A note on the Poisson limit for the number of system failures of a monotone system

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Abstract

It has been formally proved that the number of failures of a highly available monotone system is asymptotic Poisson distributed if the lifetime distributions are exponentially distributed. Intuitively it seems clear that it is possible to generalize this result to the non-exponential case as long as the components are highly available. But formal asymptotic results are rather difficult to establish. Strict conditions have to be imposed to establish the results, to the system structure and the component lifetime and downtime distributions. So further research is needed to obtain general results. This paper reviews the literaure in the field and restructures and simplifies the so-called Szász approach for proving the asymptotic Poisson limit.

1 Introduction

We consider a binary, monotone system Φ comprising n independent components, observed in the time interval $[0, \infty)$. Each component generates an alternating renewal process. The distribution of the number of system failures, and the probability of no failures in particular, is an informative performance measure from a safety and operational point of view. The computation of this measure has therefore been given much attention in the literature. It is however very difficult to establish exact formulae for these measures. Therefore much focus has been placed on asymptotic analysis. Assuming exponential lifetime distributions and using the theory of regenerative processes, it has been shown that the number of system failures, suitably normalized, converges in distribution to a homogeneous Poisson process, see e.g. [1, 2, 4, 8, 9, 16]. Different normalizing factors are used,

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including the asymptotic failure rate of the system, λ_Φ. In [4] the accuracy of the Poisson (exponential) approximation is studied for different normalizing factors.

Monte Carlo simulations and results obtained for phase type distributions indicate that the number of failures are approximately Poisson distributed also for non-exponential lifetime distributions as long as the components are highly available, cf [1], Section 4.7.1. But formal asymptotic results are rather difficult to establish, see for example [1, 10, 11, 12, 14, 17]. Strict conditions have to be imposed to establish the results, to the system structure and the component lifetime and downtime distributions. Also the general approach of showing that the compensator of the counting process converges in probability, is difficult to apply in our setting, cf. Section 3 below.

Szász [14, 15] formulated conditions for when a Poisson process is an asymptotic limit for a parallel system of two identical components with general lifetime distributions. The lifetime distributions are fixed, whereas the repair times converge to 0. His approach for establishing the desired result seems promising, but we have not yet been able to formulate a general result using this approach. Szász's original proofs are rather complicated and it possible to restructure the results and proofs such that it is easier to see what are the critical assumptions. This new structure is presented in Section 3 of the present paper. Hopefully, this formulation and the paper as a whole can provide a useful basis for further research in the field. At the end of the paper we give a remark concerning the critical assumptions made in Section 3. First we present the model and state some basic results to be used in Section 3.

2 Model and some basic results

In this section we introduce the set-up to be used in the asymptotic analysis in Section 3. We also state some basic results from alternating renewal processes and counting process theory. See [1] for proofs of these results.

Let $X_t(i)$ be a binary stochastic process with right-continuous sample paths representing the state of component i, $i = 1, 2, \ldots, n$; $X_t(i) = 1$ if component i is functioning at time t and $X_t(i) = 0$ if component i is not functioning at time t. We assume that all components are functioning at time 0, i.e., $X_0(i) = 1$. The process $X_t(i)$ is an alternating renewal process. Let T_{im}, $m = 1, 2, \ldots$, represent the length of the mth operation period of component i, and let R_{im}, $j = 1, 2, \ldots$, represent the length of the mth repair time for component i, see Figure 1. For $i = 1, 2, \ldots, n$ we assume that (T_{im}), $m = 1, 2, \ldots$, and (R_{im}), $m = 1, 2, \ldots$, are i.i.d. sequences of positive random variables. We denote the probability distributions of T_{ij} and R_{ij} by $F_i(t)$ and $G_i(t)$, respectively, and assume that they have finite means:

$$\mu_{F_i} = ET_{ij} \qquad \mu_{G_i} = ER_{ij}.$$

To simplify the presentation, we also assume that $F_i(t)$ is a continuous distribution, i.e., F_i has a density function f_i and failure rate function λ_i. We denote by $A_i(t)$ the availability of component i at time t, i.e., $A_i(t) = P(X_t(i) = 1)$, and

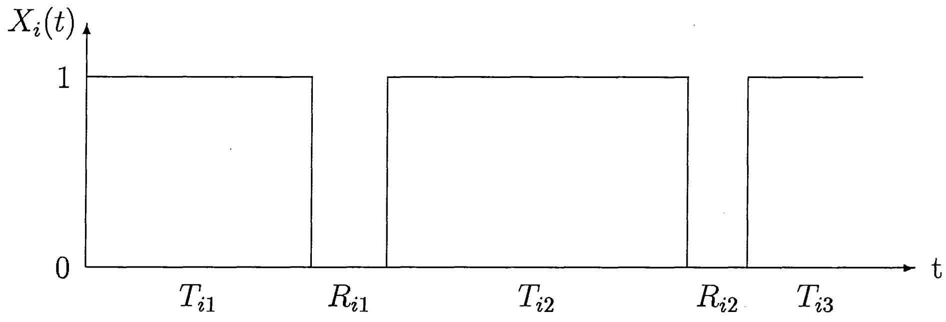

Figure 1: Time evolution of a failure and repair process for component i starting at time $t = 0$ in the operating state.

by A_i the limiting availability of component i, i.e.,

$$A_i = \frac{\mu_{F_i}}{\mu_{F_i} + \mu_{G_i}}.$$

Let $N_t(i)$ denote the number of failures of component i in $[0, t]$, and let $M_i(t) = EN_t(i)$ denote the mean value function. It can be shown that

$$\bar{A}_i(t) = M_i(t) - G_i * M_i(t), \tag{1}$$

where $\bar{A}_i = 1 - A_i$ and * denotes convolution. Note that we also have

$$M(t) = \sum_{i=1}^{\infty} F * (F * G)^{i-1}. \tag{2}$$

The counting process $N_t(i)$ has a stochastic intensity process

$$\eta_t(i) = \lambda(\beta_t(i))X_t(i),$$

where $\beta_t(i)$ is the backward recurrence time at time t, i.e., the relative age of component i at time t. If $M_i(t)$ has a density $m_i(t)$, then

$$EN_t(i) = M_i(t) = \int_0^t m_i(s)\,ds = \int_0^t E\eta_s(i)\,ds$$

and

$$m_i(t) \leq [\sup_{s \leq t} \lambda_i(s)] A_i(t), \tag{3}$$

$$\bar{A}_i(t) \leq [\sup_{s \leq t} \lambda_i(s)] \int_0^t \bar{G}_i(s) ds \leq [\sup_{s \leq t} \lambda_i(s)] \mu_{G_i}. \tag{4}$$

Let $\Phi : \{0,1\}^n \to \{0,1\}$ be the structure function of the system. We assume that this function is monotone, i.e., $\Phi(\mathbf{x})$ (where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$) is a non-decreasing function in each argument x_i, and $\Phi(\mathbf{1}) = 1$ and $\Phi(\mathbf{0}) = 0$ where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$.

At time t the states of the components are given by

$$\mathbf{X}_t = (X_t(1), X_t(2), \ldots, X_t(n)).$$

If t is fixed we often simplify the notation and omit the t. We assume the n processes are independent. The reliability function of Φ is denoted $h(\mathbf{p})$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $p_i = P(X_i = 1)$. We have $h(\mathbf{p}) = E\Phi(\mathbf{X}) = P(\Phi(\mathbf{X}) = 1)$. The reliability function when it is given that $X_i = x_i$, is denoted $h(x_i, \mathbf{p})$, i.e., $h(x_i, \mathbf{p}) = E[\phi(\mathbf{X})|X_i = x_i]$.

Let N_t denote the number of system failures in $[0, t]$. It is well-known that

$$\lim_{t\to\infty} \frac{EN_t}{t} = \sum_{i=1}^{n} \frac{h(1_i, \mathbf{A}) - h(0_i, \mathbf{A})}{\mu_{F_i} + \mu_{G_i}} = \lambda_\Phi \quad (5)$$

where the last equality is given by definition and $\mathbf{A} = (A_1, A_2, \ldots, A_n)$. We refer to λ_Φ as the system failure rate. The counting process N_t has a stochastic intensity $\lambda_\Phi(t)$ given by

$$\lambda_\Phi(t) = \sum_{i=1}^{n} [\Phi(1_i, \mathbf{X}_t) - \Phi(0_i, \mathbf{X}_t)]\eta_t(i).$$

In the following we write $\bar{b}$ in place of $1 - b$ for any quantity b taking values in $[0, 1]$, and $\mathbf{B}$ for a vector $(B_1, B_2, \ldots, B_n)$.

3 Asymptotic Analysis

We consider now for each component i a sequence (F_{ij}, G_{ij}), $j = 1, 2, \ldots,$ of distributions satisfying certain conditions. We will formulate conditions which ensure that (N_{t/λ_Φ}) converges in distribution to a Poisson process with intensity 1. To simplify notation, we normally omit the index j.

To prove that (N_{t/λ_Φ}) converges in distribution to a Poisson process with parameter 1, it is sufficient to show that the compensator A_{t/λ_Φ} of N_{t/λ_Φ} converges in probability to t, i.e.,

$$A_{t/\lambda_\Phi} = \int_0^{t/\lambda_\Phi} \lambda_\Phi(s)ds = \int_0^{t/\lambda_\Phi} \sum_{i=1}^{n} [\Phi(1_i, \mathbf{X}_s) - \Phi(0_i, \mathbf{X}_s)]\eta_s(i) \xrightarrow{P} t.$$

This follows by applying a general result from Daley and Vere-Jones [7], p. 552.

Thus to establish the asymptotic Poisson limit result for our model, it is sufficient to show that $E|A_{t/\lambda_\Phi} - t| \to 0$. This leads however to some conditions which are rather difficult to meet. It seems easier to use a more direct approach similar to the one carried out by Szász [14, 15] and Kaplan [10] for a parallel system of two identical components.

Let $S_{i\Phi}$ denote the time of the ith system failure. Assume the following limiting results hold true

$$EN_{t/\lambda_\Phi} \to t \quad (6)$$

$$\sup_{s\in[0,t]} |E[N_{t/\lambda_\Phi}|S_{1\Phi} = s/\lambda_\Phi] - [1 + (t - s)]| \to 0 \quad (7)$$

$$\sup_{s\in[0,t]} |E[N_{t/\lambda_\Phi} - N_{s/\lambda_\Phi}|S_{k\Phi} = s/\lambda_\Phi] - (t - s)| \to 0 \quad (8)$$

$$\sup_{u\in[s,t]} |E[N_{t/\lambda_\Phi} - N_{s/\lambda_\Phi}|S_{k\Phi} = s/\lambda_\Phi, S_{(k+1)\Phi} = u/\lambda_\Phi] \quad (9)$$
$$-[1 + (t - u)]| \to 0.$$

Then (N_{t/λ_Φ}) converges in distribution to a Poisson process with parameter 1. To see this, note first that the family (N_{t/λ_Φ}) is relatively compact (tight), since

$$\sup_j P(N_{t/\lambda_\Phi} \geq k) \leq \frac{1}{k} \sup_j EN_{t/\lambda_\Phi}.$$

Thus each subsequence j' of j has a subsubsequence j'' such that (the finite dimensional distributions of) $(N_{j'',t/\lambda_\Phi})$ converge weakly to (the finite dimensional distributions of) W_t (say) as $j'' \to \infty$ (cf. Daley and Vere-Jones, [7]). We must show that (W_t) is a Poisson process with parameter 1. To this end, let w_i denote the time of the jumps of the process W, and consider the identity

$$\begin{aligned} EN_{t/\lambda_\Phi} &= \int_0^t E[N_{t/\lambda_\Phi}|S_{1\Phi} = s/\lambda_\Phi]dP(S_{1\Phi} \leq s) \\ &= \int_0^t \{E[N_{t/\lambda_\Phi}|S_{1\Phi} = s/\lambda_\Phi] - [1 + (t-s)]dP(S_{1\Phi} \leq s) \\ &\quad - \int_0^t [1 + (t-s)]dP(S_{1\Phi} \leq s). \end{aligned}$$

Letting $j'' \to \infty$, and using (6) and (7), we obtain

$$t = \int_0^t [1 + (t-s)]dP(w_1 \leq s). \tag{10}$$

Let $L(x)$ denote the Laplace transform of the distribution of w_1, i.e.,

$$L(x) = \int_0^\infty e^{-xs} dP(w_1 \leq s).$$

Then by taking the Laplace transform of both sides (10), we get

$$\frac{1}{x} = L(x) + \frac{1}{x} L(x),$$

which means that $L(x) = 1/(1+x)$. We can conclude that w_1 is the exponential distribution with parameter 1, i.e., $P(w_1 \leq s) = 1 - e^{-s}$.

To obtain the distribution of w_2, let $t \geq s$. Then we obtain

$$\begin{aligned} &E[N_{t/\lambda_\Phi} - N_{s/\lambda_\Phi}|S_{1\Phi} = s/\lambda_\Phi] \\ &= \int_0^t E[N_{t/\lambda_\Phi} - N_{s/\lambda_\Phi}|S_{1\Phi} = s/\lambda_\Phi, S_{2\Phi} = y/\lambda_\Phi]dP(S_{2\Phi} \leq y/\lambda_\Phi|S_{1\Phi} = s/\lambda_\Phi). \end{aligned}$$

Taking the limit as $j'' \to \infty$, we obtain

$$t - s = \int_s^t [1 + (t-y)]dP(w_2 \leq y|w_1 = s)$$

and this gives that $w_2 - w_1$ is independent of w_1 and has the same distribution. By induction we can show that $w_i - w_{i-1}$ are i.i.d. and the desired result is proved.

Below we give sufficient conditions for when (6) holds true. We shall restrict attention to a parallel system of two components.

Parallel System of Two Components

First we introduce some necessary notation.

Let $i, i' \in \{1,2\}, i \neq i'$, denote different component indices, and let

$$g_i(\mathbf{A}) = h(1_i, \mathbf{A}) - h(0_i, \mathbf{A}) = \bar{A}_{i'}.$$

Hence we can write

$$\lambda_\Phi = \sum_{i=1}^{2} g_i(\mathbf{A}) \frac{1}{\mu_{F_i} + \mu_{G_i}}.$$

Let

$$\lambda_i^s = \sup_{0 \leq t \leq s} \lambda_i(t).$$

We have the following result.

Theorem 1 *Consider a parallel system Φ of two components. Assume that for all $s < \infty$,*

$$\sum_{i=1}^{2} \lambda_i^s \mu_{G_i} \to 0 \ \ (j \to \infty). \tag{11}$$

Assume also that there exist a decomposition $M_i = M_{i1} + M_{i2}$ for $j \geq j_0, i = 1,2$, such that

a) The measures M_{i1} are bounded, and relatively compact (tight), i.e., for every $\epsilon > 0$ there exists a constant c_1 such that

$$\sup_{j \geq j_0} M_{i1}[c_1, \infty) < \epsilon.$$

where $M_{i1}[c_1, \infty) = \lim_{t\to\infty} M_{i1}(t) - M_{i1}(c_1)$.

b) M_{i2} is absolutely continuous with a bounded density m_{i2}, i.e.,

$$\sup_{j \geq j_0} \sup_t m_{i2}(t) < \infty,$$

and

$$\lim_{j_0 \to \infty, c \to \infty} \sup_{j \geq j_0} \sup_{t \geq c} (\mu_{F_i} + \mu_{G_i}) |m_{i2}(t) - \frac{1}{\mu_{F_i} + \mu_{G_i}}| = 0.$$

Furthermore, assume that for $i = 1,2$ and $x > 0$,

$$\frac{L_{F_i}(x\lambda_\Phi)(1 - L_{G_i}(x\lambda_\Phi))}{1 - L_{F_i}(x\lambda_\Phi) L_{G_i}(x\lambda_\Phi)} = \bar{A}_i(1 + o_x(1)), \quad (j \to \infty). \tag{12}$$

where $o_x(1) \to 0$ as $j \to \infty$. Then as $j \to \infty$

$$EN_{t/\lambda_\Phi} \to t. \tag{13}$$

Proof. Let c be a positive constant. Then we can write

$$\begin{aligned} EN_{t/\lambda_\Phi} &= EN_c + \sum_{i=1}^{2} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))dM_{i1}(s) \\ &\quad + \sum_{i=1}^{2} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))m_{i2}(s)ds. \end{aligned}$$

It follows that

$$\begin{aligned} |EN_{t/\lambda_\Phi} - t| &\leq EN_c + \sum_{i=1}^{2} M_{i1}[c,\infty) \\ &\quad + \sum_{i=1}^{2} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))|m_{i2}(s) - \frac{1}{\mu_{F_i}+\mu_{G_i}}|ds \\ &\quad + |\sum_{i=1}^{2} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))\frac{1}{\mu_{F_i}+\mu_{G_i}}ds - t|. \end{aligned}$$

Let $\epsilon > 0$ be given. Choose c and j so large that $M_{i1}[c,\infty) \leq \epsilon$ and

$$(\mu_{F_i}+\mu_{G_i}) \sup_{t\geq c} |m_{i2}(t) - \frac{1}{\mu_{F_i}+\mu_{G_i}}| \leq \epsilon.$$

Then we obtain

$$\begin{aligned} \limsup_{j\to\infty} |EN_{t/\lambda_\Phi} - t| &\leq \limsup_{j\to\infty} EN_c + n\epsilon \\ &\quad + \epsilon \limsup_{j\to\infty} \sum_{i=1}^{n} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))\frac{1}{\mu_{F_i}+\mu_{G_i}}ds \\ &\quad + \limsup_{j\to\infty} |\sum_{i=1}^{n} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))\frac{1}{\mu_{F_i}+\mu_{G_i}}ds - t|. \end{aligned}$$

Thus the conclusion of the theorem, (13), holds if

$$EN_c \to 0 \quad (j\to\infty) \tag{14}$$

$$\sum_{i=1}^{2} \int_c^{t/\lambda_\Phi} g_i(\mathbf{A}(s))\frac{1}{\mu_{F_i}+\mu_{G_i}}ds \to t, \quad (j\to\infty). \tag{15}$$

First we establish (14). From (4) we have $\bar{A}_i(s) \leq \lambda_i^s \mu_{G_i}$. This gives

$$\begin{aligned} EN_c &= \sum_{i=1}^{2} \int_0^c g_i(\mathbf{A}(s))\, dM_i(s) \\ &\leq \sum_{i=1}^{2} \int_0^c \bar{A}_{i'}(s)\, dM_i(s) \\ &\leq (\sum_{i=1}^{2} \lambda_i^c \mu_{G_i}) \sum_{i=1}^{2} M_i(c)) \end{aligned}$$

which converges to 0 as $j \to \infty$ in view of (11) and the fact that

$$\limsup_{j\to\infty} M_i(c) < \infty.$$

It remains to show (15), or alternatively $v(t) \to t$ as $j \to \infty$, where

$$v(t) = \sum_{i=1}^{2} \int_0^{t/\lambda_\Phi} g_i(\mathbf{A}) \frac{1}{\mu_{F_i} + \mu_{G_i}} ds.$$

From (2) we obtain

$$L_{M_i}(x) = \frac{L_{F_i}(x)}{1 - L_{F_i}(x) L_{G_i}(x)},$$

and using that $\bar{A}_i = M_i - G_i * M_i$ (remember (1)) we obtain by taking Laplace transform

$$\begin{aligned}
L_v(x) &= \int_0^\infty e^{-xt} dv(t) \\
&= \sum_{i=1}^{2} \int_0^\infty e^{-xt} \frac{1}{\lambda_\Phi} \bar{A}_{i'}(t/\lambda_\Phi) \frac{1}{\mu_{F_i} + \mu_{G_i}} dt \\
&= \sum_{i=1}^{2} \frac{1}{\mu_{F_i} + \mu_{G_i}} \int_0^\infty e^{-sx\lambda_\Phi} \bar{A}_{i'}(s) ds \\
&= \sum_{i=1}^{2} \frac{1}{\mu_{F_i} + \mu_{G_i}} \frac{1}{x\lambda_\Phi} \int_0^\infty e^{-sx\lambda_\Phi} d(\bar{G}_{i'} * M_{i'})(s) \\
&= \sum_{i=1}^{2} \frac{1}{\mu_{F_i} + \mu_{G_i}} \frac{1}{x\lambda_\Phi} \frac{L_{F_{i'}}(x\lambda_\Phi)(1 - L_{G_{i'}}(x\lambda_\Phi))}{1 - L_{F_{i'}}(x\lambda_\Phi) L_{G_{i'}}(x\lambda_\Phi)}.
\end{aligned}$$

Now letting $j \to \infty$, we get

$$L_v(x) \to \frac{1}{x},$$

using assumption (12). Hence

$$v(t) \to t,$$

noting that

$$\int_0^\infty e^{-xs} ds = \frac{1}{x}.$$

The conclusion of the theorem follows. ∎

Similar results can be established for (7)-(9) but it is more complicated. We will just indicate the arguments for $S_{1\Phi} = s/\lambda_\Phi$. Given $S_{1\Phi} = s/\lambda_\Phi$ and the state vector $\mathbf{x}$ with component i inducing system failure, we can consider a new process $\mathbf{X}^0$ starting at s/λ_Φ with the state of the components being represented by alternating renewal processes, with consecutive interarrival distributions

$$\begin{aligned}
&G_i, F_i, G_i, \ldots, \\
&G_{i'}^0, F_{i'}, G_{i'}, \ldots,
\end{aligned}$$

where $G_{i'}^0$ is the conditional downtime distribution of component i' given that the first system failure occurs at s/λ_Φ and is caused by component i. The uptimes and downtimes are all independent.

Let M_i^0 denote the renewal failure process associated with component i and let $A_i^0(t)$ denote the corresponding availability at time t. If component i did not cause system failure, we have $M_i^0 = M_i * G_i^0$ and hence $M_i(t) - M_i^0(t) = (M_i * \bar{G}_i^0)(t)$. Furthermore $A_i^0(t) = (A_i * G_i^0)(t)$. Thus

$$A_i(t) - A_i^0(t) = (A_i * \bar{G}_i^0)(t).$$

If component i caused system failure the above equations hold with G_i^0 replaced by G_i.

Now assume the same conditions as in Theorem 1. We will state sufficient conditions for $EN_{t/\lambda_\Phi}^0 \to 0$ as $j \to \infty$. We obtain, writing $g_i(s)$ instead of $g_i(\mathbf{A}(s))$, and $g_i^0(s)$ instead of $g_i^0(\mathbf{A}(s))$,

$$\begin{aligned}
EN_{t/\lambda_\Phi}^0 &= EN_{t/\lambda_\Phi} + EN_c^0 - EN_c \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) dM_i^0(s) - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) dM_i(s) \\
&= EN_{t/\lambda_\Phi} + EN_c^0 - EN_c \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) dM_{i1}^0(s) - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) dM_{i1}(s) \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) m_{i2}^0(s)\, ds - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) m_{i2}(s)\, ds \\
&= EN_{t/\lambda_\Phi} + EN_c^0 - EN_c \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) dM_{i1}^0(s) - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) dM_{i1}(s) \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) \frac{1}{\mu_{F_i} + \mu_{G_i}}\, ds \\
&\quad + \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) (m_{i2}^0(s) - \frac{1}{\mu_{F_i} + \mu_{G_i}})\, ds \\
&\quad - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) \frac{1}{\mu_{F_i} + \mu_{G_i}}\, ds \\
&\quad - \sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i(s) (m_{i2}(s) - \frac{1}{\mu_{F_i} + \mu_{G_i}})\, ds.
\end{aligned}$$

Thus, in view of Theorem 1 and its proof, we need to prove that

$$M_{i1}^0 \text{ are relatively compact} \tag{16}$$

$$EN_c^0 \to 0 \tag{17}$$

$$\sum_{i=1}^2 \int_c^{t/\lambda_\Phi} g_i^0(s) (m_{i2}^0(s) - \frac{1}{\mu_{F_i} + \mu_{G_i}})\, ds \to 0 \tag{18}$$

$$\sum_{i=1}^2 \int_c^{t/\lambda_\Phi} (g_i^0(s) - g_i(s)) \frac{1}{\mu_{F_i} + \mu_{G_i}}\, ds \to 0. \tag{19}$$

In the special case below we will study these conditions closer. Note that the convergence of EN_{u/λ_Φ}^0 is uniform in $u \in [0, t]$ since EN_{u/λ_Φ}^0 is a non-decreasing function in u and its limit u is continuous.

Special Case

Assume that F_i is fixed and not depending on j, and the distributions $G_{ij}(t)$ vary such that $\mu_{G_i} \to 0$ as $j \to \infty$. This means that $\bar{G}_i(t) \to 0$ for all $t > 0$ noting that $\bar{G}_i(t) \leq \mu_{G_i}/t$.

From Szász [14, 15] and Kaplan [10] we know then that a decomposition $M_i = M_{i1}+M_{i2}$ exists satisfying the conditions a) and b) of Theorem 1. Assuming that F_i is a spread out distribution, i.e., there exists an $a \in \{1, 2, \ldots\}$ such that we may write $F_i^{*a} = p_i F_{i1} + q_i F_{i2}$ where $p_i > 0$, $p_i + q_i = 1$, and F_{i1} has a density which is continuous and vanishes outside a finite interval, we have

$$\begin{aligned} M_{i1} &= F_i * (\sum_{r=0}^{a-1} (F_i * G_i)^{*r}) * (\sum_{l=0}^{\infty} q_i^l (F_{i2} * G_i^{*a})^{*l}) \\ M_{i2} &= F_i * (p_i M_{i1} * G_i^{*a} * F_{i1}) * (\sum_{l=0}^{\infty} (F_i * G_i)^{*al}. \end{aligned}$$

We omit the proof since it is very long and technical. Using the Laplace transform technique it is however not difficult to show that we do in fact have $M_i = M_{i1}+M_{i2}$ in this case. Also $M_{i1}(-\infty, \infty) = a/p_i$ and as $j \to \infty$

$$M_{i1}(x) \to F_i * (\sum_{r=0}^{a-1} F_i^{*r}) * (\sum_{l=0}^{\infty} q_i^l F_{i2}^{*l})(x)$$

uniformly in x. For the full proof, reference is made to [10, 14, 15].

Thus for satisfying the conditions of Theorem 1 it remains to prove that the condition (12) holds true. Using Taylor's formula on $exp\{-xs\lambda_\Phi\}$ in the Laplace transforms L_{F_i} and L_{G_i}, we can prove (12), assuming that

1. F_i has finite second order moment, i.e., $ET_i^2 < \infty$, $i = 1, 2$.
2. The square coefficient of R_i is bounded, i.e.,
$$\sup_j \frac{ER_i^2}{\mu_{G_i}^2} < \infty.$$
3. G_i has the NBU property, i.e., $\bar{G}_i(t+u) \leq \bar{G}_i(t)\bar{G}_i(u)$ for all $t, u \geq 0$.

Use of Taylor's formula gives

$$\begin{aligned} L_{F_i}(x\lambda_\Phi) &= \int_0^\infty e^{-sx\lambda_\Phi} dF_i(s) = 1 - x\lambda_\Phi \mu_{F_i} O(1) \\ L_{G_i}(x\lambda_\Phi) &= \int_0^\infty e^{-sx\lambda_\Phi} dG_i(s) = 1 - x\lambda_\Phi \mu_{G_i} O(1) \end{aligned}$$

and furthermore

$$L_{G_i}(x\lambda_\Phi) = 1 - x\lambda_\Phi \mu_{G_i} + x^2 \lambda_\Phi^2 ER_i^2 O(1),$$

where $O(1)$ is a bounded function in j. Hence

$$\begin{aligned}
\frac{L_{F_i}(x\lambda_\Phi)(1-L_{G_i}(x\lambda_\Phi))}{1-L_{F_i}(x\lambda_\Phi)L_{G_i}(x\lambda_\Phi)} &= \frac{x\lambda_\Phi\mu_{G_i}+x^2\lambda_\Phi^2(ER_i^2+\mu_{G_i})O(1)}{x\lambda_\Phi\mu_{F_i}+x\lambda_\Phi\mu_{G_i}+\lambda_\Phi^2 O(1)} \\
&= \frac{\bar{A}_i+\frac{1}{\mu_{G_i}+\mu_{F_i}}\lambda_\Phi(ER_i^2+\mu_{G_i})O(1)}{1+\frac{1}{\mu_{G_i}+\mu_{F_i}}\lambda_\Phi O(1)} \\
&= \frac{\bar{A}_i[1+\frac{1}{\mu_{G_i}}\lambda_\Phi(ER_i^2+\mu_{G_i})O(1)}{1+\frac{1}{\mu_{G_i}+\mu_{F_i}}\lambda_\Phi O(1)} \\
&= \bar{A}_i[1+o(1)].
\end{aligned}$$

Thus (12) holds true.

It remains to show (16)-(19). From the NBU assumption it follows that

$$\bar{G}_i^0(t)\le \bar{G}_i(t).$$

We have

$$g_{i'}^0(t)=\bar{A}_i^0(t)=1-(A_i*G_i^0)(t)=\bar{G}_i^0(t)+\bar{A}_i(t)-(\bar{A}_i*\bar{G}_i^0)(t)$$

and

$$M_i^0(t)=(M_i*G_i^0)(t)\le M_i(t).$$

Also $M_{i1}^0(t)$ has the same asymptotic limit as $M_{i1}(t)$ when $j\to\infty$. So clearly, M_{i1}^0 is relatively compact.

From (3) we have $m_i(t)\le\lambda_i^t$, and using this inequality we obtain

$$\begin{aligned}
EN_c^0 &= \sum_{i=1}^2\int_0^c \bar{A}_{i'}^0(s)\,dM_i^0(s)=\sum_{i=1}^2\int_0^c \bar{A}_{i'}^0(s)\,(m_i*G_i^0)ds \\
&\le \sum_{i=1}^2\lambda_i^c\int_0^c \bar{A}_{i'}^0(s)\,ds.
\end{aligned}$$

Consequently, (17) holds true if $\bar{A}_i^0(s)\to 0$ as $j\to\infty$. By observing that $\bar{G}_i^0(t)\le \bar{G}_i(t)\to 0$ and

$$(\bar{A}_i*G_i^0)(t)=\int_0^t \bar{A}_i(t-s)dG_i^0(s)\le\ \lambda_i^t\mu_{G_i}\to 0,$$

we have proved (17).

To establish (18), we will show that m_{i2}^0 has the same properties as m_{i2}. We have $m_{i2}^0=m_{i2}*G_i^0$ and thus we only need to show that

$$\lim_{j_0\to\infty,c\to\infty}\sup_{j\ge j_0}\sup_{t\ge c}(\mu_{F_i}+\mu_{G_i})|m_{i2}^0(t)-\frac{1}{\mu_{F_i}+\mu_{G_i}}|=0. \qquad (20)$$

We obtain for $t \geq c' > a > 0$,

$$
\begin{aligned}
&|m_{i2}^0(t) - \frac{1}{\mu_{F_i} + \mu_{G_i}}| \\
\leq\ & |\int_0^t (m_{i2}(t-s) - \frac{1}{\mu_{F_i} + \mu_{G_i}}) dG_i^0(s)| + \bar{G}_i^0(t) \frac{1}{\mu_{F_i} + \mu_{G_i}} \\
\leq\ & \int_0^a |m_{i2}(t-s) - \frac{1}{\mu_{F_i} + \mu_{G_i}}| dG_i^0(s) \\
& + \int_a^t |m_{i2}(t-s) - \frac{1}{\mu_{F_i} + \mu_{G_i}}| dG_i^0(s) + \bar{G}_i^0(a) \frac{1}{\mu_{F_i} + \mu_{G_i}} \\
\leq\ & \sup_{u \geq c'-a} |m_{i2}(u) - \frac{1}{\mu_{F_i} + \mu_{G_i}}| \\
& + [\sup_s \{m_{i2}(s)\} + \frac{1}{\mu_{F_i} + \mu_{G_i}}] \bar{G}_i^0(a) + \bar{G}_i^0(a) \frac{1}{\mu_{F_i} + \mu_{G_i}},
\end{aligned}
$$

which proves (20). Hence (18) follows if we can prove (19). We know that

$$g_{i'}^0(s) - g_{i'}(s) = \bar{A}_i^0(s) - \bar{A}_i(s) = \bar{G}_i^0(t) - (\bar{A}_i * \bar{G}_i^0)(s)$$

and the desired conclusion follows by noting that $\bar{G}_i^0(s) \leq \bar{G}_i(s)$,

$$\bar{G}_i^0(s) \leq \bar{G}_i(s) \leq \frac{ER_i^2}{s^2},$$

$ER_i^2 \to 0$, $\int_c^\infty (1/s^2)\, ds < \infty$, and using the Laplace transform method on $(\bar{A}_i * \bar{G}_i^0)(t)$. From the proof of Theorem 1 it is seen that we simply has to multiply the transform established there by $(1 - L_{G_i^0}(x\lambda_\Phi))$, which converges to 0. The same holds true if replace G_i^0 by G_i. Thus we have proved that all conditions are met, and we can conclude that (N_{t/λ_Φ}) converges to a Poisson process with intensity 1.

Remark 1 *We would like to extend to above result in two directions; to a general monotone system and situations where the repair time distributions G_i are fixed and the life time distributions F_i vary. However, this seems very difficult. We have so far not been able to obtain general results in these directions. One problem is related to the limit (15). It seems difficult to generalize this result to more general systems than a parallel system of two components. If the repair time distributions are supposed to be fixed and the uptime distributions vary we need to decompose M_i in another way than used above and then probably a new approach is required.*

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