The Light of the TARDIS

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Abstract

In 2013 a new metric called TARDIS was introduced. This metric was published by Tippet and Tsang, and described a time machine. This thesis uses numerical methods to explore properties of this new metric. We start by visualizing geodesics in two and three dimensions, then we see that there is a really high blueshift when the light rays hit the borders of our time machine, but when we drop an approximation made by Tippet and Tsang, we find that the blueshift almost disappears. We use the Darmois-Israel junction conditions to see that it is possible for this time machine to travel anywhere in time and space. Finally, we calculate the energy-momentum tensor corresponding to this metric and find that it requires exotic matter, which makes it hard to construct this time machine in real life.

Acknowledgements

"There's a lot of things you need to get across this universe. Warp drive... wormhole refractors... You know the thing you need most of all? You need a hand to hold."

- The Doctor, Season 6, Episode 6

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Also I would like to thank Realistforeningen for making the years I have spent studying fun and exciting.

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Chapter 1

Introduction

"The universe is big. It's vast and complicated and ridiculous. And sometimes, very rarely, impossible things just happen and we call them miracles."
- The Doctor, Season 5, Episode 12

1.1 General relativity

This year, 2015, we celebrate the 100th anniversary of Einstein's general relativity. From the premise that matter warps spacetime and the warping of spacetime tells matter how to move, we have discovered many things that not even Einstein could foresee. We have discovered the big bang, black holes, dark matter and dark energy to mention a few things. The simple premise and the infinite new discoveries it has brought us has led some people to call it the perfect theory. The Einstein equation

$$\mathrm{G}_{\mu\nu} = 8\pi G \mathrm{T}_{\mu\nu} \tag{1}$$

is the equation that tells how spacetime curves given a specific mass distribution. G is the Einstein tensor that describes the spacetime, G is Newton's gravitational constant and T is the stress energy tensor that describes the mass distribution that we want to explore.

A number of analytic solutions to this equation has been found, the most famous are probably the Minkowski solution, because of its simplicity often used as a benchmark when comparing different solutions, the Schwarzschild solution which describes the spacetime outside a spherical mass distribution like a star or planet and the Friedmann-Robertson-Walker (FRW) metric that tries to describe the whole universe. All the solutions have different ways to describe distances in spacetime, this distance is called the metric and is the defining feature of a solution. The metrics above look like this:

Minkowski: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$

Schwarzschild: $\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$, $r_s = 2GM$

FRW: $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$

In all these cases the light speed $c = 1$, as this simplifies the equations considerably.

The Minkowski metric describes a spacetime with no curvature, i.e. no gravitation. The Minkowski spacetime is often referred to as flat spacetime, it is used as a benchmark for other metrics trying to describe the universe with gravity. It is the spacetime where special relativity is formulated.

The Schwarzchild metric describes the vacuum (space) outside a massive, non-rotating, non-charged object. $r_s = 2GM$ and is called the Schwarzschild radius, M is the mass of the object and G is as before the gravitational constant. When $r = r_s$ we have a coordinate singularity and the Schwarzschild metric breaks down. If we get the whole mass of an object inside the Schwarzschild radius many believe we get a black hole. The Schwarzschild radius is called the event horizon and is the radius of a black hole. The event horizon is a point of no return, inside the event horizon nothing can escape from the black hole, due to the strong gravitational forces.

The last of the metrics is particularly important as it is often called the standard model of modern cosmology. It describes a homogeneous, isotropic expanding or contracting universe. The latest observational evidence suggest that the universe is almost perfectly homogeneous and isotropic, when averaged over large scales, thus a FRW spacetime. The metric written above is for flat space, the more general involves a few more factors to include for the curvature of space, but is not very different.

A note on coordinates: There are several different kinds of coordinate systems, the normal Cartesian coordinate system with t,x,y and z is what is most used in this thesis. It is important to note that I use $(-,+,+,+)$ as the signature of the metrics, not $(+,-,-,-)$. There is no significant difference in using one and not the other, the choice is made purely by personal preference. The Schwarzchild metrics is an example of spherical coordinate system, that has the awkward property that when $r = r_s$ we divide by zero, and we get a singularity. We see that when $r < r_s$, the signed is flipped and some believe that within a black hole time and space change properties, but as the metric describes the vacuum outside the body not the body itself this can not be relied on. It was discovered that the singularity at $r = r_s$ is a coordinate singularity meaning that it could disappear if we change our coordinate system (Penrose, 1965). These coordinates are called Eddington-Finkelstein-coordinates and works both inside and outside $r = r_s$.

There are other types of coordinate systems like the cylindrical coordinates or the two dimensional polar coordinates, that are commonly used. We will see a little of the cylindrical coordinates when discussing the Gödel metric, but there are no singularities there. Though everything could be described in the same coordinate system, the math can become so much easier with the correct choice of coordinates.

The theory of general relativity has answered many questions about the universe and is today the leading theory on gravity. Even with the problem around dark energy and dark matter(Planck Collaboration, 2014) it still holds its ground as one of the most important scientific theories. Its so vital that when people develop alternative gravity theories they always look towards general relativity, instead of coming up with something new and exciting[1]. One of the more exotic features of the general theory of relativity is that it does not exclude the possibility for time travel, which is the topic for our next section.

1.2 Time travel

Time travel has fascinated mankind for a long time and that general relativity does not forbid such behavior is like a dream come true for science fiction writers and fans. Time travel in general relativity was first introduces by Kurt Gödel in 1949, with a metric

[1] As general relativity behaves perfectly in our own solar system, we have to believe that it is correct on those scales, which indicates that we have to use it as a basis when we want to modify gravity

known as the Gödel metric(Gödel, 1949). It introduced a concept called closed timelike curve (CTC) which is a path a particle can take in spacetime that returns it to its start point in both space and time. Even though Gödel was the first to identify a CTC in his own metric, this does not mean he was the first to construct such a metric, many metrics had possibilities for CTCs, but it was not discovered before many years later. The Tipler cylinder(Tipler, 1974) and wormholes are solution that had the possibility for CTCs, but it was not discovered before Gödel introduced the concept.

The Tipler cylinder was first introduced by Lanczos in 1924(Lanczos, 1924), but it was not before 1974 that Tipler discovered that it would be possible to create CTCs in this spacetime. The Tipler cylinder is a massive, infinitely long cylinder that rotates, the rotation causes a frame dragging effect that tilts the world lines in such a way that they now point backwards in a spacetime diagram. Wormholes were first discovered by Ludwig Flamm in 1916 and were constructed from the Schwarzschild metric, which had only been discovered a few months before. It connects two regions of space with Schwarzschild black holes at each end. It was later discovered that it is impossible to keep this kind of wormhole stable. In 1988 however Michael Morris and Kip Thorne discovered a way to create stable wormholes with exotic matter(Morris and Thorne, 1988).

The trouble with backward time travel is that it leads to paradoxes, which some physicists insist must mean that time travel is physically impossible. Some examples of paradoxes that could arise from backwards time travel is:

Bootstrap paradox: If you on your 30th birthday decide to invent a time machine and suddenly an older you shows up and give you the plans to make a time machine, you build your time machine and the first trip you do is back to yourself on your 30th birthday and deliver the plans to the younger you. This is slightly problematic as the time machine has no point of origin, no one invented it.

Grandfather paradox: If you decide to travel back in time and kill your grandfather when he was a toddler, you have prevented one of your parents birth thus also your own birth, making it impossible for you to have traveled back to kill your grandfather.

Many physicists have made attempts to avoid such paradoxes, this has led us to the Novikov self-consistency principle(Friedman et al., 1990) which states that if an event exists that would give rise to a change in the past, then the probability of that event is zero. It will be impossible to create time paradoxes if this is correct. Stephen Hawking also tried to avoid paradoxes with the chronology protection conjecture(Hawking, 1992). He is certain that once we have a theory to combine quantum physics with general relativity, time travel would be impossible on all but submicroscopic scales. Even though some prominent physicists have stated that time travel is impossible their conjectures have not been confirmed, which means that it is still possible that we could travel back in time at some point.

When discussing paradoxes it is natural to look to the philosophers as they should be the experts(Smith, 2013). Their starting point is that changing the past should be impossible. The grandfather paradox arises from changing the past, so either you cannot travel back in time or it is impossible to kill your grandfather once you have traveled back in time. One solution is that your attempt to kill your grandfather fails for some commonplace reason, the gun might jam, a noise might distract you etc. The grandfather paradox itself is not a strong enough limitation to completely rule out the possibility of time travel.

The trouble with CTC's is that it is very difficult to make them stable, many of the solutions containing them require exotic matter. Exotic matter has negative energy density, and some would call it unphysical. Even though it is unphysical on our macroscopic levels, the quantum physicists have discovered a way they can make it through the Casimir effect. It is still very impractical and would require some tremendous effort to make this work on macroscopic scales if that is even possible. You can read more about this in section 2.2.4.

1.3 Doctor Who

My thesis is based on a research article by Tippett and Tsang (2013), which in turn is inspired by the TV-series Doctor Who. It is natural to have a short introduction of one of the best TV shows of all time, before we look at the article itself.

The popular science fiction show Doctor Who has existed for almost 52 years and is the longest running science fiction show in the world. It features an alien time lord calling himself the Doctor, that travels in time and space with human companions. The program started as an educational show to learn children about science and history as the episodes often features aliens trying to change history. His time machine that looks like a blue police box is called TARDIS.

The series did not become really popular before they removed the educational part, and made it a proper science fiction show. Before this change some episodes had no science fiction elements at all apart from some time travel at the beginning of the episode. After they changed the show aliens, remote planets and absurd story lines became a common part of the British Saturday. The Doctor and his companions solves problems that arises around in the universe, and often battles alien races that wants to create chaos like the Daleks, the Cybermen and the Weeping Angels.

The Doctor has the ability to regenerate when he should have died. The "new" doctor has all the memories of the previous Doctor, but new personality and body. This has led to several people playing the Doctor and after 800 episodes there has been twelve main doctors in the series.

The TARDIS, Time and Relative Dimensions In Space, can transport the Doctor and his companions anywhere in time and space. In the series there are a lot of explanations to what the TARDIS can or cannot do, and the ship even has its own personality. The most important feature of the TARDIS after the ability to travel anywhere in time and space is that it is bigger on the inside, the small blue exterior hides a massive inside. It is one of the most characteristic elements of the series, as the Doctor changes over the series, but the TARDIS remains the same. It is therefore natural to call the metric based on Doctor Who and his time travels for TARDIS, even though it does not behave in exactly the same way. Our TARDIS requires traveling, while the Doctors TARDIS can spontaneously disappear and appear any place, it is commented on in one episode that for a spaceship the TARDIS does remarkably little flying.

Doctor Who is a significant part of popular culture in Britain and there are several

Figure 1: Picture of the outside of the TARDIS

expressions that are commonly used in English that comes from Doctor Who, probably most used is TARDIS-like, that means anything that seems larger on the inside than on the outside. The phrase ”watching from behind the sofa” originates from the earlier episodes where children wanted to see the show but not the frightening parts.

Enough talk about the show let's now talk about the spacetime based on this show.

1.4 The goal of this thesis

My work will mainly focus on the TARDIS metric (Tippett and Tsang, 2013), which was published in October 2013 at the same time as the 50th anniversary of the series Doctor Who. As the name of the solution suggests this metric's main focus is to enable you to build a time machine. This is the first solution in general relativity that allows what we would call a time machine, a box that can travel anywhere in time.

My main goal for this thesis is to visualize how the geodesics in this metric look. This has already been done for two dimensions in the original article, but it is a nice starting point as I will look at more than two dimensions. In similar metrics we have seen a very high blueshift and I will see if this also happens in the TARDIS metric. To do this I need several parts that I will calculate in the visualization part, so the first part has to be correct. Later I will find out if it will be possible to travel anywhere in time and space with this metric and lastly we will take a quick look at the energy-momentum tensor to see if this solution requires exotic matter. In the original article they use an approximate version of the metric, I will try to visualize how the metric will look if we use a proper top hat function, not an approximation. I will also calculate the blueshift and the Einstein tensor for this proper top hat function.

Chapter 2

Theory

"You want weapons? We're in a library! Books! The best weapons in the world!"
- The Doctor, Season 2, Episode 2

2.1 A Brief Introduction to General Relativity

2.1.1 The Metric

As previously mentioned a solution of the Einstein equations is often called a metric, and determines the local structure of spacetime. A metric looks like

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{2}$$

where g is the metric tensor, μ and ν are indices running from 0 to 3. The metric tensor can be viewed as a 4×4 matrix where the different elements contains all the geometric and causal structure of spacetime. We see an example of Einstein summation convention where repeated indices are summed over. As an example we can look at the spatially flat FRW metric where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix} \tag{3}$$

which gives as before

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{4}$$

and $c = 1$ as before.

The metric tensor contains absolutely all information about the spacetime, distance, volume, curvature, future and past. It is therefore the key component to calculate other interesting things about the spacetime. In the next sections we will look at different properties that are calculated from the metric tensor.

2.1.2 The geodesic equation

A geodesic in general relativity is the path of a particle would take in spacetime if not affected by any force, where gravity now is spacetime not a force. A geodesic is the equivalent of a straight line in that spacetime. As a planet is not affected by other forces as it revolves around the sun, its path could be viewed as a geodesic. The geodesics are often used to view how a spacetime "looks"

To calculate the geodesics we need the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}. \tag{5}$$

Where we again have used the Einstein summation convention. Here τ is a just a scalar parameter of motion and $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. Commonly we use $\mathrm{d}s$ or proper time as $\mathrm{d}\tau$, but we are just calculating the geodesics for photons and as $\mathrm{d}s = 0$ for photons we cannot use it. My choice later in the thesis is just evenly spaced numbers or how time behaves in Minkowski space. The Christoffel symbols are directly calculated from the metric tensor:

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \tag{6}$$

where $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$.

The Christoffel symbols are extremely important as they are used in almost every quantity used in general relativity as we will see a little bit later when we tackle the Einstein equation.

With the geodesic equation we can look at different spacetimes and always know how the particles will move. In my thesis it is often used to visualize how the spacetime looks.

2.1.3 The Einstein equation

The Einstein equation is the equation that all spacetimes have to fulfill and it looks like before:

$$\mathrm{G}_{\alpha\beta} = 8\pi G \mathrm{T}_{\alpha\beta} \tag{7}$$

The $\mathrm{G}_{\alpha\beta}$ represents the curvature of spacetime e. g. the metric, while $\mathrm{T}_{\alpha\beta}$ represents the matter and energy content of spacetime. Together with the geodesic equation this is the core of the mathematical formulation of general relativity, we can calculate everything that is related to gravity. A simple explantion of how the Einstein equation works is: If you want to understand gravity you have to understand the geometry of the universe.

The left side of this equation can be split up

$$\mathrm{G}_{\alpha\beta} = \mathrm{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathrm{R} \tag{8}$$

where $\mathrm{R}_{\alpha\beta}$ is the Ricci curvature tensor and R is the Ricci scalar that look like:

$$\mathrm{R}_{\alpha\beta} = \Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\alpha\mu} \tag{9}$$

and

$$R = g^{\alpha\beta} R_{\alpha\beta}, \tag{10}$$

where we use the notation $,\mu = \frac{\partial}{\partial x^\mu}$.

We see that the curvature of the Einstein equation is only dependent on the metric via the Christoffel symbols, not on any other properties. The right side can be calculated from the left side, but we have to have some assumptions on how the right side should behave. The most common thing however is to calculate the left side, as it is much easier to measure the contents of the universe than the curvature.

The stress energy tensor,$T^{\alpha\beta}$, on the right side contains information on energy and momentum in spacetime. It can be viewed as a 4×4 matrix and is the source of the gravitational field, the first component in this tensor, T^{00}, is the energy density and the rest of the diagonal, T^{ii}, is pressure. The rest of the tensor has no really simple physical description. The most common stress energy tensor is that of a perfect fluid, as it is a good approximation of both the interior of a star and a homogeneous, isotropic universe:

$$T^{\alpha\beta} = (\rho + p)u^{\alpha}u^{\beta} + pg^{\alpha\beta} \tag{11}$$

where ρ is energy density, p is pressure and u^{α} is the four-velocity.

2.2 Closed Timelike Curves

As mentioned before closed timelike curves are paths particles take through spacetime, that returns them to the same place and time as before. To get a better understanding of what CTCs are we have to introduce a new concept.

2.2.1 World lines

A world line of an object is the path of that object as it travels through spacetime. It is easily visualized through a Minkowski/spacetime diagram, as we see in figure 2. The axes are space on the x-axis and time on the y-axis, and we have different names for the different parts of the diagram. The blue line is exactly 45° to the time and space axes and is called light-like as this is how light moves. The time-like is for particles that move slower than the speed of light, this is the world we live in, and lastly we have space-like that is generally considered unphysical as this is matter that moves faster than light.

If we look at the line-element we see that: $ds^2 = 0$ is light-like, $ds^2 < 0$ is time-like and $ds^2 > 0$ is space-like. To create CTCs we have to have space-like world lines as we will see later when I present an example. My world lines later in this thesis are the geodesics of light, they start out light-like, but as my metric has curvature the light cones become time-like and even space-like at some point.

2.2.2 The Gödel universe

Closed timelike curves was first introduced by Gödel(Gödel, 1949) in 1949 because he wanted to prove that time does not have behave intuitively in general relatively, which he certainly achieved. We will now look at some of the properties of the Gödel universe(Buser

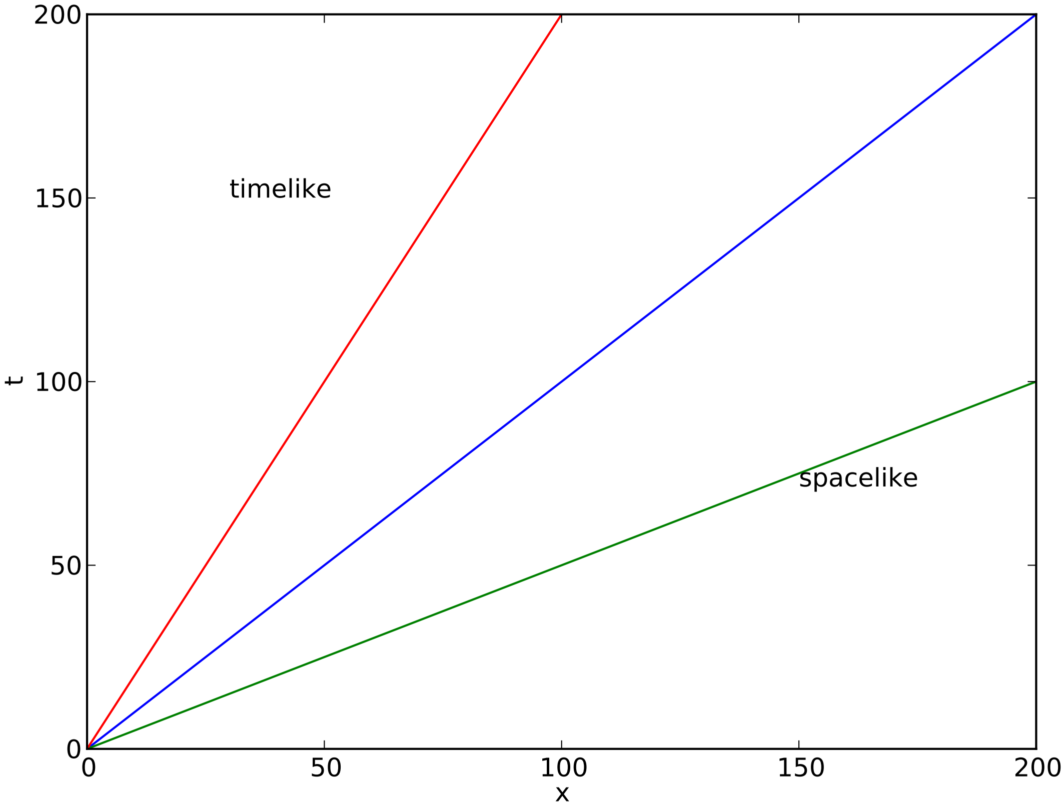


Figure 2: Trivial world lines in special relativity (no curvature), the one on the left is time-like, the middle is light-like and the right is space-like

et al., 2013). First off we have the metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & \frac{r^2}{\sqrt{2}a} & 0 \\ 0 & \frac{1}{1+(\frac{r}{2a})^2} & 0 & 0 \\ r^2\frac{1}{\sqrt{2}a} & 0 & r^2\big(1-(\frac{r}{2a})^2\big) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

where we use cylindrical coordinates, t, r, ϕ and z. In this metric we have Minkowski space in the limit $a \to \infty$, $ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2$.

We see that this metric has off-diagonal elements which makes it a little more difficult than our examples, but it is very similar to the TARDIS metric. The off-diagonal is one of the key components to enable us to create CTCs as it is these elements that causes the worldlines to twist, and that twist causes the worldlines to reach negative values of time and overlap with other worldlines. This causes space-like world lines/geodesics and enables us to travel in time, it is however not the geodesics that make up the CTCs in the Gödel universe, an acceleration has to be applied to the object. Figure 3 shows the world lines for different values of of the Gödel radius $r_G = 2a$ and we see clearly the twisting of the geodesics. The Gödel radius is important because it defines the point where the term before $\mathrm{d}\phi^2$ is 0, where there is a Cauchy horizon, a place where the metric changes from spacelike to timelike.

The Gödel universe also has some carefully chosen values in stress-energy tensor. It only require two terms in the stress-energy tensor, one representing the matter density of a homogeneous distribution of swirling dust particles, and the second is for a nonzero cosmological constant. The values of these to terms has to match perfectly, so that the value of the cosmological constant must be chosen to match the density of the dust. If we interpret the dust particles as galaxies we get a cosmological model of a rotating universe. There are some aspects of the Gödel universe that makes it unlike our universe: Both the rotation and the fact that there is no expansion makes it unlikely that this metric describes our universe.

2.2.3 The Alcubierre drive

The Alcubierre drive was proposed in 1994 by the physicist Miguel Alcubierre(Alcubierre, 1994) as an example of how a spacecraft could travel faster-than-light. By a local expansion of spacetime behind the spacecraft and an opposite contraction in front of it, motion faster than the speed of light as seen by an outside observer is possible. This metric is reminiscent of the "warp drive" we see in Star Trek and the TARDIS metric I work with in this thesis. As Alcubierre based his metric on how the spaceships in Star Trek moves and the TARDIS metric is based on this metric, it is not very surprising.

The metric for the Alcubierre drive looks like:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \big(\mathrm{d}x - v_s f(r_s)\mathrm{d}t\big)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{13}$$

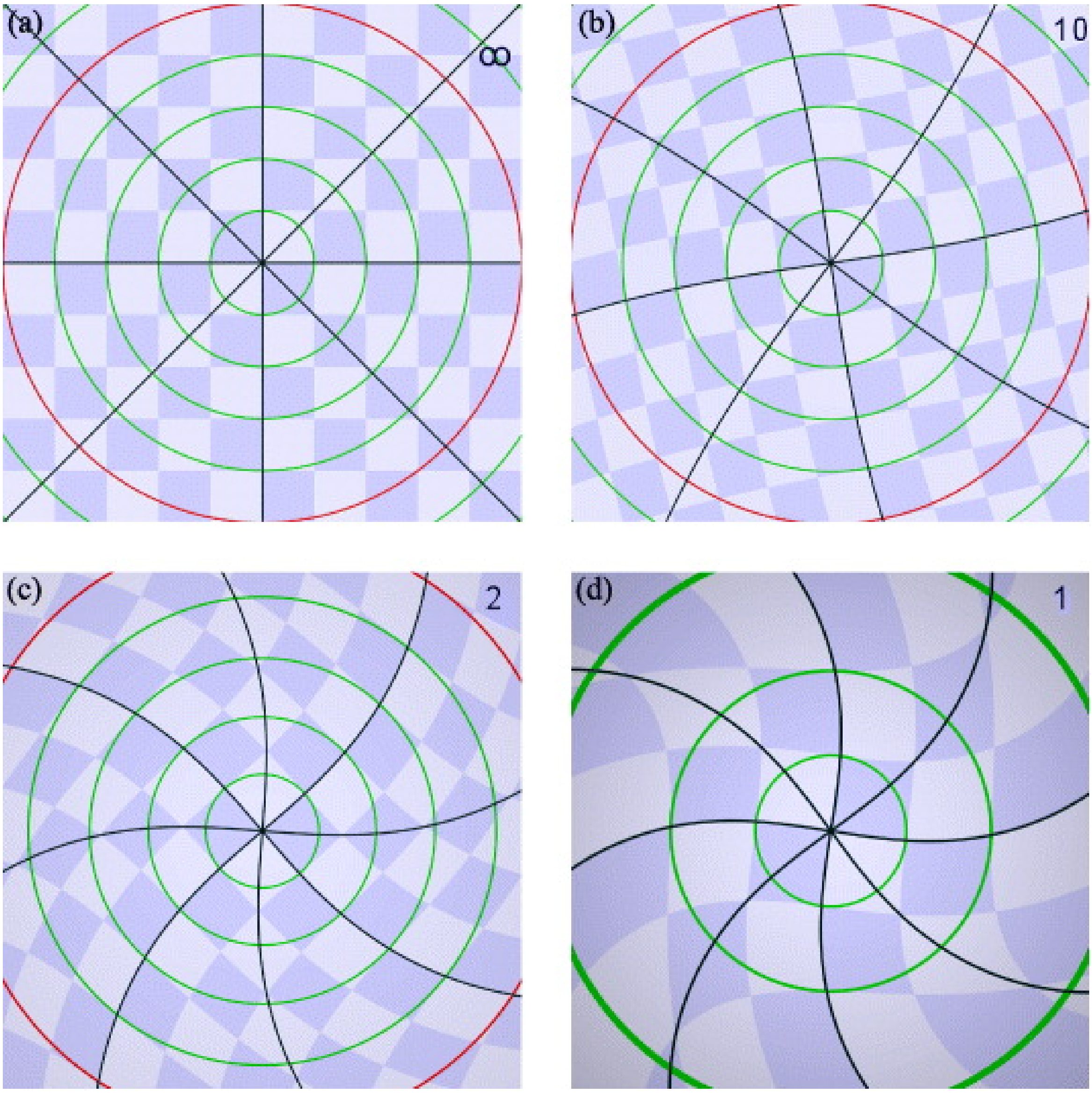


Figure 3: The world lines/geodesics for light in the Gödel universe, the vertical axis is t while the horizontal is x and the number in the top right corner is r_G. This figure is from (Buser et al., 2013)

where $f(r_s)$ is a top hat function defining the walls of the warp bubble, and v_s is the velocity of the spaceship. We see that as in the TARDIS metric in section 3.1 the space outside the bubble is Minkowski space.

It is possible to construct CTCs in the Alcubierre metric, as seen in Everett (1996), but it is a little hard and requires a little modification of the metric, but the same assumptions are used. The real problem with this metric is that there is a lot of blueshift when it decelerates from superluminal to subluminal speeds, this would kill any traveler in this metric if not proper countermeasures have been installed (McMonigal et al., 2012). Of course it also suffers from the same problems as the TARDIS metric and wormholes, you need exotic matter to construct the edges. Alcubierre states that this should not be seen as a problem as it is well known that quantum field theory permits the existence of negative energy densities under special circumstances and proposes the Casimir effect as a possible solution.

2.2.4 Practical problems experienced when traveling in time

In this section I will elaborate on the practical problems experienced when we try to travel in time, not philosophical problems like the paradoxes. First off is the fact that we seem to need exotic matter to create stable CTCs. Exotic matter has negative energy density(negative mass) and is considered unphysical. If we look at the right hand side of the Einstein equation 7, we see the stress energy tensor $T_{\mu\nu}$ and there we find the energy density. We can calculate what it has to be from the left hand side of the Einstein equation and we see that we need exotic matter to create wormholes(Morris et al., 1988), the Alcubierre drive(Alcubierre, 1994) and other stable CTCs.

Exotic matter sometimes called negative matter must not be confused with anti-matter, anti-matter is observed in our physical universe and has positive mass and energy. When anti-matter reacts with ordinary matter it creates energy as both masses are added together, when exotic matter reacts with ordinary matter they cancel each other out and are never seen again, not even as energy. As Einstein's energy equation

$$E^2 = m^2c^4 + p^2c^2 \tag{14}$$

allows for negative masses there were some scientists that thought anti-matter was the particles with negative mass from this equation, but this has since been refuted as we have measured their positive mass.

As mentioned in the previous section the problem with exotic matter could be solved with quantum physics. The Casimir effect(Casimir and Polder, 1948) could create negative energy densities in the space between to electrically neutral plates. The plates have to be really, really close to each other as this is a quantum effect, which makes this really impractical and almost impossible to achieve at a macroscopic level. To create enough exotic matter to maintain a decent sized wormhole it was first believed you need plates that are several light years long, which is somewhat impractical. However it has been shown that you can construct stable wormholes with an arbitrarily small amount of exotic matter(Visser et al., 2003).

Another problem that was first observed in the Alcubierre metric is that when you decelerate from superluminal speed you get a heavy blueshift (McMonigal et al., 2012). Both photons and particles with mass will get an insane amount of energy and both you and your destination would be blasted into oblivion. The bubble you travel within catches particles in its edges and they get an awful lot of energy built up, that has no way to be relieved before the bubble decelerates. There is no easy fix for this except traveling at subluminal speeds, but as that would be boring we should consider other ways to fix this.

Chapter 3

Methods and results

"Never ignore coincidence. Unless, of course, you're busy. In which case, always ignore coincidence.

- The Doctor, Season 5, Episode 12

3.1 Properties of the TARDIS metric

We start first with the most important part, the metric itself(Tippett and Tsang, 2013)

$$\mathrm{d}s^2 = \left[1 - h(x,y,z,t)\left(\frac{2t^2}{x^2+t^2}\right)\right](-\mathrm{d}t^2 + \mathrm{d}x^2) + h(x,y,z,t)\left(\frac{4xt}{x^2+t^2}\right)\mathrm{d}x\mathrm{d}t + \mathrm{d}y^2 + \mathrm{d}z^2 \quad (15)$$

where h is the top-hat function with $h(x,y,z,t) = 1$ inside the bubble and $h(x,y,z,t) = 0$ outside, we see that outside the bubble we have Minkowski spacetime as this is flat and the easiest to operate with in an experimental spacetime. The function h also describes how our time machine looks like, we have used

$$h(x,y,z,t) = H\Big(R^4 - y^4 - z^4 - [x^2 + t^2 - A^2]^2\Big) \quad (16)$$

where H is the Heaviside function, which is approximated in Tippett and Tsang (2013) as

$$H(x) = \frac{1}{2} + \frac{\tanh(\alpha x)}{2}. \quad (17)$$

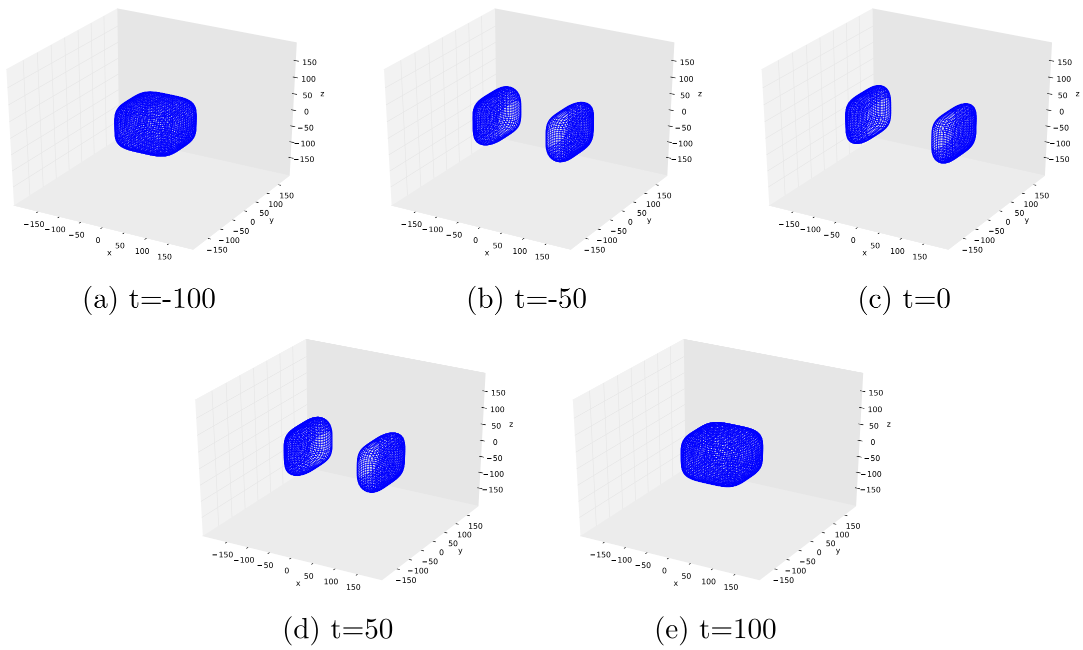
The x in the Heaviside function describes some of the properties of the TARDIS, it has a boxy spheroidal shape described by R moving along a circle of radius A in the x-t plane. In my thesis I use the values $A = 100$, $R = 70$ and $\alpha = \frac{1}{6000000}$ which are the same as Tippet and Tasng use.

3.2 Visualization of the TARDIS

My first task when starting on my thesis was to try and visualize how this metric would look, considering that it is not very straightforward looking metric. First I started with the TARDIS itself, the box we use for our travels. I start with the equation that defines the boundaries of the TARDIS equation 16:

$$R^4 - y^4 - z^4 - (x^2 + t^2 - A^2)^2 = 1$$

When I solve this equation for x with different values of y,z and t we get figure 4, we see that the box for an external observer splits into two different boxes that moves apart from each other. For the external observer this behavior is similar to the creation and annihilation of an electron-positron pair. We often describe a positron as an electron moving backwards in time(Tippett and Tsang, 2013).



(a) t=-100 (b) t=-50 (c) t=0

(d) t=50 (e) t=100

Figure 4: The boundaries of the bubble at different times, as seen by an external observer. See `https://www.dropbox.com/s/4nft2p7mdy989cz/movie.gif?dl=0` for an animation of the bubble.

With the first obstacle dealt with, we start with the real deal; visualize the light geodesics in the TARDIS metric. It is already done in the original article, but I want to make sure I get the same result as they do. First I start by calculating all the Christoffel symbols using equation 15 and 6. They are long and not very interesting, but you can look at them in the appendix. With the Christoffel symbols we get our geodesic equations, which are four second order differential equations.

$$\begin{aligned}
\ddot{x}^0 &= -\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \\
\ddot{x}^1 &= -\Gamma^1_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \\
\ddot{x}^2 &= -\Gamma^2_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \\
\ddot{x}^3 &= -\Gamma^3_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta
\end{aligned} \tag{18}$$

To solve these four equations we have to split them into eight first order differential equations, by denoting $\dot{x} = u$ we get

$$\begin{aligned}
\dot{u}^0 &= -\Gamma^0_{\alpha\beta}u^\alpha u^\beta \\
\dot{u}^1 &= -\Gamma^1_{\alpha\beta}u^\alpha u^\beta \\
\dot{u}^2 &= -\Gamma^2_{\alpha\beta}u^\alpha u^\beta \\
\dot{u}^3 &= -\Gamma^3_{\alpha\beta}u^\alpha u^\beta \\
\dot{x}^0 &= u^0 \\
\dot{x}^1 &= u^1 \\
\dot{x}^2 &= u^2 \\
\dot{x}^3 &= u^3
\end{aligned} \tag{19}$$

that are easily solved by the standard fourth order Runge-Kutta method that can be found in the appendix B. I have to choose a value for my time parameter τ, and my choice is the simplest that it is possible I just count from 1 to 600 with 10000 points. This is the time parameter that corresponds to an observer standing in the Minkowski metric. The initial conditions have to obey the following

$$g_{\mu\nu}u^\mu u^\nu = 0 \tag{20}$$

so we cannot choose whatever initials conditions that comes to mind. The way I choose is by choosing a starting velocity for the one of the parameters and calculate the other velocity through the equation above. When I choose $u^0 = 1$ I get $u^1 = 1$ which is pretty nice. In the beginning I only visualize two dimensions because this makes things considerably easier and it is what is done in the main article, so I can compare. I start the light rays in $t = -300, 300$ where I change the x and at $x = -300, 300$ where I change t.

I use Fortran for the Runge-Kutta method as this goes really fast, I save my photon paths in different text files and use Python to plot my results, the result can be viewed in figure 5. This is not very informative so I have zoomed in figure 6, and we see that some of the lightrays does not behave nicely, so I cut the lines when the derivative of the solutions with respect to τ became too large, and I got figure 7. We see that the last figure I generate is almost exactly the same as it is in Tippett and Tsang's article.

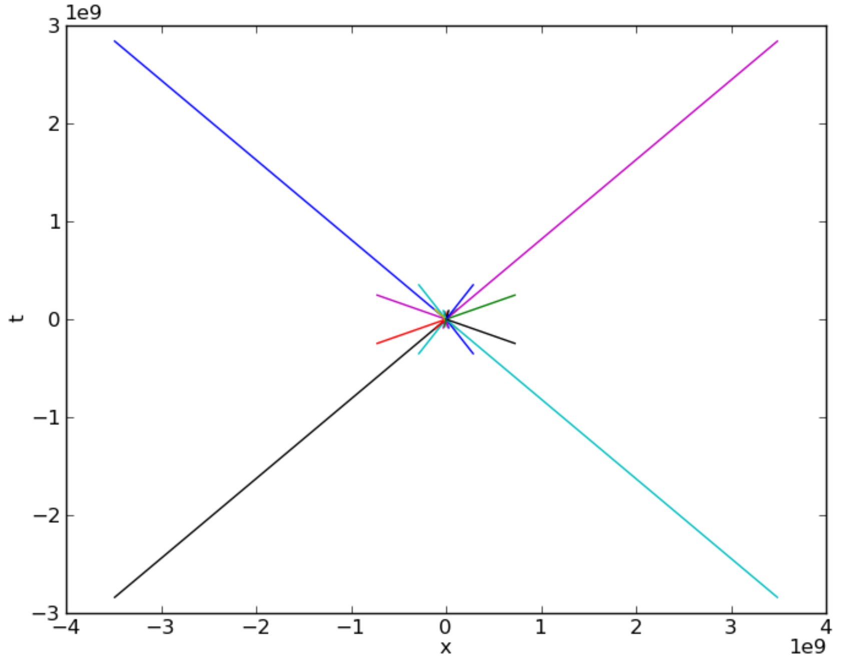


Figure 5: Light geodesics in the TARDIS metric.

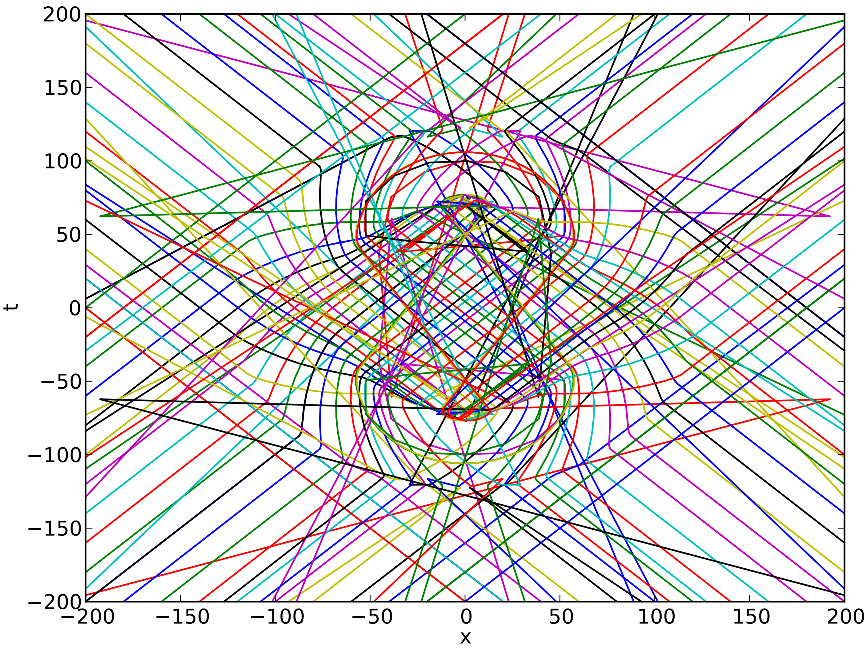


Figure 6: Light geodesics in the TARDIS metric.

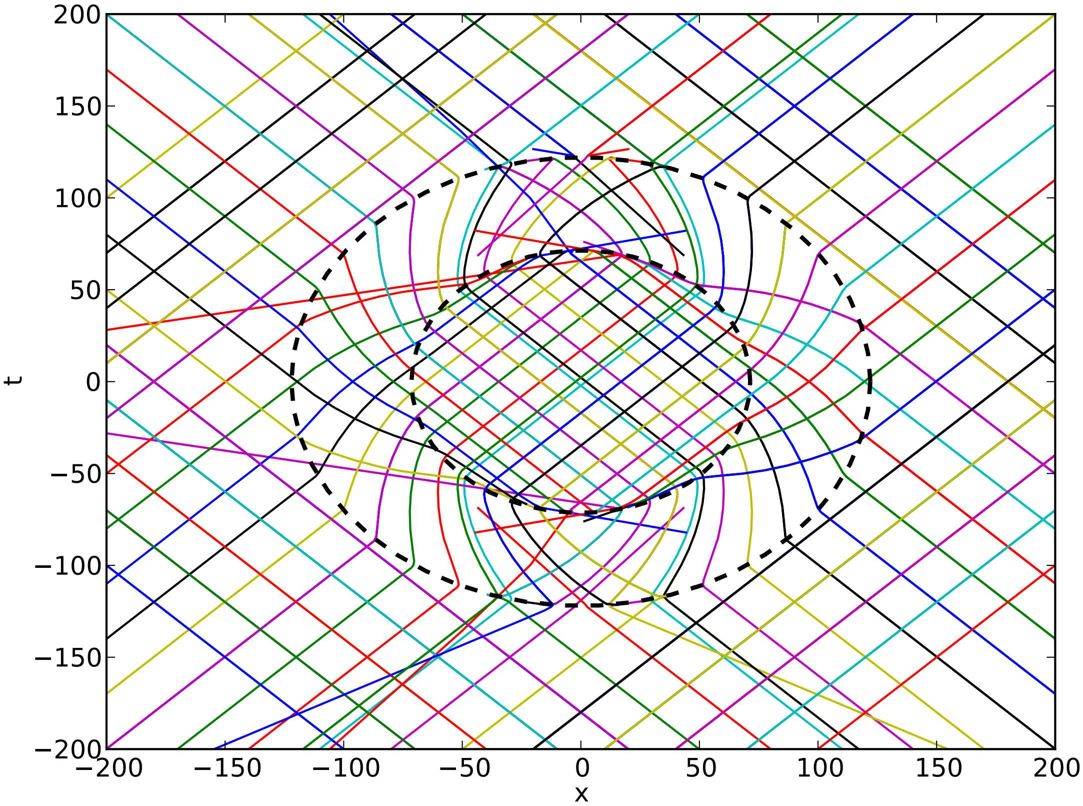


Figure 7: Light geodesics in the TARDIS metric.

3.2.1 Visualization of more than two dimensions

Now we will do something that Ben Tippett and David Tsang did not do in their article which is to visualize their metric for more than the t and x coordinate. We still have to use equation 20 to calculate the different initial velocities, which is not as easy as before. The equation becomes:

$$(u^0)^2 = (u^1)^2 + (u^2)^2 + (u^3)^2 \tag{21}$$

my choice for initial conditions have been

$$\begin{aligned} u^0 &= 1 \\ u^1 &= \frac{1}{\sqrt{2}} \\ u^2 &= \frac{1}{\sqrt{2}} \end{aligned} \tag{22}$$

if I only include t,x and y, and

$$\begin{aligned} u^0 &= 1 \\ u^1 &= \frac{1}{\sqrt{3}} \\ u^2 &= \frac{1}{\sqrt{3}} \\ u^3 &= \frac{1}{\sqrt{3}} \end{aligned} \tag{23}$$

if I include z. Otherwise this is pretty much as before, except that I have to include all the differential equations.

In the figures 8, 9 and 10 you can see how the TARDIS behaves with the various coordinates in two dimensions. As the y and z coordinate should behave the same I have not bothered including the z coordinate in these plots two dimensional plots and decided to save it for the three dimensional plots in the next paragraph.

In figure 11 you can see a three dimensional plot of the TARDIS metric, it is hard to get proper information from this plots, so I have only included one. We see the same here as we did for the two dimensional case, light becomes distorted when it hits the walls of the TARDIS and our geometrical objects are hardly recognizable.

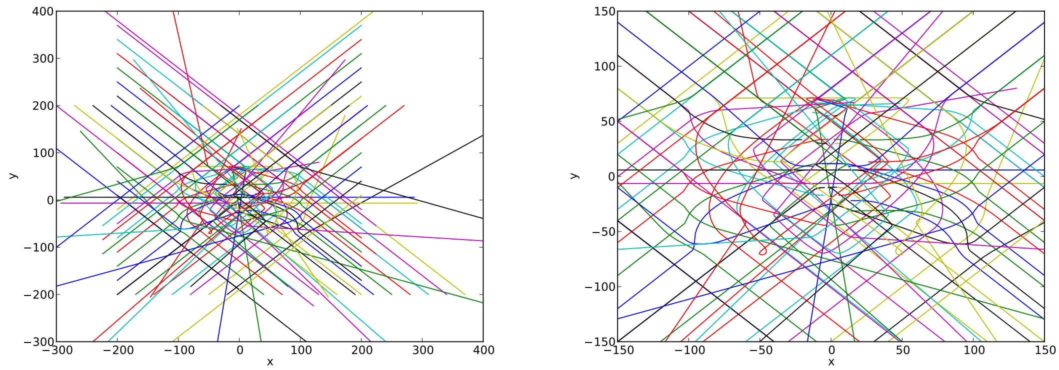


Figure 8: Light geodesics in the TARDIS metric now in x and y coordinates, the figure on the right is zoomed in on the interesting parts.

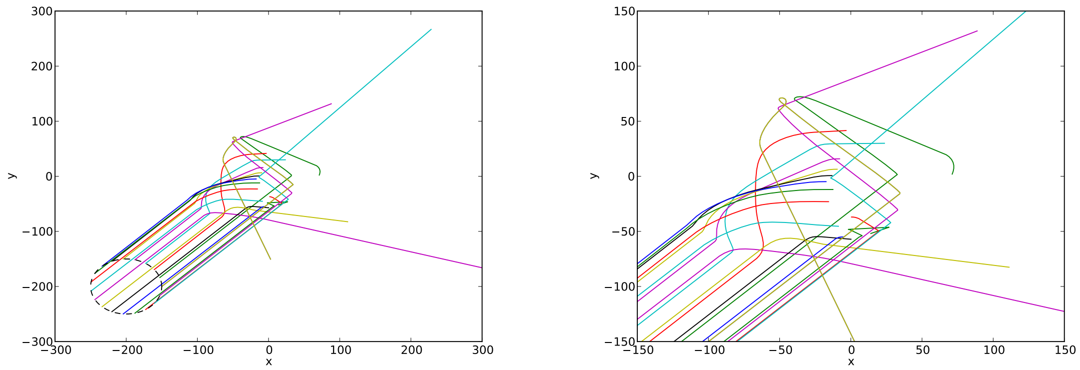


Figure 9: Here we see the deformation of a circle once it hits the borders of the TARDIS. The figure on the right is zoomed in on the boundaries of the TARDIS.

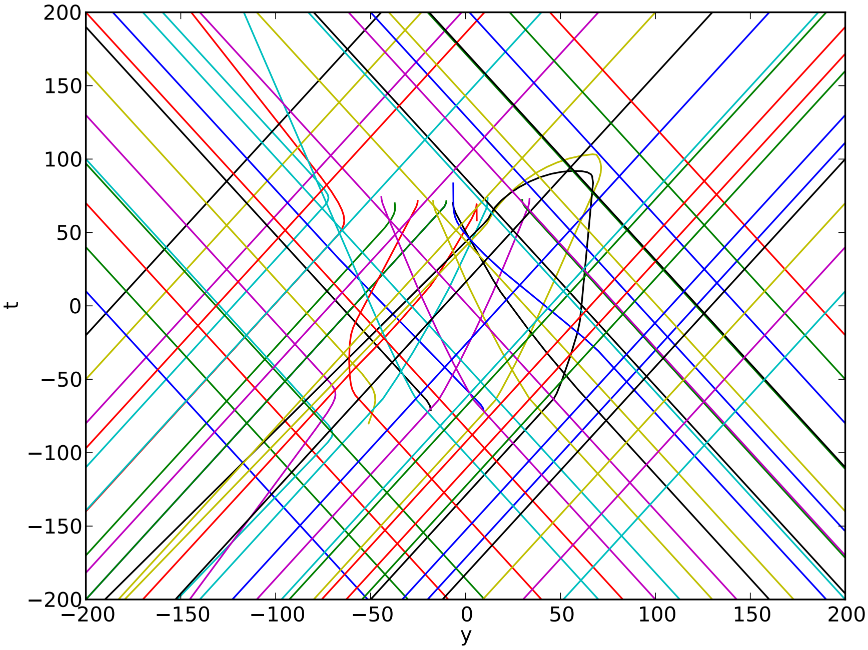


Figure 10: Light geodesics in the TARDIS metric now in t and y coordinates, these are the worldlines from before only with y instead of x.

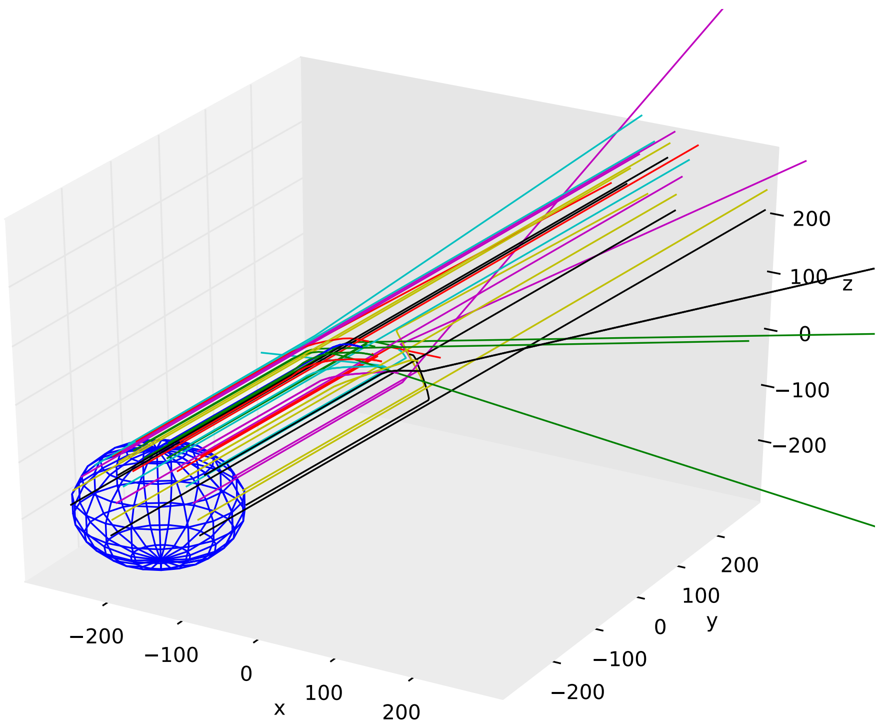


Figure 11: The deformation of a sphere in the three dimensional TARDIS metric.

3.3 Blueshift

We have seen that light is distorted when it hits the walls of our TARDIS, can this affect the energy of the photon? In the Alcubierre metric(Alcubierre, 1994), that is very similar to the TARDIS metric, it has been shown that the metric itself can give rise to heavy blueshifting of light and boost the energy of massive particles(McMonigal et al., 2012). This is of course something we have to check as The Doctor and his companions probably want to be alive when they reach their destination, and not fried as they world be in Alcubierre's warp drive(Star Trek).

I found a method to check this in the book General Relativity(Hobson et al., 2006) where they give us the equation for energy.

$$\begin{aligned} E &= p_\mu u_E^\mu \\ E &= p_\mu u_R^\mu \end{aligned} \tag{24}$$

This is directly equivalent with the energy expression for non-relativistic energy, where $E = mv^2 = pv$. In general relativity we use the four-momentum p^μ and four-velocity u^μ, these include the rest velocity and momentum of our object, so $E = p_\mu u^\mu$ would describe both the potential and kinetic energy of that object. We know that $E = h\nu$, so we end up with this equation.

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu u_R^\mu}{p_\mu u_E^\mu}. \tag{25}$$

Here ν is the frequency, p is the momentum and u is the velocity of the particles. Subscript R is where the receiver resides, whereas subscript E indicates the emitter's position. This formula unfortunately only works for photons and I found no, for me, understandable way to check this for massive particles in this metric.

We see in equation 25 that we need a way to calculate the momentum of our photons. Fortunately this is also given in the book:

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} - \Gamma^\nu_{\mu\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0 \tag{26}$$

This is just the geodesic equation, but $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = p^\mu$ which gives us the below equation

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \Gamma^\nu_{\mu\rho} p_\nu p^\rho \tag{27}$$

we see this is pretty similar how velocity behaves. Because we are only after a ratio we do not have to guess a correct initial value for the momentum, so we just choose it to be one. We start the light rays as before outside the TARDIS where Minkowski space rules and shoot light rays at the TARDIS. From the figures 12, 13, 14 and 15 we see that the blueshift is really high. Even though the y-axis is negative it still indicates blue shift, as redshift would be between 0 and 1.

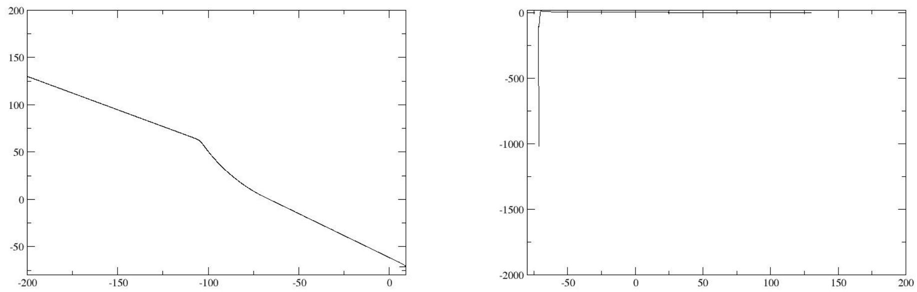


Figure 12: The left figure shows the path the light is following, while the right picture shows the blueshift along the line. The left figure has t on the vertical axis and x on the horizontal, the right figure has t on the horizontal axis and frequencyshift on the vertical.

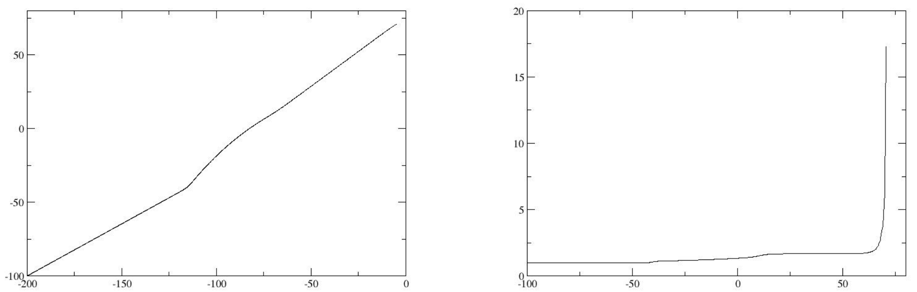


Figure 13: The left figure shows the path the light is following, while the right picture shows the blueshift along the line. The left figure has t on the vertical axis and x on the horizontal, the right figure has t on the horizontal axis and frequencyshift on the vertical.

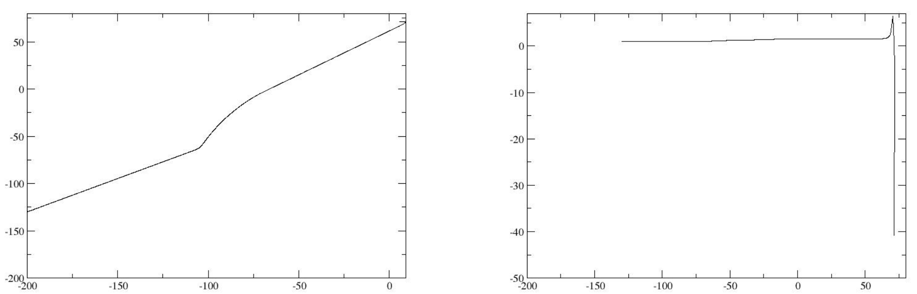


Figure 14: The left figure shows the path the light is following, while the right picture shows the blueshift along the line. The left figure has t on the vertical axis and x on the horizontal, the right figure has t on the horizontal axis and frequencyshift on the vertical.

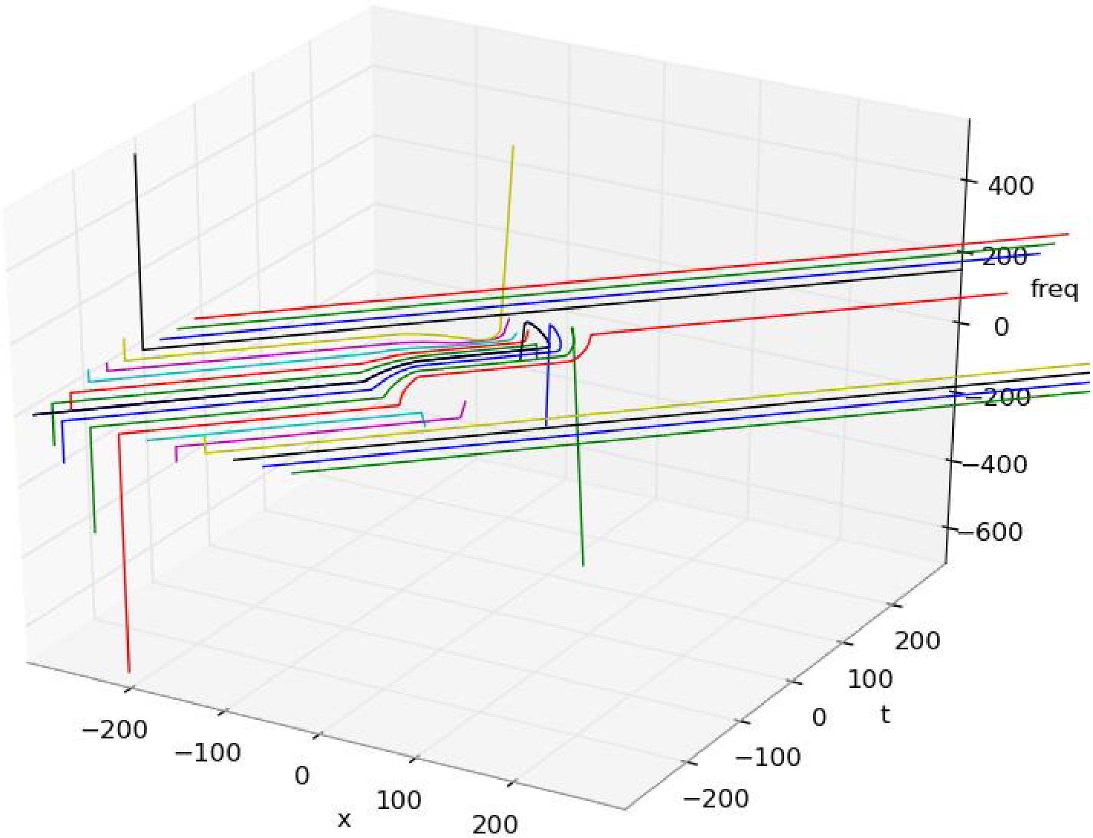


Figure 15: This figure shows the blueshift as a function of x and t for several lightrays. The light rays start in $x = t = -200$ to $x = t = 0$

3.4 Can we travel anywhere in time and space?

A time machine like the TARDIS that could not travel everywhere in time and space, would be horribly boring, that is why we have to check if it really could travel anywhere we would want it to go. The TARDIS we use can only move in a limited area, but we can expand this area by joining it with a TARDIS moving in a different area.

This is done by using the Israel-Darmois junction conditions(Misner et al., 1973). The junction conditions have to be fulfilled if two metrics are to be combined. The conditions state that both the metric and the extrinsic curvature have to be continuous in the junctions between the metrics. The extrinsic curvature measures the curvature of an object relative to the metric.

The definition of extrinsic curvature is

$$K_{ij} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial n}. \tag{28}$$

Note that we only use Latin indices meaning that we count from 1 and only use the spatial part of the metric. The metric has to be diagonal for the extrinsic curvature to be correct, so we have to diagonalize it. We use singular value decomposition to diagonalize it and we get

$$\begin{pmatrix} -\sqrt{a^2+b^2} & 0 & 0 & 0 \\ 0 & \sqrt{a^2+b^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

where $a = -1 + h\frac{2t^2}{x^2+t^2}$ and $b = h\frac{2xt}{x^2+t^2}$.

The n from the definition of extrinsic curvature is the normal vector and replaces t from the metric. The metric can now be written as:

$$\mathrm{d}s^2 = (n\dot{n})^{-1}\mathrm{d}n^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{30}$$

where $\mathrm{d}n^2 = \sqrt{a^2+b^2}\mathrm{d}t^2$.

We then get an extrinsic curvature that looks like

$$\begin{aligned} K_{ij} &= -\frac{1}{2}\frac{\partial g_{ij}}{\partial n} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t}\frac{\partial t}{\partial n} \\ &= -\frac{1}{2}\frac{\partial g_{ij}}{\partial t}\frac{1}{(a^4+b^2)^{1/4}}. \end{aligned} \tag{31}$$

Because K_{11} is the only element that does not become zero we get

$$K_{11} = -\frac{1}{2}\frac{1}{(a^4+b^2)^{1/4}}\left(\frac{\partial a}{\partial t}2a + \frac{\partial b}{\partial t}2b\right)\frac{1}{2\sqrt{a^2+b^2}} \tag{32}$$

where $\frac{\partial a}{\partial t} = h'(t)\frac{2t^2}{x^2+t^2} + \frac{4htx^2}{(x^2+t^2)^2}$ and $\frac{\partial b}{\partial t} = h'(t)\frac{2xt}{x^2+t^2} + \frac{2hx^3-2hxt^2}{(x^2+t^2)^2}$.

I tested if the metric and extrinsic curvature were continuous when the midpoints was $x = 2A$, $t = 0$ and $x, t = 0$, where $A = 100$ is the constant from the Heaviside function in equation 16. This is really simple and the two metrics should meet in $x = A, t = 0$, when I did the calculations both the extrinsic curvature and the metric where continuous. Meaning that for this simple example we can stitch together the metric. I also tried to start in a circle around $x, t = 0$, with radius $2A$. The metric and the extrinsic curvature fits perfectly through the whole circle, indicating that it is possible to travel anywhere in time and space, as we have extended the metric to be valid for other regions.

3.5 Correction terms

In eq.15 we see that the metric is symmetric if we flip the sign of any of the coordinates, this means that observers can travel in either clockwise or counter-clockwise direction. This makes serious problems where $g_{tt} = g_{xx} = 0$ and $g_{tx} = g_{xt} = 0$. There is no unique and consistent way for the lightcones to tip. This is solved by adding a term which forces the lightcones to twist in a deliberate way.

With the correction term added the metric looks like this:

$$\begin{aligned} ds^2 =& \left[1 - h(x,y,z,t)\big(\frac{2t^2}{x^2+t^2}\big)\right](-dt^2 + dx^2) \\ &+ h(x,y,z,t)\big(\frac{4xt}{x^2+t^2}\big)dxdt + dy^2 + dz^2 \\ &+ 4t^3 h(x,y,z,t)W(x,y,z,t)dxdt \end{aligned} \tag{33}$$

where

$$\begin{aligned} W(x,y,z,t) =& \frac{1}{2}\Bigg(\tanh\big(\frac{x^2 - t^2(2h(x,y,z,t)-1)+20}{t^2}\big) \\ &- \tanh\big(\frac{x^2 - t^2(2h(x,y,z,t)-1)-20}{t^2}\big)\Bigg). \end{aligned} \tag{34}$$

This new metric is not pretty to work with, but I have visualized it as I did with the original metric, with exactly the same method as previously. The Christoffel symbols for this new metric are also displayed in the appendix. Figure 16 show how the light cones behave in the corrected TARDIS metric, I have plotted the corrected TARDIS in several dimensions and in figure 17 we see the geodesics in x and y, figure 18 y and t and figure 19 shows how a sphere is distorted when it hits the TARDIS, the correction however works as it should as we clearly see that the geodesics all bend to one side, not all sides as before.

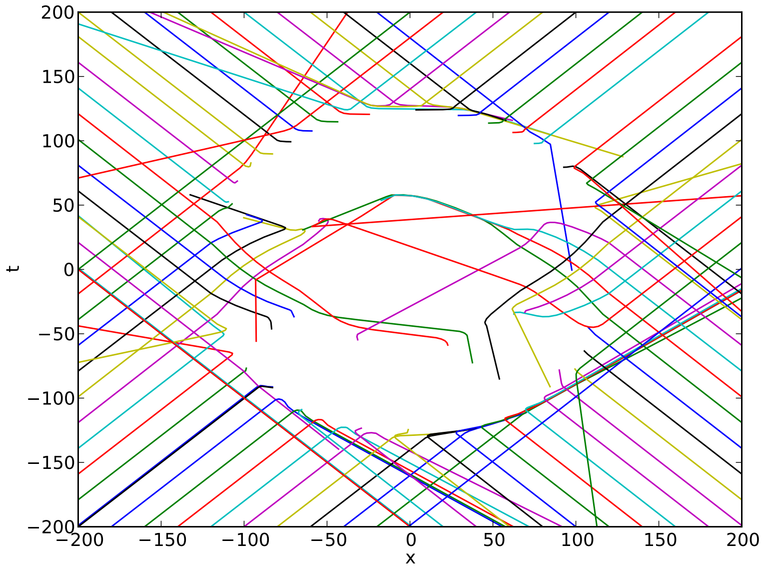


Figure 16: Geodesics in the corrected TARDIS metric in 2D, x and t

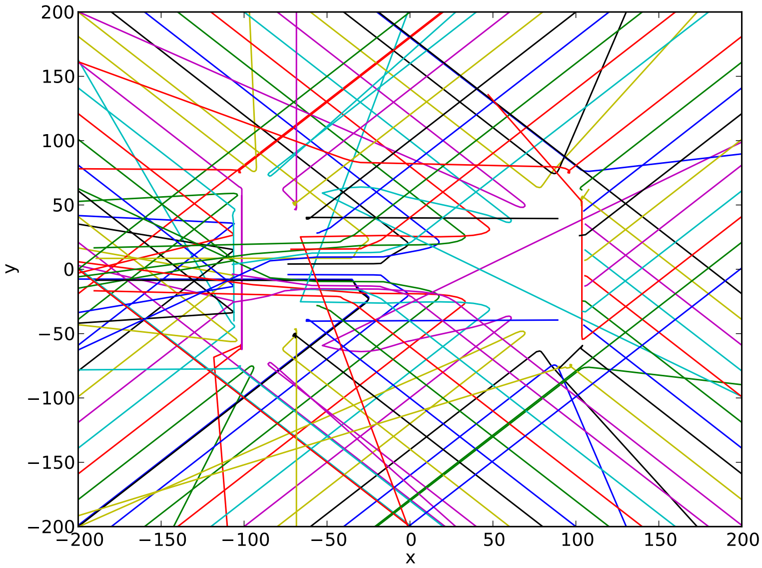


Figure 17: Geodesics in the corrected TARDIS metric in x and y

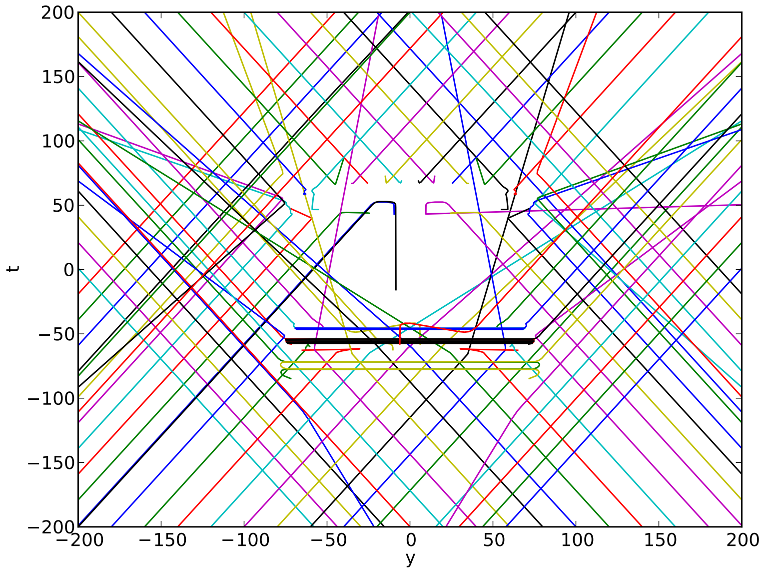


Figure 18: Geodesics in the corrected TARDIS metric in y and t, like 17 this does not look very nice.

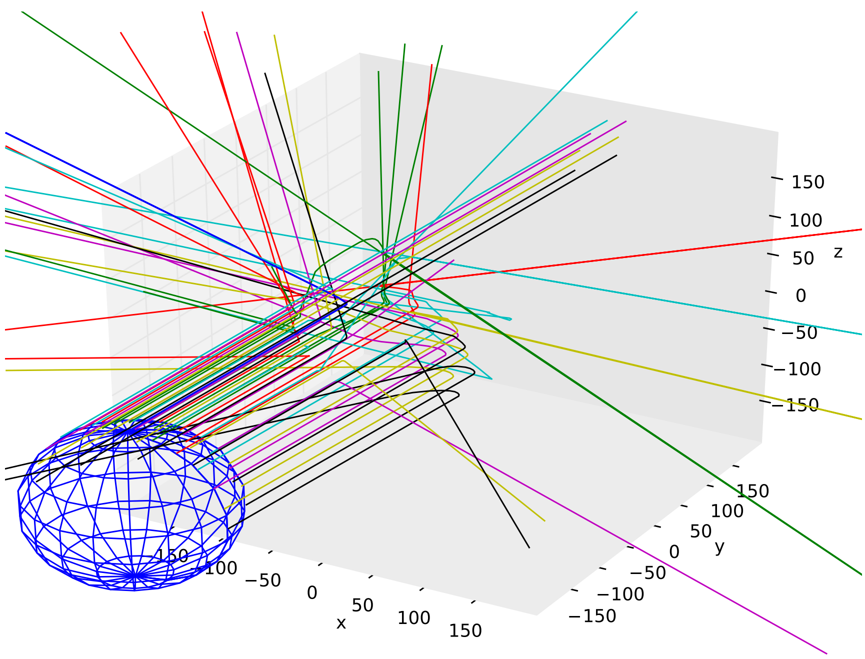


Figure 19: Geodesics in the corrected TARDIS metric in three dimensions, x, y and z. There is an even bigger distortion here than in the original metric, we see however that the geodesics have a tendency to rotate to the left when they hit the TARDIS wall.

3.5.1 Blueshift with the correction terms

Despite the weirdness of the geodesics I will try to calculate what the blueshift looks in this corrected metric. We use the same method as in section 1.2 where we used equation 25 to calculate the frequency shift from one place to another. In the figures 20, 21, 22 and 23 you can see the same light rays as before and their new blueshift and path. The blueshift has just gotten worse which does not bode well for the Doctor and his companions.

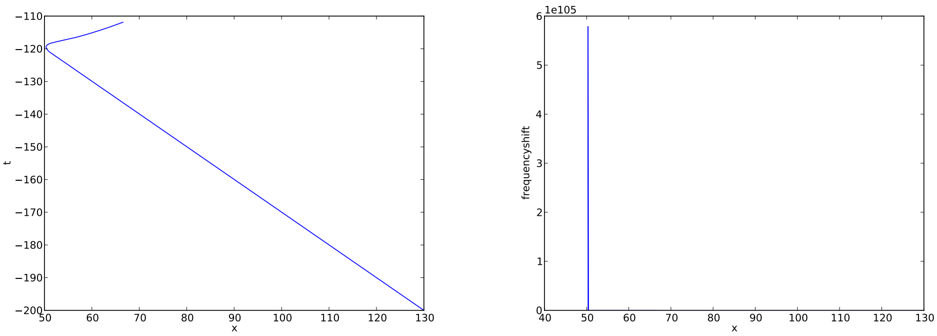


Figure 20: The left figure shows the path the light is following, while the right picture shows the blueshift along the line.

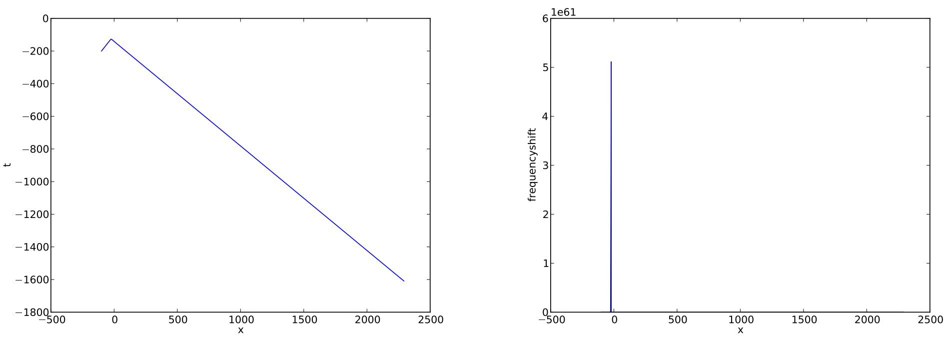


Figure 21: The left figure shows the path the light is following, while the right picture shows the blueshift along the line.

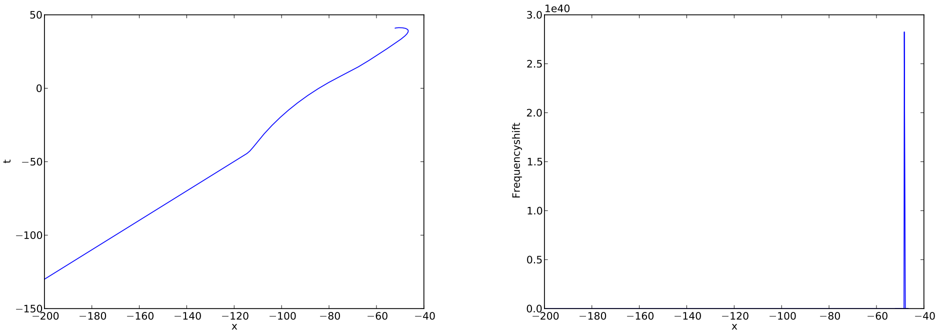


Figure 22: The left figure shows the path the light is following, while the right picture shows the blueshift along the line.

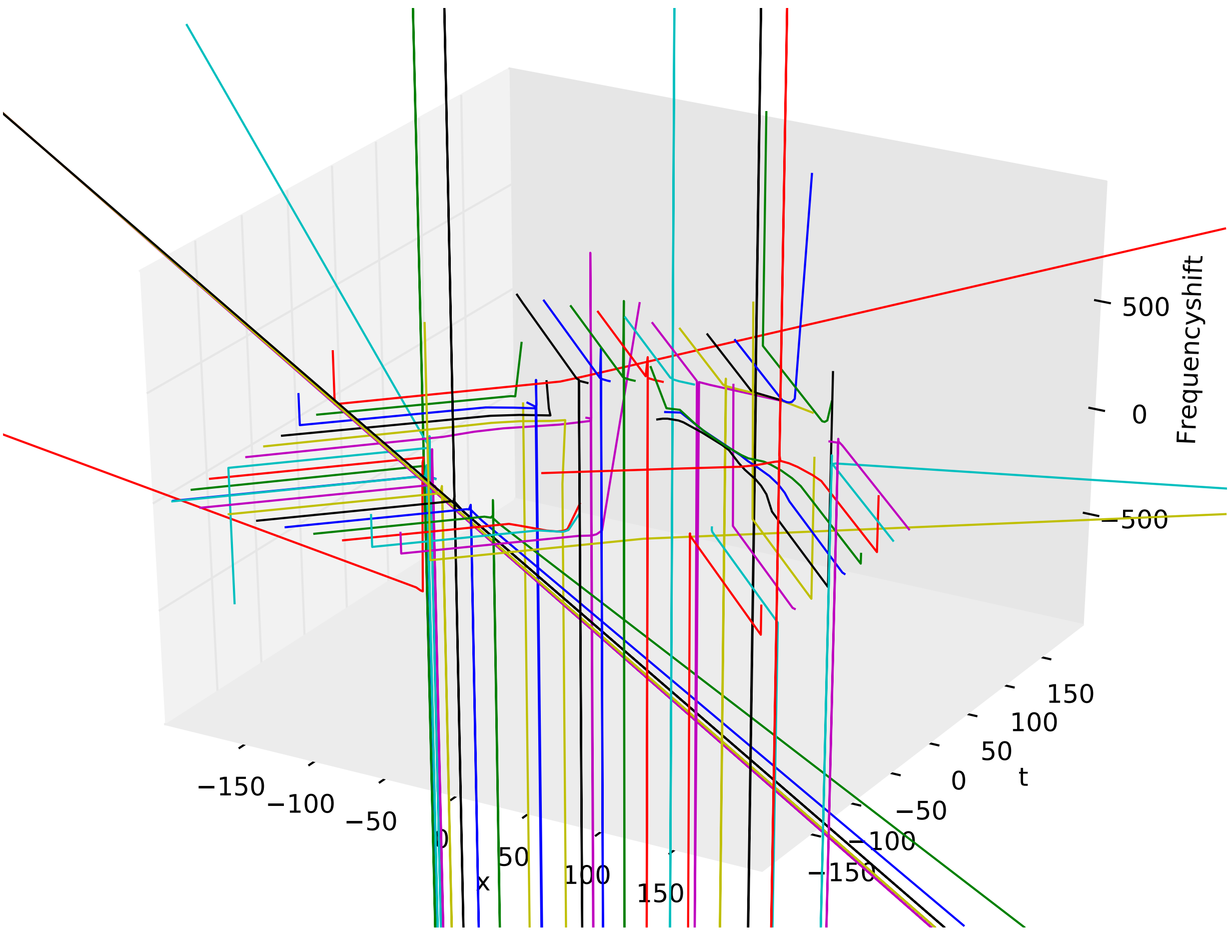


Figure 23: This figure shows the blueshift as a function of x and t for several lightrays, it is a little chaotic but the main theme is that the blueshift is high.

3.6 Proper top-hat function

To see if the blueshift will go towards infinity we use a proper top hat function instead of the approximation from the original article. I now choose a two dimensional top hat that becomes 1 when we are between $x^2 + t^2 = 100^2 + 70^2$ and $x^2 + t^2 = 100^2 - 70^2$ and 0 anywhere else, this is easily done with an "if"-test in our program. I will here use Python not Fortran and use a Euler-Cromer method, not the Runge- Kutta I used previously.

As I only consider the two dimensional metric our metric tensor becomes

$$g_{\mu\nu} = \begin{pmatrix} \frac{t^2-x^2}{x^2+t^2} & \frac{2xt}{x^2+t^2} \\ \frac{2xt}{x^2+t^2} & \frac{x^2-t^2}{x^2+t^2} \end{pmatrix} = g^{\mu\nu} \tag{35}$$

inside the TARDIS, while outside the TARDIS we have

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g^{\mu\nu}. \tag{36}$$

As before we need the Christoffel symbols, so we have to derivate the different elements in the matrices. The last matrix all the derivatives becomes zero, so it is not very interesting. The other matrix on the other hand bring out many interesting properties. In this last matrix we have to include the top hat function as we need its derivatives:

$$g_{\mu\nu} = \begin{pmatrix} -1 + h(x,t)\frac{2t^2}{x^2+t^2} & \frac{2xth(x,t)}{x^2+t^2} \\ \frac{2xth(x,t)}{x^2+t^2} & 1 - h(x,t)\frac{2t^2}{x^2+t^2} \end{pmatrix} \tag{37}$$

The derivative of the top hat function is the Dirac-delta function which I have written as

$$\delta(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \tag{38}$$

where $a \to 0$.

The derivatives of the matrix becomes

$$\frac{g_{\mu\nu}}{\mathrm{d}t} = \begin{pmatrix} \frac{2t(t(t^2+x^2)\delta(t)+2x^2)}{(t^2+x^2)^2} & \frac{2x(t(t^2+x^2)\delta(t)+(x^2-t^2))}{(t^2+x^2)^2} \\ \frac{2x(t(t^2+x^2)\delta(t)+(x^2-t^2))}{(t^2+x^2)^2} & -\frac{2t(t(t^2+x^2)\delta(t)+2x^2)}{(t^2+x^2)^2} \end{pmatrix} \tag{39}$$

$$\frac{g_{\mu\nu}}{\mathrm{d}x} = \begin{pmatrix} \frac{2t^2((t^2+x^2)\delta(x)-2x)}{(t^2+x^2)^2} & \frac{2t(x(t^2+x^2)\delta(x)+(t^2-x^2))}{(t^2+x^2)^2} \\ \frac{2t(x(t^2+x^2)\delta(x)+(t^2-x^2))}{(t^2+x^2)^2} & -\frac{2t^2((t^2+x^2)\delta(x)-2x)}{(t^2+x^2)^2} \end{pmatrix} \tag{40}$$

The Christoffel symbols derived from these can be found in the appendix.

The equation for the blueshift is the same as before

$$\Delta\nu = \frac{p_\mu u^\mu}{p_\nu u^\nu}, \tag{41}$$

and as before we start a light ray where there is Minkowski metric and no curvature to change the light rays before they hit the TARDIS.

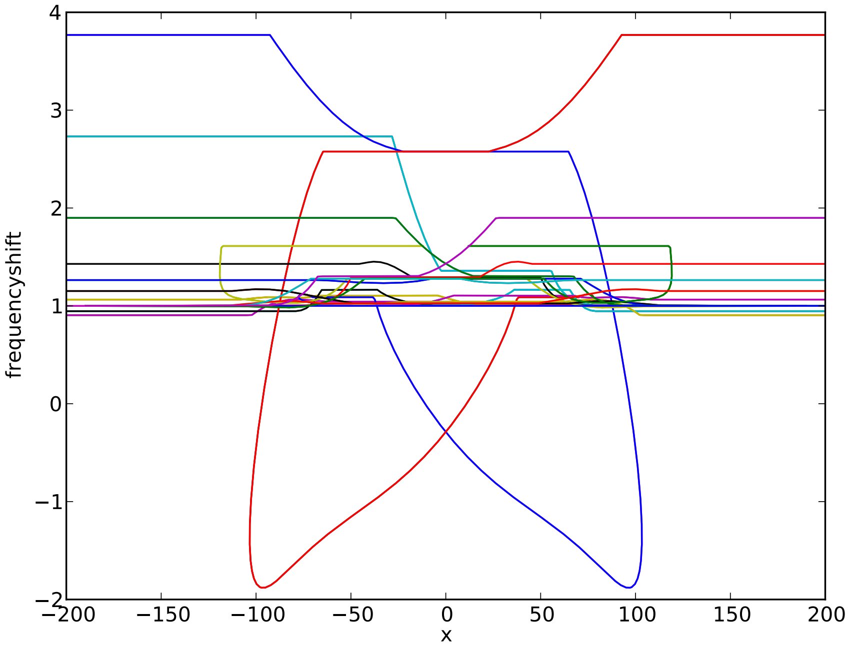


Figure 24: The blueshift compared to x, with different initial x and t positions

As you see in the figures 24 and 25 there is a slight blue- or redshift, but not as intense as it was in the numerical case. The most important feature of these to plots is to show that whatever initial position you choose you will never get the really high blueshift we saw earlier. You can also see that the lightrays behaves slightly different in figure 26 and that they loop around the center of the plot. The figures 27, 28 and 29 shows some photon paths with the frequencyshift on the z axis, so that it does not seem as messy as in the first plots, where 40 lightrays are in the same plot.

All the previous plots are all with an 'a'-value of $\frac{1}{100000}$ in the δ-function, now we will look at what happens when we change the value of a. All the way to $\frac{1}{10000}$ nothing is different in the solutions, but when it gets higher we get different results. As we see in figure 30 the solution is not that different. but if we look at figure 31 we see that the frequency shift is very different, and that with a very large a, we get almost infinite blueshift.

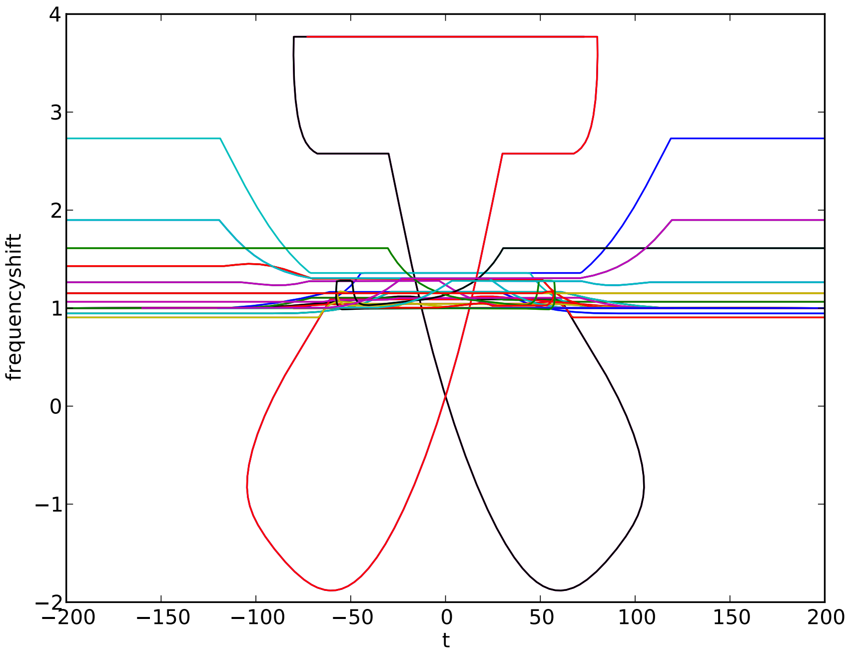


Figure 25: The blueshift compared to t, with different initial x and t positions

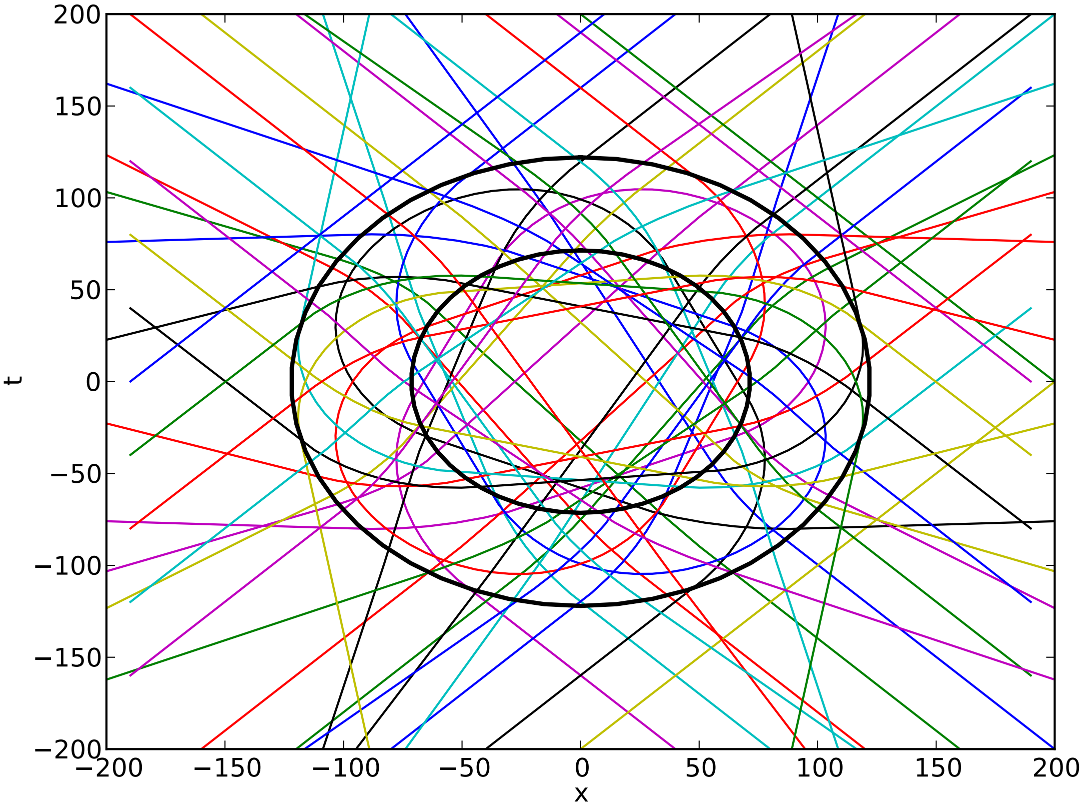


Figure 26: The geodesics fra a proper top hat solution of the TARDIS

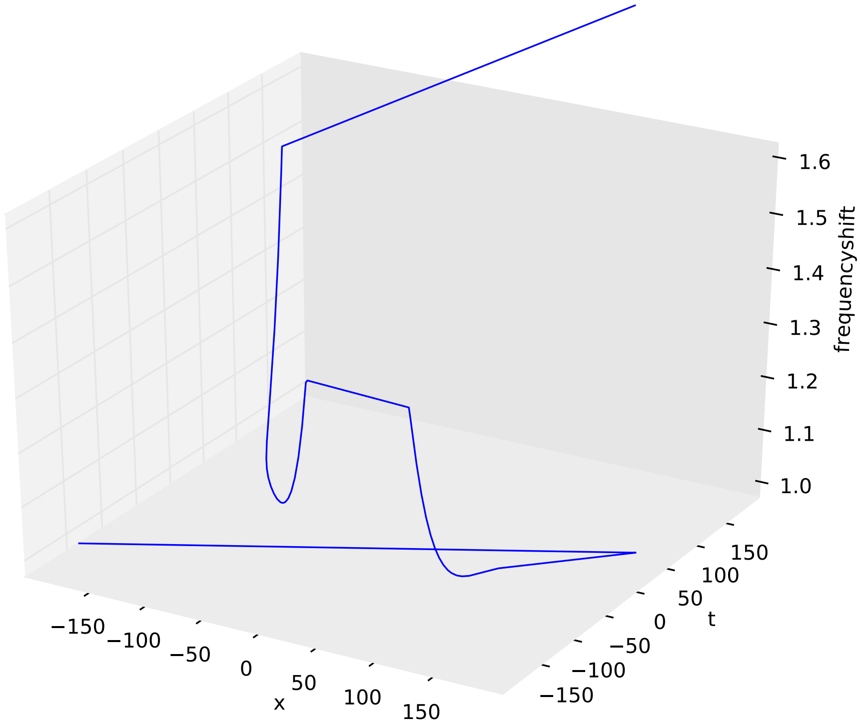


Figure 27: The blueshift at certain x and t values

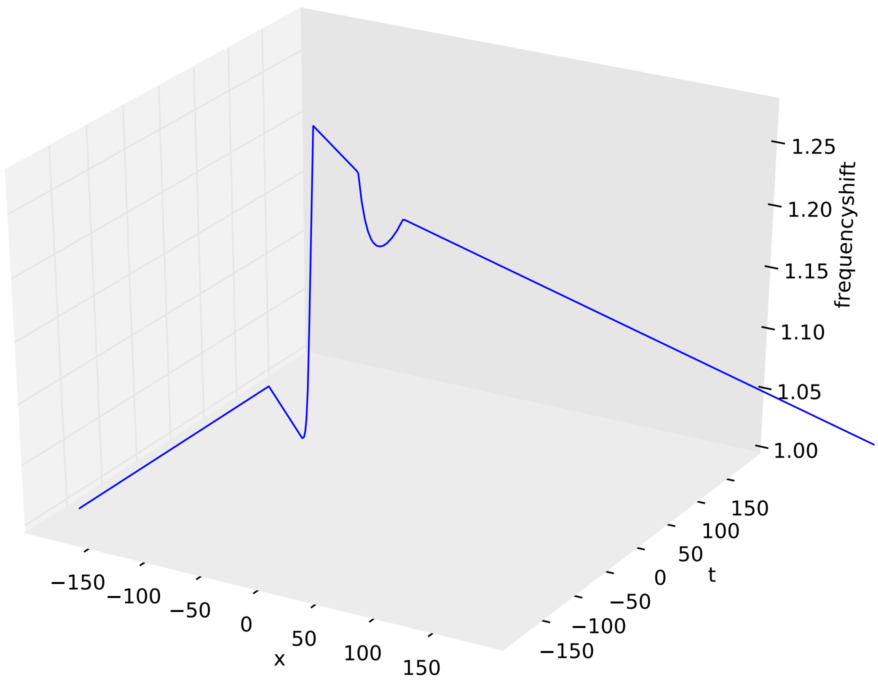


Figure 28: The blueshift at certain x and t values

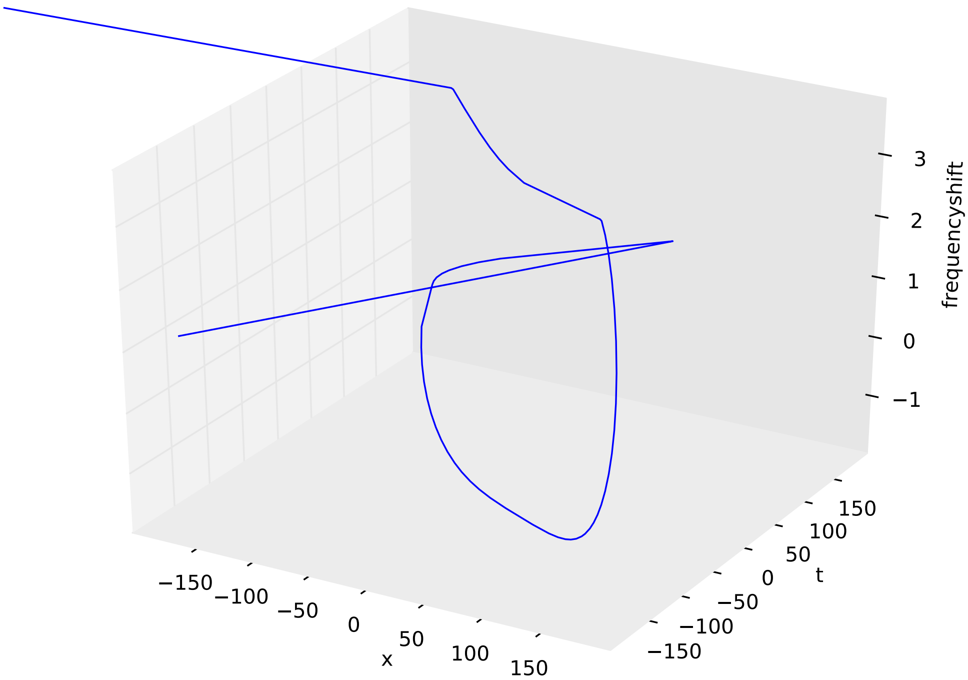
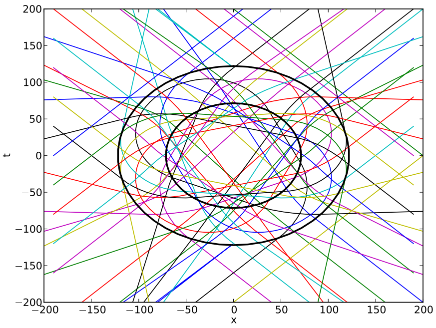
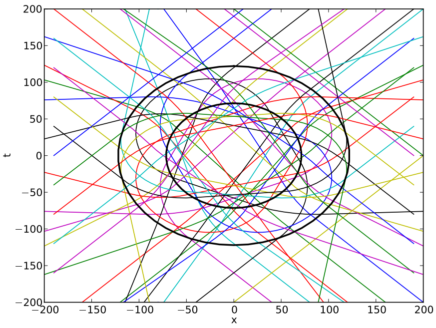


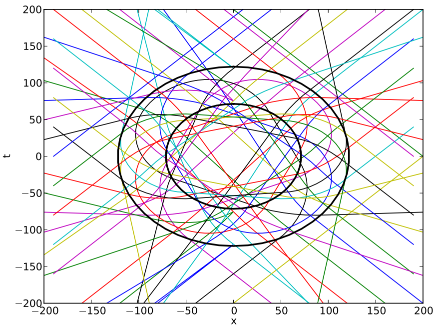
Figure 29: The blueshift at certain x and t values



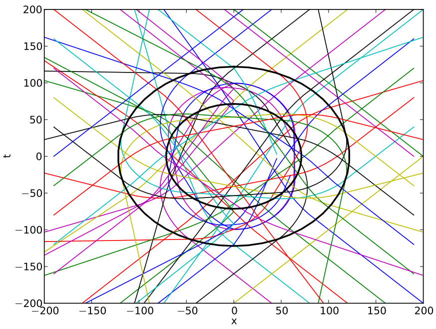
(a) $a = \frac{1}{5000}$



(b) $a = \frac{1}{500}$

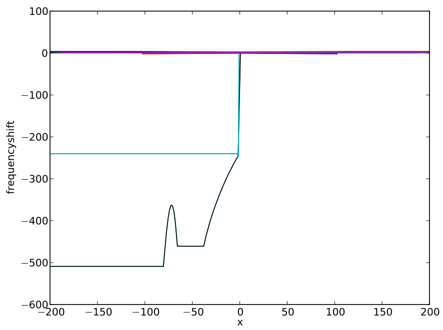


(c) $a = \frac{1}{50}$

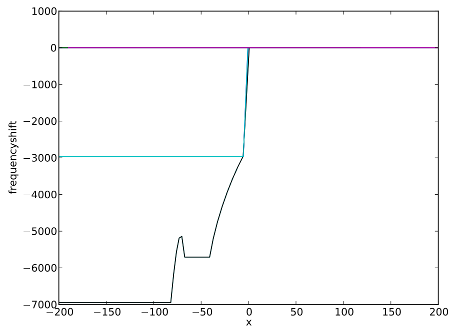


(d) $a = \frac{1}{5}$

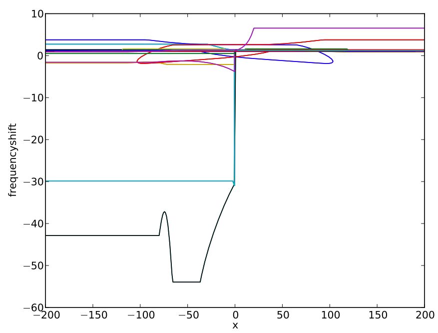
Figure 30: Solutions with a proper top hat function for four different a-values



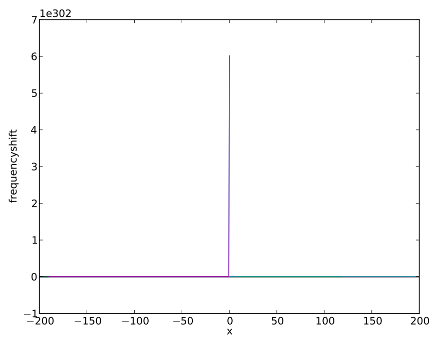
(a) $a = \frac{1}{5000}$



(b) $a = \frac{1}{500}$



(c) $a = \frac{1}{50}$



(d) $a = \frac{1}{5}$

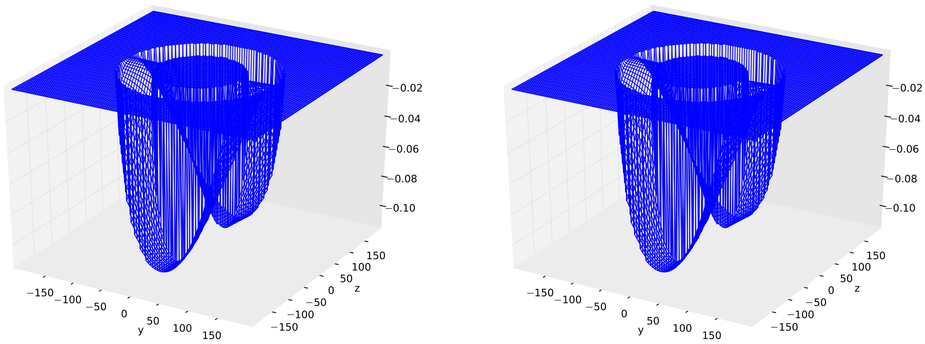
Figure 31: The blueshift plotted against x, where we see that there is more frequency shift than before

3.7 The energy-momentum tensor

We have previously talked about negative energy density/mass and how that is not possible. In this section we will calculate the Einstein tensor for the TARDIS metric with a proper top-hat function. There is a slight hope that maybe the use of that top hat function may eliminate the need for exotic matter, as it removed the blueshift. However Hawking "proved" that you need exotic matter to construct a time machine smaller than the universe(Hawking et al., 2002). The Einstein equation tells us that if we calculate the Einstein tensor, we have also calculated the stress energy tensor as there is only a proportionality factor separating them.

To calculate the Einstein tensor we need the metric 15 and the Christoffel symbols which we combine into the Ricci tensor 9 and the Ricci scalar 10. From equation 8 we see how we should use them to combine them into the Einstein tensor. Now that we have all the theory we have to use a few hours to calculate the various tensors and put our computer skills to good use and make decent plots of the Einstein tensor.

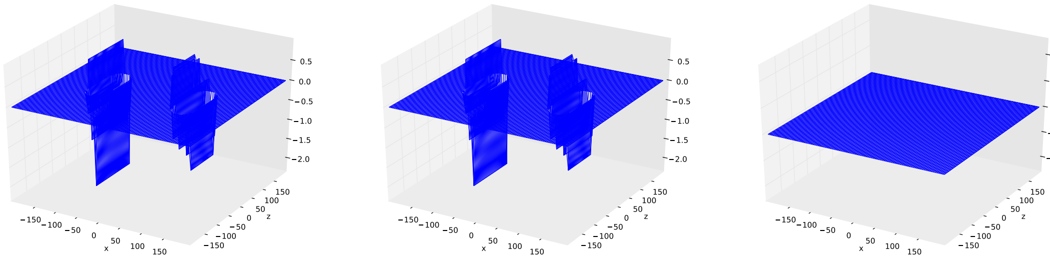
We clearly see from the figures 32, 33, 34 and 35 that we still need exotic matter as expected.



(a) T_{zz} along the slice $y = 0, z = 0$

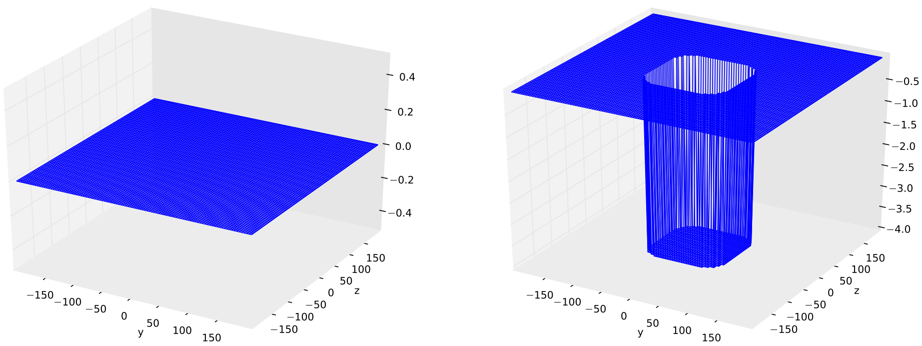
(b) T_{yy} along the slice $y = 0, z = 0$

Figure 32



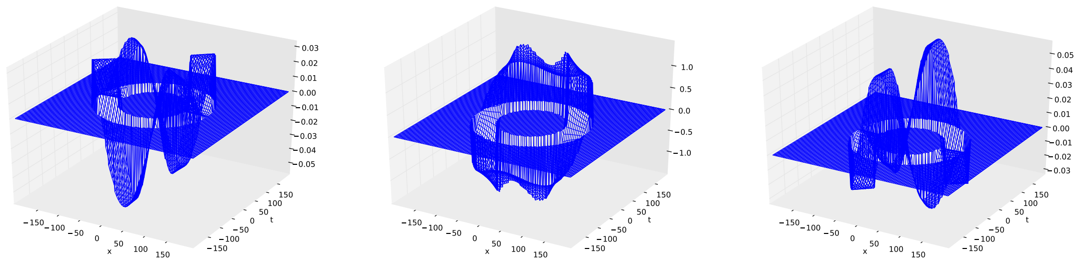
(a) T_{zz} along the slice $y = 0,t = 0$ (b) T_{yy} along the slice $y = 0,t = 0$ (c) T_{xz} along the slice $y = 0,t = 0$

Figure 33



(a) T_{xz} along the slice $t = 0,x = 100$ (b) T_{xy} along the slice $t = 0,x = 100$

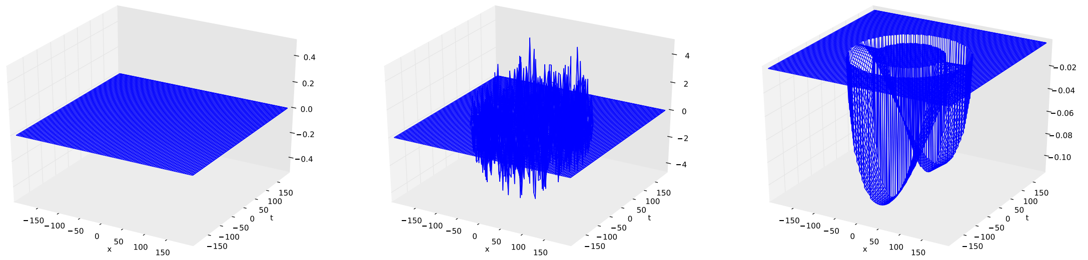
Figure 34



(a) T_{tt} along the slice $y = 0, z = 30$

(b) T_{ty} along the slice $y = 0, z = 30$

(c) T_{xx} along the slice $y = 0, z = 30$



(d) T_{xz} along the slice $y = 0, z = 30$

(e) T_{yy} along the slice $y = 0, z = 30$

(f) T_{zz} along the slice $y = 0, z = 30$

Figure 35

3.8 A quick summary of my results

3.8.1 Visualization

As you have seen in the previous section my two dimensional plot was very similar to the plot in the original article. It is however unfortunate that I have to cut some of the lines, but it seems that is what they have done in the original article. When we cut the geodesics we miss some features, like the fact that some of them turn around in time and we have a Cauchy horizon.

My three dimensional plots we have no way of verifying that they are correct, but as they are based directly on the two dimensional solution there is nothing indicating that something could be wrong. We clearly see from both all plots provided in the section above that the light rays become deflected when they hit the boundaries of the TARDIS. When using the extra term in the metric,the light rays get deflected in a more deliberate way, they all favored to go towards lower values of x.

3.8.2 Blueshift

We have seen that the blueshift is really high when using the approximation of the top hat function, both for the original metric and the one with the extra terms. When we used the proper top hat function however the really high blueshift vanished. This was really not expected as we now have sharper edges for our time machine, which according to my hypothesis should have given a higher blueshift.

3.8.3 Other results

We have seen that it is possible for the TARDIS to travel anywhere in time and space.

The energy-momentum tensor confirms that we need exotic matter to make this space time.

Chapter 4

Discussion

'"Do what I do. Hold tight and pretend it's a plan!"
- The Doctor, Season 7, Christmas Special

In the visualization we had to cut the light rays when their derivative became too big, because otherwise they would become really large. It seems that they did this in the original article also, as you can see that some of the lines just end. They have not explicitly stated that they have cut the geodesics, but when my metric is almost exactly the same as the one in the paper that should indicate something has been done.

We also see that the blueshift is really high when entering and exiting the TARDIS boundaries, we do not even have to accelerate our TARDIS. The fact that this almost disappear when using a proper top-hat function not an approximation may be because when using the proper top hat function there are fewer places where it is possible to divide by zero. There is still a possibility that we get serious blueshifting when using massive particles or the TARDIS is accelerating.

The correction that we explore may be more physically correct, as the world lines now twist in a deliberate way, not just randomly. The new problem now is that it generates more blueshift than it did previously, which is not good for the Doctor and his companions. The new metric also looks awful, but this is however not a reason to throw away the metric, even though I very much would like it to be.

4.1 Is it possible to make this metric into a space ship?

The blueshift with a proper top hat function indicates that the doctor and his companions will not be blasted into oblivion by photons when traveling, so that is one concern taken care of. We have also seen that it is possible for the TARDIS to travel anywhere in time and space, which is practical when you want to use it as a space ship. The only concerns are the Hawking's chronology protection conjecture and the need for exotic matter

4.2 Questions for further consideration

Some questions for future consideration has already been proposed in the original article, but here are some more that could prove useful if we want to explore the metric a little more:

1. As this whole article has been created for photons, it would be nice to see how particles with mass would behave. Both the blueshift and visualization would benefit from the extra information.

2. Tippett and Tsang (2013) claim that this metric is not affected by Hawking's chronology protection conjecture, they have not expanded upon this and it would be interesting to check if this is really true.

Chapter 5

Conclusion

“Rose, before I go, I just want to tell you: you were fantastic. Absolutely fantastic. And you know what? So was I.”
- The Doctor, Season 1, Episode 13

This thesis has demonstrated several aspects of the TARDIS metric, it shows how the light rays move in this universe, the blueshift that arises from traveling photons, the ability to travel anywhere in time and space and calculates the Einstein tensor to show that exotic matter is needed. There are several other aspects of this metric that could have been explored, but due to time limitations there are properties that have to remain undiscovered.

When the metric came in 2013 it was about time that Doctor Who got its own solution in general relativity. It took 50 years for Doctor Who, while Star Trek only needed 28 years to get its own spacetime. The TARDIS metric is also the first metric that allows for the creation of a proper time machine and that it happened 98 years after the birth of general relativity is a little bit sad. There are a few issues that have to be resolved before we get a working time machine and maybe future physics will stop it all, but with todays physics there is still some hope.

The hypothesis and conjectures that does not allow time travel are almost all based on future physics that we do not know are valid. The physicists that propose these anti time travel conjectures are real heavy weight physicist and give validity to the field of time travel. In the paper, Tippett and Tsang (2013) claim that this metric is not affected by the chronology protection conjecture, but it is a little difficult to understand how they can claim that when it is based on physics yet to be discovered.

It would be nice with a metric that would not require exotic matter, nor blast itself and its destination into oblivion. It may not exist such a metric, but maybe it would be possible to overcome these obstacles in the future.

In the end we do not know what will happen in the future and maybe time will be looked upon differently one day. Today however we cannot say more than; time travel may be possible one day. That day maybe we will look upon time as the Doctor did: ”People assume that time is a strict progression of cause to effect, but actually from a non-linear, non-subjective viewpoint, it is more like a big ball of wibbly-wobbly, timey-wimey stuff.”

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Chapter A

Christoffel symbols

All nonzero Christoffel symbols are listed here, the symbols are symmetric in the lower indices, $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$, so I will only list them up once.

$$\Gamma^0_{00} = \frac{t(4(ct^2x^2 + x^4)h^2 - t(t^2+x^2)^2h'(t) + 2(t^2+x^2)h(-x^2 - ct^2xh'(x) + (t^3 + 2tx^2)h'(t)))}{(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \tag{42}$$

$$\Gamma^0_{01} = \frac{t^2(-4(ct^2x + x^3)h^2 - c(t^2+x^2)^2h'(x) + 2(t^2+x^2)h(ct^2h'(x) + x(c - th'(t))))}{c(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \tag{43}$$

$$\Gamma^0_{02} = \frac{t^2(-1+2h)h'(y)}{t^2+x^2-4t^2h+4t^2h^2} \tag{44}$$

$$\Gamma^0_{03} = \frac{t^2(-1+2h)h'(z)}{t^2+x^2-4t^2h+4t^2h^2} \tag{45}$$

$$\begin{aligned}\Gamma^0_{11} =& \frac{t(4t^2(ct^2+x^2)h^2 - (t^2+x^2)^2(2cxh'(x) + th'(t))}{c^2(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \\ &+ \frac{2(t^2+x^2)h(-ct^2 - x^2 + cx^2 + ct^2xh'(x) + t^3h(t)))}{c^2(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)}\end{aligned} \tag{46}$$

$$\Gamma^0_{12} = \frac{-txh'(y)}{c(t^2+x^2-4t^2h+4t^2h^2)} \tag{47}$$

$$\Gamma^0_{13} = \frac{-txh'(z)}{c(t^2+x^2-4t^2h+4t^2h^2)} \tag{48}$$

$$\begin{aligned}\Gamma^1_{00} =& \frac{c(-4(-1+c)t^4xh^2 - t(t^2+x^2)^2(cth'(x) - 2xh'(t))}{(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \\ &+ \frac{2(t^2+x^2)h(ct^4h'(x) + x(-t^2 + ct^2 + x^2 - t^3h'(t))))}{(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)}\end{aligned} \tag{49}$$

$$\Gamma^1_{01} = \frac{t(-4(-1+c)t^2x^2h^2 - t(t^2+x^2)^2h'(t) + 2(t^2+x^2)h(-x^2+ct^2xh'(x)+t^3h'(t)))}{(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \tag{50}$$

$$\Gamma^1_{02} = \frac{ctxh'(y)}{t^2+x^2-4t^2h+4t^2h^2} \tag{51}$$

$$\Gamma^1_{03} = \frac{ctxh'(z)}{t^2+x^2-4t^2h+4t^2h^2} \tag{52}$$

$$\Gamma^1_{11} = \frac{t^2(-4(-1+c)x^3h^2 - c(t^2+x^2)^2h'(x) + 2(t^2+x^2)h(c(t^2+2x^2)h'(x)+x(c+th'(t))))}{c(t^2+x^2)^2(t^2+x^2-4t^2h+4t^2h^2)} \tag{53}$$

$$\Gamma^1_{12} = \frac{t^2(-1+2h)h'(y)}{t^2+x^2-4t^2h+4t^2h^2} \tag{54}$$

$$\Gamma^1_{13} = \frac{t^2(-1+2h)h'(z)}{t^2+x^2-4t^2h+4t^2h^2} \tag{55}$$

$$\Gamma^2_{00} = \frac{-c^2t^2h'(y)}{t^2+x^2} \tag{56}$$

$$\Gamma^2_{01} = \frac{-cxth'(y)}{t^2+x^2} \tag{57}$$

$$\Gamma^2_{11} = \frac{t^2h'(y)}{x^2+t^2} \tag{58}$$

$$\Gamma^3_{00} = \frac{-c^2t^2h'(z)}{t^2+x^2} \tag{59}$$

$$\Gamma^3_{01} = \frac{-cxth'(z)}{t^2+x^2} \tag{60}$$

$$\Gamma^3_{11} = \frac{t^2h'(z)}{x^2+t^2} \tag{61}$$

A.1 Christoffel symbols for the corrected metric

$$\begin{aligned}\Gamma^0_{00} = {} & t((-t^2 - x^2 + 2t^2h)(2x^2h + t(t^2 + x^2)h'(t)) + 2h(x + t^2(t^2 + x^2)W)t(t^2 + x^2)\times \\ & (-cth'(x) + 2(x + t^2(t^2 + x^2)W)h'(t)) + 2h(x((-1 + c)t^2 + x^2) + 3t^2(t^2 + x^2)^2W + t^3(t^2 + x^2)^2W'(t))) \\ & /(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))\end{aligned} \tag{62}$$

$$\begin{aligned}\Gamma^0_{01} = & - t^2(4xh^2(ct^2 + x^2 + t^2x(t^2 + x^2)W) + c(t^2 + x^2)h'(x) \\ & + 2(t^2 + x^2)h(-cx - ct^2h'(x) + t(x + t^2(t^2 + x^2)W)h'(t))) \\ & /c(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))\end{aligned} \tag{63}$$

$$\Gamma^0_{02} = \frac{t^2((-1 + 2h(1 + 2t^2xW + t^4(t^2 + x^2)W^2))h'(y) + 2t^2h^2(x + t^2(t^2 + x^2)W)W'(y))}{t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2)} \tag{64}$$

$$\Gamma^0_{03} = \frac{t^2((-1 + 2h(1 + 2t^2xW + t^4(t^2 + x^2)W^2))h'(z) + 2t^2h^2(x + t^2(t^2 + x^2)W)W'(z))}{t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2)} \tag{65}$$

$$\begin{aligned}\Gamma^0_{11} = & t(4t^2h^2(ct^2 + x^2 + ct^2x(t^2 + x^2)W + ct^2(t^2 + x^2)W'(x)) - (t^2 + x^2)^2(2c(x + t^2(x^2 + t^2)W)h'(x) \\ & + th'(t)) + 2(t^2 + x^2)h(-ct^2 - x^2 + cx^2 + ct^2(x + t^2(t^2 + x^2)W)h'(x) - ct^2(t^2 + x^2)^2W'(x) + t^3h'(t))) \\ & /c^2(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))\end{aligned} \tag{66}$$

$$\Gamma^0_{12} = \frac{-t((x + t^2(t^2 + x^2)W)h'(y) + t^2h(t^2 + x^2 - 2t^2h)W'(y))}{c(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))} \tag{67}$$

$$\Gamma^0_{13} = \frac{-t((x + t^2(t^2 + x^2)W)h'(z) + t^2h(t^2 + x^2 - 2t^2h)W'(z))}{c(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))} \tag{68}$$

$$\begin{aligned}\Gamma^1_{00} = & c(2t^2h(x + t^2(t^2 + x^2)W)(2x^2h + t(t^2 + x^2)h'(t)) - (-t^2 - x^2 + 2t^2h)(t(t^2 + x^2)\times \\ & (cth'(x) + 2(x + t^2(t^2 + x^2)W)h'(t)) + 2h(x((-1 + c)t^2 + x^2) + 3t^2(t^2 + x^2)^2W + t^3(t^2 + x^2)^2W'(t)))) \\ & /(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))\end{aligned} \tag{69}$$

$$\begin{aligned}\Gamma^1_{01} = & t(-4t^2xh^2((-1 + c)x + ct^2(t^2 + x^2)W) - t(t^2 + x^2)^2h'(t) \\ & + 2(t^2 + x^2)h(-x^2 + ct^2(x + t^2(t^2 + x^2)W) + t^3h'(t))) \\ & /(t^2 + x^2)^2(t^2 + x^2 - 4t^2h + 4h^2(t^2 + 2t^4xWt + t^6(t^2 + x^2)W^2))\end{aligned} \tag{70}$$

$$\Gamma^1_{02} = \frac{ct((x+t^2(t^2+x^2)W)h'(y)+t^2h(t^2+x^2-2t^2h)W'(y))}{(t^2+x^2-4t^2h+4h^2(t^2+2t^4xWt+t^6(t^2+x^2)W^2))} \tag{71}$$

$$\Gamma^1_{03} = \frac{ct((x+t^2(t^2+x^2)W)h'(z)+t^2h(t^2+x^2-2t^2h)W'(z))}{(t^2+x^2-4t^2h+4h^2(t^2+2t^4xWt+t^6(t^2+x^2)W^2))} \tag{72}$$

$$\begin{aligned}\Gamma^1_{11} =& t^2(c(-t^2-x^22t^2h)(-2xh+(t^2+x^2)h'(x))+2h(x+t^2x^2+t^2)W(2h(x^2+ct^2(t^2+x^2)^2W'(x)) \\ &+(t^2+x^2)(2c(x+t^2(t^2+x^2)W)h'(x)+th'(t)))) \\ &/c(t^2+x^2)^2(t^2+x^2-4t^2h+4h^2(t^2+2t^4xWt+t^6(t^2+x^2)W^2))\end{aligned} \tag{73}$$

$$\Gamma^1_{12} = \frac{t^2((-1+2h(1+2t^2xW+t^2(t^2+x^2)W^2))h'(y)+2t^2h^2(x+t^2(t^2+x^2)W)W'(y))}{(t^2+x^2-4t^2h+4h^2(t^2+2t^4xWt+t^6(t^2+x^2)W^2))} \tag{74}$$

$$\Gamma^1_{13} = \frac{t^2((-1+2h(1+2t^2xW+t^2(t^2+x^2)W^2))h'(y)+2t^2h^2(x+t^2(t^2+x^2)W)W'(y))}{(t^2+x^2-4t^2h+4h^2(t^2+2t^4xWt+t^6(t^2+x^2)W^2))} \tag{75}$$

$$\Gamma^2_{00} = \frac{-c^2t^2h'(y)}{t^2+x^2} \tag{76}$$

$$\Gamma^2_{01} = \frac{-ct((x+t^2(t^2+x^2)W)h'(y)+t^2(t^2+x^2)hW'(y)}{t^2+x^2} \tag{77}$$

$$\Gamma^2_{11} = \frac{t^2h'(y)}{x^2+t^2} \tag{78}$$

$$\Gamma^3_{00} = \frac{-c^2t^2h'(z)}{t^2+x^2} \tag{79}$$

$$\Gamma^3_{01} = \frac{-ct((x+t^2(t^2+x^2)W)h'(z)+t^2(t^2+x^2)hW'(z)}{t^2+x^2} \tag{80}$$

$$\Gamma^3_{11} = \frac{t^2h'(z)}{x^2+t^2} \tag{81}$$

A.2 Christoffel symbols for the proper top hat function

$$\begin{aligned}\Gamma^0_{00} = {} & \frac{1}{2}\frac{t^2 - x^2}{x^2 + t^2}\frac{2t(t(t^2 + x^2)\delta(t) + x^2)}{(t^2 + x^2)^2} \\ & + \frac{xt}{x^2 + t^2}\left(\frac{4x(t(t^2 + x^2)\delta(t) + (x^2 - t^2))}{(t^2 + x^2)^2} - \frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2}\right)\end{aligned} \tag{82}$$

$$\begin{aligned}\Gamma^0_{10} = {} & \frac{1}{2}\frac{t^2 - x^2}{x^2 + t^2}\frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2} \\ & - \frac{xt}{x^2 + t^2}\frac{2t(t(x^2 + t^2)\delta(t) + 2x^2)}{(x^2 + t^2)^2}\end{aligned} \tag{83}$$

$$\begin{aligned}\Gamma^0_{11} = {} & \frac{1}{2}\frac{t^2 - x^2}{x^2 + t^2}\left(\frac{4t(x(t^2 + x^2)\delta(x) + (t^2 - x^2))}{(x^2 + t^2)^2} + \frac{2t(t(x^2 + t^2)\delta(t) - 2x^2}{(x^2 + t^2)^2}\right) \\ & - \frac{xt}{x^2 + t^2}\frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2}\end{aligned} \tag{84}$$

$$\begin{aligned}\Gamma^1_{00} = {} & \frac{xt}{x^2 + t^2}\frac{2t(t(t^2 + x^2)\delta(t) + x^2)}{(t^2 + x^2)^2} \\ & + \frac{1}{2}\frac{x^2 - t^2}{x^2 + t^2}\left(\frac{4x(t(t^2 + x^2)\delta(t) + (x^2 - t^2))}{(t^2 + x^2)^2} - \frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2}\right)\end{aligned} \tag{85}$$

$$\begin{aligned}\Gamma^1_{10} = {} & \frac{xt}{x^2 + t^2}\frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2} \\ & + \frac{1}{2}\frac{x^2 - t^2}{x^2 + t^2}\frac{2t(t(x^2 + t^2)\delta(t) + 2x^2)}{(x^2 + t^2)^2}\end{aligned} \tag{86}$$

$$\begin{aligned}\Gamma^1_{11} = {} & \frac{xt}{x^2 + t^2}\left(\frac{4t(x(t^2 + x^2)\delta(x) + (t^2 - x^2))}{(x^2 + t^2)^2} + \frac{2t(t(x^2 + t^2)\delta(t) - 2x^2}{(x^2 + t^2)^2}\right) \\ & + \frac{1}{2}\frac{x^2 - t^2}{x^2 + t^2}\frac{2t^2((t^2 + x^2)\delta(x) - 2x)}{(x^2 + t^2)^2}\end{aligned} \tag{87}$$

Chapter B

Numerical methods

Numerical methods let us solve ordinary differential equations from a given initial value.

B.1 Forward Euler

Forward Euler is the simplest numerical method and uses only one step each iteration to get the next value of the solution. The initial value looks like

$$y'(t) = f(t, y(t)), y(t_0) = y_0. \tag{88}$$

Then we have to choose a value h that is the size of our steps and $t_n = t_0 + nh$ and $t_{n+1} = t_n + h$ and we get

$$y_{n+1} = y_n + hf(t_n, y_n). \tag{89}$$

As this is an approximation of a solution there will be some error, but if we choose a small enough h the error will be insignificant.

B.2 Runge-Kutta 4

This numerical method works principally the same way that Forward Euler does, but there are several steps for each iteration. For the same case as last time

$$y'(t) = f(t, y(t)), y(t_0) = y_0 \tag{90}$$

it looks like

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
t_{n+1} &= t_n + h \\
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\
k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \\
k_4 &= f(t_n + h, y_n + hk_3).
\end{aligned} \tag{91}$$

Runge-Kutta 4 has a much smaller error than Forward Euler, so you can use a larger h-value and it handles sharp changes in the function better.