Testing rational expectations in vector autoregressive models. *

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Abstract

Assuming that the solutions of a set of restrictions on the rational expectations of future values can be represented as a vector autoregressive model, we study the implied restrictions on the coefficients. Nonstationary behavior of the variables is allowed, and the restrictions on the cointegration relationships are spelled out. In some interesting special cases it is shown that the likelihood ratio statistic can easily be computed.

Keywords: VAR-models, cointegration, rational expectations.

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1 Introduction

Expectations play a central role in many economic theories. But the incorporation of this kind of variables in empirical models rises many problems. The variables are in many cases unobserved either because data on expectations are unavailable, or because there may often be reason to suspect that the available data on expectations are unreliable. There are also problems connected with the validity. Economic agents may benefit from not revealing their real expectations. Some sort of proxies must therefore be used.

One possibility when the models contain stochastic elements, is to use conditional expectations in the probabilistic sense given some previous information. When this information is all available past and present information contained in the variables of the model, rational expectation is the usual denomination. Another, perhaps more precise, name is model consistent expectations. Then the aspect that the expectations mean conditional expectations in the model the analysis is based upon, is emphasized. This is an idea originally introduced by Muth [12] and [13]. However, since rational expectation seems to be the common name of this type of expectations, we shall stick to this usage in the following.

It is well known that dynamic models containing rational expectations of future values have a multitude of solutions. In a recent paper Baillie [2] advocated a procedure for testing restrictions between future rational expectations of a set of variables by assuming that the solutions could be described by a vector autoregressive (VAR) model. He then expressed the restrictions implied by the postulated relationships between the expectations as restrictions on the coefficients of the VAR model.

In this paper we shall follow the same approach. However, Baillie also allowed for non-stationary behavior of the variables that could be eliminated by first transforming the variables using known cointegrating relationships. Thus some knowledge about how the variables cointegrate is necessary. At this point we shall pursue another line. Starting out with the VAR model we only assume that the variables are integrated of order one. It turns out, as one can expect, that the restrictions on the expectations entail restrictions on the cointegration relationships. In addition some restrictions on the short run part of the model must be satisfied.

These implications can be tested by invoking the results of Johansen [8] and [9] and of Johansen and Juselius [10] and [11]. In general it seems that a two step procedure must be used, but in an interesting special case it is possible to find the likelihood ratio test. What is also of interest, is that this test is easy to compute involving by now well known reduced rank regression procedures.

The paper is organized as follows: In the next section we state the type of relationships between the expectations we shall consider, and derive the implications for the VAR model when the expectations are considered to be rational in the sense described earlier. In section 3 we treat the special case where a likelihood ratio test can be developed. Finally, assuming that the variables are integrated of order 1 we discuss the asymptotic distribution of the tests.

2 The form of the restrictions.

We assume that the $p \times 1$ vectors of observations are generated according to the vector autoregressive (VAR) model

$$X_t = A_1 X_{t-1} + \cdots + A_k X_{t-k} + \mu + \Phi D_t + \epsilon_t, \;\; t = 1, \ldots, T \tag{1}$$

where $X_{-k+1}, \ldots, X_0$ are assumed to be fixed and $\epsilon_1, \ldots, \epsilon_T$ are independent, identically distributed Gaussian vectors, with mean zero and covariance matrix Σ. The vectors $D_t, t = 1, \ldots, T$ consists of centered seasonal dummies. The model (1) can be reparameterized as

$$\Delta X_t = \Pi X_{t-1} + \Pi_2 \Delta X_{t-1} + \cdots + \Pi_k \Delta X_{t-k+1} + \mu + \Phi D_t + \epsilon_t, \; t = 1, \ldots, T \tag{2}$$

where $\Pi = A_1 + \cdots + A_k - I, \Pi_i = -(A_i + \cdots + A_k), i = 2, \ldots, k$.

To allow for nonstationary behavior of $\{X\}_{t=1,2,\ldots}$ we assume that the matrix Π has reduced rank $0 < r < p$ and thus may be written

$$\Pi = \alpha \beta', \tag{3}$$

where α and β are $p \times r$ matrices of full rank. This model, which we shall use as starting point, has been treated extensively see e.g. Johansen [8] and [9], and Johansen and Juselius [10] and [11]. We remind that the parameters α and β are unidentified because of the multiplicative form in (3).

In our treatment of rational expectations we shall, as explained in the introduction, elaborate upon ideas similar to those exposed by Baillie [2]. The set of restrictions we consider is of the form

$$E_t \sum_{j=0}^{\infty} c_j' X_{t+j} + c_{-1}' X_{t-1} + \cdots + c_{-k+1}' X_{t-k+1} + c = 0. \tag{4}$$

Here E_t denotes conditional expectation in the probabilistic sense taken in model (1) given the variables $X_1, \ldots, X_t$. The $p \times q$ matrices $c_i, i = -k+1, \ldots$ are known matrices, possibly equal to zero. The $q \times 1$ matrix c can contain unknown parameters and is of the form $c = H\omega$ where the $q \times s$ matrix H is known, and ω is an $s \times 1$ vector consisting of unknown parameters, $0 \leq s \leq q$. Note that we allow lagged values of X_t to be included in the restrictions.

There are a number of interesting economic hypotheses that are subsumed in the formulation (4). We only mention three, but refer to the paper by Baillie [2] mentioned above for a more thorough discussion.

Example 1. Let X_t denote the vector $(\pi_{1t}, \pi_{2t}, d_t, i_{1,t}, i_{2,t})'$ where $i_{1,t}$ and $i_{2,t}$ denote domestic and foreign interest rate respectively , $\pi_{1,t}$ and $\pi_{2,t}$ are the domestic and foreign inflation rate and d_t is the depreciation of own currency. Two hypotheses of interest are the uncovered interest parity hypothesis which can be formulated as

$$i_{1,t} - i_{2,t} = E_t d_{t+1},$$

and equality of the expected real interest rates

$$i_{1,t} - E_t \pi_{1,t+1} = i_{2,t} - E_t \pi_{2,t+1}.$$

These hypotheses have the form (4) where $c = c_j = 0, j = 2, 3, \dots$ and where c_0 and c_1 are given by the matrices

$$c_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \textit{ and } c_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \square$$

Example 2. Campbell and Shiller [4] studied a present value model for two variables Y_t and y_t having the form

$$Y_t = \gamma(1-\delta) \sum_{j=0}^{\infty} \delta^j E_t y_{t+j} + c,$$

where γ is a coefficient of proportionality, δ a discount factor and c a constant that may be unknown. This relation is of the form (4), which can be seen by taking $c_j = \delta^{j-1} c_1, j = 2, 3, \dots$.

In a related paper Campbell [3] treated a system with $X_t = (y_{kt}, y_{lt}, co_t)'$ where y_{kt} and y_{lt} are capital and labor income respectively and co_t is consumption. The permanent income hypothesis he investigated is of form

$$co_t = \gamma[y_{kt} + (1-\delta) \sum_{j=0}^{\infty} \delta^j E_t y_{l,t+j}].$$

Thus in the case where γ and δ are known, under the hypothesis, these are examples of the hypotheses that can be cast in the form (4). $\square$

Example 3. In a study of money demand Cuthbertson and Taylor [5] considered restrictions of the form

$$(m-p)_t = \lambda(m-p)_{t-1} + (1-\lambda)(1-\lambda D) \sum_{j=0}^{\infty} (\lambda D)^j E_t \gamma' z_{t+i},$$

where $m-p$ is real money balances, and $\gamma' z$ are the determinants of the long-run real money demand. The restrictions are deduced from a model where agents minimize the expected discounted present value of an infinite-period cost function measuring both the cost of being away from the long run equilibrium and the cost of adjustment, conditional on information at time t. The constant λ, which satiesfies $0<\lambda<1$, depends on the relative importance of the two cost factors.

Taking $X_t=(m_t-p_t,z_t')'$ and $c_{-1}=(-\lambda,0,\ldots,0)'$, $c_0=(1,-(1-\lambda)(1-\lambda D)\gamma')'$, $c_1=(0,-\lambda D(1-\lambda)(1-\lambda D)\gamma')'$ and $c_j=(\lambda D)^{j-1}c_1, j=2,\ldots$ we see that this is a situation covered by the assumption (4) if λ, D and γ are known. A recent application of a similar model to the demand for labor can be found in Engsted and Haldrup [7]. □

The model in (1) can, as is well known, be written on the so-called companion form as

$$Z_t=AZ_{t-1}+e_1\otimes\mu+e_1\otimes\Phi D_t+e_1\otimes\epsilon_t, \tag{5}$$

where $Z_t=(X_t',\ldots,X_{t-k}')$, $e_1\otimes\epsilon_t$ is the Kronecker product of the $k\times 1$ unit vector $e_1=(1,0,\ldots,0)'$ and ϵ_t, and A is the $pk\times pk$ matrix

$$A=\begin{pmatrix} A_1\cdots & A_k \\ I_{p(k-1)} & 0 \end{pmatrix}.$$

Denoting the (i_1,i_2) block of the $pk\times pk$ matrix A^j by $A^j_{i_1,i_2}, i_1,i_2=1,\ldots,k$, we have the following

Lemma 1 *With the notations defined above*

$$A^j_{11}+\cdots+A^j_{1k}-I=C_j\alpha\beta'.$$

The $p\times p$ matrices C_j, j=1,... are defined recursively by $C_j=(A^{j-1}_{11}+C_{j-1})\alpha\beta'$ with $C_1=I$ and $A^1_{11}=A_1$.

Proof. By straightforward algebra and the reduced rank condition

$$\begin{aligned} A^j_{11}+\cdots+A^j_{1k} &= (A^{j-1}_{11}A_1+A^{j-1}_{12})+(A^{j-1}_{11}A_2+A^{j-1}_{13})+\cdots+A^{j-1}_{11}A_k \\ &= A^{j-1}_{11}(A_1+\cdots+A_k)+A^{j-1}_{12}+\cdots+A^{j-1}_{1k} \\ &= A^{j-1}_{11}(A_1+\cdots+A_k-I)+A^{j-1}_{11}+\cdots+A^{j-1}_{1k} \\ &= A^{j-1}_{11}\alpha\beta'+A^{j-1}_{11}+\cdots+A^{j-1}_{1k}. \end{aligned}$$

Now the Lemma follows by induction. For $j=1$ the Lemma is just the reduced rank condition (3). If the lemma is true for j, then by this assumption and the identity above

$$\begin{aligned} A^{j+1}_{11}+\cdots+A^{j+1}_{1k}-I &= A^j_{11}\alpha\beta'+(A^j_{11}+\cdots+A^j_{1k}-I) \\ &= (A^j_{11}+C_j)\alpha\beta'. \quad \square \end{aligned}$$

Since

$$E_t Z_{t+j} = A^j Z_t + \sum_{l=1}^{j} A^{j-l}(e_1 \otimes \mu) + \sum_{l=1}^{j} A^{j-l}(e_1 \otimes \Phi D_{t+l}),$$

it follows that

$$c_j' E_t X_{t+j} = c_j' A_{11}^j X_t + \cdots + c_j' A_{1k}^j X_{t-k} + c_j' \sum_{l=1}^{j} A_{11}^{j-l} \mu + c_j' \sum_{l=1}^{j} A_{11}^{j-l} \Phi D_{t+l},$$

for $j > 0$. Furthermore, $c_0' E_t X_t = c_0' X_t$. Hence by inserting into (4)

$$\sum_{j=1}^{\infty} c_j' A_{11}^j + c_0' = 0 \quad , \quad \sum_{j=1}^{\infty} c_j' A_{1i}^j + c_{-i+1}' = 0, \;\; i = 2, \ldots, k$$

$$(\sum_{j=1}^{\infty} c_j' \sum_{l=1}^{j} A_{11}^{j-l})\mu + c = 0 \quad , \quad (\sum_{j=1}^{\infty} c_j' \sum_{l=1}^{j} A_{11}^{j-l})\Phi = 0.$$

By Lemma 1,

$$\sum_{j=1}^{\infty} c_j' A_{11}^j + c_0' = \sum_{j=1}^{\infty}(c_j'(A_{11}^j + \cdots + A_{1k}^j - I) + c_j') + \sum_{i=-k+1}^{0} c_i'$$
$$= \sum_{j=1}^{\infty} c_j' C_j \alpha\beta' + \sum_{j=-k+1}^{\infty} c_j' = 0.$$

Thus we have when $1 \leq q \leq r$,

Proposition 1 . *The restrictions on the coefficients in the reduced rank VAR model implied by the hypothesis (4) are equivalent to*

$$\begin{aligned} &(i) && \beta\alpha' \textstyle\sum_{j=1}^{\infty} C_j' c_j = -\sum_{j=-k+1}^{\infty} c_j && (6)\\ &(ii) && \textstyle\sum_{j=1}^{\infty} c_j' A_{1i}^j = -c_{-i+1}', i = 2, \ldots, k, &&\\ & && \textstyle\sum_{j=1}^{\infty} c_j' \sum_{l=1}^{j} A_{11}^{j-l} \mu = -H\omega, \sum_{j=1}^{\infty} c_j' \sum_{l=1}^{j} A_{11}^{j-l} \Phi = 0, && \end{aligned}$$

where C_j and $A_{1i}^j, i = 1, \ldots, k, j = 1, 2, \ldots$ are as defined in Lemma 1 and $A_{11}^0 = I$.

The infinite sums appearing in the expressions above are all assumed to exist. In case they do not converge, the restriction (6) does not make sense. In many special situations convergence is no problem. The eigenvalues of A then have all modulus less than or equal to 1, and the sum $\sum_{j=-k+1}^{\infty} c_j$ either consists of a finite sum of nonzero terms or of exponentially decreasing terms.

One can also remark that the conditions of the first part of the proposition may be formulated as $\sum_{j=-k+1}^{\infty} c_j \in sp(\beta)$, i.e. the vector $\sum_{j=-k+1}^{\infty} c_j$ must belong to the space spanned by the columns of β. Also by multiplying both sides with the matrix $(\beta'\beta)^{-1}\beta'$, one has the following restrictions on the adjustment parameters α: $\alpha' \sum_{j=1}^{\infty} C_j' c_j = -(\beta'\beta)^{-1}\beta' \sum_{j=-k+1}^{\infty} c_j$.

The restrictions on the α parameters and the conditions in the second part of Proposition 1 are in general non-linear in terms of the parameters of the VAR model in (1) or (2). In the particular case where $c_2 = c_3 = \cdots = 0$, the conditions in Proposition 1 simplify since $C_1 = A_{11}^0 = I$ and the terms involving the other $C_j's$ disappear.

Corollary 1 *If $c_i = 0, i = 2, \ldots,$ the conditions of Proposition 1 take the form in terms of the model (2):*

$$\begin{aligned} &(i) \qquad \beta\alpha' c_1 = -\textstyle\sum_{j=-k+1}^{1} c_j \\ &(ii) \quad c_1'\Pi_i = \textstyle\sum_{j=i}^{k} c_{-j+1}', i = 2, \ldots, k, \\ &\qquad\quad c_1'\mu = -H\omega \text{ and } c_1'\Phi = 0. \end{aligned} \tag{7}$$

That restrictions like those of example 1 are covered by Corollary 1 is evident. What may not be so obvious, is that the restrictions in the other two examples, where $c_j = \delta^{j-1}c_1, j = 2, \ldots,$ are also covered. To see that, write the restrictions (4) as

$$c_0'X_t + E_t c_1' X_{t+1} + \delta \sum_{j=2}^{\infty} \delta^{j-2} E_t c_1' X_{t+j} + \sum_{j=1}^{k-1} c_{-j}' X_{t-j} + c = 0. \tag{8}$$

Using iterated conditional expectations in a similar expression at time $t+1$, multiplying by δ and subtracting from (8) yields

$$\begin{aligned} &(c_0 - \delta c_{-1})' X_t + (c_1 - \delta c_0)' E_t X_{t+1} + \\ &\sum_{j=1}^{k-2} (c_{-j} - \delta c_{-(j+1)})' X_{t-j} + c_{-k+1}' X_{t-k+1} + (1-\delta)c = 0, \end{aligned}$$

which shows that also restrictions in examples 2 and 3 have a form covered by Corollary 1.

In the next section we shall derive the likelihood ratio test for restrictions of this particular type. To discuss this problem the following result turns out to be useful. We introduce the notation that if a is a $p \times q$ matrix of full rank q, then $a_\perp$ is a $p \times (p-q)$ matrix so that the square matrix $(a, a_\perp)$ is nonsingular. Also let $\overline{a} = a(a'a)^{-1}$. Then the result can be formulated as:

Proposition 2 *The $p \times p$ matrix Π has reduced rank r and satisfies*

$$\Pi' b = d \tag{9}$$

where b and d are $p \times q$ matrices of full rank if and only if Π has the form

$$\Pi = \bar{b}d' + \overline{b_\perp}\eta\xi'\overline{d}_\perp' + \overline{b_\perp}\Theta\overline{d}' \tag{10}$$

where η and ξ are matrices of dimension $(p-q)\times(r-q)$ and of full rank $(r-q)$.

Proof. Assuming that (9) is true we consider

$$(b, b_\perp)'\Pi\,(d, d_\perp) = \begin{bmatrix} b'\Pi d & b'\Pi d_\perp \\ b_\perp'\Pi d & b_\perp'\Pi d_\perp \end{bmatrix} = \begin{bmatrix} d'd & 0 \\ b_\perp'\Pi d & b_\perp'\Pi d_\perp \end{bmatrix}.$$

If Π has rank r, then $r \geq q$, which is the rank of $d'd$. Then $b_\perp'\Pi d_\perp$ must have rank $r-q$, and can be written $b_\perp'\Pi d_\perp = \eta\xi'$ for matrices of rank $r-q$. If we define the $(p-q)\times q$ matrix Θ as $\Theta = b_\perp'\Pi d$, we get the representation

$$\Pi = (\bar{b}, \overline{b_\perp})\begin{bmatrix} d'd & 0 \\ \Theta & \eta\xi' \end{bmatrix}\left(\bar{d}, \overline{d_\perp}\right)' = \bar{b}d' + \overline{b_\perp}\Theta\overline{d'} + \overline{b_\perp}\eta\xi'\overline{d_\perp}'.$$

which proves one part of the proposition.

Next assume that Π can be represented as in (10). Then $b'\Pi = d'$. That the rank is reduced can be seen from

$$(b, b_\perp)'\Pi\,(d, d_\perp) = \begin{bmatrix} d'd & 0 \\ \Theta & \eta\xi' \end{bmatrix},$$

which has rank equal to $rank(d'd) + rank(\eta\xi') = q + (r-q) = r.\square$

3 Derivation of the maximum likelihood estimators and the likelihood ratio test in a special case.

We consider a situation similar to the one covered by Corollary 1, i.e $c_j = 0, j = 2, 3, \ldots$, and c_0 and c_1 are known $p\times q$ matrices. For simplicity we also assume that $c_{-2} = \cdots = c_{-k} = 0$, so that the restrictions only involve one lagged variable. Also we make the additional assumption that $b = c_1$ and $d = -(c_{-1} + c_0 + c_1)$ are of full rank. Let $a = b_\perp$.

Using the results of Proposition 2 in model (2) with b and d as just defined yields the equation

$$\begin{aligned} \Delta X_t &= \bar{b}d'X_{t-1} + \bar{a}\eta\xi'\overline{d_\perp}'X_{t-1} + \bar{a}\Theta\bar{d}'X_{t-1} \\ &+ \Pi_2\Delta X_{t-1} + \cdots + \Pi_k\Delta X_{t-k-1} + \mu + \Phi D_t + \epsilon_t. \end{aligned} \tag{11}$$

By multiplying (11) with a' and b' we get after taking the restrictions in Corollary 1 into account

$$
\begin{aligned}
a'\Delta X_t &= \eta\xi'\overline{d_\perp}'X_{t-1} + \Theta\overline{d}'X_{t-1} && (12)\\
&+ a'\Pi_2\Delta X_{t-1} + \cdots + a'\Pi_k\Delta X_{t-k-1} \\
&+ a'\mu + a'\Phi D_t + a'\epsilon_t \\
b'\Delta X_t &= d'X_{t-1} + c'_{-1}\Delta X_{t-1} - H\omega + b'\epsilon_t. && (13)
\end{aligned}
$$

We thus end up with a model (12)-(13) being equivalent to the reduced rank model (2) satisfying the restrictions (i) and (ii) of Corollary 1. The $(p-q)\times(p-q)$ matrix $\eta\xi'(d_\perp' d_\perp)^{-1}$ in front of $d_\perp' X_{t-1}$ has rank $r-q$ and is therefore of reduced rank. It contains $(p-q)(r-q)+(p-q-r+q)(r-q) = 2(p-q)(r-q)-(r-q)^2$ parameters. The matrix $\Theta(d'd)^{-1}$ in front of $d'X_{t-1}$ contains $(p-q)q$ parameters. These correspond to the parameters of Π of (2) taking the restrictions (i) of Corollary 1 into account. Also we see how the restrictions (ii) of Corollary 1 are incorporated, since no parameters except in the constant terms and the covariance matrix are allowed in (13).

The parameters of the VAR model (2) with the restrictions (3) and (4) imposed can thus up to a reparametrization be estimated from the system (12)-(13) where the reduced rank matrix is of rank $(r-q)$.

In order to estimate the parameters of this model we consider the conditional model of $a'\Delta X_t$ given $b'\Delta X_t$ and past information. Using similar results as in Johansen [9], this model may be written

$$
\begin{aligned}
a'\Delta X_t &= \eta\xi'(d_\perp' d_\perp)^{-1}d_\perp' X_{t-1} + \rho(b'\Delta X_t - c'_{-1}\Delta X_{t-1}) \\
&+ (\Theta(d'd)^{-1} - \rho)d'X_{t-1} && (14)\\
&+ a'\Pi_2\Delta X_{t-1} + \cdots + a'\Pi_k\Delta X_{t-k-1} \\
&+ (\rho H\omega + a'\mu) + a'\Phi D_t + u_t,
\end{aligned}
$$

where the $(p-q)\times q$ matrix ρ is defined by $\rho = a'\Sigma b(b'\Sigma b)^{-1}$ and the errors are $u_t = (a' - \rho b')\epsilon_t$. Note that they are independent of the errors $b'\epsilon_t$ of (13).

We intend to find the maximum likelihood estimators and the maximal value of the likelihood by considering separately the marginal model given by (13), and the conditional model (14) described above. Due to the independence of the errors the likelihood factorizes. What must furthermore be established, is that the parameters of the two parts are variation free.

The parameters of the marginal model are ω and $b'\Sigma b = \Sigma_{22}$. The parameters of the conditional model are $\eta, \xi, \rho, \gamma = (\Theta(d'd)^{-1} - \rho), \psi_i = a'\Pi_i, i = 2, \ldots, k, \Phi_0 = a'\Phi, \phi = (\rho H\omega + a'\mu)$ and $\Sigma_{11\cdot 2} = a'\Sigma a - a'\Sigma b(b'\Sigma b)^{-1}b'\Sigma a$. It is well known that Σ_{22} is variation free with ρ and $\Sigma_{11\cdot 2}$. What needs some closer attention is the parameter ω which is common to both systems. Writing

$$\begin{aligned}\mu &= a(a'a)^{-1}a'\mu + b(b'b)^{-1}b'\mu \\ &= a(a'a)^{-1}\mu_1 - b(b'b)^{-1}H\omega,\end{aligned}$$

we see that $\mu_1 = a'\mu$ is independent of ω. Since $a'\mu = \phi - \rho H\omega$, any particular value ω may have, will not influence the value μ_1 can take since ϕ will not be restricted in any way.

The estimation of the conditional system is carried out by first regressing the variables $a'\Delta X_t$ and $\overline{d_\perp}'X_{t-1}$ on $b'\Delta X_t - c'_{-1}\Delta X_{t-1}, d'X_{t-1}, \Delta X_{t-1}, \ldots, \Delta X_{t-k+1}, D_t$ and 1, $t = 1, \ldots, T$. In the case where $r = q$ only the variable $a'\Delta X_t$ is used as regressand. Defining the residuals as $R_{1t\cdot 2}$ and $R_{2t\cdot 2}$ the equation (14) takes the form

$$R_{1t\cdot 2} = \eta\xi' R_{2t\cdot 2} + error.$$

Define the $(p-q) \times (p-q)$ matrices $S_{ij\cdot 2}, i, j = 1, 2$ by

$$S_{ij\cdot 2} = \frac{1}{T}\sum_{t=1}^{T} R_{it\cdot 2}R'_{jt\cdot 2}, i, j = 1, 2. \tag{15}$$

By now well known arguments the maximum likelihood estimators of ξ is given by $\tilde{\xi} = (v_1, \ldots, v_{r-q})$ where $v_1, \ldots, v_{p-q}$ are eigenvectors in the eigenvalue problem

$$|\lambda S_{22\cdot 2} - S_{21\cdot 2}S_{11\cdot 2}^{-1}S_{12\cdot 2}| = 0, \tag{16}$$

which has solutions $\tilde{\lambda}_1 > \ldots > \tilde{\lambda}_{p-q}$. Here the normalization $\tilde{\xi}'S_{22\cdot 2}\tilde{\xi} = I_{r-q}$ is used. The estimator of η is given by

$$\tilde{\eta} = S_{12\cdot 2}\tilde{\xi}.$$

We now consider the form of the likelihood ratio test of the restrictions (4) in the VAR model (2) with the reduced rank condition (3) imposed.

The part of the maximized likelihood function stemming from the conditional model is

$$L_{1\cdot 2,max}^{-2/T} = |S_{11\cdot 2}|\prod_{i=1}^{r-q}(1 - \tilde{\lambda}_i)/|a'a|.$$

The part stemming from the marginal model (13) follows from results for standard multivariate Gaussian models, and equals

$$L_{2,max}^{-2/T} = |\tilde{\Sigma}_{22}|/|b'b|,$$

where

$$\tilde{\Sigma}_{22} = \frac{1}{T}\sum_{t=1}^{T}(b'\Delta X_t - d'X_{t-1} - c'_{-1}\Delta X_{t-1} + H\tilde{\omega})(b'\Delta X_t - d'X_{t-1} - c'_{-1}\Delta X_{t-1} + H\tilde{\omega})', \tag{17}$$

and $\tilde{\omega}$ is the maximum likelihood estimator for ω. Hence the maximum value of the likelihood function is given by $L_{H,max}^{-2/T} = |\tilde{\Sigma}_{22} S_{11 \cdot 2}| \prod_{i=1}^{r-q}(1 - \tilde{\lambda}_i)/|b'b||a'a|$.

In Johansen and Juselius [10] it is shown that the maximum value of the likelihood in the reduced rank model defined by (2) and (3) is given by $L_{max}^{-2/T} = |S_{00}| \prod_{i=1}^{r}(1 - \hat{\lambda}_i)$, where $S_{00}, \hat{\lambda}_i, i = 1, \ldots, r$ arise from maximizing the likelihood in a manner similar to the one described above. In this case only the restriction (3) is taken into account.

Collecting the results above we have:

Proposition 3 *Consider the rational expectation restrictions of the form (4) with c_{-1}, c_0 and c_1 known, and $c_i = 0$ otherwise. Assume that $b = c_1, a = c_{1\perp}$ and $d = -(c_{-1} + c_0 + c_1)$ have full rank. The likelihood ratio statistic of a test for the restrictions (4) in the reduced rank VAR model satisfying (3) against a VAR model satisfying only the reduced rank condition (3), is*

$$
\begin{aligned}
-2lnQ &= Tln|S_{11 \cdot 2}| - \sum_{i=1}^{r} ln(1 - \hat{\lambda}_i) + Tln|\tilde{\Sigma}_{22}| \\
&- Tln|S_{00}| + \sum_{i=1}^{r-q} ln(1 - \tilde{\lambda}_i) - Tln(|b'b||a'a|),
\end{aligned}
$$

where $\tilde{\Sigma}_{22}, S_{11 \cdot 2}$ and $\tilde{\lambda}_i, i = 1, \ldots, r - q$ are given by (15), (16) and (17), and $S_{00}, \hat{\lambda}_i, i = 1, \ldots, r$ are estimates from the VAR model (2) satisfying (3).

It should be fairly clear how to cope with restrictions on further lags than one. The form of such restrictions will have an impact on (13) and (14) which means that one of the regressors must be redefined. Furtermore (17) has to be modified appropriately.

4 The asymptotic distribution of the test statistics.

So far no mention has been made of the distribution of the estimators and test statistics. To do so one has to introduce some further conditions. Let $\Pi(z)$ denote the characteristic polynomial of the VAR model (2), i.e. $\Pi(z) = (1 - z) - z\Pi - (1 - z)z\Pi_2 - \cdots + (1 - z)z^{k-1}\Pi_k$, and let $-\Psi$ equal the derivative of Π evaluated at $z = 1$. Under the condition that $|\Pi(z)| = 0$ implies that $|z| > 1$ or $z = 1$, the restriction (3) and the condition that $\alpha'_\perp \Psi \beta_\perp$ has rank $p - r$, Johansen [8] derived an explicit representation of X_t in terms of the errors. In particular the vector ΔX_t and the rows of $\beta' X_t$ are stationary vectors. Therefore, the columns of β are the cointegrating vectors in the sense of Engle and Granger [6].

Using these results one can find the asymptotic distribution of the estimators of α, β and the other unknown parameters, see Johansen [8] or Ahn and Reinsel [1]. Properly normalized the distribution of the estimators of β converge at the rate T^{-1} towards a mixed Gaussian distribution. The distributions of the estimators of the other parameters converge at a rate $T^{-1/2}$. The asymptotic distribution of these estimators is a multivariate Gaussian distribution, except for the distribution of the estimator for the constant term, which is more complicated. The asymptotic covariance matrix of the estimators of β and of the other parameters is block diagonal.

This has the consequence that a test on the β parameters and the rest may be carried out separately. Since the conditions derived in Proposition 1 separate in conditions on β and in conditions on the rest of the parameters, it seems natural to proceed in two steps. First we test the restrictions on β ignoring the restrictions on the other parameters, i.e. we test whether $(\sum_{j=-k+1}^{\infty} c_j) \in sp(\beta)$. This can be done by the maximum likelihood procedure developed by Johansen and Juselius [11], and amounts to carrying out a χ^2 test. If this hypothesis is not rejected, one can procede to test the restrictions on the other parameters implied by Proposition 1 treating β as known. This means that the processes involved can be transformed to stationary processes. Hence this part of the testing can be carried out following well known procedures developed for inference in stationary time series. In general the restrictions are nonlinear as pointed out in section 2.

As shown in the previous section there are interesting situations where it is possible to carry out the test in one step. We shall indicate the asymptotic distribution in the case covered by Proposition 3. By the results referred to above the asymptotic distribution is χ^2, and the degrees of freedom is the difference between the number of free parameters in the general case and the number of parameters under the hypothesis. Since there are $pr + (p-r)r + (k-1)p^2 + p + 3p + p(p+1)/2$ in the model (2) satisfying (3) when the seasonal pattern is quarterly, and the formulation (13)-(12) has $(p-q)r + (p-r)(r-q) + (k-1)p(p-q) + s + (p-q) + 3(p-q) + p(p+1)/2$ parameters, the degrees of freedom are $rq + (p-r)q + (k-1)pq - s + 4q$.

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