ISBN 82-553-0366-9
Mathematics
No 15 - November 22 1978

REGULAR ABELIAN BANACH ALGEBRAS OF LINEAR MAPS OF OPERATOR ALGEBRAS

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PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

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(revised edition)

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This is a corrected and expanded version of an ealier preprint with the same title.

Abstract

If H is a Hilbert space we study regular abelian Banach subalgebras of $B(B(H))$, and mainly algebras generated by maps of the form $x \to axb$ with a and b belonging to an abelian C^*-algebra. Main emphasis is put on the study of the Gelfand transform of maps in these abelian Banach algebras; in particular two versions of positive definiteness of the transforms are shown to be important.

1. Introduction

In recent years it has become apparent that spectral theory for linear maps of von Neumann algebras is intimately connected with Fourier analysis. The present paper is an attempt at obtaining a deeper understanding of this relationship. If $B(H)$ denotes the von Neumann algebra of all bounded linear operators on the Hilbert space H into itself, we shall study abelian Banach subalgebras of $B(B(H))$ - the Banach algebra of bounded linear maps of $B(H)$ into itself. Thus in the process we shall obtain some insight into the extremely complicated Banach algebra $B(B(H))$. The main difficulty encoundered in this Banach algebra is the bad behaviour of its norm. Recall that a theorem of Grothendieck [7] identifies $B(B(H))$ as a Banach space with $(B(H) \hat{\otimes} \mathcal{T})^*$, where $\mathcal{T}$ is the trace class operators on H with the trace norm, and $\hat{\otimes}$ is the projective tensor product of Banach spaces. We shall therefore try to avoid the norm as much as possible and shall restrict attention to maps which are ultraweakly continuous and which map the Hilbert-Schmidt operators $\mathcal{H}$ into themselves, and as operators on $\mathcal{H}$ are normal operators. Such maps will be called <u>operator normal</u>. Furthermore we shall have to require that our abelian Banach algebras will have a well behaved Gelfand theory. We have partly for this reason and partly because this case contains most of the interesting examples, restricted attention to regular abelian Banach algebras of operator normal maps. Then the restriction to $\mathcal{H}$ is a concrete isometric representation of the Gelfand transform. In particular it should be noted that since abelian C^*-algebras are semi-simple our abelian Banach algebras will automatically be semi-simple.

With these preliminaries we are now ready to give an outline of the paper. If G is a locally compact abelian group represented as

*-automorphisms of a von Neumann algebra, Arveson, Borchers and Connes [2,3,5] developed the theory of spectral subspaces. In § 2 we shall generalize to regular abelian Banach algebras acting continuously on a locally convex topological vector space, much of that part of the theory of spectral subspaces which does not depend essentially on the group structure of the dual group of G.

In §3 we prove the basic general results on operator normal maps. We assume the operator normal map φ is contained in a regular abelian Banach subalgebra of $B(B(H))$. Then it follows from §2 that the spectrum of φ in $B(B(H))$ is the same as the spectrum of $\varphi|\mathcal{H}$ in $B(\mathcal{H})$. A consequence of this is that if the spectrum of φ in $B(B(H))$ is contained in the unit circle and $\varphi(1) = 1$ then φ is either a *-automorphism or a *-anti-automorphism.

In §4 we give examples of regular and nonregular abelian Banach subalgebras of $B(B(H))$. If $a,b \in B(H)$ we denote by L_a and R_b the maps $x \to ax$ and $x \to xb$ respectively. Then L maps every C^*-subalgebra of $B(H)$ isometrically into $B(B(H))$. If we denote by $a \otimes b$ the map L_aR_b, we can imbed the algebraic tensor product of two abelian C^*-algebras A and B into $B(B(H))$. The norm is a cross-norm, so the closure $A \otimes B$ is a regular abelian Banach subalgebra of $B(B(H))$ consisting of operator normal maps.

If G is a locally compact abelian group and α a continuous representation of G into the automorphism group of $B(H)$, and $\mu \in M(G)$ - the bounded Borel measures on G, $\alpha_\mu \in B(B(H))$, where $\alpha_\mu(x) = \int_G \alpha_t(x)d\mu(t)$. Then the image of $L^1(G)$ has as closure in $B(B(H))$ a regular abelian Banach algebra consisting of operator normal maps. However, the image of $M(G)$ need not have regular closure.

If f is a complex function on a product space $X \times X$ we say f is positive definite if whenever $\gamma_1, \dots, \gamma_n \in X$ then the $n \times n$ matrix $(f(\gamma_i, \gamma_j))$ is positive. This concept is useful in order to study maps in $A \otimes A$, where A is an abelian C^*-algebra, because the spectrum $Sp(A \otimes A)$ can be identified with $SpA \times SpA$. It is shown in §5 that if $\varphi \in A \otimes A$ then φ is a positive map if and only if φ is completely positive, and that this in turn is equivalent to the Gelfand transform $\hat{\varphi}$ of φ being a positive definite function on $SpA \times SpA$. In addition it is pointed out that if φ furthermore satisfies $\varphi(1) = 1$ then $Tr(\varphi(x)) = Tr(x)$ for all trace class operators x. The section is concluded by noting that the case $A \otimes A$ includes the examples $\alpha(L^1(G))$ exhibited in §4, so that our results for $\varphi \in A \otimes A$ are applicable to maps of the form α_f, $f \in L^1(G)$.

In the following two sections we study the converse type of problem, namely, given a map in $B(B(H))$, when can we conclude that it belongs to an algebra of the form $A \otimes A$? In the infinite dimensional case we can only reach conclusions like the map belongs to the point-ultraweak closure of $A \otimes A$. Note that if H is finite dimensional, then every map in $A \otimes A$ has a complete set of eigenvectors in the Hilbert-Schmidt operators $\mathcal{H}$ consisting of rank 1 operators. In §6 we show a converse to this result for positive maps.

Since a positive map $\varphi \in A \otimes A$ is completely positive it has a decomposition $\varphi = V^* \pi V$, where V is a bounded linear map of H into a Hilbert space K, and π is a $*$-representation of $B(H)$ on K. In §7 we show that if $\varphi(1) = 1$, φ restricted to A is the identity, and the above decomposition is in a suitably nice position, then φ is an average over automorphisms in $A \otimes A$, hence in particular φ belongs to the point-ultraweak closure of $A \otimes A$.

The last result is relevant in the study of a certain class of $n \times n$ matrices, namely the closed convex set K_n of matrices spanned by the positive rank 1 matrices of the form $(z_i \bar{z}_j)$, where $|z_1| = \ldots = |z_n| = 1$. Let (e_{ij}) denote the usual matrix units for the $n \times n$ matrices M_n, so that if $a = (\alpha_{ij}) \in M_n$ then $a = \Sigma \alpha_{ij} e_{ij}$. Let D_n be the diagonal matrices, so D_n is spanned by $e_{11}, \ldots, e_{nn}$. With a as above and $\tilde{a} = \Sigma \alpha_{ij} e_{ii} \otimes e_{jj} \in D_n \otimes D_n$, then $\tilde{a}(b) = a * b$, is the Hadamard product of a and the matrix b. In §8 we give characterizations for a matrix a to belong to K_n in terms of properties of the Hadamard product with a and also in terms of the existence of certain positive definite functions on $\mathbb{Z}^n$.

Finally, in §9 we show that a map $\varphi \in A \otimes A$ is of the form α_μ described in §4, where μ is a Borel probability measure on a compact abelian group, if $(\hat{\varphi}(\gamma_i, \gamma_j)) \in K_n$ whenever $\gamma_1, \ldots, \gamma_n \in \mathrm{Sp}A$. Thus this stronger form of positive definiteness implies the stronger result that $\varphi = \alpha_\mu$ rather than just positive.

The author is happy to express his indebtness to Jørgen Vesterstrøm for pointing out serious errors in early versions of Proposition 8.1.

2. Spectral subspaces

Let X be a locally convex topological co
Let A be a regular abelian semi-simple Banach
complex numbers with an approximate unit consis
whose Gelfand transforms are real and with comp
[13,14]. We assume X is a left A-module via
$(a,x) \to ax$, which is separately continuous and
variables. Our typical example will be when A
represented into the algebra of continuous line

If $S \subset A$ and $Y \subset X$ we let

$$S^{\perp} = \{x \in X : ax = 0 \quad \text{for all} \quad a \in S\}$$
$$Y_{\perp} = \{a \in A : ay = 0 \quad \text{for all} \quad y \in Y\}$$

Clearly $Y_{\perp}$ is a closed ideal in A. We let
maximal ideal space in A, identified with the
characters on A. SpA is given the hull-kern

If $a \in A$ we denote by

$$Z(a) = \{\gamma \in \mathrm{Sp}A : \gamma(a) = \hat{a}(\gamma) = 0\} .$$

If $F \subset \mathrm{Sp}A$ is a closed subset we let

$$j(F) = \{a \in A : Z(a) \ \text{contains a neighborhoo} \\ \text{and support } \hat{a} \text{ is compact}\} .$$

We recall from [13, 25 D] that $j(F)$ is the smal
whose hull is F. We denote by

$$X(A,F) = j(F)^{\perp} .$$

Then $X(A,F)$ is a closed subspace of X, calle
<u>subspace</u> of F. Finally if $x \in X$ we denote

$$Sp(x) = h(\{x\}_\perp) ,$$

the hull of the annihilator of x in A. $Sp(x)$ is a closed subset of SpA. Furthermore $Sp(x) = \emptyset$ if and only if $x = 0$. Indeed, $h(\{x\}_\perp) = \emptyset$ if and only if $\{x\}_\perp = A$ [13, 25 D Corollary], if and only if $ax = 0$ for all $a \in A$, if and only if $x = 0$, since the representation $A \times X \to X$ is faithful in both variables.

Lemma 2.1 Let A and X be as above. Let F be a closed subset of A, $a \in A$ and $x \in X$. Then we have

(i) If $Z(a)$ contains a neighborhood of $Sp(x)$ then $ax = 0$.

(ii) $x \in X(A,F)$ if and only if $Sp(x) \subset F$.

(iii) If $\operatorname{supp} \hat{a} \subset F$ then $ax \in X(A,F)$.

Proof (i) By assumption $h(\{x\}_\perp)$ is contained in the interior of $Z(a)$. By the assumption on approximate unit in A there is $b \in A$ such that $\hat{b}$ has compact support and $\|ab-a\| < \varepsilon$ for given $\varepsilon > 0$. Then $h(\{x\}_\perp)$ is contained in the interior of $Z(ab)$, so $ab \in j(h(\{x\}_\perp))$. By [13, 25 D] $ab \in \{x\}_\perp$, i.e. $abx = 0$. Since $\varepsilon > 0$ is arbitrary and $c \to cx$ is continuous on A, $ax = 0$.
(ii) Suppose $Sp(x) \subset F$. If $a \in j(F)$ then $Z(a)$ contains a neighborhood of $Sp(x)$, so that $ax = 0$ by (i). Thus $x \in j(F)^\perp = X(A,F)$. Conversely, let $x \in X(A,F)$. Then $\{x\}_\perp \supset (j(F)^\perp)_\perp \supset j(F)$. Thus $h(\{x\}_\perp) \subset h(j(F)) = F$ [13, 25 D].
(iii) Suppose $\gamma \notin \operatorname{supp} \hat{a}$. Then, since A is regular, there is $b \in A$ such that $\hat{b}(\gamma) \neq 0$ while $ab = 0$. Thus $b(ax) = ba(x) = 0$. But then $\gamma \notin Sp(ax)$, so we have shown $Sp(ax) \subset \operatorname{supp} \hat{a}$. Now use (ii).

We denote by $\tilde{A}$ the algebra A with the identity map of X

adjoined, and we consider X as an $\tilde{A}$-module as well. Note that by [14, 2.7.3] $\tilde{A}$ is regular, and we can consider SpA as a subset of $Sp\tilde{A}$.

<u>Lemma 2.2</u> Let F be a compact subset of SpA. Then

(i) $X(\tilde{A},F) \supset X(A,F)$.

(ii) If $a \in A$ and $\hat{a}(\gamma) = 1$ for all γ in a neighborhood of F then $ax = x$ for all $x \in X(A,F)$.

<u>Proof</u> (i) Let

$$i(F) = \{a \in A : Z(a) \text{ contains a neighborhood of } F\}.$$

Then $i(F) \supset j(F)$, so $i(F)^{\perp} \subset j(F)^{\perp}$. Let $x \in X(A,F)$, and $a \in i(F)$. Then $ax = 0$ by Lemma 2.1 so $x \in i(F)^{\perp}$, and $i(F)^{\perp} = j(F)^{\perp}$. However, $X(\tilde{A},F) \supset i(F)^{\perp}$ since $\operatorname{supp} \hat{a}$ is compact in $Sp\tilde{A}$ for all $a \in \tilde{A}$.

(ii) Let ι denote the identity in $\tilde{A}$. Then $\hat{a} - \hat{\iota}$ is zero in a neighborhood of F. Let $x \in X(A,F)$. By (i) $x \in X(\tilde{A},F)$, so by Lemma 2.1 $(a-\iota)x = 0$, i.e. $ax = x$.

We say a subset Y of X is <u>bounded</u> if for each absorbing neighborhood V of 0 in X there is $\varepsilon > 0$ such that $\varepsilon Y \subset V$. The following result is a generalization of [5, 2.3.5].

<u>Proposition 2.3</u> Let V be an absorbing neighbourhood of 0 in X, and let Y be a bounded subset of X such that $a(Y) \subset \|a\| Y$ for all $a \in A$. Let $\gamma_0 \in SpA$ and $a_1,\dots,a_n \in A$. Then there is a compact neighborhood N of γ_0 in SpA such that

$$a_i x - \hat{a}_i(\gamma_0) x \in V \quad \text{for all } x \in Y \cap X(A,N), \quad i = 1,\dots,n.$$

Proof Since V is an absorbing neighborhood of 0 in X and Y is bounded, there exists $\varepsilon > 0$ such that $\varepsilon Y \subset V$. Thus by our assumption on Y, $a(Y) \subset V$ whenever $\|a\| < \varepsilon$. Let N be a compact neighborhood of γ_0 and $a \in A$ such that $\hat{a}(\gamma) = 1$ for $\gamma \in N_1$. For each $i \in \{1,\dots,n\}$ let $b_i \in A$ be defined by $\hat{b}_i(\gamma) = (\hat{a}_i(\gamma) - \hat{a}_i(\gamma_0))\hat{a}(\gamma)$. Then $\hat{b}_i(\gamma_0) = 0$, and $\hat{b}_i(\gamma) = \hat{a}_i(\gamma) - \hat{a}_i(\gamma_0)$ on N_1. From the regularity of A there is $c \in A$ such that $\max_i \|b_i c\| < \varepsilon$ and $\hat{c}(\gamma) = 1$ for all γ in a neighborhood N_2 of γ_0. Let N be a compact neighborhood of γ_0 contained in the interior of $N_1 \cap N_2$. Let $x \in Y \cap X(A,N)$. Now $\hat{c}(\gamma) = 1$ for γ in a neighborhood of N, and N contains $Sp(x)$ by Lemma 2.1. Thus $cx = x$ by Lemma 2.2, and similarly $ax = x$. We thus have $b_i cx = b_i x = a_i ax - \hat{a}_i(\gamma_0)ax = a_i x - \hat{a}_i(\gamma_0)x$. Since

$\|b_i c\| < \varepsilon$, $b_i c(Y) \subset V$. Thus $a_i x - \hat{a}_i(\gamma_0)x \in V$ for all $x \in Y \cap X(A,N)$. Q.E.D.

If E is a Banach algebra we denote by $\sigma_E(x)$ the spectrum of x as an element in E.

Corollary 2.4 Suppose X is a Banach space and that the identity operator is in A. Let $a \in A$. Then $\sigma_{B(X)}(a) = \{\hat{a}(\gamma) : \gamma \in \mathrm{Sp}A\} = \sigma_A(a)$.

Proof Given $\varepsilon > 0$ let $V = \{x \in X : \|x\| < \varepsilon\}$, and let Y be the unit ball in X. If $\gamma_0 \in \mathrm{Sp}A$, then by Lemma 2.1 (iii) $Y \cap X(A,N) \neq (0)$ for each compact neighborhood N of γ_0. Thus $\hat{a}(\gamma_0) \in \sigma_{B(X)}(a)$ by Proposition 2.3. Since $\{\hat{a}(\gamma) : \gamma \in \mathrm{Sp}A\} = \sigma_A(a)$, we have shown $\sigma_{B(X)}(a) \supset \sigma_A(a)$. The converse inclusion is immediate, since we can consider A as a Banach subalgebra of $B(X)$ containing the identity.

It should be remarked that just as in the theory of spectral subspaces of automorphisms we can introduce the auxiliary concept $R(A,E)$, cf. [2] and then prove that $X(A,E) = \cap R(A,V)$, where the intersection is taken over all closed neighborhoods V of E, see the proof of [2, Proposition 2.2]. However, we shall not need this and shall therefore not include the proof. We shall rather prove another result which we shall not need technically, but which is of importance for our understanding of spectral subspaces.

Proposition 2.5 Let B be a Banach subalgebra of A satisfying the same assumptions as A. Let $r : \mathrm{Sp}A \to \mathrm{Sp}B$ be the restriction map $\gamma \to \gamma | B$. Suppose F is a compact subset of $\mathrm{Sp}B$ such that $r^{-1}(F)$ is compact in $\mathrm{Sp}A$. Then we have $X(A,r^{-1}(F)) = X(B,F)$.

Proof To our previous notation add the subscripts A or B to distinguish between A and B. Let $x \in X(B,F)$. Then by Lemma 2.1 $h_B(\{x\}_\perp \cap B) \subset F$, hence $r^{-1}(h_B(\{x\}_\perp \cap B)) \subset r^{-1}(F)$. Therefore we have that if J_x is the ideal in A generated by $\{x\}_\perp \cap B$, then

$$\begin{aligned} r^{-1}(F) &\supset r^{-1}(h_B(\{x\}_\perp \cap B)) \\ &= \{\gamma \in \mathrm{Sp}A : \ker\gamma \supset J_x\} \\ &= h(J_x) \\ &\supset h(\{x\}_\perp), \end{aligned}$$

since $J_x \subset \{x\}_\perp$. Thus $\mathrm{Sp}_A(x) \subset r^{-1}(F)$, hence by Lemma 2.1 $x \in X(A,r^{-1}(F))$, and we have shown $X(B,F) \subset X(A,r^{-1}(F))$.

Conversely let $x \in X(A,r^{-1}(F))$; then $h_A(\{x\}_\perp) \subset r^{-1}(F)$. Let $b \in j_B(F)$. Then $Z_B(b) \supset F$. If $\gamma \in r^{-1}(F)$ then $r(\gamma) \in F$ so $\hat{b}(r(\gamma)) = 0$, hence $b \in \ker(r(\gamma)) = (\ker\gamma) \cap B$. Therefore $\hat{b}(\gamma) = 0$, so $\gamma \in Z_A(b)$, and we have shown $r^{-1}(F) \subset Z_A(b)$.

Since F is compact and $Z_B(b)$ contains a neighborhood of F, there is a compact neighborhood N of F contained in $Z_B(b)$. Since r is continuous $r^{-1}(N)$ is a neighborhood of $r^{-1}(F)$, and by the above argument $r^{-1}(N) \subset Z_A(b)$. Thus $Z_A(b)$ is a neighborhood of $r^{-1}(F)$, hence by the definition in Lemma 2.2 $b \in i_A(r^{-1}(F))$. From the proof of that lemma $i_A(r^{-1}(F))^{\perp} = j_A(r^{-1}(F))^{\perp} = X(A,r^{-1}(F))$. Since $x \in X(A,r^{-1}(F))$ it thus follows from Lemma 2.1 that $bx = 0$. Since b was arbitrary in $j_B(F)$, we have shown $X(A,r^{-1}(F)) \subset j_B(F)^{\perp} = X(B,F)$, and the proof is complete.

3. Operator normal maps

Let H be a Hilbert space and $\mathcal{T}$ and $\mathcal{H}$ the trace class and Hilbert-Schmidt operators on H respectively. We denote the inner product on $\mathcal{H}$ by $\langle x,y\rangle = \mathrm{Tr}(xy^*)$ and the norms in $\mathcal{T}$ and $\mathcal{H}$ by $\|\ \|_1$ and $\|\ \|_2$ respectively.

Definition 3.1 Let $\varphi \in B(B(H))$. We say φ is operator normal if φ is ultraweakly continuous and the restriction $\varphi|\mathcal{H}$ is a normal operator in $B(\mathcal{H})$. If moreover $\varphi|\mathcal{H}$ is self-adjoint we say φ is operator hermitian. φ is said to be a regular operator normal map if φ is contained in a regular abelian Banach subalgebra of $B(B(H))$ consisting of operator normal maps.

We denote by $\|\varphi\|_2$ the norm of $\varphi|\mathcal{H}$ whenever $\varphi|\mathcal{H} \in B(\mathcal{H})$. Note that when φ is ultraweakly continuous then its adjoint map restricts to a map $\varphi^* \in B(\mathcal{T})$ with norm $\|\varphi^*\| = \|\varphi\|$.

Lemma 3.2 Let $\varphi \in B(B(H))$ be regular and operator normal, and denote by ψ the adjoint in $B(\mathcal{H})$ of $\varphi|\mathcal{H}$. Then $\psi|\mathcal{T} = \varphi^*$, and $\|\varphi\|_2 \leq \|\varphi\|$.

Proof Let $x \in \mathcal{T}$ and $y \in \mathcal{H}$. Then $\langle\psi(x),y\rangle = \langle x,\varphi(y)\rangle = \langle\varphi^*(x),y\rangle$, so $\psi(x) = \varphi^*(x)$. Let A be a regular abelian Banach subalgebra of $B(B(H))$ consisting of operator normal maps such that $\varphi \in A$. Let r denote the restriction map $\psi \to \psi|\mathcal{H}$ of A into $B(\mathcal{H})$. Then r is continuous. Indeed, if (ψ_n) is a sequence in A converging to ψ, and $r(\psi_n)$ converges to ψ' in $B(\mathcal{H})$ then clearly $\psi(x) = \psi'(x)$ for each $x \in \mathcal{H}$. Thus the graph of r is closed, so r is continuous by the closed graph theorem. Since r is an isomorphism of A into $B(\mathcal{H})$ it follows that

$\sigma_A(\varphi) \supset \sigma_{B(\mathcal{H})}(\varphi)$, hence the spectral radius of φ in $B(\mathcal{H})$ is not larger than the spectral radius s of φ in A. But $\|\varphi\|_2$ equals the spectral radius of φ in $B(\mathcal{H})$, so $\|\varphi\|_2 \leq s$. By the minimality of the spectral radius norm in a regular abelian Banach algebra [14, 3.7.7] we have $\|\varphi\|_2 \leq s \leq \|\varphi\|$, Q.E.D.

<u>Theorem 3.3</u> Let A be a regular abelian Banach subalgebra of $B(B(H))$ consisting of operator normal maps. Then the map $\hat{\varphi} \to \varphi|\mathcal{H}$ is an isometric isomorphism of $\{\hat{\varphi} : \varphi \in A\}$ onto $\{\varphi|\mathcal{H} : \varphi \in A\}$, which extends to an isomorphism of $C(SpA)$ onto the closure of $\{\varphi|\mathcal{H} : \varphi \in A\}$ in $B(\mathcal{H})$, where $C(SpA)$ denotes the continuous complex functions on SpA vanishing at infinity.

<u>Proof</u> Let $\alpha(\hat{\varphi}) = \varphi|\mathcal{H}$ for $\varphi \in A$. Then clearly α is an isomorphism of $\{\hat{\varphi} : \varphi \in A\}$ onto $\{\varphi|\mathcal{H} : \varphi \in A\}$. Let $r(\varphi) = \varphi|\mathcal{H}$. By Lemma 3.2 r is norm decreasing on A, hence if χ is a character on the norm closure of $r(A)$ in $B(\mathcal{H})$ then $\chi \circ r \in SpA$. Thus for $\varphi \in A$ we have

$$\|\varphi\|_2 = \sup_\chi |\chi \circ r(\varphi)| \leq \sup_{\gamma \in SpA} |\gamma(\varphi)| = \|\hat{\varphi}\|,$$

and α is norm decreasing. However, $\|\hat{\varphi}\|$ is the spectral radius of φ in A, so by the minimality of the spectral radius [14, 3.7.7], $\|\hat{\varphi}\| \leq \|\varphi\|_2$. Thus $\|\hat{\varphi}\| = \|\varphi\|_2$, and the theorem follows.

<u>Corollary 3.4</u> If φ is a regular operator normal map in $B(B(H))$ then $\sigma_{B(B(H))}(\varphi) = \sigma_{B(\mathcal{H})}(\varphi|\mathcal{H})$.

<u>Proof</u> φ is contained in a regular abelian Banach subalgebra of $B(B(H))$ consisting of operator normal maps and containing the identity map. Thus the corollary follows from Corollary 2.4 and Theorem 3.3.

The next result will not be used in the sequel but is included because its proof is a good illustration of the techniques and ideas involved.

Proposition 3.5 Let φ be a regular operator normal map in the unit ball of $B(B(H))$ such that $\varphi(1) = 1$ and such that its spectrum in $B(B(H))$ is contained in the unit circle. Then φ is either a $*$-automorphism or $*$-anti-automorphism of $B(H)$.

Proof By Corollary 3.4 the spectrum of $\varphi|\mathcal{H}$ in $B(\mathcal{H})$ is contained in the unit circle, so $\varphi|\mathcal{H}$ normal implies $\varphi|\mathcal{H}$ is unitary. In particular, since $\varphi^{-1} \in B(B(H))$, $\varphi^{-1}|\mathcal{H}$ is the adjoint of $\varphi|\mathcal{H}$. Since $\|\varphi\| = 1$ and $\varphi(1) = 1$, φ is positive (i.e. $a \geq 0$ in $B(H)$ implies $\varphi(a) \geq 0$). Thus if $x,y \in \mathcal{T}^+$ - the positive cone in $\mathcal{T}$ - then $\varphi^*(y) \in \mathcal{T} \subset \mathcal{H}$, so

$$0 \leq \langle\varphi(x),y\rangle = \langle x,\varphi^*(y)\rangle = \langle x,\varphi^{-1}(y)\rangle,$$

hence $\varphi^{-1}(y) \geq 0$. Thus $\varphi^{-1} : \mathcal{T}^+ \to \mathcal{T}^+$. Since φ^{-1} is norm continuous on $B(H)$, $\varphi^{-1} : C(H)^+ \to C(H)^+$, where $C(H)$ denotes the compact operators on H, using that $\mathcal{T}^+$ is norm dense in $C(H)^+$. Let B be the C^*-algebra $\mathbb{C}1 + C(H)$. Then φ^{-1} is a positive linear map of B carrying 1 on itself. Since φ is operator normal, $\varphi : \mathcal{H} \to \mathcal{H}$, hence by continuity, $\varphi : C(H) \to C(H)$. Thus φ is also a positive linear map of B into itself preserving the identity, so that φ is an order-isomorphism of B onto itself, hence is either a $*$-automorphism or a $*$-anti-automorphism [9]. By ultraweak continuity of φ the desired result follows.

We shall need the next result in the next section.

<u>Lemma 3.6</u> Let $(\varphi_\nu)_{\nu\in J}$ be a uniformly bounded net of regular operator normal maps, which converges poinwise ultraweakly to a map $\varphi \in B(B(H))$. Then we have:

(i) $\varphi|\mathcal{H} \in B(\mathcal{H})$.

(ii) $\varphi_\nu|\mathcal{H} \to \varphi|\mathcal{H}$ weakly in $B(\mathcal{H})$, so in particular $(\varphi_\nu|\mathcal{H})^* \to (\varphi|\mathcal{H})^*$ weakly.

(iii) If the φ_ν pairwise commute then $\varphi|\mathcal{H}$ is normal.

(iv) If $\varphi_\nu \to \varphi$ in norm then φ is ultraweakly continuous.

Consequently, if (iii) and (iv) hold then φ is operator normal.

<u>Proof</u> Choose $K \geq 0$ such that $\|\varphi_\nu\| \leq K$ for all $\nu \in J$. By Lemma 3.2 $\|\varphi_\nu\|_2 \leq \|\varphi_\nu\| \leq K$, so $(\varphi_\nu|\mathcal{H})_{\nu\in J}$ is a uniformly bounded net in $B(\mathcal{H})$. Thus there is a subnet $(\varphi_\alpha)_{\alpha\in I}$ such that $(\varphi_\alpha|\mathcal{H})_{\alpha\in I}$ converges weakly to an operator $\psi \in B(\mathcal{H})$, i.e.

$$\langle\psi(x),y\rangle = \lim_\alpha \langle\varphi_\alpha(x),y\rangle = \lim_\alpha \mathrm{Tr}(\varphi_\alpha(x)\, y^*)$$

for all $x,y \in \mathcal{H}$. Now $(\varphi_\alpha)_{\alpha\in I}$, being a subnet of the converging net $(\varphi_\nu)_{\nu\in J}$, converges pointwise ultraweakly to φ. Thus if $y \in \mathcal{T}$

$$\langle\psi(x),y\rangle = \lim_\alpha \mathrm{Tr}(\varphi_\alpha(x)y^*) = \mathrm{Tr}(\varphi(x)y^*) = \langle\varphi(x),y\rangle,$$

and $\psi(x) = \varphi(x)$ for all $x \in \mathcal{H}$, so $\varphi : \mathcal{H} \to \mathcal{H}$. Furthermore $\varphi_\nu|\mathcal{H} \to \varphi|\mathcal{H}$ weakly since each converging subnet does. Since $\|\varphi_\nu\|_2 \leq K$ for all ν, $\|\varphi\|_2 \leq K$, hence $\varphi|\mathcal{H} \in B(\mathcal{H})$. This proves (i) and (ii).

Now assume all the φ_ν commute, and let $M \subset B(\mathcal{H})$ be the abelian von Neumann algebra generated by all the maps $\varphi_\nu|\mathcal{H}$. Since $\varphi_\nu|\mathcal{H} \to \varphi|\mathcal{H}$ weakly, $\varphi|\mathcal{H} \in M$. Since $(\varphi_\nu|\mathcal{H})^* \to (\varphi|\mathcal{H})^*$ weakly, we have by Lemma 3.2 that $(\varphi|\mathcal{H})^*|\mathcal{T} = \varphi^*$. Since $(\varphi|\mathcal{H})^* \in M$, $\varphi|\mathcal{H}$ is normal, and (iii) follows.

If $\varphi_\nu \to \varphi$ in norm, then $\omega\circ\varphi_\nu \to \omega\circ\varphi$ in norm for each $\omega \in B(H)_*$; hence $\omega\circ\varphi$ is ultraweakly continuous for each $\omega \in B(H)_*$, and φ is itself ultraweakly continuous. This concludes the proof of (iv) and therefore of the lemma.

4. Examples of regular algebras

The most easily obtained examples of regular abelian algebras of operator normal maps are of the form $x \to ax = L_a x$ and $x \to xa = R_a x$, where a belongs to an abelian C^*-algebra A. Both L_a and R_a are isometric isomorphisms since A is abelian. When A is not abelian L_a is still an isometric isomorphism, so that every C^*-algebra $A \subset B(H)$ has a canonical isometric imbedding in $B(B(H))$.

We denote the map $L_a R_b = R_b L_a$ by $a \otimes b$ for $a,b \in B(H)$. Taking linear combinations we can in this way consider the algebraic tensor product $B(H) \odot B(H)$ as a subset of $B(B(H))$ consisting of ultraweakly continuous maps, which restrict to bounded operators in $B(\mathcal{H})$. If $x,y \in \mathcal{H}$, then $\langle L_a x,y\rangle = \mathrm{Tr}(ax\,y^*) = \mathrm{Tr}(x(a^* y)^*) = \langle x,a^* y\rangle$, so $L_a^* = L_{a^*}$ and similarly $R_a^* = R_{a^*}$. Thus the restriction map $B(H) \odot B(H) \to B(\mathcal{H})$ is $*$-preserving when $B(H) \odot B(H)$ has the $*$-operation $(\Sigma a_i \otimes b_i)^* = \Sigma a_i^* \otimes b_i^*$. Note that since R_b is anti-isomorphic in b the imbedding of $B(H) \odot B(H)$ into $B(B(H))$ is not an algebraic isomorphism. However, if A and B are abelian subalgebras of $B(H)$, then the imbedding of $A \odot B$ in $B(B(H))$ is a $*$-isomorphism.

Lemma 4.1 The norm on $B(B(H))$ restricts to a cross norm on $B(H) \odot B(H)$.

Proof Let $a,b \in B(H)$. Then clearly $\|a \otimes b\| \leq \|a\|\,\|b\|$. To show the converse inequality let $\varepsilon > 0$ and choose unit vectors $\xi,\eta \in H$ such that $\|a\xi\| \geq \|a\| - \varepsilon$ and $\|b\eta\| \geq \|b\| - \varepsilon$. Let v be a partial isometry of rank 1 such that $vb\eta = \|b\eta\|\xi$. Then $\|avb\eta\| = \|b\eta\|\,\|a\xi\| \geq (\|b\|-\varepsilon)(\|a\|-\varepsilon)$, hence $\|a\otimes b\| \geq \|a\|\,\|b\|$.

Proposition 4.2 Let A and B be abelian C^*-subalgebras of $B(H)$. Then the closure $A \otimes B$ of $A \odot B$ in $B(B(H))$ is a regular abelian Banach subalgebra consisting of operator normal maps.

Proof By Lemma 3.6 each map in $A \otimes B$ is operator normal. The rest is immediate from Lemma 4.1 and a result of Tomiyama [21].

Remark 4.3 By Proposition 4.2 each map of the form $a \otimes b$ with a and b normal, is regular in the sense of Definition 3.1. It can be shown that even more is true, namely that the Banach subalgebra of $B(B(H))$ generated by $a \otimes b$ is regular.

If G is a locally compact abelian group we denote by $M(G)$ its measure algebra, consisting of all bounded Borel measures with convolution as multiplication and $*$-operation $\tilde{\mu}(E) = \overline{\mu(-E)}$. We write multiplication in G and its dual $\hat{G}$ additively. I am indebted to G.K. Pedersen for discussions which led to Proposition 4.6.

Lemma 4.4 Let G be a locally compact abelian group and $t \to u_t$ a continuous unitary representation of G on the Hilbert space H. Let $\alpha_t(x) = u_t x u_t^*$, $x \in B(H)$. Then for each $\mu \in M(G)$, α_μ defined by $\alpha_\mu(x) = \int \alpha_t(x) d\mu(t)$, is an operator normal map such that $(\alpha_\mu | \mathcal{H})^* = \alpha_{\tilde{\mu}} | \mathcal{H}$.

Proof It is easy to see that $t \to \alpha_t | \mathcal{H}$ is a continuous unitary representation, cf. [19]. Thus $\alpha_\mu | \mathcal{H} \in B(\mathcal{H})$. If $x, y \in \mathcal{H}$ we have

$$
\begin{aligned}
\langle \alpha_\mu(x), y \rangle &= \int \langle \alpha_t(x), y \rangle d\mu(t) \\
&= \int \langle x, \alpha_{-t}(y) \rangle d\mu(t) \\
&= \int \langle x, \alpha_t(y) \rangle d\mu(-t) \\
&= \langle x, \int \alpha_t(y) d\overline{\mu(-t)} \rangle \\
&= \langle x, \alpha_{\tilde{\mu}}(y) \rangle .
\end{aligned}
$$

Thus $(\alpha_\mu | \mathcal{H})^* = \alpha_{\tilde{\mu}} | \mathcal{H}$. Since $\alpha_\mu \circ \alpha_{\tilde{\mu}} = \alpha_{\mu * \tilde{\mu}} = \alpha_{\tilde{\mu} * \mu} = \alpha_{\tilde{\mu}} \circ \alpha_\mu$, α_μ commutes with its adjoint, so $\alpha_\mu | \mathcal{H}$ is a normal operator. Finally, it follows from [2] that α_μ is ultraweakly continuous, hence α_μ is operator normal.

<u>Lemma 4.5</u> Let G be a locally compact abelian group. Then the map $T : M(G) \to B(L^\infty(G))$ defined by $T_\mu(f) = \mu * f$ for $f \in L^\infty(G)$, is an isometric isomorphism into.

<u>Proof</u> It is well known and easy that T is an isomorphism into $B(L^\infty(G))$. Moreover, it is shown in the proof of [12, 3.4.1] that T_μ is a continuous multiplier of $L^\infty(G)$ endowed with the weak-* topology induced by the elements in $L^1(G)$, and furthermore that the adjoint map T_μ^* is a continuous multiplier of $L^1(G)$. By [12, 0.1.1] $\|T_\mu^*\| = \|\mu\|$, hence $\|T_\mu\| = \|\mu\|$.

<u>Proposition 4.6</u> Let G be a locally compact abelian group and $H = L^2(G)$. Then there is a canonical isometric isomorphism of $M(G)$ into the operator normal maps in $B(B(H))$ such that $\alpha_{\tilde{\mu}} | \mathcal{H} = (\alpha_\mu | \mathcal{H})^*$.

<u>Proof</u>. Let λ be the regular representation of G on H, and let S be the *-isomorphism of $L^\infty(G)$ into $B(H)$ defined by $S_f g = fg$ for $g \in L^2(G)$. Let $\alpha_t(x) = \lambda_t x \lambda_{-t}$ for $x \in B(H)$. By Lemma 4.4 α_μ is operator normal, and $\alpha_{\tilde{\mu}} | \mathcal{H} = (\alpha_\mu | \mathcal{H})^*$. Furthermore, $\alpha_\mu(S_f) = \alpha_{\mu * f}$ for each $f \in L^\infty(G)$. Indeed, let $g \in L^2(G)$ and $s,t \in G$. Then we have, with $g_t(u) = g(u-t), u \in G$,

$$\begin{aligned}
&(\alpha_t(S_f)g)(s) = (\lambda_t(S_f(\lambda_{-t}g)))(s) \\
&= (\lambda_t(S_f g_{-t}))(s) = f(s-t)\, g(s) \\
&= (f_t g)(s) = (S_{f_t} g)(s)
\end{aligned}$$

hence $\alpha_t(S_f) = S_{f_t}$. Let $g,h \in L^2(G)$; then we have, using the Fubini theorem,

$$\begin{aligned}
<\alpha_\mu(S_f)g,h> &= \int<\alpha_t(S_f)g,h>d\mu(t) \\
&= \int\int(\alpha_t(S_f)g)(s)\overline{h(s)}\, ds\, d\mu(t) \\
&= \int\int f(s-t)g(s)\overline{h(s)}\, ds\, d\mu(t) \\
&= \int g(s)\overline{h(s)}\, (\int f(s-t)d\mu(t))ds \\
&= \int g(s)\overline{h(s)}(\mu * f)(s)ds \\
&= <S_{\mu * f}g,h> ,
\end{aligned}$$

and $\alpha_\mu(S_f) = S_{\mu * f}$ as asserted. From the definition of α_μ it is clear that $\|\alpha_\mu\| \leq \|\mu\|$. However, we have just shown that $\alpha_\mu : S_{L^\infty(G)} \to S_{L^\infty(G)}$, and since S is an isometry, we have

$$\|\alpha_\mu(S_f)\| = \|S_{\mu * f}\| = \|\mu * f\| .$$

By Lemma 4.5 we thus have

$$\|\alpha_\mu\| \geq \sup_{\|S_f\|=1} \|\alpha_\mu(S_f)\| = \sup_{\|f\|_\infty=1} \|\mu * f\| = \|\mu\| ,$$

hence $\|\alpha_\mu\| = \|\mu\|$, and we are through.

<u>Corollary 4.7</u> Let G be a locally compact abelian group and $H = L^2(G)$. Then there is a canonical isometric isomorphism of $L^1(G)$ onto a regular abelian subalgebra of $B(B(H))$ consisting of operator normal maps.

<u>Proof</u> Restrict α in Proposition 4.6 to $L^1(G)$, and use that $L^1(G)$ is regular.

If H is a finite dimensional Hilbert space it is obvious that every operator normal map in $B(B(H))$ is regular. However, if H is infinite dimensional this appears to be false.

Corollary 4.8 If H is a separable infinite dimensional Hilbert space, there exists an operator normal map φ in $B(B(H))$ such that the Banach subalgebra of $B(B(H))$ generated by φ is nonregular.

Proof Let G be a nondiscrete locally compact abelian group such that $L^2(G)$ is separable, and identify $L^2(G)$ with H. Then $M(G)$ is a nonregular abelian Banach algebra, since $\hat{G}$ in its natural imbedding in $\mathrm{Sp}\, M(G)$ is nondense, while the vanishing of of a Fourier transform $\hat{\mu}$, $\mu \in M(G)$, on $\hat{G}$ implies $\mu = 0$. Let A be the isometric image of $M(G)$ in $B(B(H))$ constructed in Proposition 4.6. Then A is nonregular, so by [14, 3.7.4] there exists an element $\varphi \in A$ such that the Banach subalgebra of A generated by φ is nonregular.

5. The algebra $A \otimes A$

For the rest of the paper we shall mainly study the regular abelian Banach subalgebra $A \otimes A$ of $B(B(H))$ where A is an abelian C^*-algebra. Our results indicate that its relationship to abstract harmonic analysis is quite profound. In the present section we shall study maps in $A \otimes A$ whose Gelfand transforms are positive definite, defined as follows. If X is a set and f a complex function on $X \times X$ we say f is <u>positive definite</u> if whenever $\gamma_1, \ldots, \gamma_n$ are n elements in X then the $n \times n$ matrix $(f(\gamma_i, \gamma_j))$ is positive.

Recall from [21] that $\mathrm{Sp}A \otimes A$ can be identified with $\mathrm{Sp}A \times \mathrm{Sp}A$. We shall therefore write elements in $\mathrm{Sp}A \otimes A$ as pairs (γ, γ') with $\gamma, \gamma' \in \mathrm{Sp}A$. We denote by $C(\mathrm{Sp}A \otimes A)$ the continuous complex functions on $\mathrm{Sp}A \otimes A$, vanishing at infinity if $\mathrm{Sp}A \otimes A$ is noncompact, and by α the canonical isomorphism of $C(\mathrm{Sp}A \otimes A)$ onto the norm closure $\mathcal{A}$ of $\{a|\mathcal{H} : a \in A \otimes A\}$ described in Theorem 3.3. We denote by $\mathcal{H}^+$ and $\mathcal{H}_{s.a.}$ the positive and self-adjoint Hilbert-Schmidt operators respectively. An operator $a \in B(\mathcal{H})$ is said to be <u>positivity preserving</u> (respectively <u>hermitian preserving</u>) if $a(\mathcal{H}^+) \subset \mathcal{H}^+$ (resp. $a(\mathcal{H}_{s.a.}) \subset \mathcal{H}_{s.a.}$). If $C(\mathrm{Sp}A \otimes A)$ has the cone of positive definite functions and $\mathcal{A}$ the cone of positivity preserving operators we next show that the isomorphism α is an order-isomorphism.

<u>Theorem 5.1</u> Let A be an abelian C^*-algebra acting on the Hilbert space H. Let α be the canonical isomorphism of $C(\mathrm{Sp}A \otimes A)$ onto the norm closure of $\{a|\mathcal{H} : a \in A \otimes A\}$, and let $f \in C(\mathrm{Sp}A \otimes A)$. Then f is positive definite if and only if

$\alpha(f)$ is a positivity preserving operator in $B(\mathcal{H})$.

Lemma 5.2 Let $f \in C(\mathrm{Sp}A \otimes A)$. Then if $\alpha(f)$ is hermitian preserving then $f(\gamma,\gamma') = \overline{f(\gamma',\gamma)}$ for all $\gamma,\gamma' \in \mathrm{Sp}A$. In particular, if $\gamma_1,\ldots,\gamma_n \in \mathrm{Sp}A$ then the $n \times n$ matrix $(f(\gamma_i,\gamma_j))$ is self-adjoint.

Proof Assume first $\alpha(f)$ is the restriction to $\mathcal{H}$ of a map $\varphi \in A \odot A$, say $\varphi = \sum_{i=1}^{n} a_i \otimes b_i$, $a_i, b_i \in A$. Then for $x \in \mathcal{H}$ we have $\Sigma a_i x^* b_i = \varphi(x^*) = \varphi(x)^* = \Sigma b_i^* x^* a_i^*$, so that $\Sigma a_i \otimes b_i = \Sigma b_i^* \otimes a_i^*$ on $\mathcal{H}$. But then

$$
\begin{aligned}
(\gamma,\gamma')(\Sigma a_i \otimes b_i) &= (\gamma,\gamma')(\Sigma b_i^* \otimes a_i^*) \\
&= \Sigma \overline{\gamma(b_i)}\ \overline{\gamma'(a_i)} \\
&= \overline{(\gamma',\gamma)(\Sigma a_i \otimes b_i)},
\end{aligned}
$$

so that $f(\gamma,\gamma') = \overline{f(\gamma',\gamma)}$ in this case.

In the general case choose a sequence (φ_n) in $A \odot A$ such that the restrictions to $\mathcal{H}$ converge to $\alpha(f)$ in $B(\mathcal{H})$. Say $\varphi_n = \Sigma a_{in} \otimes b_{in}$. Let $\varphi_n^+ = \Sigma b_{in}^* \otimes a_{in}^*$, so that $\psi_n = \frac{1}{2}(\varphi_n + \varphi_n^+) \in A \odot A$. If $x \in \mathcal{H}$ then $\|\varphi_n^+(x) - \alpha(f)(x)\|_2 = \|\varphi_n(x^*)^* - \alpha(f)(x^*)^*\|_2 = \|\varphi_n(x^*) - \alpha(f)(x^*)\|_2 \to 0$ uniformly for $\|x\|_2 \leq 1$. Thus $\psi_n \to \alpha(f)$ in norm in $B(\mathcal{H})$. By Theorem 3.3 $\hat{\psi}_n \to f$ in supnorm, so

$$
f(\gamma,\gamma') = \lim_n \hat{\psi}_n(\gamma,\gamma') = \lim_n \overline{\hat{\psi}_n(\gamma',\gamma)} = \overline{f(\gamma',\gamma)},
$$

Q.E.D.

Proof of Theorem 5.1 Assume $\alpha(f)$ is positivity preserving, and let $\gamma_1,\ldots,\gamma_n \in \mathrm{Sp}A$. If B is the weak closure of A then every character of B restricts to a character of A, and $A \otimes A \subset B \otimes B$ as subalgebras of $B(B(H))$. Thus in order to show that the $n \times n$ matrix $(f(\gamma_i, \gamma_j))$ is positive, we may assume $A = B$, i.e. A is a von Neumann algebra. Let $\varepsilon > 0$. Now $\alpha(f)$ can be approximated in $\| \|_2$-norm by restriction of maps in $A \odot A$. and each operator in A can be approximated in norm by linear combinations of mutually orthogonal projections. We can therefore find mutually orthogonal projections $e_1, \dots, e_n$, $e_{n+1}, \dots, e_m$ in A with sum 1 such that $\gamma_i(e_i) = 1$, $i = 1, \dots, n$, and constants λ_{ij}, $i,j \in \{1, \dots, m\}$, such that if ψ denotes the restriction of $\Sigma \lambda_{ij} e_i \otimes e_j$ to $\mathcal{H}$ then

$$(1) \qquad \| \alpha(f) - \psi \|_2 < \varepsilon .$$

Furthermore, if we replace ψ by $\frac{1}{2}(\psi + \psi^+)$, cf. Lemma 5.2, we may by that lemma assume ψ is hermitian preserving.

Let V_i be the closed subset of SpA corresponding to e_i under the Gelfand transform. By Proposition 2.3 there is a compact neighborhood N_{ij} of (γ_i, γ_j) in $\mathrm{SpA} \otimes A$ such that $N_{ij} \subset V_i \times V_j$, $i,j \in \{1, \dots, n\}$, and

$$(2) \qquad \| \psi(x) - \hat{\psi}(\gamma_i, \gamma_j) x \|_2 < \varepsilon$$

for all $x \in X(A \otimes A, N_{ij})$ with $\|x\|_2 \leq 1$, where $X = \mathcal{H}$. Choose compact neighborhoods W_i of γ_i such that $W_i \times W_j \subset N_{ij}$, $i,j \in \{1, \dots, n\}$, and let f_i be the projection in A corresponding to the characteristic function χ_{W_i} of W_i. Let now p_k be one of the projections f_i, $e_i - f_i$, $i = 1, \dots, n$, and e_i for $i = n+1, \dots, m$, and renumber them so that $p_i = f_i$ for $i = 1, \dots, n$. We can thus write

$$\psi = \Sigma \mu_{kl} \, p_k \otimes p_l | \mathcal{H} ,$$

where $\mu_{kl} \in \{\lambda_{ij} : i,j \in \{1, \dots, m\}\}$.

By Lemma 2.2 $\mu_{kk}p_k = \psi(p_k) = \hat{\psi}(\gamma_k,\gamma_k)p_k$ for $k = 1,\dots,n$. Since $\operatorname{supp} \widehat{p_i \otimes p_j} = \operatorname{supp} \chi_{W_i} \times \chi_{W_j} = W_i \times W_j$, we have by Lemma 2.1 that

(3) $$p_i \mathcal{H} p_j \subset X(A \otimes A, W_i \times W_j),$$

$i,j \in \{1,\dots,n\}$.

Let $q_i \leq p_i$ be a 1-dimensional projection, $i = 1,\dots,n$, and as above adding $p_i - q_i$ for $i = 1,\dots,n$, and p_i for $i = n+1,\dots,m$ to the q_i, we can write

$$\psi = \Sigma \rho_{rs} q_r \otimes q_s | \mathcal{H},$$

where $\rho_{rs} \in \{\lambda_{ij} : i,j \in \{1,\dots,m\}\}$, and $q_1,\dots,q_n$ are 1-dimensional. Choose partial isometries q_{rs} of rank one with domain q_s and range q_r such that $(q_{rs})_{1\leq r,s\leq n}$ is a set of matrix units, $q_{rr} = q_r$. Let $q = \sum_{r=1}^{n} q_r$, and let M denote the factor $B(H)_q$ of type I_n spanned by the q_{rs}. If M_n is the $n \times n$ complex matrices, then the map $\Sigma a_{rs} q_{rs} \to (a_{rs})$ is a *-isomorphism, hence an isometry of M onto M_n. Let e be the 1-dimensional projection $e = \frac{1}{n} \Sigma q_{rs}$ in M. By (1)

$$\|\alpha(f)(e) - \psi(e)\|_2 < \varepsilon,$$

hence, since $\psi(e)$ is self-adjoint, $\|x\| \leq \|x\|_2$ for $x \in \mathcal{H}_{s.a}$ and $\alpha(f)(e) \geq 0$,

(4) $$\psi(e) + \varepsilon q \geq 0.$$

By (2) and (3)

$$\|\psi(q_{rs}) - \hat{\psi}(\gamma_r,\gamma_s)q_{rs}\|_2 < \varepsilon$$

Thus we have

(5) $$\|\psi(e) - \frac{1}{n}\Sigma\hat{\psi}(\gamma_r,\gamma_s)q_{rs}\|_2 \leq \frac{1}{n}\Sigma\|\psi(q_{rs}) - \hat{\psi}(\gamma_r,\gamma_s)q_{rs}\|_2 < n\varepsilon.$$

By Lemma 5.2 the operator $\Sigma\hat{\psi}(\gamma_r,\gamma_s)q_{rs}$ is self-adjoint. Thus by (4) and (5)

(6) $$\frac{1}{n}\Sigma\hat{\psi}(\gamma_r,\gamma_s)q_{rs} \geq (-n\varepsilon-\varepsilon)q .$$

If $a = (a_{ij})$ is a matrix in M_n then its norm is majorized by $\Sigma|a_{ij}|$. Indeed, $|a_{ij}| \leq \|a\|$ for all i,j, so we have $\|a\|^2 = \|a^*a\| \leq \mathrm{Tr}(a^*a) = \|a\|_2^2 = \Sigma|a_{ij}|^2 \leq \|a\|\Sigma|a_{ij}|$. Thus from (1) we have

$$\|(f(\gamma_r,\gamma_s)) - (\hat{\psi}(\gamma_r,\gamma_s))\| \leq \Sigma|f(\gamma_r,\gamma_s) - \hat{\psi}(\gamma_r,\gamma_s)|$$
$$\leq n^2\|\alpha(f) - \psi\|_2 < n^2\varepsilon .$$

If we combine this with (6) we have since $(f(\gamma_r,\gamma_s))$ is self-adjoint

$$(f(\gamma_r,\gamma_s)) \geq (-n^2-n^2-n)\varepsilon .$$

Since ε is arbitrary $(f(\gamma_r,\gamma_s)) \geq 0$, and we have shown f is positive definite.

Conversely, assume f is positive definite. Let B denote the weak closure of A and let $\tilde{\gamma}$ be the restriction to A of $\gamma \in \mathrm{Sp}B$. Thus $(f(\tilde{\gamma}_i,\tilde{\gamma}_j))$ is positive for all $\gamma_1,\ldots,\gamma_n \in \mathrm{Sp}B$.

In order to show $\alpha(f)$ is positivity preserving it suffices to show $\alpha(f)(p) \geq 0$ for each 1-dimensional projection p in $\mathcal{H}$. For this it suffices to show that for each unit vector ξ in H and $\varepsilon > 0$ there is a nonnegative real number a such that

(7) $$|\langle\alpha(f)(p)\xi,\xi\rangle - a| < \varepsilon .$$

We let p,ξ and $\varepsilon > 0$ be given.

Choose mutually orthogonal projections $e_1,\ldots,e_n$ in B and λ_{ij}, $i,j \in \{1,\ldots,n\}$ such that if ψ is the restriction of $\Sigma\lambda_{ij}\, e_i \otimes e_j$ to $\mathcal{H}$ then $\|\psi-\alpha(f)\|_2 < \varepsilon/2$. Choose $\gamma_i \in \mathrm{Sp}B$ such that $\gamma_i(e_i) = 1$. Since α is an isometry we have

$$|f(\tilde{\gamma}_i,\tilde{\gamma}_j) - \lambda_{ij}| < \varepsilon/2 .$$

Let ψ' be the restriction of $f(\tilde{\gamma}_i,\tilde{\gamma}_j)e_i \otimes e_j$ to $\mathcal{H}$. Then $\|\alpha(f)-\psi'\|_2 \leq \|\alpha(f)-\psi\|_2 + \|\psi-\psi'\|_2 < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$.

Let $\xi_i = e_i\xi$ and let η be a unit vector such that p is the projection on the subspace it spans. Then we have

$$\begin{aligned}
<\psi'(p)\xi,\xi> &= \Sigma f(\tilde{\gamma}_i,\tilde{\gamma}_j)<e_i\, p\, e_j\xi,\xi> \\
&= \Sigma f(\tilde{\gamma}_i,\tilde{\gamma}_j)<p\ \xi_j,p\ \xi_i> \\
&= \Sigma f(\tilde{\gamma}_i,\tilde{\gamma}_j)<<\xi_j,\eta>\eta,<\xi_i,\eta>\eta> \\
&= \Sigma f(\tilde{\gamma}_i,\tilde{\gamma}_j)<\xi_j,\eta>\overline{<\xi_i,\eta>} \\
&\geq 0 ,
\end{aligned}$$

since $(f(\tilde{\gamma}_i,\tilde{\gamma}_j)) \geq 0$. If q is the 1-dimensional projection on the subspace spanned by ξ then

$$\begin{aligned}
|<\alpha(f)(p)\xi,\xi>-<\psi'(p)\xi,\xi>| &= |<(\alpha(f)-\psi')p,q>| \\
&\leq \|\alpha(f)-\psi'\|_2\|p\|_2\|q\|_2 < \varepsilon .
\end{aligned}$$

Thus with $a = <\psi'(p)\xi,\xi>$ the proof is complete.

Recall that if φ is a linear map from one C^*-algebra M into another N, then φ is said to be <u>positive</u> if $\varphi(x) \geq 0$ for each $x \geq 0$ in M. φ is said to be <u>completely positive</u> if $\varphi \otimes \iota_n : M \otimes M_n \to N \otimes M_n$ is positive for each n, where ι_n is the identity map on M_n.

<u>Corollary 5.3</u> Let A be an abelian C^*-algebra acting on the Hilbert space H. Let $\varphi \in A \otimes A$. Then the following conditions are equivalent:

(i) φ is positive.

(ii) φ is completely positive.

(iii) $\hat{\varphi}$ is positive definite on $\mathrm{Sp}A \otimes A$.

Proof (ii) $\Rightarrow$ (i) is trivial. Since φ is ultraweakly continuous, $\varphi|\mathcal{H}$ is positivity preserving if and only if φ is positive. Thus (i) $\Leftrightarrow$ (iii) is immediate from Theorem 5.1. To show (iii) $\Rightarrow$ (ii) let $n \in \mathbb{N}$ be given. Let $\mathbb{C}_n$ denote the scalar operators in M_n. Then $\varphi \otimes \iota_n$ belongs to $(A \otimes \mathbb{C}_n) \otimes (A \otimes \mathbb{C}_n) \subset B(B(H \otimes \mathbb{C}^n))$. We can identify $\operatorname{Sp}(A \otimes \mathbb{C}_n)$ with $\operatorname{Sp} A$ via $\gamma \otimes 1 \to \gamma$. Thus $\widehat{\varphi \otimes \iota_n}$ is positive definite if and only if $\hat{\varphi}$ is positive definite. By (i) $\Leftrightarrow$ (iii) $\varphi \otimes \iota_n$ is positive. Q.E.D.

Lemma 5.4 Let A be an abelian C^*-algebra acting on the Hilbert space H, and let α denote the canonical isomorphism of $C(\operatorname{Sp} A \otimes A)$ onto the C^*-subalgebra of $B(\mathcal{H})$ generated by $a|\mathcal{H}$, $a \in A \otimes A$. Let $\varphi \in A \otimes A$ satisfy $\varphi(1) = 1$, and let f be a continuous positive definite function on $\operatorname{Sp} A \otimes A$ such that $f(\gamma,\gamma) = 1$ for all $\gamma \in \operatorname{Sp} A$. Then we have:

(i) $\hat{\varphi}(\gamma,\gamma) = 1$ for all $\gamma \in \operatorname{Sp} A$.

(ii) If ψ is an operator normal map in $B(B(H))$ such that $\psi|\mathcal{H} = \alpha(f)$, then ψ is positive and $\psi(1) = 1$.

Proof (i) Let $\gamma \in \operatorname{Sp} A$, and let $\varepsilon > 0$. As in the proof of Theorem 5.1 there is $\psi = \Sigma \lambda_{ij}\, e_i \otimes e_j$ such that $\|\psi - \varphi\| < \varepsilon$, where (e_i) is an orthogonal family of projections in the weak closure of A with sum 1 such that $\gamma(e_1) = 1$. In particular $\lambda_{11} = \hat{\psi}(\gamma,\gamma)$.

Now

$$\varepsilon > \|\psi(1) - \varphi(1)\| = \|\Sigma \lambda_{ii}\, e_i - 1\| \geq |\lambda_{11} - 1| .$$

By Lemma 3.2 we thus have

$$\begin{aligned} |\hat{\varphi}(\gamma,\gamma) - 1| &\leq |\hat{\varphi}(\gamma,\gamma) - \hat{\psi}(\gamma,\gamma)| + |\lambda_{11} - 1| \\ &\leq \|\varphi - \psi\| + \varepsilon < 2\varepsilon . \end{aligned}$$

Since ε is arbitrary (i) follows.

(ii) As in the proof of Theorem 5.1 we may assume A is a maximal abelian von Neumann algebra. Let f and ψ be as in (ii). Choose $\psi' = \Sigma\lambda_{ij}\, e_i \otimes e_j$ as above such that $\|\psi-\psi'\|_2 < \varepsilon$. Let $\gamma_i \in \mathrm{Sp}\,A$ s tisfy $\gamma_i(e_i) = 1$. Then for all i,

$$\varepsilon > |f(\gamma_i,\gamma_i) - \hat{\psi}'(\gamma_i,\gamma_i)| = |1-\lambda_{ii}| .$$

Thus $\|\psi'(1)-1\| = \|\Sigma\lambda_{ii}\, e_i - 1\| < \varepsilon$. Modifying ψ' we can assume $\lambda_{ij} = f(\gamma_i,\gamma_j) = \hat{\psi}'(\gamma_i,\gamma_j)$, and in particular $\lambda_{ii} = 1$, so $\psi'(1) = 1$.

Using Lemma 8.4 below it is immediate that $\hat{\psi}'$ is positive definite, hence by Corollary 5.3 ψ' is positive, and by construction $\psi'(1) = 1$. In particular $\|\psi'\| = 1$ [15]. Choose a sequence of such maps ψ_n such that $\|\psi_n-\psi\|_2 < 1/n$, $\|\psi_n\| = \|\psi_n(1)\| = 1$. Let $\tilde{\psi}$ be a point-ultraweak limit point in $B(B(H))$ of the sequence (ψ_n), cf. [11]. Then $\tilde{\psi}$ is positive, $\tilde{\psi}(1) = 1$, $\tilde{\psi}|\mathcal{H} = \psi|\mathcal{H}$. Let $\varepsilon > 0$ and $1/n < \varepsilon$. Then $\psi_n = \Sigma\lambda_{ij}\, e_i \otimes e_j$ for suitable λ_{ij} and e_i. Let e be a 1-dimensional projection, $e \leq e_1$. Then $\psi_n(e) = e$, so that $\|\psi(e)-e\|_2 < \varepsilon$. In particular $\|\psi(e)-e\| < \varepsilon$, so $\|\psi(e)\| \geq \|e\| - \varepsilon = 1-\varepsilon$. Thus if $C(H)$ denotes the compact operators on H then $\|\psi|C(H)\| \geq 1$. But $\tilde{\psi}(1) = 1$, so $\|\tilde{\psi}\| = 1$ [15]. Therefore $\|\tilde{\psi}|C(H)\| \leq 1$. Since $\psi|C(H) = \tilde{\psi}|C(H)$, and ψ is ultraweakly continuous $\|\psi\| \leq 1$, hence equal to 1.

Again by ultraweak continuity of ψ, ψ is a positive map because $\psi|C(H)$ is positive being equal to $\tilde{\psi}$ on $C(H)$. Thus $\|\psi(1)\| = \|\psi\| = 1$ [15].

Let $a \in A$, $x \in B(H)$, then $\psi_n(ax) = a\,\psi_n(x)$, so that $\tilde{\psi}(ax) = a\,\tilde{\psi}(x)$. Thus if $x \in C(H)$ we have $\psi(ax) = \tilde{\psi}(ax) = a\,\tilde{\psi}(x) = a\,\psi(x)$, and symmetrically $\psi(xa) = \psi(x)a$. Since ψ is ultraweakly continuous we therefore have $\psi(a) = a\,\psi(1) = \psi(1)a$.

In particular $\psi(1) \in A' = A$, since A is assumed to be maximal abelian.

Suppose $\psi(1) \neq 1$. Then there exists a nonzero projection $e \in A$ such that $\|e\psi(1)\| < 1$. We now apply the preceding part of the proof to Ae, $B(eH)$, $\psi_e = \psi|Ae$, $\psi_{ne} = \psi_n|Ae$, $\tilde{\psi}_e = \tilde{\psi}|Ae$. Since the set of Hilbert-Schmidt operators on eH equals $e\mathcal{H}e$ and $C(eH) = eC(H)e$ all the previous assumptions and arguments hold when we restrict attention to $B(eH)$ as above. But then the previous argument shows $\|\psi_e\| = \|\psi_e(e)\| = \|\psi(e)\| = \|e\psi(1)\| < 1$, while $\|\psi_e\| = \|\tilde{\psi}_e\| = \|\tilde{\psi}_e(e)\| = \|e\tilde{\psi}(e)\| = \|e\| = 1$, a contradiction. Thus $\psi(1) = 1$ as asserted. Q.E.D.

<u>Proposition 5.5</u> Let A be an abelian C^*-algebra acting on a separable Hilbert space H. Let $\varphi \in A \otimes A$ be positive and $\varphi(1) = 1$. Then we have

(i) $\varphi|\mathcal{T} \in B(\mathcal{T})$.

(ii) $Tr(\varphi(x)) = Tr(x)$ for all $x \in \mathcal{T}$.

(iii) φ^* has a unique extension to a positive operator normal map ψ in $B(B(H))$ such that $\psi(1) = 1$.

<u>Proof</u> We may assume A is a von Neumann algebra. From the proof of Theorem 5.1 there is a sequence (φ_n) of maps in $A \otimes A$ of the form $\Sigma\lambda_{ij}e_i \otimes e_j$ in the algebraic tensor product $A \odot A$ / converging in norm to φ such that φ_n is positive, $\varphi_n(1) = 1$, and the e_i's are mutually orthogonal projections. Let $x \in \mathcal{T}$ be positive. Then $\varphi_n(x) \to \varphi(x)$ uniformly. Since Tr is lower semicontinuous being the countable sum of vector states, and $Tr(\varphi_n(x)) = Tr(x)$, we have

$$Tr(\varphi(x)) \leq \overline{\lim_n}\, Tr(\varphi_n(x)) = \overline{\lim_n}\, Tr(x) = Tr(x).$$

Thus $\|\varphi(x)\|_1 \leq \|x\|_1$ for $x \in \mathcal{T}^+$. By polarization $\varphi|\mathcal{T} \in B(\mathcal{T})$ and has norm less than or equal to 4. Thus (i) follows.

Since $\varphi|\mathcal{T} \in B(\mathcal{T})$, φ^* has a unique extension to an ultra-weakly continuous map ψ in $B(B(H))$ such that $\psi^* = \varphi|\mathcal{T}$. Furthermore, if $f \in C(\mathrm{Sp}\,A \otimes A)$ is defined by $f(\gamma,\gamma') = \overline{\hat{\varphi}(\gamma,\gamma')}$, then $f(\gamma,\gamma) = 1$ for all $\gamma \in \mathrm{Sp}\,A$. If α in the canonical isomorphism of $C(\mathrm{Sp}\,A \otimes A)$ onto the norm closure $\mathcal{A}$ of $\{\rho|\mathcal{H} : \rho \in A \otimes A\}$ in $B(\mathcal{H})$, then $\alpha(f) = \psi|\mathcal{H}$, because $\alpha(f)$ is the adjoint of $\varphi|\mathcal{H}$ in $\mathcal{A}$. In particular ψ is operator normal. Thus f and ψ satisfy the assumptions of Lemma 5.4 (ii), hence $\psi(1) = 1$, and (iii) follows. But then, if $x \in \mathcal{T}$ we have $\mathrm{Tr}(\varphi(x)) = \langle 1,\varphi(x)\rangle = \langle\psi(1),x\rangle = \langle 1,x\rangle = \mathrm{Tr}(x)$, so (ii) follows. Q.E.D.

We conclude this section by showing how the obtained results are applicable to representations of locally compact abelian groups as automorphisms of $B(H)$.

<u>Lemma 5.6</u> Let G be a locally compact abelian group and $t \to u_t$ a continuous unitary representation of G on the Hilbert space H. Let $\alpha_t(x) = u_t x u_t^*$ for $x \in B(H)$, and let A denote the abelian von Neumann algebra generated by $\{u_t : t \in G\}$. Then for each $f \in L^1(G)$, we have $\alpha_f \in A \otimes A$.

<u>Proof</u> Let $\varepsilon > 0$ and assume $\|f\|_1 \leq 1$. Let K be a compact subset of G such that $\int_{\complement K}|f(t)|dt < \varepsilon/4$. Let $\varphi = \Sigma a_i \chi_{E_i}$ be a simple function with support in K such that $\|\varphi - f\|_1 < \varepsilon/2$, say $\|\varphi\|_1 \leq 2$. From Stone's theorem we can find mutually orthogonal

projections $e_1,\dots,e_r$ in A and $\gamma_1,\dots,\gamma_r \in \hat{G}$ such that

$$\|u_t - \sum_{j=1}^{r} \overline{\langle\gamma_j,t\rangle} e_j\| < \varepsilon/8 \qquad \text{for } t \in K .$$

Then for $t \in K$ we have

$$\begin{aligned}
&\|\alpha_t - \Sigma\overline{\langle\gamma_j,t\rangle}\langle\gamma_k,t\rangle e_j \otimes e_k\| \\
\leq\ &\|(u_t - \Sigma\overline{\langle\gamma_j,t\rangle} e_j) \otimes u_t^*\| + \|\Sigma\overline{\langle\gamma_j,t\rangle} e_j \otimes (u_t^* - \Sigma\langle\gamma_k,t\rangle e_k)\| \\
<\ &\varepsilon/4 .
\end{aligned}$$

Thus we have for $x \in B(H)$, with $m(E)$ the Haar measure of a set $E \subset G$,

$$\begin{aligned}
&\|\int\varphi(t)\alpha_t(x)dt - \sum_{jk}(\sum_i(a_i\int_{E_i}\overline{\langle\gamma_j,t\rangle}\langle\gamma_k,t\rangle dt))e_j\, x\, e_k\| \\
=\ &\|\sum_i a_i\int_{E_i}(\alpha_t(x) - \sum_{jk}\overline{\langle\gamma_j,t\rangle}\langle\gamma_k,t\rangle e_j\, x\, e_k)dt\| \\
\leq\ &\sum_i |a_i|\int_{E_i}\|\alpha_t(x) - \sum_{jk}\overline{\langle\gamma_j,t\rangle}\langle\gamma_k,t\rangle e_j\, x\, e_k\| dt \\
<\ &\Sigma|a_i|\varepsilon/4\|x\|m(E_i) \\
=\ &\|\varphi\|_1\|x\|\varepsilon/4 \\
\leq\ &\varepsilon/2\|x\| .
\end{aligned}$$

Let $c_{jk} = \sum_i a_i \int_{E_i}\overline{\langle\gamma_j,t\rangle}\langle\gamma_k,t\rangle dt$. Then we have

$$\begin{aligned}
\|\alpha_f - \Sigma c_{jk}\, e_j \otimes e_k\| &\leq \|\alpha_f - \alpha_\varphi\| + \|\alpha_\varphi - \Sigma c_{jk}\, e_j \otimes e_k\| \\
&\leq \|f-\varphi\|_1 + \varepsilon/2 \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon .
\end{aligned}$$

Since ε is arbitrary $\alpha_f \in A \otimes A$. Q.E.D.

Let G be a locally compact abelian group, and f a continuous complex function on G. If E is a closed subset of G we say f is positive definite on E if the $n \times n$ matrix $(f(g_i - g_j))$ is positive whenever $g_1,\dots,g_n \in E$.

Proposition 5.7 Let G be a locally compact abelian group and $t \to u_t$ a continuous unitary representation of G on the Hilbert space H. Let $\mathrm{Sp}\, u$ denote the spectrum of $t \to u_t$ in the dual group $\hat{G}$, and let $\alpha_t(x) = u_t x u_t^*$ for $x \in B(H)$. Then if $f \in L^1(G)$ the following three conditions are equivalent:

(i) α_f is positive

(ii) α_f is completely positive

(iii) $\hat{f}$ is positive definite on $\mathrm{Sp}\, u$.

Proof Let A_0 denote the C^*-algebra generated by $\{u_g = \int g(t)u_t dt : g \in L^1(G)\}$. Then $\mathrm{Sp}\, A_0 = \mathrm{Sp}\, u$. Indeed, let P_γ be the projection valued measure on $\hat{G}$ such that by Stone's theorem $u_t = \int_{\hat{G}} \overline{\langle\gamma, t\rangle} d P_\gamma$. Let $g \in L^1(G)$. Then $u_g = \int_{\hat{G}} \hat{g}(\gamma) d P_\gamma$ so $\|u_g\| = \|\hat{g} | \mathrm{Sp}\, u\|_\infty$. By density of the Fourier transforms in $C(\hat{G})$ we obtain a $*$-isomorphism of A_0 on $C(\mathrm{Sp}\, u)$, and the assertion follows. By Lemma 5.6 $\alpha_f \in A \otimes A$, where A is the weak closure of A_0. If $g \in C(\hat{G})$ let $\tilde{g} \in C(\hat{G} \times \hat{G})$ be defined by $\tilde{g}(\gamma, \gamma') = g(\gamma - \gamma')$. Then it is immediate that $\hat{f}$ is positive definite on $\mathrm{Sp}\, u$ if and only if $\tilde{\hat{f}}$ is positive definite on $\mathrm{Sp}\, u \times \mathrm{Sp}\, u$. In particular it follows from Corollary 5.3 that if α_f is positive then $\tilde{\hat{f}}$ is positive definite on $\mathrm{Sp}\, A \otimes A$, hence by restriction $\tilde{\hat{f}}$ is positive definite on $\mathrm{Sp}\, A_0 \otimes A_0$, and so $\hat{f}$ is positive definite on $\mathrm{Sp}\, u$. Conversely, if $\hat{f}$ is positive definite on $\mathrm{Sp}\, u$ then $\tilde{\hat{f}}$ is positive definite on $\mathrm{Sp}\, A_0 \otimes A_0$. From the proof of Theorem 5.1 we see that α_f is positive, and so completely positive by Corollary 5.3. Thus (iii) $\Rightarrow$ (ii), and the proof is complete.

6. Maps with pure point spectra

In this section we shall study the case when an operator normal map has pure point spectrum when restricted to the trace class operators $\mathcal{T}$. In the finite dimensional case the result is a characterization of those identity preserving positive maps which belong to an algebra of the form $A \otimes A$.

Theorem 6.1 Let φ be an operator normal positive map in $B(B(H))$ such that $\varphi(1) = 1$. Suppose $\varphi|\mathcal{T}$ is a bounded operator on $\mathcal{T}$ such that the eigenvectors of $\varphi|\mathcal{T}$ of rank 1 form a total set in $\mathcal{T}$. Then φ is completely positive, and there is a totally atomic maximal abelian von Neumann algebra A on H such that φ belongs to the point-ultraweak closure of $A \otimes A$.

We divide the proof into some lemmas. The first has the same conclusion as Proposition 5.5 and shows in particular that in the finite dimensional case $\mathrm{Tr}(\varphi(x)) = \mathrm{Tr}(x)$ whenever φ is an operator normal map such that $\varphi(1) = 1$.

Lemma 6.2 Let φ be an operator normal map in $B(B(H))$ such that $\varphi(1) = 1$. Suppose $\varphi|\mathcal{T} \in B(\mathcal{T})$ and that the eigenvectors of $\varphi|\mathcal{T}$ form a total set in $\mathcal{T}$. Then $\mathrm{Tr}(\varphi(x)) = \mathrm{Tr}(x)$ for all $x \in \mathcal{T}$.

Proof Let S be a total set of eigenvectors of $\varphi|\mathcal{T}$. For each $x \in S$, $x \in \mathcal{H}$ and is an eigenvector for φ and thus for φ^*, since $\varphi|\mathcal{H}$ is normal. If $\varphi(x) = \lambda x$ then $\bar{\lambda}\langle x,1\rangle = \langle\varphi^*(x),1\rangle = \langle x,\varphi(1)\rangle = \langle x,1\rangle$, hence $\langle x,1\rangle = \langle\varphi(x),1\rangle = 0$ if $\lambda \neq 1$, and $\langle x,1\rangle = \langle\varphi(x),1\rangle$ if $\lambda = 1$.

Thus $\langle x,1\rangle = \langle\varphi(x),1\rangle$ for all x in the linear span

T of S. Since T is dense in $\mathcal{T}$ by assumption and $\varphi|\mathcal{T} \in B(\mathcal{T})$, $<x,1> = <\varphi(x),1>$ for all $x \in \mathcal{T}$. Q.E.D.

<u>Lemma 6.3</u> Let φ be as in Theorem 6.1 and let S be a total set in $\mathcal{T}$ consisting of eigenvectors of rank 1 for $\varphi|\mathcal{T}$. If e is a projection in $B(H)$ such that $\varphi(e) = e$, then $\{exe : x \in S\}$ is a total set of eigenvectors of rank 1 for $\varphi|e\mathcal{T}e$.

<u>Proof</u> From an unpublished result of Broise it follows that $\varphi(exe) = e\varphi(x)e$ for all $x \in B(H)$. A simple proof in our case goes as follows: Let ρ be a state of $B(H)$ with support in e. Then $\rho \circ \varphi$ is a state of $B(H)$ with support in e. Thus $\rho(\varphi(x)) = \rho(\varphi(exe))$ for all $x \in B(H)$. Since this holds for all such ρ, $e\varphi(x)e = \varphi(exe)$.

Let $x \in B(H)$, $\varphi(x) = \lambda x$, then $\varphi(exe) = e\varphi(x)e = \lambda exe$, so exe is an eigenvector for φ. Finally, since S is total in $\mathcal{T}$, and the map $y \to eye$ is norm decreasing on $\mathcal{T}$, it is clear that the set $\{exe : x \in S\}$ is total in $e\mathcal{T}e$.

<u>Lemma 6.4</u> Let φ be as in Theorem 6.1. Suppose x is a rank 1 operator with $\|x\| = 1$ such that $\varphi(x) = x$. Then either x is a scalar multiple of a projection, or x is a partial isometry such that the C^*-algebra M generated by x is isomorphic to the complex 2×2 matrices, and φ restricted to M is the identity map.

<u>Proof</u> If x is a scalar multiple of a normal operator then, since x is of rank 1, x is already a scalar multiple of a projection. We may thus assume x is a partial isometry such that $p = x^*x \neq xx^* = q$, and p and q are 1-dimensional projections.

From Kadison's Schwarz inequality [10] applied to $x+x^*$ and $i(x-x^*)$ we have, cf. [18, Lemma 7.3]

$$\varphi(p+q) = \varphi(x^*x+xx^*) \geq \varphi(x)^*\varphi(x) + \varphi(x)\varphi(x)^*$$
$$= x^*x + xx^* = p+q \ .$$

From Lemma 6.2 we have $\mathrm{Tr}(\varphi(p+q)) = \mathrm{Tr}(p+q)$, hence by the faithfulness of the trace, $\varphi(p+q) = p+q$. Since x is of rank 1 and $p \neq q$ the C^*-algebra M generated by x is isomorphic to the complex 2×2 matrices. Furthermore, the identity of M is $e = p \vee q$. Thus there exist positive constants α and β such that $e \leq \alpha(p+q) \leq \beta e$. In particular, if $x \geq 0$ is in the unit ball of M, then $0 \leq \varphi(x) \leq \varphi(e) \leq \alpha\varphi(p+q) = \alpha(p+q) \leq \alpha\beta e$. Thus $\varphi(x) \in M$, since $M = B(H)_e$, and $\varphi|M$ is a positive linear map of M into itself of norm 1. In particular $0 \leq \varphi(e) \leq e$, and again by Lemma 6.2 $\varphi(e) = e$. Thus φ preserves the identity of M.

Now x, x^*, and e are linearly independent in M. For if there are complex numbers γ, δ such that $\gamma x + \delta x^* = e$, then multiplication of this equation respectively from the left and right by x yields the equations $\gamma x^2 + \delta xx^* = x$, and $\gamma x^2 + \delta x^*x = x$. Thus $\delta xx^* = \delta x^*x$, so that $\delta = 0$, and x, x^*, e are linearly independent as asserted. Since they all are eigenvectors for φ with eigenvalue 1, it follows that the eigenspace $N \subset M$ for the eigenvalue 1 is at least of dimension 3.

Suppose $\varphi|M$ is not the identity, then $\dim N = 3$. Since S is total in $\mathcal{T}$ the set $\{eye : y \in S\}$ is a total set of eigenvectors in M by Lemma 6.3. Since $\varphi|M$ is operator normal, there is thus $y \in S$ such that $eye \neq 0$ and $\varphi(eye) = \lambda\, eye$ with $\lambda \neq 1$. We have thus found an eigenvector z for $\varphi|M$ of rank 1, $\|z\| = 1$, and $\varphi(z) = \lambda z$, $\lambda \neq 1$. Now z^* is an eigenvector with eigenvalue $\bar{\lambda}$. If z is not a scalar multiple of z^* they span a subspace of M of dimension 2 , which is orthogonal to N . This is impossible since $\dim M = 4$. We may thus assume z is self-adjoint, hence a scalar multiple of a projection; hence we may assume z is a projection. Since $e-z$ is orthogonal to z in M , $e-z \in N$. Thus $e-z = \varphi(e-z) = \varphi(e) - \varphi(z) = e - \lambda z$, and we have shown $\lambda = 1$, contrary to assumption. Thus $\varphi | M$ is the identity. Q.E.D.

<u>Lemma 6.5</u> Let φ be as in Theorem 6.1. Then there exists a 1-dimensional projection p such that $\varphi(p) = p$.

<u>Proof</u> Let S be a total set of eigenvectors for $\varphi | \mathcal{T}$ of rank 1 . If no eigenvector in S has eigenvalue 1 then for all $x \in S$, $\varphi(x) = \lambda x$ with $\lambda \neq 1$. Then by Lemma 6.2 $\lambda \operatorname{Tr}(x) = \operatorname{Tr}(\varphi(x)) = \operatorname{Tr}(x)$, so $\operatorname{Tr}(x) = 0$. In particular $\operatorname{Tr}(x) = 0$ for all x in the linear span R of S . But S is assumed to be total in $\mathcal{T}$, so R is dense in $\mathcal{T}$. But then $\operatorname{Tr}(x) = 0$ for all $x \in \mathcal{T}$, which is a contradiction. Thus there is $x \in S$ with $\varphi(x) = x$. An application of Lemma 6.4 completes the proof.

<u>Proof of Theorem 6.1</u> We first show that there is an orthogonal family $(p_j)_{j \in J}$ of 1-dimensional projections with sum 1 such that $\varphi(p_j) = p_j$. By Zorn's lemma let $(p_j)_{j \in J}$ be a maximal such family. By Lemma 6.5 it is nonempty. Let $q = 1 - \sum_{j \in J} p_j$. Since φ is ultraweakly continuous $\varphi(q) = q$. If $q \neq 0$, φ restricted to $B(H)_q$ has by Lemma 6.3 exactly the same properties as φ has as a map in $B(B(H))$. Thus by Lemma 6.5 there is a 1-dimensional projection $p \leq q$ such that $\varphi(p) = p$.

This contradicts the maximality of the family $(p_j)_{j\in J}$, so $q = 0$, and $\sum_{j\in J} p_j = 1$. Let A denote the totally atomic maximal abelian von Neumann algebra generated by $(p_j)_{j\in J}$. For each finite subset $I \subset J$ let $q_I = \sum_{j\in I} p_j$. Then the net $(q_I)_{I\subset J}$ is monotone increasing, so $q_I \to 1$ ultrastrongly, and $\dim q_I = \operatorname{card} I$. We show $\varphi|B(H)_{q_I}$ belongs to $A_I \otimes A_I$, where A_I is the finite dimensional algebra generated by p_j , $j \in I$.

For every pair $p_i \neq p_j$, $i,j \in I$, there are x,y in S such that $p_i x p_j \neq 0$ and $p_j y p_i \neq 0$. Let $e = p_i + p_j$. Then in particular $exe \neq 0 \neq eye$, and by Lemma 6.3 exe and eye are rank 1 eigenvectors for $\varphi|B(H)_e$. We have thus found four eigenvectors of rank 1 for $\varphi|B(H)_e$, and $\varphi(e) = e$. Since two of them are p_i and p_j , and the other two are scalar multiplies of partial isometries between them, we have shown that if we multiply the chosen eigenvectors exe and eye by suitable scalars for all pairs $i,j \in I$, we have found a set of eigenvectors for $\varphi|B(H)_{q_I}$ consisting of a complete set of matrix units for $B(H)_{q_I}$. Thus $\varphi|B(H)_{q_I}$ is of the form

$$\varphi \mid B(H)_{q_I} = \sum_{i,j\in I} \lambda_{ij}\, p_i \otimes p_j \in A_I \otimes A_I .$$

Since $A_I \otimes A_I \subset A \otimes A$ and $q_I \to 1$ ultrastrongly, $q_I x q_I \to x$ ultrastrongly, so ultraweakly for all $x \in B(H)$. Furthermore $(\varphi|B(H)_{q_I}) \circ q_I \otimes q_I \in A \otimes A$. Thus we have by the above formula

$$\varphi(x) = \lim_I \varphi(q_I x q_I) = \lim_I (\varphi \mid B(H)_{q_I})(q_I x q_I)$$

and φ belongs to the point-ultraweak closure of $A \otimes A$. Note that since $\varphi|B(H)_{q_I}$ is positive and belongs to $A_I \otimes A_I$, it is completely positive by Corollary 5.3. Thus φ , being the point-ultraweak limit of completely positive maps, is itself completely positive. The proof is complete.

Remark 6.6 The last theorem gives a necessary condition for an operator normal positive map φ to be completely positive in terms of spectral properties. It might be belived that there is a converse to the theorem. However, the following example shows that a regular operator normal completely positive map φ such that $\varphi(1) = 1$ need not have a basis of eigenvectors of rank 1.

Let $H = \mathbb{C}^2$, let $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\varphi = \frac{1}{2}(a \otimes a + b \otimes b)$. Since a and b are self-adjoint unitaries, φ is operator hermitian and completely positive, being the convex sum of two $*$-automorphisms. An orthogonal basis of eigenvectors is

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $\varphi(x_1) = x_1$, $\varphi(x_4) = -x_4$, $\varphi(x_2) = \varphi(x_3) = 0$. Thus the eigenvalues ± 1 have multiplicities 1, and every eigenvector with eigenvalue ± 1 has rank 2.

7. Some completely positive maps

If φ is a completely positive map of a C^*-algebra M into $B(H)$ then there exist a Hilbert space H', a bounded linear map V of H into H', and a $*$-representation π of M on H' such that $\varphi(x) = V^*\pi(x)V$ for all $x \in M$ [17]. We say $V^*\pi V$ is a Stinespring decomposition for φ. If M and N are von Neumann algebras we denote by $M \bar{\otimes} N$ their von Neumann algebra tensor product.

Theorem 7.1 Let H be a separable Hilbert space and $\varphi \in B(B(H))$ ultraweakly continuous, positive, and $\varphi(1) = 1$. Let A be a maximal abelian von Neumann algebra acting on H. Then the following two conditions are equivalent:

(i) There exist a probability space $(X, \mathcal{B}, \mu)$ and a measurable map u of X into the unitary group of A such that

$$\varphi(x) = \int_X u(\zeta)\, x\, u(\zeta)^* d\mu(\zeta), \qquad x \in B(H).$$

(ii) $\varphi(x) = x$ for all $x \in A$, and φ is completely positive with a Stinespring decomposition $V^*\pi V$ with π normal such that there exist a Hilbert space K and an abelian von Neumann algebra B acting on K with the following properties:

(1) $V : H \to H' = H \otimes K$

(2) $\pi(B(H)) \subset B(H) \bar{\otimes} B$

(3) $\pi(B(H))' \cap (B(H) \bar{\otimes} B) = \mathbb{C} \bar{\otimes} B$

(4) $VV^* \in \mathbb{C} \bar{\otimes} B(K)$

(5) $\pi(A) = A \bar{\otimes} \mathbb{C}$

In particular, if the above conditions are satisfied then φ belongs to the point-ultraweak closure of $A \otimes A$.

Proof (ii) $\Rightarrow$ (i). Assume φ satisfies the conditions in (ii). Since π is normal $\pi(B(H))$ is a von Neumann algebra isomorphic to $B(H)$. Let N denote the von Neumann algebra generated by $\pi(B(H))$ and $\mathbb{C} \bar{\otimes} B$. We show $N = B(H) \bar{\otimes} B$. Indeed, by condition (2) $N' \cap (B(H)\bar{\otimes}B) = \mathbb{C} \bar{\otimes} B$. Since $B(H) \otimes B$ is of type I it is a normal von Neumann algebra [6, ch. III, 7, ex. 13]. Since $N \subset B(H) \bar{\otimes} B$ and N contains the center $\mathbb{C} \bar{\otimes} B$ of $B(H) \bar{\otimes} B$, we have

$$\begin{aligned} N &= (N' \cap (B(H) \bar{\otimes} B))' \cap (B(H) \bar{\otimes} B) \\ &= (\mathbb{C} \otimes B)' \cap (B(H) \bar{\otimes} B) \\ &= B(H) \bar{\otimes} B , \end{aligned}$$

as asserted.

Let e be a minimal projection in $B(H)$. Then $\pi(e)$ is an abelian projection with central carrier 1 in $B(H) \bar{\otimes} B$. Indeed, let $(e_n)_{n\in\mathbb{N}}$ be an orthogonal sequence of minimal projections in $B(H)$ with sum 1 such that $e_1 = e$. Then $\sum_{n=1}^{\infty} \pi(e_n) = 1$. Since $e \sim e_n$ for all n as projections in $B(H)$, $\pi(e) \sim \pi(e_n)$ for all n. In particular their central carriers are equal, so must be the identity. Let $a = \sum_{i=1}^{n} \pi(x_i)(1 \otimes b_i) \in N$ with $x_i \in B(H)$, $b_i \in B$, $i = 1,\dots,n$. Then we have

$$\begin{aligned} \pi(e)a\pi(e) &= \Sigma\, \pi(e)\pi(x_i)\pi(e)(1 \otimes b_i) \\ &= \Sigma\, \pi(ex_ie)(1 \otimes b_i) \\ &= \pi(e)\, \Sigma\, \mathrm{Tr}(x_ie)1 \otimes b_i \\ &\in \pi(e)(\mathbb{C} \bar{\otimes} B) . \end{aligned}$$

Since by the preceding paragraph operators like a are ultraweakly dense in $B(H) \bar{\otimes} B$,

$$\pi(e)(B(H) \bar{\otimes} B)\pi(e) = \pi(e)(\mathbb{C} \bar{\otimes} B) ,$$

so that $\pi(e)$ is an abelian projection as asserted.

For each $n \in \mathbb{N}$ let v_n be a partial isometry in $B(H)$ such that $v_n^* v_n = e_n$, $v_n v_n^* = e$, and $v_1 = e$. Since $e \otimes 1$ is an abelian projection with central carrier 1 in $B(H) \bar{\otimes} B$, and $\pi(e)$ is the same by the preceding paragraph, $\pi(e) \sim e \otimes 1$ in $B(H) \bar{\otimes} B$ [6, Ch. III, §3, Lemme 1]. Let u be a partial isometry in $B(H) \bar{\otimes} B$ such that $u^*u = e \otimes 1$, $uu^* = \pi(e)$. Let

$$U = \sum_{n=1}^{\infty} \pi(v_n^*)u(v_n \otimes 1),$$

where the convergence is in the strong topology. Since the supports of the $v_n \otimes 1$ and the ranges of the $\pi(v_n^*)$ are pairwise orthogonal for different n's and both span the whole space it is easy to see that U is a unitary operator in $B(H) \bar{\otimes} B$. Furthermore, a straightforward computation shows $U(e_n \otimes 1)U^* = \pi(e_n)$, and $U(v_n \otimes 1)U^* = \pi(v_n)$. Since the $*$-algebra generated by the e_n and the v_n is ultraweakly dense in $B(H)$, and π is ultraweakly continuous, $U(x \otimes 1)U^* = \pi(x)$ for all $x \in B(H)$.

By [16, 1.18.1] there exists a localizable measure space $(X, \mathcal{B}, \nu)$ such that B can be identified with $L^\infty(X,\nu)$ acting on $L^2(X,\nu)$ by pointwise multiplication. By [16, 1.22.13] we can identify $B(H) \bar{\otimes} B$ with $L^\infty(X,\nu,B(H))$ - the Banach algebra of all essentially bounded weakly-$*$ ν-locally measurable functions on (X,ν) into $B(H)$ via the map $(a \otimes f)(\zeta) = f(\zeta)a$, where f is identified with the function $f(\zeta)$ on X, and $a \in B(H)$. Furthermore $L^\infty(X,\nu,B(H))$ acts on $L^2(X,\nu,H)$ - the Hilbert space of H-valued L^2-functions on X, with inner product

$$\langle \xi, \eta \rangle = \int_X \langle \xi(\zeta), \eta(\zeta) \rangle d\nu(\zeta),$$

and action is pointwise; $(f\xi)(\zeta) = f(\zeta)\xi(\zeta)$. In particular, since $U \in B(H) \bar{\otimes} B$, U can be identified with the function $w(\zeta) \in L^\infty(X,\nu,B(H))$.

By condition (4), $VV^* \in \mathbb{C} \otimes B(K)$. Since $\varphi(1) = 1$, V is an isometry, hence there is a projection p in $B(K)$ such that $VV^* = 1 \otimes p$. We show $\dim p = 1$. For this note that since V is an isometry, $V^*B(H')V = B(H)$. Thus

$$V B(H) V^* = (1 \otimes p)(B(H) \bar{\otimes} B(K))(1 \otimes p) = B(H) \bar{\otimes} p B(K) p ,$$

and the map $x \to V x V^*$ is an isomorphism of $B(H)$ onto $B(H) \bar{\otimes} pB(K)p$. By condition (5) $\pi(A) = A \bar{\otimes} \mathbb{C}$, so there exists a $*$-automorphism α of A such that $\pi(a) = \alpha(a) \otimes 1$ for $a \in A$. Hence, if $a \in A$, then $\varphi(a) = a$ by assumption, so that

$$V a V^* = V\varphi(a)V^* = VV^*(\alpha(a) \otimes 1)VV^* = \alpha(a) \otimes p .$$

Consequently $A \bar{\otimes} \mathbb{C}p = V A V^*$. Since by assumption A is a maximal abelian subalgebra of $B(H)$, $A \bar{\otimes} \mathbb{C}p$ is maximal abelian in $VB(H)V^* = B(H) \otimes pB(K)p$. But this is only possible when $\dim p = 1$.

Let ξ_0 be a unit vector in K such that $p\xi_0 = \xi_0$. If $\xi \in H$ then $V\xi = V'\xi \otimes \xi_0$, where V' is a unitary operator in $B(H)$, as is trivially verified. Recall that $(\xi \otimes \xi_0)(\zeta) = \xi_0(\zeta)\xi$ if $\xi \in H$. Thus if $x \in B(H)$, $\xi, \eta \in H$, we have

$$\begin{aligned} \langle V^*\pi(x)V\xi, \eta\rangle &= \langle \pi(x)V'\xi \otimes \xi_0, V'\eta \otimes \xi_0\rangle \\ &= \langle U(x \otimes 1)U^*(V'\xi \otimes \xi_0), V'\eta \otimes \xi_0\rangle \\ &= \int_X \langle w(\zeta)\, x w(\zeta)^* \xi_0(\zeta)V'\xi, \xi_0(\zeta)V'\eta\rangle d\nu(\zeta) \\ &= \int_X \langle V'^* w(\zeta)\, x\, w(\zeta)^* V'\xi, \eta\rangle |\xi_0(\zeta)|^2 d\nu(\zeta) \end{aligned}$$

Letting $u(\zeta) = V'^*w(\zeta)$ and $d\mu(\zeta) = |\xi_0(\zeta)|^2 d\nu(\zeta)$ we thus have

$$\varphi(x) = \int_X u(\zeta)\, x\, u(\zeta)^*\, d\mu(\zeta)$$

and

$$\int_X d\mu(\zeta) = \int_X |\xi_0(\zeta)|^2 d\nu(\zeta) = \|\xi_0\|^2 = 1 .$$

Therefore all that remains in order to complete the proof of (i) is to show $u(\zeta)$ is a unitary operator in A a.e. (μ). Since U is unitary and V' is unitary it is clear that $u(\zeta)$ is unitary a.e. (μ). Let q be a projection in A, and let $\xi \in H$ be a unit vector. Then we have

$$\langle q\xi,\xi\rangle = \int \langle u(\zeta)q\, u(\zeta)^*\xi,\xi\rangle \, d\mu(\zeta) = \int \|u(\zeta)q\, u(\zeta)^*\xi\|^2 d\mu(\zeta).$$

Since $0 \leq \|u(\zeta)q\, u(\zeta)^*\xi\| \leq 1$ a.e. (μ) it follows that if $\xi \in q(H)$ then $\langle q\xi,\xi\rangle = 1$, hence $\|u(\zeta)q\, u(\zeta)^*\xi\| = 1$ a.e. (μ), hence $\xi \in u(\zeta)q\, u(\zeta)^*(H)$ a.e. (μ). Since this holds for all $\xi \in q(H)$, $q \leq u(\zeta)q\, u(\zeta)^*$ a.e. (μ). If $\xi \in (1-q)(H)$ then $0 = \langle q\xi,\xi\rangle$, and $u(\zeta)q\, u(\zeta)^*\xi = 0$ a.e. (μ). As above then, $1-q \leq 1-u(\zeta)q\,u(\zeta)^*$ a.e. (μ). Consequently $q = u(\zeta)q\, u(\zeta)^*$ a.e. (μ) for each projection q in A. Since H is separable, A is countably generated, so that $u(\zeta)x\, u(\zeta)^* = x$ for all $x \in A$ and all $\zeta \in X$ outside a set of μ-measure 0. Thus $u(\zeta) \in A' = A$ a.e. (μ), and the proof of (ii) $\Rightarrow$ (i) is complete.

(i) $\Rightarrow$ (ii). Let $(X,\mathcal{B},\mu)$ and u be given so that (i) holds. Let $K = L^2(X,\mu)$, and let

$$H' = H \otimes L^2(X,\mu) = L^2(X,\mu,H),$$

where the identification of $H \otimes L^2(X,\mu)$ and $L^2(X,\mu,H)$ is via $(\xi\otimes f)(\zeta) = f(\zeta)\xi$. Define a linear map $V : H \to H'$ by

$$(V\xi)(\zeta) = \xi, \quad \xi \in H,$$

and define a map π of $B(H)$ into operators on H' by

$$(\pi(x)f)(\zeta) = u(\zeta)x\, u(\zeta)^* f(\zeta), \qquad x \in B(H),\ f \in H'.$$

Then we have

$$\begin{aligned}\|\pi(x)f\|^2 &= \int_X \|u(\zeta)x\, u(\zeta)^* f(\zeta)\|^2 d\mu(\zeta)\\ &\leq \|x\|^2 \int_X \|f(\zeta)\|^2 d\mu(\zeta)\\ &= \|x\|^2\|f\|^2,\end{aligned}$$

so that $\|\pi(x)\| \le \|x\|$. Since it is trivial to verify that π is $*$-preserving, linear, multiplicative, and $\pi(1) = 1$, π is a $*$-representation of $B(H)$ on H' . Let $\xi, \eta \in H$. Then we have

$$\begin{aligned} \langle V^*\pi(x)V\xi, \eta\rangle &= \int_X \langle(\pi(x)V\xi)(\zeta), (V\eta)(\zeta)\rangle d\mu(\zeta) \\ &= \int_X \langle u(\zeta)x\, u(\zeta)^*\xi, \eta\rangle\, d\mu(\zeta) \\ &= \langle\varphi(x)\xi, \eta\rangle , \end{aligned}$$

so that $V^*\pi V$ is a Stinespring decomposition for φ .
We let $B = L^\infty(X,\mu)$ and verify conditions (1) - (5).

(1) is trivial by definition of V .

(2) By definition, if $x \in B(H)$, $f \in H'$ then $(\pi(x)f)(\zeta) = u(\zeta)x\, u(\zeta)^*f(\zeta)$. Thus $\pi(x) \in L^\infty(X,\mu,B(H))$, which equals $B(H) \bar{\otimes} L^\infty(X,\mu)$ by [16, 1.22.13], and (2) follows.

(3) Suppose $y \in \pi(B(H))' \cap (B(H) \bar{\otimes} B)$. Then $y \in L^\infty(X,\mu,B(H))$, so $y(\zeta) \in B(H)$ for $\zeta \in X$, $\zeta \to y(\zeta)$ is measurable and ess. $\sup\|y(\zeta)\| = \|y\|$. Since $y \in \pi(B(H))'$, if $x \in B(H)$

$$y(\zeta)u(\zeta)x\, u(\zeta)^* = u(\zeta)x\, u(\zeta)^*\, y(\zeta), \text{ a.e. } \mu .$$

Since $x \to u(\zeta)x\, u(\zeta)^*$ is a $*$-automorphism of $B(H)$ a.e. (μ) , $y(\zeta)w = w\,y(\zeta)$ for all $w \in B(H)$, i.e. $y(\zeta)$ is a scalar a.e. (μ). Thus $y(\zeta) = f(\zeta)1$ for some $f \in L^\infty(X,\mu)$, i.e. $y \in \mathbb{C} \bar{\otimes} B$, and (3) is proved.

(4) If $f \in H'$, $\xi \in H$ we have

$$\langle V^*f, \xi\rangle = \langle f, V\xi\rangle = \int_X \langle f(\zeta), \xi\rangle d\mu(\zeta) = \langle \int_X f(\zeta)d\mu(\zeta), \xi\rangle ,$$

hence

$$V^*f = \int_X f(\zeta)d\mu(\zeta) .$$

Let $x \in B(H)$, $f,g \in H'$. Then we have

$$\begin{aligned} \langle VV^*(x\otimes 1)f, g\rangle &= \int \langle(V^*(x\otimes 1)f)(\zeta), (V^*g)(\zeta)\rangle d\mu(\zeta) \\ &= \langle\int x f(\zeta)\, d\mu(\zeta), \int g(\zeta)d\mu(\zeta)\rangle = \langle x V^*f, V^*g\rangle . \end{aligned}$$

Similarly we have

$$<(x\otimes 1)VV^*f,g> = <V^*f,V^*(x^*\otimes 1)g> = <V^*f,x^*V^*g> ,$$

hence $(x\otimes 1)VV^* = VV^*(x\otimes 1)$ for all $x \in B(H)$, and (4) follows.

(5) By assumption $u(\zeta) \in A$ a.e. (μ). Hence for $x \in A$ and $f \in H'$ we have

$$(\pi(x)f)(\zeta) = u(\zeta)x\,u(\zeta)^*f(\zeta) = xf(\zeta) = ((x\otimes 1)f)(\zeta) ,$$

so $\pi(x) = x\otimes 1$, and (5) follows. Thus (i) $\Rightarrow$ (ii) is proved.

Finally if (i) or (ii) is satisfied then it is straightforward from (i) to show that φ belongs to the point-ultraweak closure of $A\otimes A$. Q.E.D.

In the finite dimensional case part (ii) of the above theorem has a much simpler form. Recall that if n is a natural number we denote by M_n the complex $n\times n$ matrices and D_n the diagonal $n\times n$ matrices.

<u>Corollary 7.2</u> Let $\varphi \in B(M_n)$ be a positive map. Then the following two conditions are equivalent.

(i) There exist a probability space $(X,\mathcal{B},\mu)$ and a measurable map u of X into the unitary group of D_n such that

$$\varphi(x) = \int_X u(\zeta)x\,u(\zeta)^*d\mu(\zeta) , \qquad x\in M_n .$$

(ii) $\varphi(x) = x$ for all $x \in D_n$, and φ is completely positive with a Stinespring decomposition $V^*\pi V$ satisfying the following three conditions:

(1) There exists a Hilbert space K such that $V : \mathbb{C}^n \to \mathbb{C}^n\otimes K$.

(2) There exists an abelian von Neumann algebra B on K such that $\pi(x) \in M_n\bar{\otimes} B$ for all $x \in M_n$.

(3) $VV^* \in \mathbb{C}\bar{\otimes} B(K)$.

Proof (i) ⇒ (ii) is immediate from Theorem 7.1. In order to show the converse we have to show that conditions (3) and (5) in Theorem 7.1 are redundant when H is finite dimensional.

Let (e_{ij}) be the natural matrix units in M_n, so that e_{ii} is a projection in D_n of dimension 1. Let $f_{ij} = \pi(e_{ij})$. Then $f_{ii} \sim f_{11}$ in $M_n \bar{\otimes} B$, and $\Sigma f_{ii} = 1$, so each f_{ii} has central carrier 1. Let ψ denote the canonical center valued trace on $M_n \bar{\otimes} B$ [5, Ch. III, § 4, Theorème 3]. Then $\psi(f_{ii}) = {}^1/n\, 1 = \psi(e_{ii} \otimes 1)$, so $f_{ii} \sim e_{jj}$ for all i,j. In particular f_{ii} is an abelian projection in $M_n \bar{\otimes} B$ for each i. From the proof of (ii) ⇒ (i) in Theorem 7.1 there is a unitary operator $U \in M_n \bar{\otimes} B$ such that $U(x \otimes 1)U^* = \pi(x)$ for $x \in M_n$. In particular $U(M_n \otimes \mathbb{C})U^* = \pi(M_n)$, so that

$$
\begin{aligned}
\pi(M_n)' \cap (M_n \bar{\otimes} B) &= U(M_n \bar{\otimes} \mathbb{C})' U^* \cap (M_n \bar{\otimes} B) \\
&= U(\mathbb{C} \bar{\otimes} B(K))U^* \cap (M_n \bar{\otimes} B) \\
&= U((\mathbb{C} \bar{\otimes} B(K)) \cap (M_n \bar{\otimes} B))U^* \\
&= U(\mathbb{C} \bar{\otimes} B)U^* \\
&= \mathbb{C} \bar{\otimes} B ,
\end{aligned}
$$

and (3) follows.

To show (5) we notice that $VV^* = 1 \otimes p$ for a projection p in $B(K)$. Since V is an isometry, $n \dim p = \dim(1 \otimes p) = \dim VV^* = n$, hence $\dim p = 1$. Then it follows as in the proof of (ii) ⇒ (i) in Theorem 7.1 that $\varphi(x) = \int u(\zeta) x\, u(\zeta)^* d\mu(\zeta)$ with $u(\zeta) \in D_n$ a.e. (μ). In particular, $U \in (D_n \bar{\otimes} \mathbb{C})' = D_n \bar{\otimes} B(K)$ and $\pi(x) = x \otimes 1$ for $x \in D_n$. Thus (5) follows. Q.E.D.

8. Hadamard products of matrices

As before we let M_n denote the complex $n \times n$ matrices. If $a = (\alpha_{ij})$ and $b = (\beta_{ij})$ belong to M_n their Hadamard product is defined as

$$a * b = (\alpha_{ij}\beta_{ij}) .$$

We refer the reader to [20] for a survey on this matrix product. If D_n denotes the diagonal matrices in M_n we shall first see that the study of maps in $D_n \otimes D_n$ is the same as that of the Hadamard product. Then we shall characterize a certain class of matrices by means of their Hadamard product.

Let (e_{ij}) denote the usual matrix units in M_n, so if $a = (\alpha_{ij}) \in M_n$ then $a = \Sigma \alpha_{ij} e_{ij}$. Let $\tilde{a}$ denote the map $\tilde{a} = \Sigma \alpha_{ij} e_{ii} \otimes e_{jj}$ in $D_n \otimes D_n$, considered as a subalgebra of $B(M_n)$. Then if $b = (\beta_{ij}) \in M_n$ we have

$$\tilde{a}(b) = \Sigma \alpha_{ij} e_{ii}\, b\, e_{jj} = \Sigma \alpha_{ij}\beta_{ij} e_{ij} = a * b .$$

If we identify $\mathrm{Sp}\, D_n$ with the set $\{1,\dots,n\}$, so that $\hat{c}(i) = \gamma_i$ whenever $c = \Sigma \gamma_i e_{ii} \in D_n$, then for a and $\tilde{a}$ as above we have $\hat{\tilde{a}}(i,j) = \alpha_{ij}$. Thus $\hat{\tilde{a}}$ is a positive definite function on $\mathrm{Sp}\, D_n \times \mathrm{Sp}\, D_n$ if and only if a is positive. It is then clear that Theorem 5.1 is the infinite dimensional analogue of the classical result that $a \geq 0$ if and only if $a * b \geq 0$ for all $b \geq 0$ [20, Theorem 3.1].

In the next section we shall give an abstract characterization of maps in $A \otimes A$ of the form α_μ, cf. § 4. For this we shall need a stronger property than positive definiteness, namely we shall need that the matrices $(\hat{\varphi}(\gamma_i, \gamma_j))$ considered in § 5 belong to a restricted class of positive matrices. We next give some equivalent definitions of this class of matrices.

<u>Proposition 8.1</u> Let $a = (\alpha_{ij})$ be a positive matrix in M_n such that $\alpha_{ii} = 1$, $i = 1,\dots,n$. Then the following four conditions are equivalent:

(i) $a \in \overline{\text{conv}}\{(z_i\bar{z}_j) \in M_n : |z_1| = \cdots = |z_n| = 1\}$.

(ii) There exist a continuous unitary representation u of the n-dimensional torus T^n into the diagonal matrices D_n and a Borel probability measure μ on T^n such that

$$a * b = \int_{T^n} u(z) b\, u(z)^* d\mu(z)\,, \qquad b \in M_n\,.$$

(iii) The map $b \to a * b$ in $B(M_n)$ is completely positive with a Stinespring decomposition satisfying the conditions in Corollary 7.2 (ii).

(iv) There exists a positive definite function f on $\mathbb{Z}^n$ such that

$$f((\delta_{i1}-\delta_{j1},\dots,\delta_{in}-\delta_{jn})) = \alpha_{ij}\,, \qquad 1 \le i,j \le n\,,$$

where $\delta_{kl} = 0$ if $k \neq l$ and 1 if $k = l$.

<u>8.2 Notation</u> We denote by K_n the closed convex hull of the rank 1 matrices $(z_i\bar{z}_j)$ such that $z = (z_1,\dots,z_n) \in T^n$.

<u>Proof of Proposition 8.1</u> We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let $a \in K_n$. Let u be the continuous unitary representation of T^n with values in D_n given by $u(z) = \Sigma z_i e_{ii}$, where $z = (z_1,\dots,z_n) \in T^n$. Since M_n is finite dimensional and K_n is convex and compact, its extreme boundary ∂K_n is closed. Furthermore, since each matrix $(z_i\bar{z}_j)$, $z \in T^n$, is of rank 1, they are all in ∂K_n. Consequently $\partial K_n = \{(z_i\bar{z}_j) : z \in T^n\}$. From convexity theory [1, 1.4.8] there is a Borel probability measure ν

on ∂K_n such that

$$a = \int_{\partial K_n} \zeta d\nu(\zeta) .$$

We map T^n onto ∂K_n via the map $z = (z_1,\dots,z_n) \to (z_i\bar{z}_j)$. Since the map $b = (\beta_{ij}) \to \Sigma\beta_{ij}e_{ii}\otimes e_{jj} = \tilde{b}$ is a continuous isomorphism of M_n given the Hadamard product and $D_n \otimes D_n$ it follows easily that there exists a Borel probability measure μ on T^n such that

$$\tilde{a} = \int_{T^n} u(z) \otimes u(z)^* d\mu(z) ,$$

hence (ii) follows.

(ii) $\Rightarrow$ (iii). This is immediate from Corollary 7.2.

(iii) $\Rightarrow$ (iv). By Corollary 7.2 there exists a probability space $(X,\mathcal{B},\mu)$ such that

$$\tilde{a} = \int_X u(\zeta) \otimes u(\zeta)^* d\mu(\zeta) ,$$

where $\zeta \to u(\zeta)$ is a measurable map of X into the unitary group of D_n. For each $\zeta \in X$ let f_ζ denote the function on $\mathbb{Z}^n$ defined by

$$f_\zeta(m_1,\dots,m_n) = \prod_{i=1}^{n} u(\zeta)_i^{m_i} ,$$

where $u(\zeta) = \sum_{i=1}^{n} u(\zeta)_i e_{ii}$. Since we have chosen u so that $u(\zeta)$ is unitary for each ζ, f is a character of $\mathbb{Z}^n$. Furthermore

$$\begin{aligned}\hat{\tilde{a}}(i,j) = \alpha_{ij} &= \int_X u(\zeta)_i \overline{u(\zeta)_j}\, d\mu(\zeta) \\ &= \int_X f_\zeta((\delta_{i1} - \delta_{j1},\dots,\delta_{in} - \delta_{jn}))d\mu(\zeta) .\end{aligned}$$

Thus the function f on $\mathbb{Z}^n$ defined by

$$f(m_1,\dots,m_n) = \int_X f_\zeta(m_1,\dots,m_n)d\mu(\zeta) ,$$

is the required function.

(iv)$\Rightarrow$(i). Let f be a positive definite function on $\mathbb{Z}^n$ satisfying (iv). By Bochner's theorem [12,36 A] there exists a Borel probability measure μ on $\hat{\mathbb{Z}}^n = T^n$ such that

$$f(m_1-k_1,\ldots,m_n-k_n) = \int_{T^n} \prod_{i=1}^{n} z_i^{m_i-k_i}\, d\mu(z)$$

where $z = (z_1,\ldots,z_n)$, and $m_i, k_i \in \mathbb{Z}$. In particular we have

$$\alpha_{ij} = \int_{T^n} \prod_{k=1}^{n} z_k^{\delta_{ik}-\delta_{jk}}\, d\mu(z) = \int_{T^n} z_i \bar{z}_j\, d\mu(z)\,.$$

Thus $a \in K_n$, and the proof is complete.

A positive $n\times n$ matrix $a = (\alpha_{ij})$ such that $\alpha_{ii} = 1$ is often called a correlation matrix, see [20]. In a forthcoming paper J.P.R. Christensen and J. Vesterstrøm [4] give another characterization for a correlation matrix to belong to K_n. Furthermore they show that K_n is properly contained in the set of correlation matrices when $n \geq 4$. The latter result was also known to U. Haagerup, at least for some n. I am much indebted to Vesterstrøm for pointing out mistakes in early versions of Proposition 8.1. In the sequel we shall need the following results on K_n.

<u>Lemma 8.3</u> Let $a = (\alpha_{ij}) \in M_n$ be positive and $|\alpha_{ij}| = 1$ for all i,j. Then there exists $z \in T^n$ such that $\alpha_{ij} = z_i\bar{z}_j$.

<u>Proof</u> If $1 \leq n \leq 2$ the lemma is trivial. Assume $n \geq 3$ and let $z_i = \overline{\alpha_{1i}}$. Since a is positive, so is the 3×3 matrix

$$\begin{pmatrix} 1 & \bar{z}_i & \bar{z}_j \\ z_i & 1 & \alpha_{ij} \\ z_j & \overline{\alpha_{ij}} & 1 \end{pmatrix}.$$

Its determinant $D = -2 + 2\mathrm{Re}(\bar{z}_i z_j \alpha_{ij})$ is nonnegative since the matrix is positive. Since $|\bar{z}_i| = |z_j| = |\alpha_{ij}| = 1$, $\alpha_{ij} = z_i \bar{z}_j$.

Lemma 8.4 Let $a = (\alpha_{ij}) \in M_n$. Let J_i be a finite set with r_i elements, $i = 1,\dots,n$, such that the integers $\{1,2,\dots,r\}$ where $r = \sum_{i=1}^{n} r_i$, is the disjoint union of the J_i. Let $b \in M_r$ be the matrix (β_{kl}) where $\beta_{kl} = \alpha_{ij}$ if $k \in J_i$, $l \in J_j$. Then b is positive if a is positive, and $b \in K_r$ if $a \in K_n$.

Proof Assume a positive. Let $(\xi_1,\dots,\xi_r) \in \mathbb{C}^r$. Then

$$\sum_{k,l} \beta_{kl} \bar{\xi}_k \xi_l = \sum_{i,j} \sum_{(k,l)\in J_i \times J_j} \beta_{kl} \bar{\xi}_k \xi_l$$

$$= \sum_{ij} \alpha_{ij} \Big(\sum_{k\in J_i} \bar{\xi}_k \Big)\Big(\sum_{l\in J_j} \xi_l \Big),$$

which is nonnegative since a is positive. Thus b is positive.

If $a \in K_n$ then by Proposition 8.1 there is a Borel probability measure μ on T^n such that $a = \int_{T^n} (z_i \bar{z}_j) d\mu(z)$, hence $\alpha_{ij} = \int_{T^n} z_i \bar{z}_j \, d\mu(z)$. For each $z \in T^n$ let $b(z)$ be the $r \times r$ matrix $(\beta_{kl}(z))$ with $\beta_{kl}(z) = z_i \bar{z}_j$ if $k \in J_i$, $l \in J_j$. Then $b = \int_{T^n} b(z) d\mu(z)$. Since $b(z)$ is positive by the first part of the proof, it is in K_n by Lemma 8.3. Thus $b \in K_n$ by Proposition 8.1.

It is a well known and a very useful fact that a self-adjoint $n \times n$ matrix is positive if and only if all its submatrices symmetric about the diagonal have nonnegative determinants. A natural analogous problem is: Find an integer k depending on n and complex functions $f_1,\dots,f_k$ in n^2 variables such that a correlation matrix $a = (\alpha_{ij})$ belongs to K_n if and only if $f_l(\alpha_{11}, \dots, \alpha_{nn}) \geq 0$ for $l = 1,\dots,k$.

9. A Bochner theorem for positive maps

In this section we study the problem of when a map $\varphi \in A \otimes A$ is of the form α_μ, cf. §4, where μ is a Borel probability measure on a locally compact abelian group. For this we shall need a stronger condition of positive definiteness of $\hat{\varphi}$ than the one used in §6. Recall from 8.2 that
$K_n = \overline{\mathrm{conv}}\{(z_i\bar{z}_j) \in M_n : z = (z_1,\dots,z_n) \in T^n\}$.

Definition 9.1 Let X be a set and f a complex function on $X \times X$. We say f is strongly positive definite if whenever $\gamma_1,\dots,\gamma_n \in X$ then the matrix $(f(\gamma_i,\gamma_j)) \in K_n$. If $(X,\mathcal{A},\nu)$ is a σ-finite measure space we say $f \in L^\infty(X \times X, \nu\times\nu)$ is essentially strongly positive definite if there is a set $N \in \mathcal{A}$ of ν-measure zero such that f is strongly positive definite on $(X \setminus N) \times (X \setminus N)$.

In the above definition we have intrinsically assumed that $f(\gamma,\gamma) = 1$ for all (respectively almost all) $\gamma \in X$. However from Lemma 5.4 we know that if $\varphi \in A \otimes A$ is positive then $\varphi(1) = 1$ if and only if $\hat{\varphi}(\gamma,\gamma) = 1$ for all $\gamma \in \mathrm{Sp}\,A$.

Remark 9.2 If A and B are abelian C^*-algebras such that $A \subset B$ and $\varphi \in A \otimes A$ is such that $\hat{\varphi} \in C(\mathrm{Sp}\,A \times \mathrm{Sp}\,A)$ is strongly positive definite then $\hat{\varphi}$ considered as a function in $C(\mathrm{Sp}\,B \times \mathrm{Sp}\,B)$ is also strongly positive definite. Indeed, let ι denote the inclusion map of $A \otimes A$ into $B \otimes B$. Then its adjoint map restricts to a continuous map r of $\mathrm{Sp}\,B \otimes B$ into $\mathrm{Sp}\,A \otimes A$ such that if $f \in C(\mathrm{Sp}\,A \otimes A)$ then $\iota(f)(\gamma) = f(r(\gamma))$ for $\gamma \in \mathrm{Sp}\,B \otimes B$. If $\hat{\varphi}$ considered as a function in $C(\mathrm{Sp}\,A \otimes A)$ is strongly positive definite it is thus clear that $\iota(\hat{\varphi})$, which is $\hat{\varphi}$ considered as a function in $C(\mathrm{Sp}\, B \otimes B)$ is also strongly positive definite. A consequence of this remark is that we may always assume A is a von Neumann algebra in order to study maps in $A \otimes A$ which have strongly positive definite Gelfand transforms.

If $(X, \mathfrak{A}, \nu)$ is a σ-finite measure space and $f, g \in L^\infty(X,\nu)$ we identify the function $f \otimes g$ in the C^*-tensor product $L^\infty(X,\nu) \overset{*}{\otimes} L^\infty(X,\nu)$ with the function $(f\otimes g)(\gamma,\gamma') = f(\gamma)g(\gamma')$ in $L^\infty(X\times X, \nu\times\nu)$, and thus imbed $L^\infty(X,\nu) \overset{*}{\otimes} L^\infty(X,\nu)$ isometrically into $L^\infty(X\times X,\nu\times\nu)$. We consider this imbedding as an inclusion, so we can talk about functions in $L^\infty(X,\nu) \overset{*}{\otimes} L^\infty(X,\nu)$ as essentially strongly positive definite.

Proposition 9.3 Let A be a countably generated nonatomic abelian von Neumann algebra. Let T^ω be the compact abelian group which is the countable infinite product of the circle group T with itself. Let $(X, \mathfrak{A}, \nu)$ be a σ-finite measure space such that A is identified with $L^\infty(X, \mathfrak{A}, \nu)$. Let $f \in L^\infty(X,\nu) \overset{*}{\otimes} L^\infty(X,\nu)$ be essentially strongly positive definite. Then we have:

(i) There is a continuous unitary representation S of T^ω into the unitary group $\mathcal{U}(A)$ of A such that the function $u \to \langle\gamma, S(u)\rangle$ $(= S(u)(\gamma))$ is measurable for each $\gamma \in X$.

(ii) There is a state ω on the abelian C^*-algebra of bounded measurable functions on T^ω such that

$$f(\gamma,\gamma') = \omega(\langle\gamma, S(u)\rangle \overline{\langle\gamma', S(u)\rangle}) \quad \text{a.a.} \quad \gamma,\gamma' \in X .$$

Lemma 9.4 With the assumptions and notation of Proposition 9.3 let N be a measurable set of ν-measure zero such that f is strongly positive definite on $(X \setminus N) \times (X \setminus N)$. Then there exist a sequence $(\mathcal{P}_n)$ of measurable partitions of $X \setminus N$ and a sequence $(\gamma_{(\varepsilon_1,\ldots,\varepsilon_n)})$ in $X \setminus N$ with the following properties:

(i) $\mathcal{P}_n = \{P_{(\varepsilon_1,\ldots,\varepsilon_n)} : \varepsilon_i \in \{0,1\}\}$.

(ii) $P_{(\varepsilon_1,\ldots,\varepsilon_n,0)} \cup P_{(\varepsilon_1,\ldots,\varepsilon_n,1)} = P_{(\varepsilon_1,\ldots,\varepsilon_n)}$ for all ε_i .

(iii) $\gamma_{(\varepsilon_1,\ldots,\varepsilon_n)} \in P_{(\varepsilon_1,\ldots,\varepsilon_n)}$.

(iv) If $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$, $\eta = (\eta_1,\ldots,\eta_n)$ then the functions

$$f_n = \sum_{\varepsilon \in \{0,1\}^n} \sum_{\eta \in \{0,1\}^n} f(\gamma_\varepsilon,\gamma_\eta)\, \chi_{P_\varepsilon} \otimes \chi_{P_\eta}$$

are strongly positive definite, where $\chi_E \otimes \chi_F$ denotes the characteristic function of the set $E \times F \subset X \times X$.

(v) $\|f_n - f\|_\infty \to 0$.

<u>Proof</u> Let $\delta > 0$. Considering $X \setminus N$ instead of X we may assume f is strongly positive definite. Since the algebraic tensor product $L^\infty(X,\nu) \odot L^\infty(X,\nu)$ is norm dense in $L^\infty(X,\nu) \overset{*}{\otimes} L^\infty(X,\nu)$ and each function in $L^\infty(X,\nu)$ is a norm limit of simple functions, we can find a measurable partition $\{P_1,\ldots,P_n\}$ of X of sets of positive measure and $\lambda_{ij} \in \mathbb{C}$, $i,j = 1,\ldots,n$ such that if $\psi' = \Sigma \lambda_{ij}\, \chi_{P_i} \otimes \chi_{P_j}$, then $\|f-\psi'\|_\infty < \delta/2$. Deleting a set of measure zero we may assume $\sup|f(\gamma,\gamma') - \psi'(\gamma,\gamma')| < \delta/2$. Thus if $\gamma_i \in P_i$ then $|f(\gamma_i,\gamma_j) - \lambda_{ij}| < \delta/2$. Let $\psi = \Sigma f(\gamma_i,\gamma_j)\chi_{P_i} \otimes \chi_{P_j}$. Then the triangle inequality yields $\|\psi - f\|_\infty < \delta$. Since f is strongly positive definite an easy application of Lemma 8.4 shows that ψ is strongly positive definite. Furthermore we may split up the sets $P_1,\ldots,P_n$ so we may assume $n = 2^m$ for some m . A standard inductive argument now yields the sequences $(\mathcal{P}_n)$ and (f_n)

in the lemma, using the assumption that A is nonatomic. Q.E.D.

<u>Proof of Proposition 9.3</u> Let $\mathcal{P}_n$ and f_n be constructed as in Lemma 9.4. Let $Y_n = \prod_1^n \{0,1\}$ and $Y = \prod_1^\infty \{0,1\}$ ($= \{0,1\}^{\mathbb{N}}$). Let β_n be the continuous imbedding

$$\beta_n : Y_n \to Y \text{ by } \beta_n(\varepsilon_1,\ldots,\varepsilon_n) = (\varepsilon_1,\ldots,\varepsilon_n,0,0,\ldots) .$$

Let $H_n = \prod_{y \in Y_n} T_y$, where $T_y = T$ is the circle group, and let H_n have the product topology. Let $H = \prod_{y \in Y} T_y$, also with the product topology. Define a continuous imbedding $\alpha_n : H_n \to H$ by

$$(\alpha_n(u))_{(\varepsilon_1,\varepsilon_2,\ldots)} = u_{(\varepsilon_1,\ldots,\varepsilon_n)} .$$

Let $\gamma_n : H \to H_n$ by $\gamma_n(u) = u \circ \beta_n$, so that $\gamma_n(u)_{(\varepsilon_1,\ldots,\varepsilon_n)} = u_{(\varepsilon_1,\ldots,\varepsilon_n,0,0,\ldots)}$. Then we have

$$\gamma_n(\alpha_n(u)) = u \quad \text{for } u \in H_n .$$

We define a map $T_n : H_n \to \mathcal{U}(A)$ by

$$T_n(u) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)} u_{(\varepsilon_1,\ldots,\varepsilon_n)} \chi_{P_{(\varepsilon_1,\ldots,\varepsilon_n)}} ,$$

where we note that $\mathcal{U}(A)$ is identified with the L^∞-functions on X into the circle group T. Then $u_i \to u$ in H_n implies $T_n(u_i) \to T_n(u)$ pointwise as L^∞-functions, or equivalently T_n is continuous, when H_n has the product topology and $\mathcal{U}(A)$ the strong topology.

Define the map $S_n : H \to \mathcal{U}(A)$ by $S_n = T_n \circ \gamma_n$. Then S_n is continuous and easy computations show they are extensions of each other in the following sense: if $n > m$ and $u = \alpha_m(v)$, $v \in H_m$ then $S_n(u) = S_m(u) = T_m(v)$.

We now define a map $S : H \to \mathcal{U}(A)$ which extends all the S_n. Let $u \in H$. Now $(\alpha_n(H_n))_{n \in \mathbb{N}}$ is an increasing sequence of subgroups of H with union dense in H, and if $u^n = \alpha_n(\gamma_n(u))$ then $u^n \in \alpha_n(H_n)$ and $u^n \to u$ in H. Let $\gamma \in X$. From the construction of the partitions $\mathcal{P}_n$, there is a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ in Y such that $\gamma \in P_{(\varepsilon_1, \ldots, \varepsilon_n)}$ for all n. A straightforward computation shows that $S_n(u^n)(\gamma) = u_{(\varepsilon_1, \ldots, \varepsilon_n, 0, 0, \ldots)}$, so that $S_n(u^n)(\gamma) \to u_{(\varepsilon_1, \varepsilon_2, \ldots,)}$. Define $S(u)$ as the pointwise limit of the functions $S_n(u^n)$. If we write the value of $S(u)$ at γ, as $\langle\gamma, S(u)\rangle$ we have shown that $\langle\gamma, S(u)\rangle = \lim\langle\gamma, S_n(\alpha_n(\gamma_n(u)))\rangle$, hence the function $u \to \langle\gamma, S(u)\rangle$ is a pointwise limit of continuous functions on H for each $\gamma \in X$. In particular it is a measurable function on H. Furthermore, since $S_n(u^n) \to S(u)$ pointwise, $S_n(u^n) \to S(u)$ in the strong topology. Thus if $\xi, \eta \in L^2(X, \nu)$, $\langle S_n(u^n)\xi, \eta\rangle \to \langle S(u)\xi, \eta\rangle$ for all u, hence the function $u \to \langle S(u)\xi, \eta\rangle$ is the pointwise limit of continuous functions hence is measurable.

Since $L^\infty(X, \nu)$ is countably generated $L^2(X, \nu)$ is a separable Hilbert space. Also it is clear from their definitions that γ_n and T_n are multiplicative, hence so is S_n. Thus S being a pointwise limit of the S_n is multiplicative, hence S is a measurable unitary representation of H on the separable Hilbert space $L^2(X, \nu)$. But then S is strongly continuous [8, p. 347]. Thus the proof of part (i) in the proposition is complete.

To show (ii) we write ε for the element $(\varepsilon_1, \ldots, \varepsilon_n)$ in Y_n. Then f_n defines a strongly positive definite function g_n on $Y_n \times Y_n$ by $g_n = \sum_{\varepsilon, \eta \in Y_n} f(\gamma_\varepsilon, \gamma_\eta)\delta_\varepsilon \otimes \delta_\eta$, where δ_ε is the point measure with value 1 at ε. By Proposition 8.1 there is a Borel probability measure ν_n on H_n such that

$$g_n(\varepsilon, \eta) = \int_{H_n} \langle\varepsilon, u\rangle \overline{\langle\eta, u\rangle}\, d\nu_n(u) .$$

If $\gamma \in P_\varepsilon$, $\gamma' \in P_\eta$ then $f_n(\gamma,\gamma') = g_n(\varepsilon,\eta)$, and $<\varepsilon,u> = <\gamma,T_n(u)>$, $<\eta,u> = <\gamma',T_n(u)>$. Thus we have

$$f_n(\gamma,\gamma') = \int_{H_n} <\gamma,T_n(u)>\overline{<\gamma',T_n(u)>}d\nu_n(u) .$$

Let $\nu_n' = \nu_n \circ \alpha_n^{-1}$. Then ν_n' is a Borel probability measure on H. If $\gamma \in P_{(\varepsilon_1,\ldots,\varepsilon_n)}$, and $u \in H_n$ then

$$<\gamma,S(\alpha_n(u))> = <\gamma,S_n(\alpha_n(u))> = <\gamma,T_n(u)> .$$

Thus

$$f_n(\gamma,\gamma') = \int_H <\gamma,S(u)>\overline{<\gamma',S(u)>}d\nu_n'(u) .$$

Let ω_n be the state

$$\omega_n(f) = \int_H f(u)d\nu_n'(u)$$

on the abelian C*-algebra $\mathcal{A}$ of bounded Borel measurable functions on H. Let ω be a w*-limit point of the sequence (ω_n) in the state space of $\mathcal{A}$, say the subnet (ω_{n_α}) converges to ω in the w*-topology. Since the functions $u \to <\gamma,S(u)>$ belong to $\mathcal{A}$ we have for almost all $\gamma,\gamma' \in X$,

$$\begin{aligned} f(\gamma,\gamma') &= \lim_\alpha f_{n_\alpha}(\gamma,\gamma') \\ &= \lim_\alpha \int_H <\gamma,S(u)>\overline{<\gamma',S(u)>}d\nu_{n_\alpha}'(u) \\ &= \lim_\alpha \omega_{n_\alpha}(<\gamma,S(u)>\overline{<\gamma',S(u)>}) \\ &= \omega(<\gamma,S(u)>\overline{<\gamma',S(u)>}) . \end{aligned}$$

Since H can be identified with T^ω we are through.

<u>Corollary 9.5</u>. Let X be a separable compact Hausforff space and ν a finite nonatomic regular Borel measure on X with support X. Let $f \in C(X \times X)$ be strongly positive definite. Then there are a

continuous unitary representation S of T^{ω} with values in $L^{\infty}(X,\nu)$ and a state ω on the abelian C^*-algebra $\mathcal{A}$ of bounded measurable functions on T^{ω} such that the functions $u \to \langle\gamma,S(u)\rangle$ are in $\mathcal{A}$ for all $\gamma \in X$ and such that

$$f(\gamma,\gamma') = \omega(\langle\gamma,S(u)\rangle\overline{\langle\gamma',S(u)\rangle}) \qquad \text{a.e. } (\nu) .$$

<u>Proof</u> Since support ν is X, $C(X)$ is isometrically imbedded in $L^{\infty}(X,\nu)$, and f considered as an element of $L^{\infty}(X \times X,\nu \times \nu)$ is strongly positive definite. If A is the abelian von Neumann algebra $L^{\infty}(X,\nu)$ acting on $L^2(X,\nu)$ by multiplication, then an application of Proposition 9.3 yields the desired result.

<u>Remark</u> In Proposition 9.3 and Corollary 9.5 we have assumed that the abelian von Neumann algebra in question is nonatomic. This is not important. The general case can be taken care of as in the proof of Theorem 9.6 below.

We are now in position to prove the main representation theorem for positive normalized maps in $A \otimes A$, which shows that such maps which are strongly positive definite, are of the form α_{μ} as in Lemma 4.4, where the group is a closed subgroup of the infinite dimensional torus T^{ω} . This result is an answer to our initial problem, namely to obtain a deeper insight into the relationship between spectral theory of linear maps of $B(H)$ and Fourier analysis. It shows in particular that function calculus for such a map $\varphi \in A \otimes A$ corresponds to function calculus for measures in the measure algebra $M(G)$. We denote by $\mathrm{Ad}\,u$ the automorphism $u \otimes u^*$ of $B(H)$ when u is a unitary operator.

Theorem 9.6 Let A be an abelian von Neumann algebra acting on a separable Hilbert space H.. Let $\varphi \in A \otimes A$, and assume $\hat{\varphi}$ is strongly positive definite. Then there are a compact abelian group G, a continuous unitary representation S of G with values in A and a Borel probability measure μ on G such that

$$\varphi = \int_G \operatorname{Ad} S(u)\, d\mu(u) .$$

Proof We first assume A is nonatomic, and we identify the Gelfand transform map of $A \otimes A$ on $\operatorname{Sp} A \otimes A$ with the canonical imbedding $A \otimes A \to A \overset{*}{\otimes} A$. If A is identified with $L^\infty(X,\nu)$ for some σ-finite measure space $(X, \mathfrak{A}, \nu)$ an easy argument using Lemma 9.4 shows that $\hat{\varphi}$ considered as an element of $L^\infty(X \times X, \nu \times \nu)$ is essentially strongly positive definite. Thus by Proposition 9.3 there are a state ω on the abelian C^*-algebra $\mathcal{A}$ of bounded measurable functions on T^ω and a continuous unitary representation S on T^ω with values in A such that

(8) $$\hat{\varphi}(\gamma,\gamma') = \omega(\langle\gamma, Su\rangle \overline{\langle\gamma', S(u)\rangle}) \quad \text{a.a. } \gamma,\gamma' \in X .$$

Let by the Riesz representation theorem μ be the Borel probability measure on T^ω such that

$$\omega(f) = \int_{T^\omega} f(u)\, d\mu(u)$$

for f a continuous function on T^{ω}. We shall show that for each $\rho \in B(H)_*$, $x \in B(H)$ we have

(9) $$\rho(\varphi(x)) = \int_{T^{\omega}} \rho(Su\, x\, Su^*)d\mu(u).$$

Suppose first φ is of the form $\varphi = \sum_{i,j=1}^{n} \lambda_{ij}\, e_i \otimes e_j$ with $e_1,\ldots,e_n$ an orthogonal family of projections in A with sum 1. Let B be the abelian C^*-algebra they generate. Then B is a finite dimensional subalgebra of A. From the proof of Proposition 9.3 we see that the formula (8) is an integral

$$\hat{\varphi}(\gamma,\gamma') = \int_{T^{\omega}} \langle\gamma, Su\rangle \overline{\langle\gamma', Su\rangle} d\mu(u),$$

where the support of μ is in a closed subgroup H_m of T^{ω} isomorphic to a finite product of T with itself. Thus (8) can be rephrased as

$$\chi(\varphi) = \int_{T^{\omega}} \chi(Su \otimes Su^*)d\mu(u)$$

for all characters χ of $B \otimes B$. Since the characters span $(B\otimes B)^*$, and $(B\otimes B)^*$ is the set of restrictions of functionals in $(A\otimes A)^*$ to $B \otimes B$ it follows that

$$\omega(\varphi) = \int_{T^{\omega}} \omega(Su \otimes Su^*)d\mu(u)$$

for all $\omega \in (A\otimes A)^*$. If $\omega(\varphi) = \rho(\varphi(x))$ we see that (9) holds.

For φ as in the theorem we can just as in Lemma 9.4 find a sequence (φ_n) of positive maps in $A \otimes A$ of the form $\varphi_n = \Sigma \lambda_{ij}^{n}\, e_i^n \otimes e_j^n$, such that $\lambda_{ii}^n = 1$, and $\|\varphi_n - \varphi\| \to 0$. Let

$$\hat{\varphi}_n(\gamma,\gamma') = \int_{T^{\omega}} \langle\gamma, Su\rangle \overline{\langle\gamma', Su\rangle} d\mu_n(u) \qquad \text{a.a. } \gamma,\gamma' \in X.$$

As in the proof of Proposition 9.3 there is a subnet (μ_{n_α}) of

(μ_n) which converges to μ in w*-topology. Thus if $\rho \in B(H)_*$ and $x \in B(H)$, $u \to \rho(Su\,x\,Su^*)$ is continuous on T^ω, hence

$$\rho(\varphi(x)) = \lim_{n_\alpha} \rho(\varphi_{n_\alpha}(x)) = \lim_{n_\alpha} \int_{T^\omega} \rho(Su\,x\,Su^*)d\mu_{n_\alpha}(u)$$

$$= \int_{T^\omega} \rho(Su\,x\,Su^*)d\mu(u)\ .$$

In the general case when A may have minimal projections let K be a separable infinite dimensional Hilbert space and B a nonatomic abelian von Neumann algebra acting on K. Then the von Neumann algebra tensor product $C = A \bar{\otimes} B$ of A and B acts on $H \otimes K$, and is a nonatomic abelian von Neumann algebra. If A is identified with the subalgebra $A \bar{\otimes} \mathbb{C}$ of C we have $\varphi \in C \otimes C \subset B(B(H \otimes K))$. By the first part of the proof there is a probability measure μ on T^ω such that $\varphi = \int_{T^\omega} \mathrm{Ad}\, S(u) d\mu(u)$, where $\mathrm{Ad}\, Su \in \mathrm{Aut}\, B(H \otimes K)$. In order to complete the proof of the theorem we first show that $S(u) \in A$ for each $u \in T^\omega$ except for a set of μ-measure zero. If not there is a measurable set $E \subset T^\omega$ of positive measure such that $u \in E$ implies $Su \notin A$, and there is a one dimensional projection p on K such that $Su(1 \otimes p)Su^* \neq 1 \otimes p$ for all $u \in E$. Let $F = \{u \in T^\omega : Su(1 \otimes p)Su^* \neq 1 \otimes p\}$. Then $F \supset E$ and is measurable with positive measure. Let ω be a faithful normal state on $B(H)$ and ρ be the vector state on $B(K)$, $\rho(x) = \langle x\xi, \xi \rangle$, where $p\xi = \xi$. If $u \in T^\omega$ and $\omega \otimes \rho(Su(1 \otimes p)Su^*) = 1$, then $Su(1 \otimes p)Su^* \geq \mathrm{supp}(\omega \otimes \rho) = 1 \otimes p$, so if $u \in F$, then $Su(1 \otimes p)Su^* > 1 \otimes p$. Therefore two possibilities may occur. Either $\omega \otimes \rho(Su(1 \otimes p)Su^*) < 1$ for u in a subset F_1 of F with positive measure, or if not $Su(1 \otimes p)Su^* > 1 \otimes p$ for all $u \in F$ except on a set of zero measure. Since $\varphi \in A \otimes A \subset C \otimes C$ and $1 \otimes p$ belongs to the commutant of A, $\varphi(1 \otimes p) = 1 \otimes p$. Thus in the former case

$$1 = \omega \otimes \rho(1 \otimes p) = \omega \otimes \rho(\varphi(1 \otimes p))$$

$$= \int_{T^\omega} \omega \otimes \rho(Su(1 \otimes p)Su^*)d\mu(u)$$

$$< \int_{T^\omega} 1 d\mu = 1 ,$$

a contradiction. In the latter case we may find a normal state η on $B(H \otimes K)$ such that $\eta(Su(1 \otimes p)Su^*) > \eta(1 \otimes p)$ on F except on a set of zero measure. Since $\eta(Su(1 \otimes p)Su^*) = \eta(1 \otimes p)$ for $u \in T^\omega - F$ we have

$$\eta(1 \otimes p) = \eta(\varphi(1 \otimes p))$$

$$= \int_{T^\omega} \eta(Su(1 \otimes p)Su^*)d\mu(u)$$

$$> \int_{T^\omega} \eta(1 \otimes p)d\mu = \eta(1 \otimes p) ,$$

a contradiction. Thus $Su \in A$ for μ-almost all $u \in T^\omega$. Let $G = S^{-1}(\mathcal{U}(A))$. Then G is a compact abelian group and $S|G$ is a continuous unitary representation of G into $\mathcal{U}(A)$. Since $\operatorname{supp} \mu \subset G$ we are through.

Remark 9.7 We have not succeeded in proving a direct converse to Theorem 9.6 , i.e. if $\varphi \in A \otimes A$ is of the form α_μ with μ a Borel probability measure on a compact abelian group G , then $\hat{\varphi}$ is strongly positive definite. The reason for this is that it is not clear whether

$$\hat{\alpha}_\mu(\gamma,\gamma') = \int_G \hat{\alpha}_g(\gamma,\gamma')d\mu(g) . \tag{10}$$

It is clear that if $\hat{\alpha}_\mu$ is of this form and $\gamma_1,\ldots,\gamma_n \in \operatorname{Sp} A$ then

$$(\hat{\alpha}_\mu(\gamma_i,\gamma_j)) = (\int_G \hat{\alpha}_g(\gamma_i,\gamma_j)d\mu(g)) = \int_G (\hat{\alpha}_g(\gamma_i,\gamma_j))d\mu(g) ,$$

and since each matrix $(\hat{\alpha}_g(\gamma_i,\gamma_j))$ ia an extreme point of K_n, $(\hat{\alpha}_\mu(\gamma_i,\gamma_j)) \in K_n$, hence $\hat{\alpha}_\mu$ is strongly positive definite.

Conversely, if $\varphi \in A \otimes A$ is such that $\hat{\varphi}$ is strongly positive definite, so by Theorem 9.6 $\varphi = \alpha_\mu$, then (10) holds. Indeed, we assume A nonatomic, and leave it to the reader to use the techniques of the proof of Theorem 9.6 to extend the argument to the general case. Then from the proof of the theorem there exists a sequence $\varphi_n = \Sigma \lambda_{ij}^n e_i^n \otimes e_j^n$ in the algebraic tensor product $A \otimes A$ such that $\|\varphi_n - \varphi\| \to 0$ and $(\lambda_{ij}^n) \in K_m$ for some m. Furthermore

$$\varphi_n = \alpha_{\mu_n} = \int \alpha_g \, d\mu_n(g) ,$$

hence by the first part of the proof of the theorem

$$\hat{\varphi}_n(\gamma,\gamma') = \int \hat{\alpha}_g(\gamma,\gamma') \, d\mu_n(g) .$$

Let $f_{\gamma,\gamma'}(g) = \hat{\alpha}_g(\gamma,\gamma')$. Then $f_{\gamma,\gamma'}$ is a continuous function on G. Since a subnet (μ_{n_β}) of (μ_n) converges to μ in the w*-topology,

$$\hat{\varphi}_{n_\beta}(\gamma,\gamma') = \int f_{\gamma,\gamma'}(g) d\mu_{n_\beta}(g) \to \int f_{\gamma,\gamma'}(g) d\mu(g) .$$

Since also $\hat{\varphi}_{n_\beta}(\gamma,\gamma') \to \hat{\varphi}(\gamma,\gamma')$, (10) follows as asserted.

<u>Corollary 9.8</u> Let A be an abelian C^*-algebra acting on the separable Hilbert space H. Let $\varphi \in A \otimes A$ be such that $\hat{\varphi}$ is strongly positive definite. Then φ is an extreme point of the convex set $K = \{\psi \in B(B(H)) : \psi$ is positive, $\psi(1) = 1\}$ if and only if φ is a *-automorphism of $B(H)$.

<u>Proof</u> It follows from [18] that every *-automorphism is an extreme point of K. The converse is an immediate consequence of Theorem 9.6 and the fact that if μ is a point measure then φ is a *-automorphism.

Corollary 9.9 Let A be an abelian C^*-algebra acting on the separable Hilbert space H. Let $\varphi \in A \otimes A$ be such that $\hat{\varphi}$ is strongly positive definite. Thus φ has a Stinespring decomposition satisfying the conditions of Theorem 7.1 (ii).

10. Comments

There are several problems left open in the previous paragraphs. Some we have not touched because we feel they are outside the scope of the paper. To this class of problems belong the study of unbounded maps and the problem of maps of general von Neumann algebras into themselves, rather than $B(H)$. For the latter problem there are two obvious approaches. One is first to perform an analogous study of maps of semi-finite von Neumann algebras using the trace in a way similar to ours, and then to try to use Tomita theory to modify this approach to type III algebras. Another approach is to follow the line of the present paper and then to consider the von Neumann algebras in question as invariant subspaces of the maps. This approach has the drawback that it makes it only possible to study maps in $B(M)$, M a von Neumann algebra, which have nice extensions to maps in $B(B(H))$.

There is one concrete problem we have left open. In both Proposition 5.5 and Lemma 6.2 we have results to the effect that if φ is operator normal and $\varphi(1) = 1$ then $Tr(\varphi(x)) = Tr(x)$ for all $x \in \mathcal{T}$. Is this true for all (regular) operator normal maps φ such that $\varphi(1) = 1$ and $\varphi|\mathcal{T} \in B(\mathcal{T})$? A possible approach is to generalize the result in [19] and then approximate 1 ultraweakly by Hilbert-Schmidt operators x such that $\varphi(x) - x$ is "small".

The problem of computing norms seems to be extremely difficult. If φ is positive, regular, operator normal, and $\varphi(1) = 1$, then $1 \in \sigma_{B(\mathcal{H})}(\varphi|\mathcal{H})$ by Corollary 3.4, hence by Lemma 3.2 $\|\varphi\|_2 = \|\varphi\| = 1$. It is clear from Corollary 5.3 how this is related to the fact that if $f \in L^1(G)$ and $\hat{f}$ is positive definite, then $\|f\|_1 = \|\hat{f}\|_\infty$. For other maps, it is as for Fourier transforms of functions, difficult to know the norm of φ if $\|\varphi\|_2$ is known. A consequence of this is the limited set of functions we can use if we want to do functional calculus for an operator normal map φ.

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