t_1)]$$

$$+ p_i^{(4,2)}(t_1)[1-p_i^{(2,0)\{0\}}(t_2-t_1)]$$

$$q_i^{4(I)} = p_i^{(4,2)}(t_1)[1-p_i^{(2,4)\{4\}}(t_2-t_1)]$$

$$+ p_i^{(4,0)}(t_1)[1-p_i^{(0,4)\{4\}}(t_2-t_1)]$$

$$q_i^{2(I)} = p_i^{(4,0)}(t_1)\exp(-\mu_{04}(t_2-t_1))$$

We conclude this section by giving some numerical values for the availabilities and unavailabilities for the components A_1, A_2, A_3, S and L based on "questimates" of the transition intensities. More and better data is needed to get better values. A_1, A_2 and A_3 is assumed to be of the same type. The time unit is year.

Quantity	A_1, A_2, A_2	S	L
μ_{40}	1.46	2	0.04
μ_{42}	27.74	12	9.09
μ_{20}	1.46	5	0.2
μ_{04}	730	4380	10.95
μ_{24}	17520	17520	21.9
$p_i^{4(3,4)}$	0.000	0.000	0.000
$p_i^{4(0.1,0.11)}$	0.744	0.868	0.655
$p_i^{2(3,4)}$	0.232	0.135	0.910
$p_i^{2(0.1,0.11)}$	0.984	0.980	0.995
$q_i^{4(3,4)}$	0.000	0.000	0.000
$q_i^{4(0.1,0.11)}$	0.000	0.000	0.227
$q_i^{2(3.4)}$	0.000	0.000	0.000
$q_i^{2(0.1,0.11)}$	0.000	0.000	0.004

Table 5. Availabilities and unavailabilities for A_1, A_2, A_3, S and L.

4. Bounds for the availabilities and unavailabilities for the electrical power generation system.

We are forced to start by returning to multistate reliability theory.

Definition 4.1

The marginal performance processes $\{X_i(t),\ t\in[0,\infty)\}$, $i=1,\ldots,n$ are independent in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\}\subset I$ the random vectors $(X_1(t_1),\ldots,X_1(t_m)),\ldots,(X_n(t_1),\ldots,X_n(t_m))$ are independent. The marginal performance process $\{X_i(t),\ t\in[0,\infty)\}$ is associated in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\}\subset I$, the r.v.'s $X_i(t_1),\ \ldots,X_i(t_m)$ are associated.

For the definition and properties of associated r.v.'s see Barlow and Proschan (1975). As an example of the bounds in Funnemark and Natvig (1985) for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$ we give the following theorem by first introducing the $n\times M$ matrices

$$\underline{P}_\phi^{(I)} = \{p_i^{j(I)}\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \qquad \underline{Q}_\phi^{(I)} = \{q_i^{j(I)}\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}}$$

Theorem 4.2

Let (C,ϕ) be an MMS with the marginal performance processes of its components being independent and each of them associated in I. Furthermore for $j\in\{1,\ldots,M\}$ let $\underline{y}_k^j=(y_{1k}^j,\ldots,y_{nk}^j)$, $k=1,\ldots,n^j$ $(\underline{z}_k^j=(z_{1k}^j,\ldots,z_{nk}^j),\ k=1,\ldots,m^j)$ be its minimal path (cut) vectors to level j. Define

$$\ell_\phi^{j'}(\underline{P}_\phi^{(I)}) = \max_{1 \leq k \leq n^j} \prod_{i=1}^{n} p_i^{y_{ik}^j(I)} \qquad \bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}) = \max_{1 \leq k \leq m^j} \prod_{i=1}^{n} q_i^{z_{ik}^j+1(I)}$$

$$\ell_\phi^{j^*}(\underline{P}_\phi^{(I)}) = \prod_{k=1}^{m^j} \coprod_{i=1}^{n} p_i^{z_{ik}^j+1(I)} \qquad \bar{\ell}_\phi^{j^*}(\underline{Q}_\phi^{(I)}) = \prod_{k=1}^{n^j} \coprod_{i=1}^{n} q_i^{y_{ik}^j(I)}$$

$$B_\phi^j(\underline{P}_\phi^{(I)}) = \max_{j \leq k \leq M} \{\max[\ell_\phi^{j'}(\underline{P}_\phi^{(I)}), \ell_\phi^{j^*}(\underline{P}_\phi^{(I)})]\}$$

$$\bar{B}_\phi^j(\underline{Q}_\phi^{(I)}) = \max_{1 \leq k \leq j} \{\max[\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}), \bar{\ell}_\phi^{j^*}(\underline{Q}_\phi^{(I)})]\}$$

Then

$$B_\phi^j(\underline{P}_\phi^{(I)}) \leq h_\phi^{j(I)} \leq 1 - \bar{B}_\phi^j(\underline{Q}_\phi^{(I)})$$

$$\bar{B}_\phi^j(\underline{Q}_\phi^{(I)}) \leq g_\phi^{j(I)} \leq 1 - B_\phi^j(\underline{P}_\phi^{(I)})$$

Here $\coprod_{i=1}^{n} a_i \overset{\text{def}}{=} 1 - \prod_{i=1}^{n} (1-a_i)$. By specializing $M=1$ and $I = [t,t]$ the bounds reduce to the familiar ones from binary theory as given in Barlow and Proschan (1975).

To apply the theorem one has to check that the marginal performance process of each component is associated in I. When these processes are Markovian, a convenient sufficient condition for this to hold, in terms of the transition intensities, is given in Hjort, Natvig and Funnemark (1985). For the set of states of our components this condition reduces to

$$\mu_{04} \leq \mu_{24} \quad \text{and} \quad \mu_{40} \leq \mu_{20},$$

which is satisfied by the transition intensities of Table 5.

By assuming the marginal performance processes of A_1, A_2, A_3, S and L to be independent in $[0,\infty)$, using the minimal path and cut vectors of ϕ_1 and ϕ_2 in Tables 1-4 and the availabilities and unavailabilities of the components in Table 5, we arrive at the following bounds for the availabilities and unavailabilities linked to ϕ_1 and ϕ_2:

k	j	I	Bounds for $h^{j(I)}_{\phi_k}$		Bounds for $g^{j(I)}_{\phi_k}$	
			lower	upper	lower	upper
1	4	[3,4]	0.0000	1.0000	0.0000	1.0000
	4	[0.1,0.11]	0.9388	1.0000	0.0000	0.0612
	2	[3,4]	0.0313	1.0000	0.0000	0.9687
	2	[0.1,0.11]	0.9773	1.0000	0.0000	0.0227
2	4	[3,4]	0.0000	1.0000	0.0000	1.0000
	4	[0.1,0.11]	0.8515	1.0000	0.0000	0.1485
	3	[3,4]	0.0000	1.0000	0.0000	1.0000
	3	[0.1,0.11]	0.8711	1.0000	0.0000	0.1289
	2	[3,4]	0.0313	1.0000	0.0000	0.9687
	2	[0.1,0.11]	0.9717	1.0000	0.0000	0.0283
	1	[3,4]	0.0313	1.0000	0.0000	0.9687
	1	[0.1,0.11]	0.9731	1.0000	0.0000	0.0269

Table 6. Bounds for $h^{j(I)}_{\phi_1}$, $g^{j(I)}_{\phi_1}$, $h^{j(I)}_{\phi_2}$, $g^{j(I)}_{\phi_2}$

We see that the bounds are very informative for I=[0.1,0.11] corresponding to an interval of 36 days. However, for I=[3,4], corresponding to an interval of a whole year, bounds are giving close to nothing. To handle this case study and more sophisticated ones, involving for instance modular decompositions, several computer programs are developed by Sørmo (1985). Some improvements are necessary and will hopefully be carried through in the near future.

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