

Optical Microcavities and One-atom Lasers

by

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Introduction

The year 2015 is named the international year of light and light-based technologies[15]. Historically the study of light have brought about some of the high-points of physics, like Youngs double-slit experiment. Today, light-based technologys are all around, from eye surgery to fiber optics. In many of these applications laser light is used.

The term laser was coined as the acronym for "Light Amplification by stimulated emission of radiation". In a normal laser atoms and the electromagnetic field interact quantum mechanically and produces light. The light from a laser is closely centered in both wavelength and in phase, unlike many other sourced of light.

Stimulated emission is the process where an excited atom interacts with a photon and emits a new photon equal to the photon it interacted with, whereas in the process of spontaneous emission the excited atom emits a photon by itself, where the characteristics of the emitted photon is in many respect random. In a normal laser we have a very large number of atoms, which is excited by outside forces and, by stimulated emission, build up a large number of equal photons. Some of these then escapes and we have laser light.

In this thesis we will study a much more simple system. Instead of a very large number of atoms we have one atom. By using a cavity we can ensure that this atom strongly interact/couples with one particularly mode of the electromagnetic field. This also allows the energy from the electromagnetic field to be reabsorbed by the atom. This system can be solved analytically, but to actually make the cavity-atom system resemble a normal laser we need to include other channels for energy to pass through. We need first a pumping mechanism which excites the atom, and a mechanism responsible for the loss of photons from the cavity. In addition we will have a chance for the excited atom to emit a photon spontaneously. Other channels of loss are

possible to include, for instance that the atom-cavity coupling is not perfect, but this will not be talked about in this thesis.

The advantage of this system is its simplicity. Even if we need to solve the system numerically, we can do so in a reasonable time without any further approximations than the numerically ones. This is unlike a standard laser, in which a very large number of atoms would have needed to be modelled.

In chapter one we introduce the necessary background material. We start with the quantization of the electromagnetic field into harmonic oscillators. Then we introduce the coherent states of an harmonic oscillator, which is the quantum mechanical counterpart to monochromatic light. We cover the density matrix formalism necessary to deal the open quantum system, and the density matrix equation of motion, the Lindblad equation.

We find the steady state solution of the Lindblad equation by numerically means in chapter two. This allows us to investigate the parameter space of the problem, and investigate the so called quenching effect. We will talk about the phase of the photons in the steady state case.

In the third chapter we solve the full Lindblad equation numerically, in order to find out how fast the photon expectation value is reached and how fast the phase information is lost.

Chapter 1

Background Chapter

1.1 Quantization of light in a cavity

Among the greatest accomplishment of 19th century physics was the Scottish physicist James Clerk Maxwell unification of the electric and the magnetic force. The result of this work is summed up in Maxwell's four equations:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0\end{aligned}\tag{1.1}$$

Where we have the electric charge density ρ , the electric current \mathbf{j} , the vacuum permittivity ϵ_0 and the vacuum permeability μ_0 , which together gives $\epsilon_0 \mu_0 = \frac{1}{c^2}$, where c is the speed of light in vacuum.

The magnetic vector potential \mathbf{A} and the electric potential ϕ can be introduced as:

$$\begin{aligned}\mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}\end{aligned}\tag{1.2}$$

without changing anything physical.

We will consider the electromagnetic field inside a cavity which is impenetrable by the electromagnetic waves. This gives us standing waves inside

the cavity. The vector potential inside a cavity of volume V can be Fourier-expanded, in which case we have:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sum_{a=1}^2 A_{\mathbf{k}a}(t) \epsilon_{\mathbf{k}a} e^{i\mathbf{k}\cdot\mathbf{r}} \quad (1.3)$$

where $A_{\mathbf{k}a}$ is the field amplitude. To quantize the electromagnetic field we take the field amplitude and its conjugate momenta $\Pi_{\mathbf{k}a} = -\epsilon_0 E_{\mathbf{k}a}^*$ to be operators who do not commute for the same field mode

$$[\hat{E}_{\mathbf{k}a}^\dagger, \hat{A}_{\mathbf{k}'b}] = i \frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}\mathbf{k}'} \delta_{ab} \quad (1.4)$$

Since the electromagnetic field is contained by a cavity, the electromagnetic waves are standing waves. As a result of this the electromagnetic waves are discrete, with frequency determined by the length of the cavity. We will later introduce a two-level atom into the cavity, where the energy gap between the two levels have a frequency ω . We can choose the length of the cavity so that the standing wave with the lowest frequency is resonant with the atom[4]. Since all the other frequencies of the standing waves are now far from resonant with the atom's frequency, we need only consider one mode of the electromagnetic field. We will from now on remove the indices's of the field amplitude operator \hat{A} and the electric field strength \hat{E} .

The Hamiltonian of the system then becomes

$$\hat{H} = \frac{1}{2} \epsilon_0 (E^\dagger E + \omega^2 A^\dagger A) \quad (1.5)$$

If we rewrite \hat{A} and \hat{E} as:

$$\hat{A} = \sqrt{\frac{\hbar}{2\omega\epsilon_0}} (\hat{a} + \hat{a}^\dagger) \quad (1.6)$$

$$\hat{E} = i \sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\hat{a} - \hat{a}^\dagger) \quad (1.7)$$

we get the following Hamiltonian:

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{1}{2} \hbar\omega \quad (1.8)$$

By using equation (1.4) we can find that the commutators for \hat{a} and \hat{a}^\dagger are:

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (1.9)$$

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \quad (1.10)$$

This set of operators, \hat{H} , \hat{a} and \hat{a}^\dagger , and their commutators, is equal to those of a harmonic oscillator. \hat{a} is the lowering operator while \hat{a}^\dagger is the raising operator. The electromagnetic field can then be considered as a harmonic oscillator. In other words \hat{a}^\dagger creates a photon, of the mode we consider, while \hat{a} destroys one photon.

The Hamiltonian in equation (1.8) has a constant term, independent of the number of photons or their frequency. This is the vacuum energy. If we had included the infinity of field modes the electromagnetic field actually consists of, this vacuum energy would have become infinite. Since it is only the energy difference from the ground state which are physically measurable we subtract this term, and ends up with the following Hamiltonian:

$$\hat{H} = \sum_{\mathbf{k}a} \hbar\omega \hat{a}^\dagger a \quad (1.11)$$

If we rewrite the raising and lowering operator defined by equation (1.6) and (1.7) in terms of \hat{A} and \hat{E} we get the explicit expression for the lowering and raising operator:

$$\hat{a}^\dagger = \sqrt{\frac{\epsilon_0}{2\hbar\omega}} (\omega\hat{A} + i\hat{E}) \quad (1.12)$$

$$\hat{a} = \sqrt{\frac{\epsilon_0}{2\hbar\omega}} (\omega\hat{A} - i\hat{E}) \quad (1.13)$$

1.2 Coherent states

Each mode of the electro-magnetic field, of which we will only consider one, can also be described by a harmonic oscillator. The energy eigenstate representation, which is the representation most used when dealing with an harmonic oscillator, is in the case of the electromagnetic field highly non-classical. For a state $|n\rangle$ the expectation value of electric field and the vector potential is zero. This is due to the expectation value for the creation and destruction operator being zero:

$$\langle n|\hat{a}|n\rangle = \sqrt{n} \langle n|n-1\rangle = 0 \quad (1.14)$$

$$\langle n|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle n|n+1\rangle = 0 \quad (1.15)$$

Whereon, from equation (1.7) we have that the expectation value $\langle \hat{E} \rangle = 0$, and similar with the expectation value of the magnetic potential $\langle \hat{A} \rangle = 0$. This is analogous to the fact that the expectation value for the position and momentum operator is zero for an mechanical oscillator. The expectation value for the electric field is zero for any Fock state, independent of how many photons that are present. This is a very unclassical behavior. In the classical case, in the presence of electromagnetic radiation, the measured value of the electric field would generally have been non-zero, and would have varied sinusoidal with time in free space.

One of the attractive features of the laser is that its light resembles that of classical monochromatic light. Seeing as the energy eigenstate basis does not resemble classical monochromatic light we need another basis to represent “classical” light. The basis which historically was introduced with this intent was coherent states.

Coherent states can be defined as the eigenstates of the destruction operator of a harmonic oscillator. If $|z\rangle$ is a coherent state then $a|z\rangle = z|z\rangle$, where a is the destruction operator of the harmonic oscillator and z is the eigenvalue. The destruction operator does not represent a real observable, and the eigenvalue z defining the coherent state $|z\rangle$ is therefore a complex number [6].

To construct the coherent state $|z\rangle$ we apply the operator [10]

$$\hat{D}(z) = e^{z^*\hat{a}^\dagger - z^*\hat{a}} = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} \quad (1.16)$$

to the energy ground state $|0\rangle$:

$$|z\rangle = \hat{D}(z)|0\rangle \quad (1.17)$$

To see that this constructed state is an eigenstate of the destruction operator we calculate:

$$\hat{a}|z\rangle = \hat{a}\hat{D}(z)|0\rangle \quad (1.18)$$

We first find:

$$\begin{aligned} \hat{D}(z)^\dagger \hat{a} \hat{D}(z) &= e^{z^*\hat{a}} e^{-z\hat{a}^\dagger} \hat{a} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} \\ &= e^{z^*\hat{a}} (\hat{a} - z[\hat{a}^\dagger, \hat{a}]) e^{-z^*\hat{a}} \\ &= \hat{a} + z \end{aligned} \quad (1.19)$$

Going back to equation (1.18) and inserting $\hat{D}^\dagger \hat{D}$ we get:

$$\hat{D}(z)\hat{D}(z)^\dagger \hat{a}\hat{D}(z)|0\rangle = \hat{D}(z)(\hat{a} + z)|0\rangle = z\hat{D}(0)|0\rangle = z|z\rangle \quad (1.20)$$

where we have used that applying the destruction operator on the energy ground state gives zero. This result shows that any coherent states can be constructed from the energy ground state.

We can choose to write the eigenvalue z the following way:

$$z = \sqrt{\frac{\epsilon_0}{2\hbar\omega}}(\omega A_c - iE_c) \quad (1.21)$$

where E_c and A_c are real numbers. We can then find the expectation value of the electric field and the magnetic potential:

$$\langle z|\hat{E}|z\rangle = E_c \quad (1.22)$$

$$\langle z|\hat{A}|z\rangle = A_c \quad (1.23)$$

Therefore the expectation value of the electric field is proportional to the imaginary part of z , while the expectation value of the vector potential is proportional to the real part of z . For the electromagnetic field in a coherent state the expectation value of the electric field and the magnetic vector potential is not generally 0, but are determined by which coherent state the system is in. Since there exists a continuum of complex number z , so does it exist a continuum of coherent states $|z\rangle$. The electric field and the vector potential can thus have any real value, as it is in the classical case.

A coherent state $|z\rangle$ at time 0 continues to be a coherent state at all later times. The time development of a quantum state is given by:

$$|z(t)\rangle = \hat{U}(t)|z(0)\rangle \quad (1.24)$$

where

$$\hat{U}(t) = e^{\frac{-i\hat{H}t}{\hbar}} \quad (1.25)$$

We use the Hamiltonian from equation (1.11) and apply the destruction operator on the time developed state:

$$\hat{a}|z(t)\rangle = \hat{a}\hat{U}(t)|z(0)\rangle = \hat{a}e^{-i\omega\hat{a}^\dagger\hat{a}t}|z(0)\rangle \quad (1.26)$$

By expanding the exponential operator in a Taylor series we get:

$$\hat{a}|z(t)\rangle = \hat{a} \sum_n \frac{(-i\omega t)^n}{n!} (\hat{a}^\dagger \hat{a})^n |z(0)\rangle \quad (1.27)$$

Using that $\hat{a}(\hat{a}^\dagger\hat{a})^n = (\hat{a}\hat{a}^\dagger)^n\hat{a}$ we get:

$$\hat{a}|z(t)\rangle = \sum_n \frac{(-i\omega t)^n}{n!} (\hat{a}\hat{a}^\dagger)^n \hat{a}|z(0)\rangle = e^{-i\omega\hat{a}\hat{a}^\dagger t} z_0 |z(0)\rangle \quad (1.28)$$

where z_0 is the eigenvalue of $|z(0)\rangle$. Using the commuter relations between the creation and destructor operator, equation (1.9), we can rewrite equation (1.28) as the following:

$$\hat{a}|z(t)\rangle = z_0 e^{-i\omega t} e^{-i\omega\hat{H}t} |z(0)\rangle = z_0 e^{-i\omega t} |z(t)\rangle \quad (1.29)$$

We see that the coherent state $|z(0)\rangle$ transforms in time as a coherent state, with eigenvalue $z(t) = z_0 e^{-i\omega t}$. Using equation (1.21) we get the time development of the expectation values of the electric field and the magnetic vector potential:

$$E_c(t) = E_c(0) \cos \omega t + \sin \omega t \frac{A_c(0)}{\omega} \quad (1.30)$$

$$A_c(t) = E_c(0) \sin \omega t - \sin \omega t \frac{A_c(0)}{\omega} \quad (1.31)$$

From these two equations we see that the expectation values follows the behavior of its classical counterparts, where the electric field change value sinusoidal. Later in this thesis we will move into the interaction picture. In the interaction picture the Hamiltonian is separated into two parts, the free Hamiltonian H_0 and the interaction Hamiltonian H_I . The ground Hamiltonian will consists of the Hamiltonian of the harmonic oscillator, seeing as this is the Hamiltonian of the free electromagnetic field. A quantum state in the interaction picture is connected to the Schrödinger picture by

$$|\psi(t)_I\rangle = e^{\frac{iH_0 t}{\hbar}} |\psi(t)_S\rangle \quad (1.32)$$

Comparing this to the time development of the Schrödinger state given in equation (1.25) gives us that in the interaction picture the free Hamiltonian does not contribute to the time development. That means that, for a coherent state, the harmonic oscillator Hamiltonian does not contribute to the time development of the coherent state. That means that in the interaction picture, if there are no other influences than the free electromagnetic field, a coherent state remains stationary and does not move around in the phase space constructed by the real and imaginary parts of z .

We will now calculate the variance of \hat{E} for a coherent state:

$$\sigma_{\hat{E}}^2 = \langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2 \quad (1.33)$$

The expectation value of the electric field is known from equation (1.22), but we need to calculate $\langle \hat{E}^2 \rangle$:

$$\begin{aligned} \langle \hat{E}^2 \rangle &= -\frac{\hbar\omega}{2\epsilon_0} \langle z | (\hat{a} - \hat{a}^\dagger)^2 | z \rangle \\ &= -\frac{\hbar\omega}{2\epsilon_0} (\langle z | \hat{a}^2 | z \rangle + \langle z | \hat{a}^{\dagger 2} | z \rangle - \langle z | \hat{a}\hat{a}^\dagger | z \rangle - \langle z | \hat{a}^\dagger\hat{a} | z \rangle) \\ &= -\frac{\hbar\omega}{2\epsilon_0} (z^2 + z^{*2} - 2zz^* - 1) \\ &= -\frac{\hbar\omega}{2\epsilon_0} ((z - z^*)^2 - 1) \\ &= E_c^2 + \frac{\hbar\omega}{2\epsilon_0} \end{aligned} \quad (1.34)$$

The variance is then found:

$$\sigma_{\hat{E}}^2 = \langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2 = \frac{\hbar\omega}{2\epsilon_0} \quad (1.35)$$

Likewise we find $\langle \hat{A}^2 \rangle$:

$$\begin{aligned} \langle \hat{E}^2 \rangle &= \frac{\hbar}{2\omega\epsilon_0} \langle z | (\hat{a} + \hat{a}^\dagger)^2 | z \rangle \\ &= \frac{\hbar}{2\omega\epsilon_0} (\langle z | \hat{a}^2 | z \rangle + \langle z | \hat{a}^{\dagger 2} | z \rangle + \langle z | \hat{a}\hat{a}^\dagger | z \rangle + \langle z | \hat{a}^\dagger\hat{a} | z \rangle) \\ &= \frac{\hbar}{2\omega\epsilon_0} (z^2 + z^{*2} + 2zz^* + 1) \\ &= \frac{\hbar}{2\omega\epsilon_0} ((z + z^*)^2 + 1) \\ &= A_c^2 + \frac{\hbar}{2\omega\epsilon_0} \end{aligned} \quad (1.36)$$

by which we find the variance of \hat{A} :

$$\sigma_{\hat{A}}^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \frac{\hbar}{2\omega\epsilon_0} \quad (1.37)$$

Combining the two we get:

$$\sigma_{\hat{A}}^2 \sigma_{\hat{E}}^2 = \frac{\hbar^2}{4\epsilon_0^2} \quad (1.38)$$

We know that the product of variances of two incompatible observables \hat{X}_1 and \hat{X}_2 obey the following uncertainty relation [8]:

$$\sigma_{\hat{X}_1}^2 \sigma_{\hat{X}_2}^2 \geq \frac{1}{4} |\langle [\hat{X}_1, \hat{X}_2] \rangle|^2 \quad (1.39)$$

which in this case becomes:

$$\sigma_{\hat{A}}^2 \sigma_{\hat{E}}^2 \geq \frac{\hbar^2}{4\epsilon_0^2} \quad (1.40)$$

What we have is that a coherent state is a minimum uncertainty state, in that the product of the variances are equal to its theoretical minimum.

To recapitulate: A coherent state remains a coherent state at all later times, barring outside influences. A coherent state has a distinct set of expectation values, whose time developments are classical. In addition, a coherent state is a minimum uncertainty state. It is for these reason that coherent states are viewed as the quantum mechanical states which most closely resemble classical ones. In our case they are the closest quantum mechanically equivalent to monochromatic light. Since it is monochromatic light we want as output from a laser we want a photon field in the cavity which is equal to that of a coherent state.

For the next part of the chapter we will relate the coherent states to the energy eigenvector states, and look at some more of the distinct features of the coherent basis.

We can expand a coherent state in terms of the energy eigenstates:

$$|z\rangle = \hat{D}(z) |0\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} |0\rangle \quad (1.41)$$

The last operator can be expanded as $e^{-z^*\hat{a}} = \hat{I} + z^*\hat{a} + \dots$, of which \hat{I} is the only term that do not give zero. We can now write the coherent state as:

$$|z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} |z\rangle = e^{-\frac{1}{2}|z|^2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle \quad (1.42)$$

As a consequence of this we find the overlap between a coherent state and an energy eigenstate:

$$\langle n|z\rangle = e^{-\frac{1}{2}|z|^2} \frac{z^n}{\sqrt{n!}} \quad (1.43)$$

From this it can be found that the overlap between two different coherent states, $|z\rangle$ and $|z'\rangle$, is given by:

$$|\langle z|z'\rangle|^2 = e^{-|z-z'|^2} \quad (1.44)$$

This means that, unlike energy eigenstates, the coherent states are not orthonormal. If the system is in one coherent state $|z\rangle$, we have the probability given in equation (1.44) to find the state in the coherent state $|z'\rangle$. This is unlike the case of a orthonormal set like the energy eigenvector set. If the system is in the state $|n\rangle$ there are zero probability of the system being in any other energy eigenvector state. The coherent states form a over-complete basis.

Since coherent states are not orthonormal, coherent states do not fulfill the standard completeness relation. The integral over all coherent states does not give the identity matrix, instead we have:

$$\int dz |z\rangle \langle z| = \pi \hat{I} \quad (1.45)$$

where I is the identity matrix.

1.3 Density matrices

The systems first encountered within quantum mechanics are pure systems, systems which are not disturbed by outside forces. This is often a good approximation to the real world, either because the system is quite isolated, or that the size of any external force acting on the system is negligible compared to the size of the internal forces of the system. For instance one neglects gravity when considering a hydrogen atom, since the gravitational force between the proton and electron is negligible compared to the electric force between them. However, in our case with the one-atom laser we have a somewhat different problem. Our system is not isolated. A laser sends light out into the rest of the universe, and to continue to do so for longer periods of time it requires that energy is fed into the system.

A pure, isolated system, described by a quantum state, continues to be pure and can continue to be described by a quantum state. If the state is represented by a wave function we can in principle use Schrödinger's equation and find the wave function for all later times. Since our system is not isolated

but is influenced by the wider universe, the system is not pure, and we can not subscribe a single quantum state to the system. To describe a quantum system which is not a pure state, we can use the density operator formalism.

The density operator can be introduced by considering a system where classical probabilities plays a role, where the system with probability p_i is in state $|\psi_i\rangle$. The density operator can be introduced:

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i| \quad (1.46)$$

The matrix representation of the density operator, the density matrix, is in the orthonormal basis $|n\rangle$ given by:

$$\rho_{n,n'} = \sum_i p_i \langle n|\psi_i\rangle \langle \psi_i|n'\rangle \quad (1.47)$$

All physical observables can then be found by:

$$\langle \hat{A} \rangle = Tr(\hat{\rho}\hat{A}) \quad (1.48)$$

where the trace of a matrix $Tr(\hat{O})$ is equal to the sum of the diagonal elements of the matrix \hat{o} .

The density matrix fulfills three condition. Its diagonal elements must be positive, and they must be normalizable, i.e. their sum must be equal to one. This is due to the fact that the diagonal elements are probabilities. The diagonal element:

$$\hat{\rho}_{n,n'} = \sum_i p_i |\langle n|\psi_i\rangle|^2 \quad (1.49)$$

we see are the sum, over all states $|\psi_i\rangle$, of the product of the probability of the state being in state $|\psi_i\rangle$ times the probability of a state $|\psi_i\rangle$ being in state $|n\rangle$. That is the probability of finding the system in state $|n\rangle$. In addition, as seen from equation (1.47), the density matrix is hermitian.

The time development of the density matrix follows the von-Neumann equation, which in the schrödinger picture looks like:

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \quad (1.50)$$

Where ρ is the density matrix and H is the Hamiltonian of the system. Later we will use the interaction picture, where we separate the Hamiltonian into

two parts, the ground part \hat{H}_0 plus the interaction part \hat{H}_I . Transforming equation (1.50) into the interaction picture gives:

$$i\hbar \frac{\partial \rho}{\partial t} = [H_I, \rho] \quad (1.51)$$

In our case the interaction Hamiltonian will be the part of the Hamiltonian connecting the atom and the electromagnetic field, while the rest of the Hamiltonian, H_0 will consist of the energy terms from the two separate part of the system.

If we have a composite system, a system made of two subsystems, things get a little different. We call the two subsystems A and B, which have their respective basis $|n\rangle$ and $|m\rangle$. The density operator then becomes:

$$\rho = \sum_{n,m,n',m'} \rho_{n,m,n',m'} |n\rangle \langle n'| \otimes |m\rangle \langle m'| \quad (1.52)$$

If we write $|n\rangle \otimes |m\rangle$ as $|n, m\rangle$ we can write the density matrix as:

$$\rho = \sum_{n,m,n',m'} \rho_{n,m,n',m'} |n, m\rangle \langle n', m'| \quad (1.53)$$

The diagonal elements of the diagonal matrix are those diagonal in both n and m , and like earlier must be positive and normalizable to unity, and the whole density matrix is hermitian.

To find an expectation value for an observable for one of the subsystem we can not use equation (1.48) as before, but must replace the density matrix with a reduced density matrix. The reduced density matrix for system A is found by taking the partial trace over system B. The matrix elements of the reduced density matrix for system A is then given by:

$$\rho_{n,n'}^A = \sum_m \rho_{n,m,n',m} \quad (1.54)$$

where the reduced density matrix again has only two dimensions. We see that, quite reasonable, the probability of the subsystem A being in state $|n\rangle$, given by the element $\rho_{n,n}^A$, is the sum of the probabilities for subsystem A to be in state $|n\rangle$ and subsystem B being in any of its states $|m\rangle$. The expectation value for an operator O_A acting on subsystem A is the given by replacing the density matrix in equation (1.48) with the reduced density matrix for A:

$$\langle O_A \rangle = Tr(O_A \rho^A) \quad (1.55)$$

1.4 Measurement of coherence

In general we will not deal with coherent states, so we need ways to measure how coherent a general state is.

A coherent state $|z\rangle$ has the corresponding density operator (for the photon part at least) $\rho = |z\rangle\langle z|$. The photon density matrix in the orthonormal energy eigenbasis is given by $\rho_{n,n'} = \langle n|z\rangle\langle z|n'\rangle$. Using equation (1.43) the density matrix elements becomes:

$$\rho_{n,n'} = e^{-|z|^2} \frac{z^{n*} z^{n'}}{\sqrt{n!}\sqrt{n'}} \quad (1.56)$$

The most intuitive thing to check is if the diagonal elements of the photon density matrix, also the probability distribution for the number of photons in the electromagnetic field mode, is in agreement with equation (1.56). For just the diagonal elements equation (1.56) becomes

$$\rho_{n,n} = e^{-|z|^2} \frac{|z|^{2n}}{n!} \quad (1.57)$$

This distribution is a Poisson distribution. One way to check whether the photon probability distribution closely resembles a Poisson distribution is by use of the correlation function $g_2(0)$. We have that [5]:

$$g_2(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} \quad (1.58)$$

Since the commutator of the creation and annihilation operators are $[a, a^\dagger] = 1$, we can rewrite equation (1.58):

$$g_2(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle}{\langle a^\dagger a \rangle^2} \quad (1.59)$$

$a^\dagger a$ is the number operator for the photon field, therefore we have that $\langle a^\dagger a \rangle = \langle n \rangle$ and $\langle a^\dagger a a^\dagger a \rangle = \langle n^2 \rangle$. For a Poisson distribution, which we know is the photon probability distribution for a coherent state, the variance is equal to the mean, also: $\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$. For a Poisson distribution, and therefore for a coherent state, we can find the value of g_2 :

$$g_2(0) = \frac{\langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle n \rangle + \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} = 1 \quad (1.60)$$

Also, for a coherent state the function g_2 should evaluate to 1. We will use this function as a test of how coherent the photon number part of the cavity system is.

However the photon number is not the only part of the story, and while a coherent state has $g_2(0) = 1$, a state where $g_2(0) = 1$ is not necessary coherent. To find out if a state is really coherent we need also to consider the phase of the system.

The diagonal elements, also the photon probability distribution, have to do with energy, in the sense that the expectation value of the energy of the system is proportional to the expectation value of the photon number. We see that equation (1.57) only depends on the magnitude of z , not the value of the real and imaginary part of z . We saw earlier that the real and imaginary part of z is proportional to the expectation value of the magnetic vector potential and the electric field respectively. Since the diagonal elements are equal irrespective of the phase of the system, also what values of $\langle \hat{A} \rangle$ and $\langle \hat{E} \rangle$ that is most likely, that information must be in the off-diagonal elements. We will come back to phase part of the system later, at the end of chapter two. In the mean time we must keep in mind that $g_2(0)$ is only an indicator for the system having a coherent photon distribution, but the function says nothing of the phase of the system.

1.5 The Jaynes-Cummings model

The Jaynes Cummings model consists of a two-level atom interacting with an electromagnetic field mode. The energy levels of the atom are separated by the energy difference $\Delta E = \hbar\omega$, while the electromagnetic field consists of n photons with energy $\hbar\omega$. In general, these frequencies do not need to be equal, but we will set them to be so. As mentioned in an earlier section, a suitable cavity enables us to only consider one field mode of the electromagnetic field, in our case one with frequency equal to the atoms frequency. The atom and the electromagnetic field mode will interact with a strength g . This coupling strength is inversely proportional to the wavelength of the cavity [11]. A smaller cavity therefore leads to an increase in the coupling strength. The strength of the coupling between the atom and the cavity is one of the distinct features of the Jaynes-Cummings model. An excited atom may emit a photon into the cavity, but at the same time a photon cavity can be absorbed by the atom. The setup is shown in figure (1.1).

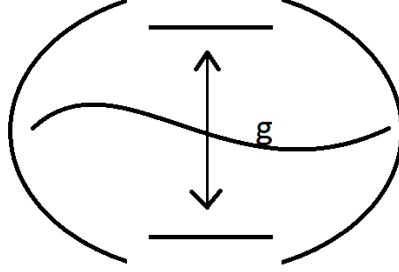


Figure 1.1: The Jaynes-Cummings model: An atom interacts with strength g with an electromagnetic field mode inside a cavity

The Hamiltonian of the system is [12]:

$$\begin{aligned} H &= H_{field} + H_{atom} + H_{interaction} \\ &= \hbar\omega a^\dagger a + \hbar\omega \frac{\sigma_z}{2} + \hbar g(a^\dagger + a)(\sigma_- + \sigma_+) \end{aligned} \quad (1.61)$$

Here a and a^\dagger are the creator and destruction operators for the electromagnetic field, where $a^\dagger a$ is the number operator. σ_- and σ_+ is the lowering and raising operator for the atom, while σ_z gives one acting on the excited state of the atom and minus one acting on the ground state of the atom. The atomic raising and lowering operator can be expanded: $\sigma_- = v^\dagger c$ and $\sigma_+ = c^\dagger v$. Here v^\dagger and v are the creation and destruction operator of the ground state of the atom, while c^\dagger and c are the creator and destruction operator on the excited state. $\sigma_+ = c^\dagger v$ means therefore to destroy the ground state while creating an excited state.

Under the rotating wave approximation we can eliminate some of the terms from the interaction part of the Hamiltonian, namely those which do not conserve the excitation number. That is $\sigma_- a$, or the one who increases the atomic energy level while adding a photon to the field mode $\sigma_+ a^\dagger$. These terms can be eliminated if the frequency of the atom and the frequency of the field mode is resonant [12]. The Hamiltonian of the system is then:

$$H = \hbar\omega a^\dagger a + \hbar\omega \frac{\sigma_z}{2} + \hbar g(a^\dagger \sigma_- + a \sigma_+) \quad (1.62)$$

We can find the time development for the Jaynes-Cummings model by finding the eigenstates of the Hamiltonian. Since equation (1.62) obeys conservation of energy, if we start the system in the state $|c, n\rangle$, we can use the basis $|c, n\rangle = |c\rangle \otimes |n\rangle$ and $|v, n+1\rangle = |v\rangle \otimes |n+1\rangle$. where n is the number of photons in the electromagnetic field mode and c, v means that the atom is in the excited state or the ground state. In this basis the Hamiltonian can be represented as the matrix:

$$H = \begin{pmatrix} \omega(n + \frac{1}{2}) & g\sqrt{n+1} \\ g\sqrt{n+1} & \omega(n + \frac{1}{2}) \end{pmatrix} \quad (1.63)$$

Its eigenvalues are found by solving the characteristic equation:

$$\det(H - \lambda I) = 0 \quad (1.64)$$

for λ , where I is the 2×2 identity matrix. This equation has the solution

$$\lambda_{\pm} = -\omega(n + \frac{1}{2}) \pm g\sqrt{n+1} \quad (1.65)$$

We find the eigenvectors by solving the equations:

$$H \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda_{\pm} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (1.66)$$

This gives the two following solutions: For λ_- we get that $X = Y$, while for λ_+ we get that $X = -Y$. We can then choose the normalizable eigenvectors to be:

$$\mathbf{v}_- = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (1.67)$$

We set the system to initially be in the state $|c, n\rangle$, in vector representation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which we see is can be expressed as the following linear combination of the two eigenstates given in equation (1.67):

$$\mathbf{x}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \mathbf{v}_- + \frac{1}{\sqrt{2}} \mathbf{v}_+ \quad (1.68)$$

The initial state is now expressed in terms of the eigenvectors of the Hamiltonian, the later time development is now given by:

$$\mathbf{x}(t) = \frac{1}{\sqrt{2}} \mathbf{v}_- e^{i\lambda_- t} + \frac{1}{\sqrt{2}} \mathbf{v}_+ e^{i\lambda_+ t} \quad (1.69)$$

The coefficient for the state $|c, n\rangle, k_{c,n}$ after a time t is given by:

$$\begin{aligned}
k_{c,n} &= \frac{1}{2}(e^{i\lambda+t} + e^{i\lambda-t}) \\
&= \frac{1}{2}(e^{i(-\omega(n+\frac{1}{2})-g\sqrt{n+1})t} e^{i(-\omega(n+\frac{1}{2})+g\sqrt{n+1})t}) \\
&= \frac{e^{-i\omega(n+\frac{1}{2})t}}{2}(e^{-ig\sqrt{n+1}t} + e^{-ig\sqrt{n+1}t}) \\
&= e^{-i\omega(n+\frac{1}{2})t} \cos(g\sqrt{n+1}t)
\end{aligned} \tag{1.70}$$

The probability of the system being in the state $|c, n\rangle$ at time t is then given by:

$$p_{c,n} = \cos^2(g\sqrt{n+1}t) \tag{1.71}$$

We see that the probability oscillates between the two state. At $t=0$ and when t equals multiples of $\frac{2\pi}{g\sqrt{n+1}}$ the system is guaranteed to be in the state $|c, n\rangle$. Likewise the system is, when t equals a multiple of $\frac{\pi}{g\sqrt{n+1}}$ the system is guaranteed to be in the state $|v, n+1\rangle$. It should be mentioned that this full oscillations we have here only happens if there is no detuning, i.e. that the frequency of the electromagnetic field mode and the difference between the atoms energy levels are precisely the same. If that had not been the case the system would never, with 100% certainty, been in the state $|v, n+1\rangle$,

1.6 The Jaynes-Cummings model:extensions

In the last section we found the time development for the Jaynes-Cummings model. That was an isolated, pure system which could be solved by finding the eigenvectors of the Hamiltonian. However, we are interested in the more complex system where the cavity is connected to the wider universe. The system has three points of contact with the rest of the world:

- Energy must be pumped into the atom, which will then be excited.
- Photons may escape the cavity, which is necessary to get laser light. This is mainly achieved by having one of the walls of the cavity by a partly transparent wall.
- The excited atom may emit a photon different than those in the cavity mode, which means that the atom falls back to the ground state while the cavity population remain stable

This is shown in figure (1.2)

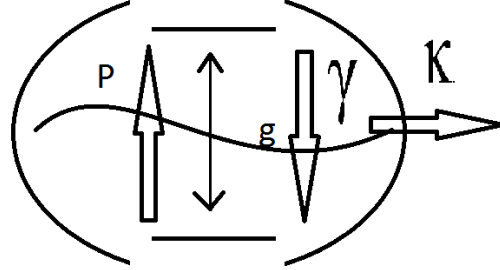


Figure 1.2: The Jaynes-Cummings model with extensions: In addition to the atom interacting with the cavity, three interactions with the wider universe have been added. One pumping the atom into the upper level, with strength P . One making the atom lose energy and fall back into its ground level, with strength γ . At last we give each photons a probability κ of escaping the cavity

This is modeled so that there is a chance P, κ and γ for, respectively, the atom gaining energy, the cavity losing energy, and the atom losing energy. This stochastic model's equation of motion is called the Lindblad equation.

The Lindblad equation is in general written as [3]:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] + \sum_{n,m} h_{n,m} \left(L_n \rho L_m^\dagger - \frac{1}{2} (\rho L_m^\dagger L_n + L_m^\dagger L_n \rho) \right) \quad (1.72)$$

The first term is the density matrix equation of motion (1.50) for an isolated system. The rest of the terms is due to influence from outside of the system. $h_{n,m}$ is the probability per unit time of the system jumping from state n to state m . The operator L_n annihilates the state n , while the operator L_m^\dagger creates a state m .

For our specific system we have the three processes mention above, and the Lindblad equation then becomes:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -i[g(a^\dagger \sigma_- + a \sigma_+), \rho] \\ & + \frac{\kappa}{2}[2a\rho a^\dagger - b^\dagger b \rho - \rho b^\dagger b] \\ & + \frac{\gamma}{2}[2v^\dagger c \rho c^\dagger v - c^\dagger c \rho - \rho c^\dagger c] \\ & + \frac{P}{2}[2c^\dagger v \rho v^\dagger c - v^\dagger v \rho - \rho v^\dagger v] \end{aligned} \quad (1.73)$$

To get an understanding of what these added terms does to the system we will examine a few examples of how these terms effect the time development of some density matrix elements.

The κ -term is due to the energy loss from the optical cavity, photons that with a probability κ exits the cavity. If we consider the change in the probability of there being n photons in the cavity, due to the κ -term, we have:

$$\langle m, n | \frac{\kappa}{2}[2a\rho a^\dagger - b^\dagger b \rho - \rho b^\dagger b] | m, n \rangle = \kappa[(n+1)\rho_{n+1, n+1}^{m, m} - n\rho_{n, n}^{m, m}] \quad (1.74)$$

where m is the atomic state, and naturally irrelevant for loss in the κ channel. $\rho_{n, n'}^{m, m'}$ is short-hand for $\langle m, n | \rho | m', n' \rangle$. We see that, due to the loss from the cavity, the probability of there being n photons increases due to the change of a photon escaping a $|m, n+1\rangle$ state, which creates a $|m, n\rangle$ state. The probability of there being n photons decrease due to the change of a photon escaping from an $|m, n\rangle$ state, which will create a $|m, n-1\rangle$ state.

The γ term is a consequence of the probability that an excited atom emits a photon, but not in the lasing mode. The probability of the atom being in a excited state, with n photons, changes due to this term as:

$$\langle c, n | \frac{\gamma}{2}[2v^\dagger c \rho c^\dagger v - c^\dagger c \rho - \rho c^\dagger c] | c, n \rangle = -\gamma \rho_{n, n}^{c, c} \quad (1.75)$$

We see that the probability of the atom being in the excited state decreases due to γ . If we calculate the same for the ground state we find that the probability increase due to γ .

The P term arises due to the pumping of energy into the system. Calculating how the probability of the atom being in an excited state due to the

P-term alone we get:

$$\langle c, n | \frac{P}{2} [2c^\dagger v \rho v^\dagger c - v^\dagger v \rho - \rho v^\dagger v] | c, n \rangle = P \rho_{n,n}^{v,v} \quad (1.76)$$

The P term increase the probability of the system being in the excited state, and if we likewise calculate this change for the ground state we get that the probability of the atom being in the ground state decrease due to P.

While there are four parameters in equation (1.73), there are only three independent parameters. If we divide all terms with g, we effectively measure all the three other parameters as a fraction of g, while we, due to dividing on g on the left side in equation (1.73), measures time in terms of g. For the rest of the thesis, we will use this when giving P, κ, γ and t as numbers.

1.7 The Standard Laser

In contrast to the two-level system we will study in this thesis, a normal laser consists of a large number of atoms. In this section we will briefly present the equation for the steady state statistics of a normal laser, taken from [10].

The atoms in the normal laser, numbering N, have three energy levels. There is N_0 atoms in the ground level, N_1 atoms in the middle level and N_2 atoms occupy the upper level. Atoms are being pumped at a rate R into the upper state from the ground state, from whereon they can decay spontaneous into the middle level at a rate Γ_{sp} , or by stimulated at a rate $n\Gamma_{st}$, where n is the number of photons in the cavity. The number $n_s = \frac{\Gamma_{sp}}{\Gamma_{st}}$ is the number of photons that must be present in the field mode for the stimulated emission to overcome the spontaneous emission. From the middle level the atom fall quickly into the ground level at a rate T. In the steady-state situation we have that the number to and from level two must balance, from where we get:

$$N_0 R = N_2 (\Gamma_{sp} + \Gamma_{st}) \quad (1.77)$$

The same can be done for the photon number:

$$N_2 \Gamma_{st} (np_{n-1} - (n+1)p_n) - \Gamma_{cav} (np_n - (n+1)p_{n+1}) = 0 \quad (1.78)$$

where Γ_{cav} is the rate of the loss of photons from the cavity.

From this equation the following can be extracted:

$$N_2 \Gamma_{st} (1 + \langle n \rangle) - \Gamma_{cav} \langle n \rangle = 0 \quad (1.79)$$

Combining equation (1.77) and (1.79) a second order equation for the mean photon number can be found:

$$\Gamma_{st}\Gamma_{cav}\langle n \rangle^2 - (NR\Gamma_{st} - \Gamma_{st}\Gamma_{cav})\langle n \rangle - NR\Gamma_{st} = 0 \quad (1.80)$$

This can be solved:

$$\langle n \rangle = \frac{1}{2} \left[(C - 1)n_s + \sqrt{(C - 1)^2 n_s^2 + 4Cn_s} \right] \quad (1.81)$$

In this the parameter C was introduced

$$C = \frac{NR\Gamma_{st}}{\Gamma_{sp}\Gamma_{cav}} \quad (1.82)$$

which is proportional to the rate of energy pumped into the system.

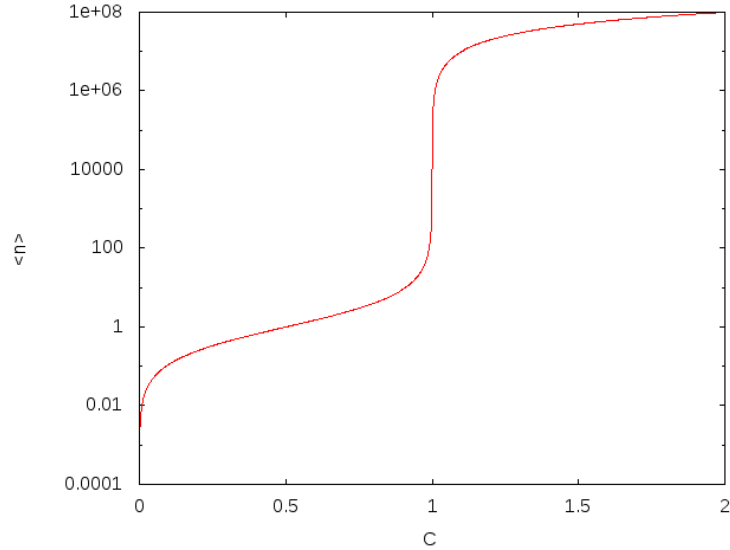


Figure 1.3: Photon expectation value as a function of C

In figure (1.3) we have plotted the expectation value of the photon number against the parameter C. We see that the photon population increases slowly when $C < 1$, but when C reaches 1 the photon number increase sharply. For $C > 1$ the growth of the photon number subdues. When $C \rightarrow \infty$ the solution simplifies to

$$\langle n \rangle = Cn_s \quad (1.83)$$

which means that for $C \gg 1$, the photon number in the photon mode increase linearly with C.

Chapter 2

The steady state solution of the two-level system

In the two previous chapters we presented the Lindblad equation and The Jaynes-Cummings model. In this chapter we will use the Lindblad equation to solve the system at equilibrium, $\frac{d\rho}{dt} = 0$. The two-level system depends, as seen earlier, on three parameters, and we will study the equilibrium situation in different regions of parameter space.

2.1 The steady state equation

The Lindblad equation has the form of a differential equation. If we express the density matrix ρ as a matrix, with the basis $|c, n\rangle$, $|v, n\rangle$, we get one differential equation for each element in the density matrix, each of them coupled with several different elements in the density matrix. We will start by looking at the diagonal elements. These elements are the elements which tells us the probability of of the system having n photons and the probability of the atom being in the ground state v or the excited state c . We will now calculate the differential equations governing the time development of the elements $\langle c, n | \rho | c, n \rangle$ and $\langle v, n | \rho | v, n \rangle$. The matrix element $\langle m, n | \rho | m', n' \rangle$ will be written as $\rho_{n,n'}^{m,m'}$.

$$\begin{aligned}
\frac{d\rho_{n,n}^{c,c}}{dt} &= \langle c, n | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v], \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |c, n\rangle \\
&= -ig[\sqrt{n+1}\rho_{n+1,n}^{v,c} - \sqrt{n+1}\rho_{n,n+1}^{c,v}] \\
&\quad + \kappa[(n+1)\rho_{n+1,n+1}^{c,c} - n\rho_{n,n}^{c,c}] \\
&\quad - \gamma\rho_{n,n}^{c,c} + P\rho_{n,n}^{v,v}
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\frac{d\rho_{n,n}^{v,v}}{dt} &= \langle v, n | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v], \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |v, n\rangle \\
&= -ig[\sqrt{n}\rho_{n-1,n}^{c,v} - \sqrt{n}\rho_{n,n-1}^{v,c}] \\
&\quad + \kappa[(n+1)\rho_{n+1,n+1}^{v,v} - n\rho_{n,n}^{v,v}] \\
&\quad + \gamma\rho_{n,n}^{c,c} - P\rho_{n,n}^{v,v}
\end{aligned} \tag{2.2}$$

We see from equation (2.1) and (2.2) that the diagonal elements $\rho_{n,n}^{c,c}$ and $\rho_{n,n}^{v,v}$ depends only on the off-diagonal elements of the form $\rho_{n+1,n}^{v,c}$ and $\rho_{n,n+1}^{c,v}$. The differential equation for these elements will now be found.

$$\begin{aligned}
\frac{d\rho_{n+1,n}^{v,c}}{dt} &= \langle v, n+1 | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v] + \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |c, n\rangle \\
&= -ig \left[\sqrt{n+1}\rho_{n,n}^{c,c} - \sqrt{n+1}\rho_{n+1,n+1}^{v,v} \right] \\
&\quad + \frac{\kappa}{2}[2\sqrt{(n+1)(n+2)}\rho_{n+2,n+1}^{v,c} - (2n+1)\rho_{n+1,n}^{v,c}] \\
&\quad - \frac{\gamma}{2}\rho_{n+1,n}^{v,c} - \frac{P}{2}\rho_{n+1,n}^{v,c}
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\frac{d\rho_{n,n+1}^{c,v}}{dt} &= \langle c, n | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v] + \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |v, n+1\rangle \\
&= -ig \left[\sqrt{n+1}\rho_{n+1,n+1}^{v,v} - \sqrt{n+1}\rho_{n,n}^{c,c} \right] \\
&\quad + \frac{\kappa}{2}[2\sqrt{(n+1)(n+2)}\rho_{n+1,n+2}^{c,v} - (2n+1)\rho_{n,n+1}^{c,v}] \\
&\quad - \frac{\gamma}{2}\rho_{n,n+1}^{c,v} - \frac{P}{2}\rho_{n,n+1}^{c,v}
\end{aligned} \tag{2.4}$$

We see from equation (2.3) and (2.4) that the elements on the form $\rho_{n+1,n}^{v,c}$ and $\rho_{n,n+1}^{c,v}$ only depends on elements of the same form as themselves or on the diagonal elements. There is therefore a subsystem of elements in the density matrix, which time-development are determined only by elements in that subgroup. These elements are of the form $\rho_{n,n}^{c,c}, \rho_{n,n}^{v,v}, \rho_{n+1,n}^{v,c}$ and $\rho_{n,n+1}^{c,v}$, where $n = 0, 1, 2, \dots$

For now we will look at the diagonal system of equations, which in the equilibrium-case can be simplified. If we add the equations (2.1) and (2.2) we get

$$\kappa(n+1)P_{n+1} = \kappa n P_n - 2g\sqrt{n+1}I_{n+1} + 2g\sqrt{n}I_n \tag{2.5}$$

where $P_n = \rho_{n,n}^{c,c} + \rho_{n,n}^{v,v}$ and $I_n = \Im(\rho_{n,n-1}^{v,c}) = \frac{1}{2i} (\rho_{n,n-1}^{v,c} - \rho_{n-1,n}^{c,v})$. We then have

$$\kappa(n+1)P_{n+1} + 2g\sqrt{n+1} = \kappa n P_n + 2g\sqrt{n} I_n \quad (2.6)$$

This has the form of a recursion formula, and since the left side of equation (2.6) is zero for $n=0$, we have:

$$\kappa n P_n + 2g\sqrt{n} I_n = 0 \quad (2.7)$$

$$I_n = \frac{-\kappa\sqrt{n}}{2g} P_n \quad (2.8)$$

If we subtract equation (2.4) from equation (2.3) and divide by $2i$ we get:

$$\begin{aligned} \kappa\sqrt{n+1}\sqrt{n+2}I_{n+2} - \left(\frac{\kappa}{2}(2n+1) + \frac{\gamma}{2} + \frac{P}{2}\right) I_{n+1} \\ + g\sqrt{n+1}p_n^c - g\sqrt{n+1}p_{n+1}^v = 0 \end{aligned} \quad (2.9)$$

We can change n to $n-1$ in equation (2.9):

$$\begin{aligned} \kappa\sqrt{n}\sqrt{n+1}I_{n+1} - \left(\frac{\kappa}{2}(2n-1) + \frac{\gamma}{2} + \frac{P}{2}\right) I_n \\ + g\sqrt{n}p_{n-1}^c - g\sqrt{n}p_n^v = 0 \end{aligned} \quad (2.10)$$

We can insert (2.8) into (2.10) and get:

$$-\frac{\kappa^2(n+1)}{2g}p_{n+1} + \left(\frac{\kappa}{2}(2n-1) + \frac{\gamma}{2} + \frac{P}{2}\right) \frac{\kappa}{2g}p_n - gp_{n-1}^c + gp_n^v = 0 \quad (2.11)$$

Using that $I_n = \Im(\rho_{n,n-1}^{v,c}) = \frac{1}{2i} (\rho_{n,n-1}^{v,c} - \rho_{n-1,n}^{c,v})$, we can rewrite equation (2.1) and (2.2):

$$\kappa(n+1)p_{n+1}^c = (\kappa n + \gamma) - Pp_n^v - 2g\sqrt{n+1}I_{n+1} \quad (2.12)$$

$$\kappa(n+1)p_{n+1}^v = (\kappa n + P)p_n^v - \gamma p_n^c + 2g\sqrt{n}I_n \quad (2.13)$$

Using equation (2.8) we can rewrite equation (2.13):

$$\kappa(n+1)p_{n+1}^v = Pp_n^v - (\kappa n + \gamma)p_n^c \quad (2.14)$$

We can in the same way rewrite equation (2.12)

$$\kappa(n+1)p_{n+1}^c = (\kappa n + \gamma)p_n^c - Pp_n^v + \kappa(n+1)(p_{n+1}^v + p_{n+1}^c) \quad (2.15)$$

Using equation (2.11) to replace $p_{n+1}^v + p_{n+1}^c$ in (2.15), we get:

$$\begin{aligned} \kappa(n+1)p_{n+1}^c = & \left(\kappa\left(n - \frac{1}{2}\right) + \frac{\gamma}{2} - \frac{P}{2} \right) + \frac{2g^2}{\kappa} p_n^v \\ & + \left(\kappa\left(2n - \frac{1}{2}\right) + \frac{3}{2}\gamma + \frac{P}{2} \right) p_n^c - \frac{2g^2}{\kappa} p_{n-1}^c \end{aligned} \quad (2.16)$$

Equations (2.14) and (2.16) are the two equations we will use to solve the equilibrium numerically.

2.1.1 A simplified set of equation

Before we solve equation (2.14) and (2.16) numerically we will study them a bit closer, by summing up over the photon number. We make the assumption that the probability p_n^c and p_n^v can be separated into $p_n^c = p_c n$ and $p_n^v = p_v n$, where p_c is the probability of the atom being in its excited state, while p_v the probability that the atom is in its ground state. This assumption means that there is no correlation between the atom and photon system. This will give us two new equations. The new equations will resemble the ones of a classical laser, and gets us an approximation of an analytically result for the mean photon number. This is parallel to Gartner's approach [5], in that both approaches start from the Lindblad equation (1.73), and through different ways end up with the same expression based on the same assumption. The assumption that the correlations between the atom and the cavity is negligible is expected to be valid for a high photon number, as mentioned in [11].

Summing up (2.14) gives

$$\sum_{n=0} \kappa(n+1)p_{n+1}^v = \sum_{n=0} P p_n^v - \sum_{n=0} \kappa n p_n^c - \gamma \sum_{n=0} p_n^c \quad (2.17)$$

Using that $p_n^v + p_n^c = p_n$ and introducing $p_c = \sum p_n^c$ we find, without any approximations in this case, that:

$$P p_v = \kappa \langle n \rangle + \gamma p_c \quad (2.18)$$

which we see is an expression for the conservation of energy. The energy is fed into the system by pushing the atom from its ground level to its excited level, which happens at a rate P . Energy can escape the system in two ways, it can escape by the excited atom, which it has the probability p_c to be in,

emitting a photon, but not into the field mode. This happens at a rate γ . The system can also lose energy to the environment by a photon escaping from the cavity. This happens at a rate κ per photon, of which we expect to have $\langle n \rangle$ of.

Equation (2.16) only goes from $n = 1$, but by multiplying every term by n the equation goes from $n = 0$. By then summing up we get:

$$-\kappa \langle n^2 \rangle = -\frac{1}{2}(\kappa + 3\gamma + P)p_c \langle n \rangle - \frac{1}{2}(-\kappa + \gamma - P)p_v n - \frac{2}{\kappa}(p_v n - p_c n - p_c) = 0 \quad (2.19)$$

From multiplying equation (2.14) by n and then summing up, we can get an expression for $\langle n^2 \rangle$:

$$\langle n^2 \rangle = p_v P n - \gamma p_c n + \kappa p_v n \quad (2.20)$$

Inserting this into equation (2.19) we get:

$$\Gamma p_c (\langle n \rangle + 1) - \Gamma p_v \langle n \rangle = \kappa \langle n \rangle \quad (2.21)$$

where we have introduced, like in reference [5] $\Gamma = \frac{4}{\kappa + \gamma + P}$. Combining these two equation we find the following:

$$2(\kappa \langle n \rangle)^2 + [\kappa + \gamma - P + \kappa(\gamma + P)\Gamma] (\kappa \langle n \rangle) - \kappa P = 0 \quad (2.22)$$

This is a second order equation which can be solve without great difficulty, which gives us an expression for the mean photon number:

$$\kappa \langle n \rangle = \frac{-\kappa - \gamma + P - \kappa(\gamma + P)\Gamma + \sqrt{\kappa^2(\gamma - P)^2\Gamma^2 + 8\kappa P}}{4} \quad (2.23)$$

In equation (2.21) we have introduced the parameter Γ . Comparing equation (2.21), which is an approximation of the one-atom two-level laser, with equation (1.79), which is for a standard laser, we notice something interesting. Both equations are of the same form. Equation (1.79) gives the energy balance for the cavity, where the photons are emitted from the atoms in the upper state, and can escape the cavity. Our approximation closely resembles this. We have the similar term by which photons are emitted from the atom in the upper level. Since we only have one photon we have the probability of the atom being in the upper level instead of the number of atoms in the upper level. We also, for our one-atom two-level system, have the term for photon escaping the cavity. In addition we have a term which is the opposite

of the atom emitting a photon, namely that it absorbs one instead. This is due to the atom and photon part of the system being strongly coupled, while in the standard laser there is the underlying assumption that the photons are not reabsorbed by the atoms. We have that the rate absorption/emitting of photons from the atom is, under the approximation we have made, is decided by the parameter Γ . This parameter has been described as an effective coupling constant for the atom-cavity subsystems [2] [1].

2.2 Numerical solutions

In this section we will solve the equations given by equation (2.14) and (2.16) numerically. This will allow us to start studying the parameter space of the problem and to check the validity of the simplified set of equations.

2.2.1 The Algorithm for solving the steady state equations

Now we have an infinite number of differential equations with an infinite number of unknowns. Luckily, we are in a position to truncate the number of these elements, and by extension the number of equations. It is reasonable to assume that the probability of there being N photons in the cavity, where N is a large enough number, is negligible and can be set to zero, along with the probabilities for there being more than N photons in the cavity. What N should be set to depends of course on the parameters. In practice we will set N to be a number, solve the equations, increase N , solve the equations again, and continue until the solution converges.

So now that we have a more reasonable number of equations we can begin to solve them. We set the elements p_N^c and p_N^v to zero. From equation (2.14) for $n = N - 1$ the equation has only two unknowns. Considering that we also have the normalization constraint we can set one of the unknown elements to any value, for instance $p_{N-1}^c = 1$, and then use this to find the other element, p_{N-1}^v . Afterwards the system must be normalized, so that $\sum p_n^v + \sum p_n^c = 1$, and this will determine p_{N-1}^c . If we now look at (2.16) for $n = N - 1$ we know every other element other than p_{N-2}^c , which can then be found. We can now return to equation (2.14) for $n = N - 2$, where we now know every element apart from p_{N-2}^v , which we can now find. Like so we can continue until all elements are found.

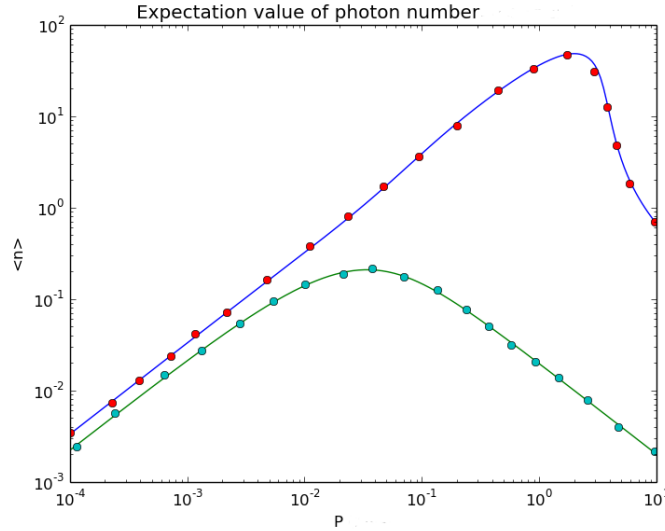


Figure 2.1: Expectation value for the photon number operator. Comparison between own data and from [5]. The Points are from [5]. For the upper line $\kappa = 0.1, \gamma = 0.2$, while for the lower line $\kappa = 0.2, \gamma = 0.1$

Now that we can find solutions we want to test this method against known results. [5] have a graph of the photon expectation value, for a given set of the parameters κ and γ , while P is being varied. We extract a series of data point from [5] using the tools on the website [13]. The comparison with our results and with [5] is shown in figure 2.1. We see that there is a clear agreement between our results and those from [5]. The graph for the upper level population taken from [5] also show a clear agreement, as shown in figure 2.2.

2.2.2 The quenching effect

One of the distinct features of this one-atomic two-level laser is the phenomenon that the photon number increase only up to a certain point, whereon it decrease back to zero. This effect is named the quenching effect [11]. Comparing this to the standard laser in figure 1.3, where the the photon number always increase, we see that the one-atomic two-level laser has a very distinct behavior.

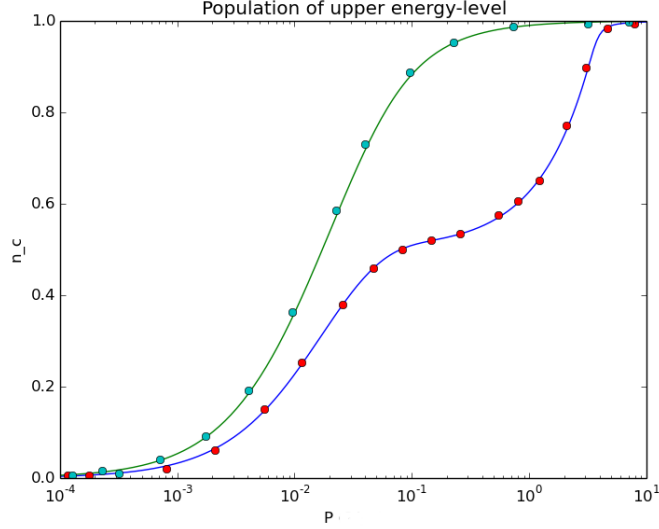


Figure 2.2: Population of the upper atomic level. Comparison between own data and from [5]. The Points are from [5]. For the lower line $\kappa = 0.1, \gamma = 0.2$, while for the upper line $\kappa = 0.2, \gamma = 0.1$

The energy balance

We will now consider the energy balance of the system, which we found an expression for last section:

$$Pp_v = \kappa \langle n \rangle + \gamma p_c \quad (2.24)$$

As we now can calculate all the quantities involved, $\langle n \rangle$, p_v and p_c , numerically, we can calculate the energy flow in and out of the system for any parameter. Figure 2.3 shows an example of the energy balance in practice. We have calculated the energy output and input through their respective channels for different values of the pump rate P . The sum of the two lower lines, representing loss of energy from the system, equal the upper line, representing the energy flowing into the system, which of course is due to energy conservation. Here we also see the root of the quenching effect. We see that the photon output, $\kappa \langle n \rangle$ to a large degree follow the energy input. However, the energy input also eventually begin to decrease. Since the pump rate P is increasing the decreasing in energy input must then be due to the other factor, p_v , the occupation number for the ground state. Referring back to figure

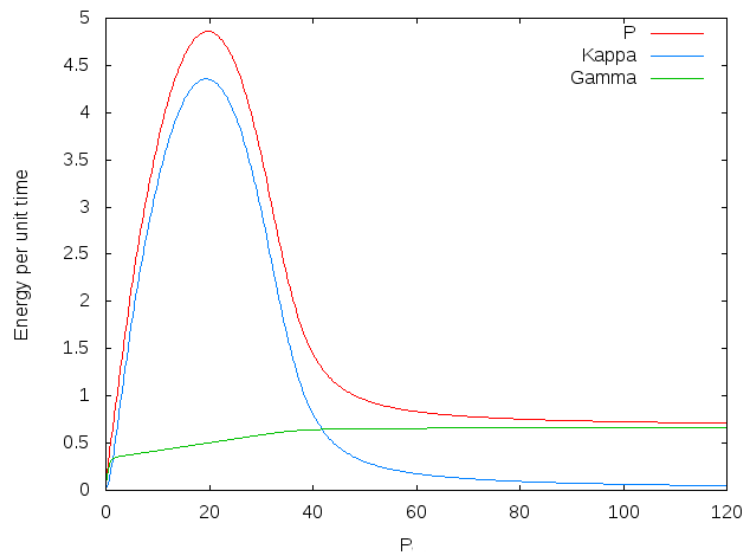


Figure 2.3: Energy input and output through the three different channels. The energy pumped into the system is proportional to P , the loss of energy from the atom is proportional to γ while the energy loss from the cavity is proportional to κ . In this case we have set $\kappa = 0.1$, $\gamma = 0.67$

2.2 we see that eventually the atomic system will be forced into its upper state, as the increased pump rate overcomes the loss through the atom-cavity coupling and the loss through other modes.

Some comments can also be in place for small values of P . From the equation giving us the energy balance, equation (2.24), we can find an expression for the photon number $\langle n \rangle$:

$$\langle n \rangle = \frac{Pp_v - \gamma p_c}{\kappa} \quad (2.25)$$

Since p_v can never be higher than one or p_c lower than zero, the photon number is bound by:

$$\langle n \rangle = \frac{P}{\kappa} \quad (2.26)$$

The photon number $\langle n \rangle$ can not reach one before the value of P reaches the value of κ . As we see in figure 2.2 the value of p_c increase sharply for small values of P (and p_v correspondingly drops sharply), which means that the value of P must be more than equal to the value of κ before $\langle n \rangle$ reaches a value greater than one.

The photon distribution

In the previous section we saw how the system have a quenching effect for a sufficient high rates of the pump rate P . But the photon expectation value must be seen in comparison with the photon probability distribution. We want to investigate is the photon system to in a coherent state, which is the quantum mechanically equivalent of monochromatic light. From the last chapter we know that the photon probability distribution of a coherent state must be a Poisson distribution, and that whether a probability distribution is a Poisson distribution can be measured by the function $g_2(0)$. For a coherent state this function must be equal to one. To investigate whether the the system is in a coherent state it is not enough to check if $g_2(0) = 1$, since we also must consider the phase of the system which must have a coherent distribution. However $g_2(0) = 1$ is one of the requirements for that the system is in a coherent state. The question of phase coherence will be talked more about later.

We will begin the study of the photon distribution by plotting the function $g_2(0)$ against the photon expectation value $\langle n \rangle$. We keep the values of κ and γ fixed while we linearly increase the pump rate P .

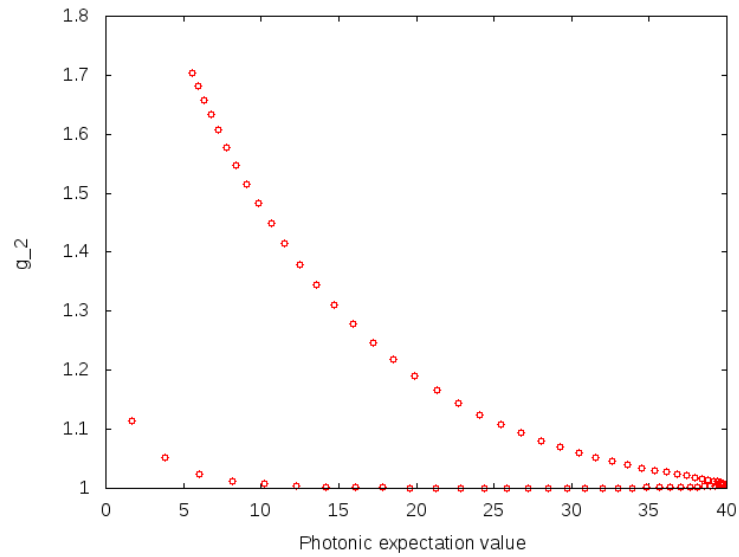


Figure 2.4: The value of the function $g_2(0)$ plotted against the photon expectation value. The value of κ and γ is fixed at 0.110 and 0.169 respectively. P increases linearly, from 0.5 to 40.0. Going anti-clock wise increases P

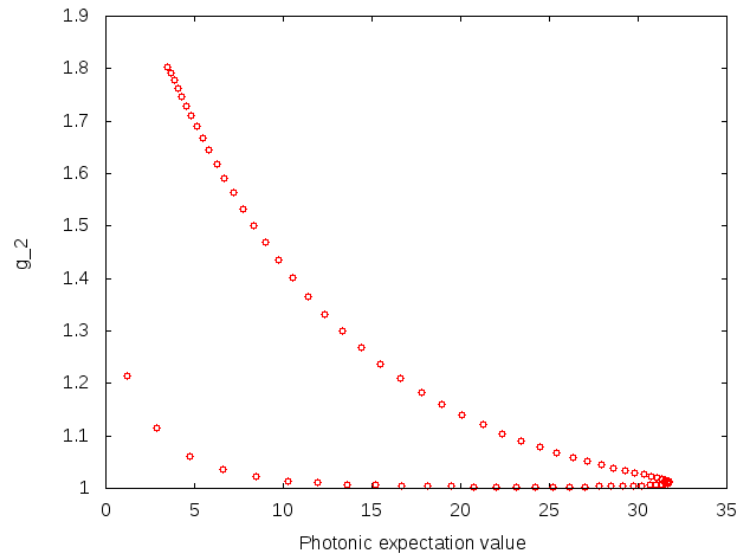


Figure 2.5: The value of the function $g_2(0)$ plotted against the photon expectation value. The value of κ and γ is fixed at 0.121 and 0.339 respectively. P increases linearly, from 0.5 to 40.0. Going anti-clock wise increases P

Here we see the behavior of the photon expectation value and the photon probability distribution in the same plot. The behavior in these plots seems to be characteristic of the behaviour in general, for values of κ and γ which allows for a high enough photon number. The system starts with a low photon number and with $g_2(0)$ larger than one, but when the photon population starts to build up the photon statistics becomes Poissonian, and stays so until the photon number reaches it maximum. From thereon the photon distribution drifts further and further away from Poissonian as the photon number recede.

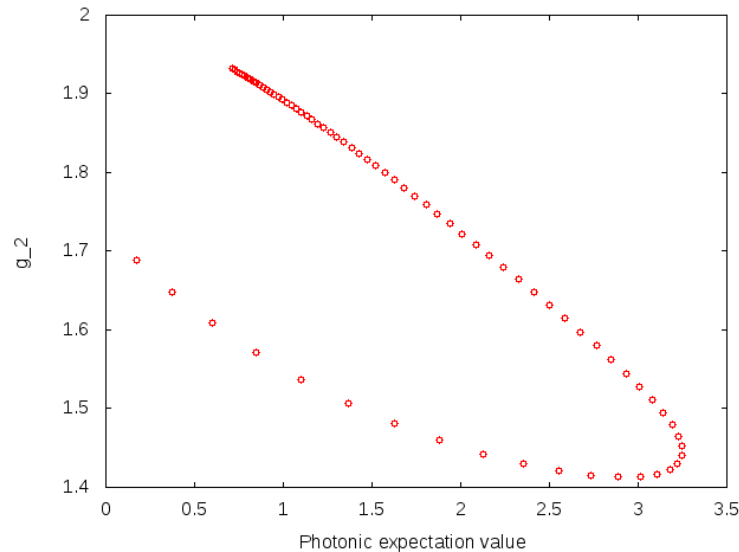


Figure 2.6: The value of the function $g_2(0)$ plotted against the photon expectation value. The value of κ and γ is fixed at 0.203 and 2.542 respectively. P increases linearly, from 0.5 to 40.0. Going anti-clock wise increases P

For situations where the value of the parameters κ and γ which strongly restrict the photon number, the overall form of the $g_2(0)-\langle n \rangle$ plot is familiar, but tells a different story. As we see in figure 2.6 the parameters κ and γ are so that the photon expectation value is at no point higher than 3.5. Similar to the high-photon case earlier, the lowest value of $g_2(0)$ is reached when the photon number still increase, but $g_2(0)$ never reaches a value close to one.

We will now show what the photon distribution looks like for the situation presented in figure 2.4. We plot the photon distribution for three different

values of P . $\kappa = 0.110$ and $\gamma = 0.169$ as in the situation in figure 2.4. We also plot the Poisson distribution for the same photon expectation value as the actual distribution. This is the form the actual distribution would have had if the system had been in a coherent state, and the value of $g_2(0)$ would have been equal to one. In figure 2.7, which is for a low value of P before $g_2(0)$

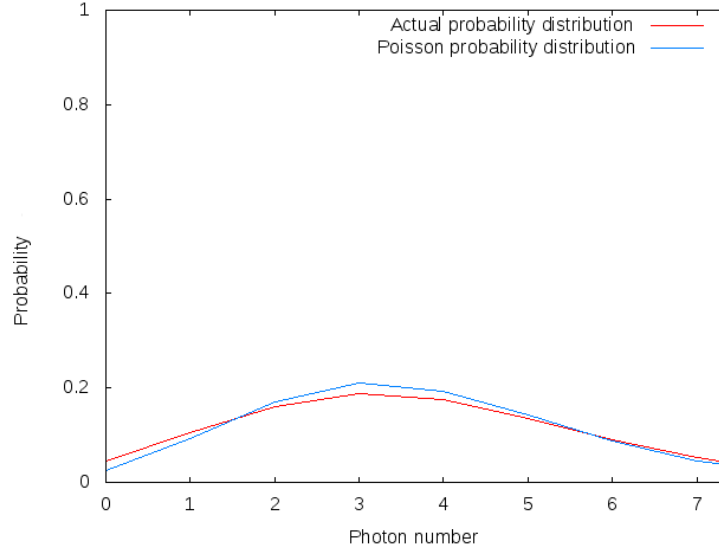


Figure 2.7: Photon distribution for $P=1.0g$, while $\kappa = 0.110$ and $\gamma = 0.169$. The photon expectation value is 3.68, while $g_2(0) = 1.05$

have reached a value close to one, we have that the actual photon probability distribution is broader than it would have been if we had had a coherent state. In figure 2.8 we have that the actual probability distribution overlaps with the coherent case. In this case the function $g_2(0)$ have reached a value of one and the photon number have reached a quite high value. In figure 2.9 we have passed the value of P which gives the highest photon number (for our fixed κ and γ). The photon number is still high, but the value of $g_2(0)$ have increased away from one. We see that the actual photon distribution again is broader than would have been the case if we have had a coherent state.

Discussion

The morale from the last section seems to be that the function $g_2(0)$ seems to be close to one when these two conditions are fulfilled:

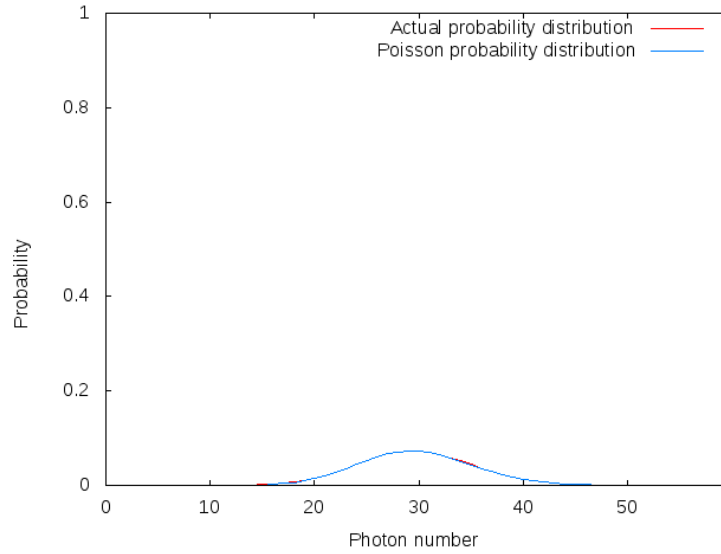


Figure 2.8: Photon distribution for $P=9.0$, while $\kappa = 0.110$ and $\gamma = 0.169$. The photon expectation value is 29.76, while $g_2(0) = 1.00$.

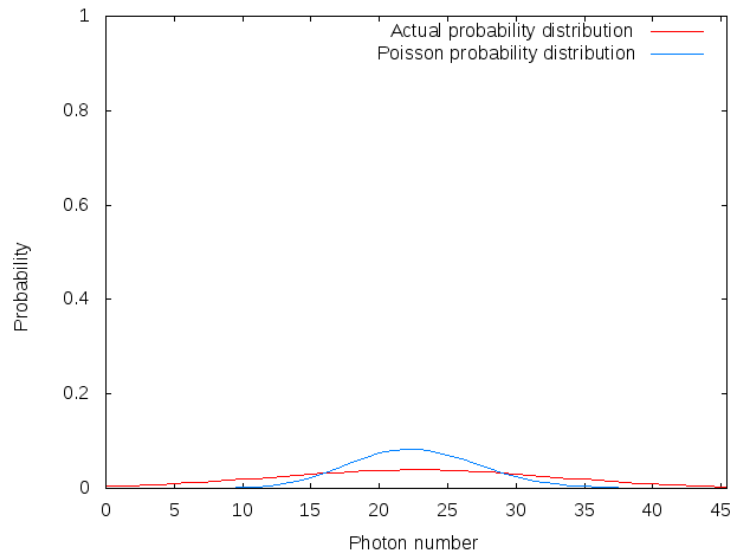


Figure 2.9: Photon distribution for $P=30.0$, while $\kappa = 0.110$ and $\gamma = 0.169$. The photon expectation value is 22.71, while $g_2(0) = 1.15$.

- $\frac{\partial \langle n \rangle}{\partial P} > 0$, also that the photon number can still be increased by increasing the pump rate.
- That the photon number is high enough. The term "high enough" may be somewhere difficult to quantify, but a photon expectation number higher than ca. 20 seems to be enough.

From figure 2.4 we can divide the behavior of the system into three different situations for different values of P . Each situation fulfills one or two of the conditions.

The first situation is the one in which the value of P is still low, and where we have a low photon number. This is the situation pictured in figure 2.7. The first condition is fulfilled, as the photon number increases while we increase P , but the photon number is too low. An explanation for the condition of a "high enough" photon number can perhaps be because we need a number of photons to build up a coherent, Poisson field.

In the second situation, portrayed in figure 2.8, the photon number $\langle n \rangle$ has been built up to a quite high population. The photon number would still have increased by increasing P , so both conditions are fulfilled. We see that the value of $g_2(0)$ is close to one.

In the third and last situation the photon number is still high, but the system is now being quenched, and the photon number is now decreasing with increased P . Here the value of the function $g_2(0)$ is increasing away from its earlier value of one.

2.2.3 The parameter space

In figure 2.10 we fix the parameter κ , while we gradually increase γ from zero. Since γ is a source of loss of energy from the system this quite naturally decreases the photon number. In figure 2.11 the situation is reversed, γ is fixed while κ is being increased. κ is the loss of energy from the cavity, so increasing this leads to a lower number of photons in the cavity. Even though we increase the rate γ in the first case more than we increase κ in the second, the photon loss is less. One way of viewing this is that the loss of energy from the atom is bound by $\gamma p_c < \gamma$, while the loss of energy from the cavity is not bound by any law of probability, but only by the photon number.

Figure 2.12 shows the photon expectation value $\langle n \rangle$ for a slice of parameter space, where the pump-rate P is fixed. Like we could expect it is the

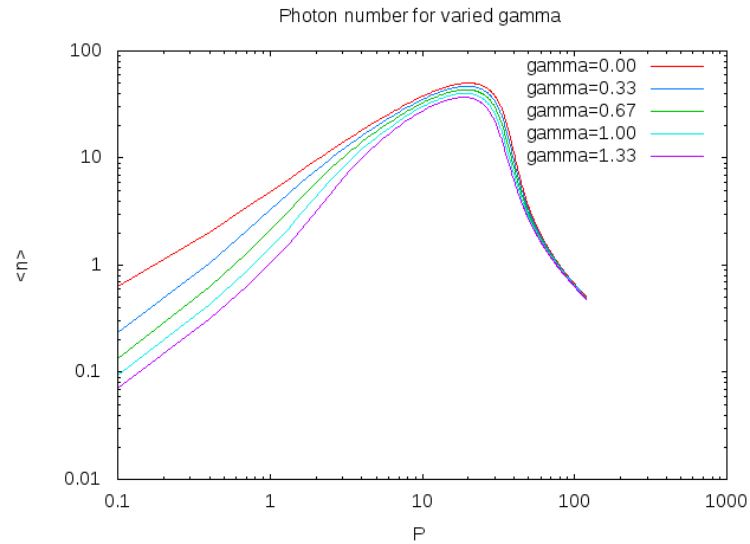


Figure 2.10: The photon expectation value is plotted as a function of P , where $\kappa = 0.1g$ and γ is varied. The photon number decreases by increased γ .

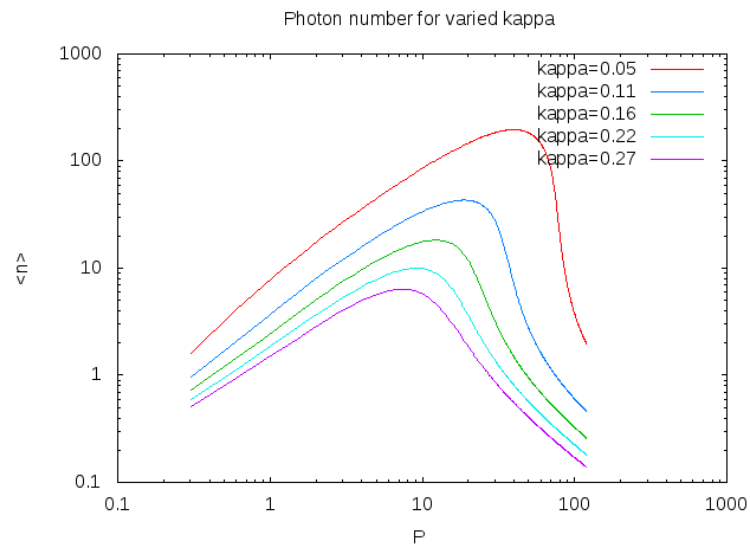


Figure 2.11: The photon expectation value is plotted as a function of P , $\gamma = 0.2g$ and κ is varied. The photon number decreases by increased κ .

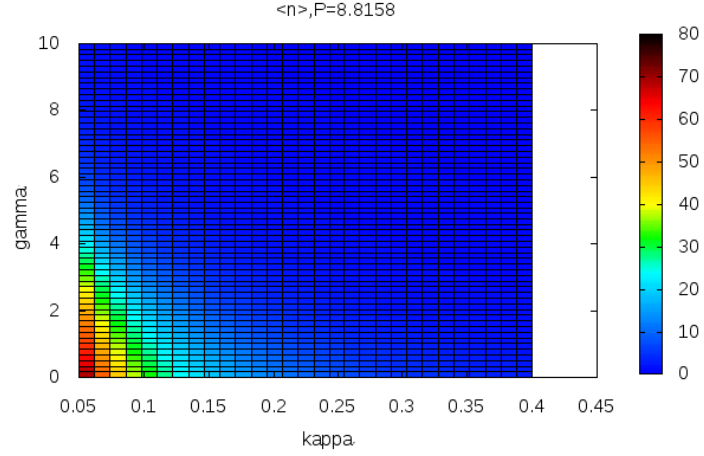


Figure 2.12: Expectation value for the number of photons for a slice of parameter space

regions where κ and γ is low that the number of photons build up, P being fixed. If we increase P , eventually N would have decrease to zero, as seen in figure 2.1.

For the simplified system we saw that the parameter Γ takes the place of the rate of energy going from the atom to the cavity (and visa versa). To have $\langle n \rangle > 1$ we must have, as is mentioned in [2], that $\Gamma > \kappa$. This give us the following equation:

$$P < \frac{4}{\kappa} - \kappa - \gamma \quad (2.27)$$

Remembering the restriction on the photon number found from the balance of energy equation, namely that:

$$\langle n \rangle < \frac{P}{\kappa} \quad (2.28)$$

We can combine these two equation and find that

$$\langle n \rangle < \frac{4}{\kappa^2} - \frac{\gamma}{\kappa} - 1 \quad (2.29)$$

Equation was found by restricting $\langle n \rangle > 1$. By this we have:

$$\gamma < \frac{4}{\kappa} - 2\kappa \quad (2.30)$$

We see that to achieve a high photon number the values of κ and γ must be restricted, that only if they are not too high will the photon number $\langle n \rangle$ be larger than one. This is what we saw in figure (2.12).

2.2.4 Checking the simplified set of equation

Earlier we found that, using the approximation that the field and the atom is not correlated, a closed form expression for the mean photon number can be found in equation (2.23). We are now in a position to check this. In

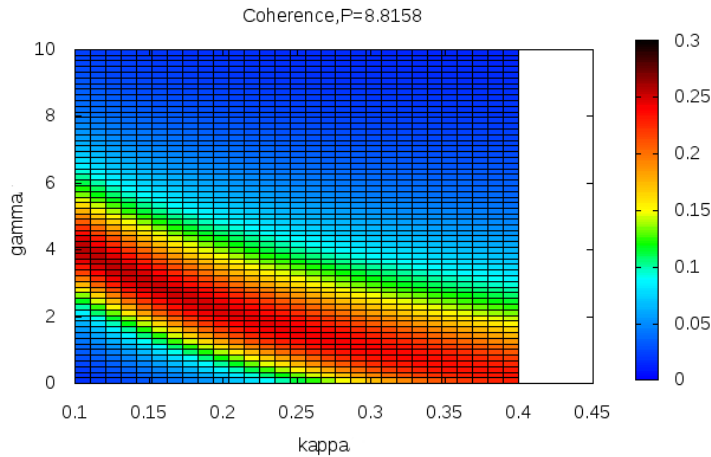


Figure 2.13: The difference in percent between the actual photon number and the photon number predicted by the simplified model

figure 2.13 we see how well the simplified model compares to the real data. Comparing this to figure 2.14 we see that the simplified model works well for a high number of photons, as expected due to the assumptions made in finding the simplified expression. The simplified expression also gives a good fit when we full solution give zero photons in the cavity, which is of less interest.

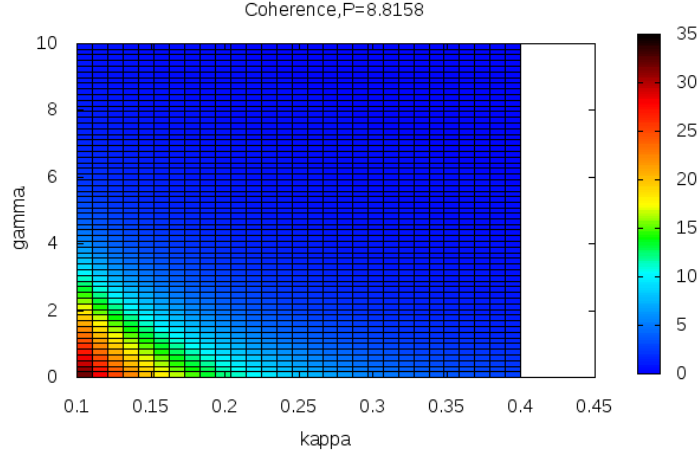


Figure 2.14: The photon number for the same area as in figure 2.13

2.2.5 Phase space

We now want to see how the photon part of the system looks like in phase space. What we want is the probability of the system being in the coherent state Z , this probability is given by $\langle Z | \rho | Z \rangle$. This is a pseudo-probability distribution, which, because of the completeness relation for coherent states, equation (1.45), is going to be integrated up to π .

The reduced density matrix for the photon part of the system, ρ^{photon} , is given by $\rho^{photon} = Tr_{atom}(\rho)$, where we take the trace over the atomic states as in equation (1.54). ρ^{photon} is then, represented in energy eigenstates, $\rho^{photon} = \sum \rho_{n,n'}^{photon} |n\rangle \langle n'|$, where $\rho_{n,n'}^{photon} = \sum_m \rho_{n,n'}^{m,m}$. For the phase-space representation we then have:

$$\langle Z | \rho | Z \rangle = \sum \rho_{n,n'} \langle Z | n \rangle \langle n' | Z \rangle \quad (2.31)$$

Since we know that

$$\langle n | Z \rangle = e^{-\frac{1}{2}|Z|^2} \frac{Z^n}{\sqrt{n!}}$$

we find that:

$$\langle Z | \rho | Z \rangle = \sum \rho_{n,n'} e^{-|Z|^2} \frac{Z^{n*} Z^{n'}}{\sqrt{n!} \sqrt{n'!}} \quad (2.32)$$

In the next section, we will show that the density matrix consist of an infinity of sets of differential equation, like the ones given by equation (2.2) to (2.4). However, all of these set are going to be completely separated. This means that if we start the system in its ground state $|v, 0\rangle$, the only non-zero elements are those of the diagonal system of elements. If we only have non-zero element in the diagonal system we can set $n' = n$ in equation (2.32)¹ and get:

$$\langle Z | \rho | Z \rangle = \sum \rho_{n,n} e^{-|Z|^2} \frac{|Z|^{2n}}{n!} \quad (2.33)$$

We see that if there is only diagonal elements in the photon density matrix the phase-space representation has no angular dependency, and depends only on the magnitude of Z , which square is the expectation value of the photon number. In other words, if we only have diagonal elements as non-zero in the photon density matrix, we have no knowledge of the phase of the system, there being equal probability in any direction. An example of this is given in figure 2.15. This form, centered around origo and rotational invariant, is according to [11] characteristic of a laser. The distance from origo depends on the number of photons, while the rotational invariance is due to phase drift. We will come back to this in the next chapter.

2.3 The non-diagonal elements

We saw earlier that we have a set of elements in the density matrix, containing among other the diagonal elements, whose time development only depends on other elements in the same set. In this section we will see that we in fact have an infinit number of sets, each of which behaves independently from the other sets. We will then show, analytically, that the steady-state solution for all non-diagonal sets is zero.

¹ ρ^{photon} consists of just diagonal elements, the diagonal elements in the diagonal set of equations are not in neither the photon or atom indicies

$$\begin{aligned}
\frac{d\rho_{n,n'}^{v,v}}{dt} &= \langle v, n | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v], \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |v, n'\rangle \\
&= -ig[\sqrt{n}\rho_{n-1,n'}^{c,v} - \sqrt{n'}\rho_{n,n'-1}^{v,c}] \\
&\quad + \kappa[\sqrt{n+1}\sqrt{n'+1}\rho_{n+1,n'+1}^{v,v} - \frac{n+n'}{2}\rho_{n,n}^{v,v}] \\
&\quad + \gamma\rho_{n,n'}^{c,c} - P\rho_{n,n'}^{v,v}
\end{aligned} \tag{2.37}$$

From equation (2.36) and (2.37) we see that the time development of the elements $\rho_{n,n'}^{c,c}$ and $\rho_{n,n'}^{v,v}$ depends only on elements of the form $\rho_{n+1,n'}^{v,c}$ and $\rho_{n,n'+1}^{c,v}$. The differential equation for these elements are

$$\begin{aligned}
\frac{d\rho_{n+1,n'}^{v,c}}{dt} &= \langle v, n+1 | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v], \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |c, n'\rangle \\
&= -ig\left[\sqrt{n+1}\rho_{n,n'}^{c,c} - \sqrt{n'+1}\rho_{n+1,n'+1}^{v,v}\right] \\
&\quad + \frac{\kappa}{2}[2\sqrt{(n+2)(n'+1)}\rho_{n+2,n'+1}^{v,c} - (n+n'+1)\rho_{n+1,n'}^{v,c}] \\
&\quad - \frac{\gamma}{2}\rho_{n+1,n'}^{v,c} - \frac{P}{2}\rho_{n+1,n'}^{v,c}
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
\frac{d\rho_{n,n'+1}^{c,v}}{dt} &= \langle c, n | \left(-i[g[b^\dagger v^\dagger c + bc^\dagger v], \rho] \right. \\
&\quad + \frac{\kappa}{2}[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\
&\quad + \frac{\gamma}{2}[2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\
&\quad \left. + \frac{P}{2}[2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \right) |v, n' + 1\rangle \\
&= -ig[\sqrt{n+1}\rho_{n+1,n'+1}^{v,v} - \sqrt{n'+1}\rho_{n,n'}^{c,c}] \\
&\quad + \frac{\kappa}{2}[2\sqrt{(n+1)(n'+2)}\rho_{n+1,n'+2}^{c,v} - (n+n'+1)\rho_{n,n'+1}^{c,v}] \\
&\quad - \frac{\gamma}{2}\rho_{n,n'+1}^{c,v} - \frac{P}{2}\rho_{n,n'+1}^{c,v}
\end{aligned} \tag{2.39}$$

We see that the time development of elements of the form $\rho_{n+1,n'}^{v,c}$ and $\rho_{n,n'+1}^{c,v}$ only depends on themselves and elements on the form $\rho_{n,n'}^{c,c}$ and $\rho_{n,n'}^{v,v}$. This system is a closed system of differential equations. In the following density matrix this set of elements, for $n' = n + 1$, are the elements blocked in.

$$\left(\begin{array}{cccccccc}
\rho_{0,0}^{v,v} & \rho_{0,0}^{v,c} & \rho_{0,1}^{v,v} & \rho_{0,1}^{v,c} & \rho_{0,2}^{v,v} & \rho_{0,2}^{v,c} & \rho_{0,3}^{v,v} & \rho_{0,3}^{v,c} & \cdots \\
\rho_{0,0}^{c,v} & \rho_{0,0}^{c,c} & \rho_{0,1}^{c,v} & \rho_{0,1}^{c,c} & \rho_{0,2}^{c,v} & \rho_{0,2}^{c,c} & \rho_{0,3}^{c,v} & \rho_{0,3}^{c,c} & \cdots \\
\rho_{1,0}^{v,v} & \rho_{1,0}^{v,c} & \rho_{1,1}^{v,v} & \rho_{1,1}^{v,c} & \rho_{1,2}^{v,v} & \rho_{1,2}^{v,c} & \rho_{1,3}^{v,v} & \rho_{1,3}^{v,c} & \cdots \\
\rho_{1,0}^{c,v} & \rho_{1,0}^{c,c} & \rho_{1,1}^{c,v} & \rho_{1,1}^{c,c} & \rho_{1,2}^{c,v} & \rho_{1,2}^{c,c} & \rho_{1,3}^{c,v} & \rho_{1,3}^{c,c} & \cdots \\
\rho_{2,0}^{v,v} & \rho_{2,0}^{v,c} & \rho_{2,1}^{v,v} & \rho_{2,1}^{v,c} & \rho_{2,2}^{v,v} & \rho_{2,2}^{v,c} & \rho_{2,3}^{v,v} & \rho_{2,3}^{v,c} & \cdots \\
\rho_{2,0}^{c,v} & \rho_{2,0}^{c,c} & \rho_{2,1}^{c,v} & \rho_{2,1}^{c,c} & \rho_{2,2}^{c,v} & \rho_{2,2}^{c,c} & \rho_{2,3}^{c,v} & \rho_{2,3}^{c,c} & \cdots \\
\rho_{3,0}^{v,v} & \rho_{3,0}^{v,c} & \rho_{3,1}^{v,v} & \rho_{3,1}^{v,c} & \rho_{3,2}^{v,v} & \rho_{3,2}^{v,c} & \rho_{3,3}^{v,v} & \rho_{3,3}^{v,c} & \cdots \\
\rho_{3,0}^{c,v} & \rho_{3,0}^{c,c} & \rho_{3,1}^{c,v} & \rho_{3,1}^{c,c} & \rho_{3,2}^{c,v} & \rho_{3,2}^{c,c} & \rho_{3,3}^{c,v} & \rho_{3,3}^{c,c} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \right) \tag{2.40}$$

This system does not overlap with the one in equation (2.35). The elements of the set where $n' = n + 2$ create a new set which have no common elements with the two other sets, and so on. We have an infinite number of sets of differential equations, which can be solved independent of the others.

We note that all elements in the same block in equation (2.35) have the same 'energy', or excitation number, $E = m + m' + n + n'$. Here m is the atomic index on the left and m' is the atomic index on the right. They are 0 if the atomic index is v and 1 if the atomic index is c . The first block in the upper left corner of equation (2.35), consisting only of the element $\rho_{0,0}^{v,v}$,

diagonal set, i.e. we have

$$\mathbf{v}_\Delta = \begin{pmatrix} \rho_{0,\Delta-1}^{v,c} \\ \rho_{0,\Delta}^{v,v} \\ \rho_{0,\Delta}^{c,c} \\ \rho_{0,\Delta}^{v,c} \\ \rho_{1,\Delta}^{c,v} \\ \rho_{0,\Delta+1}^{c,v} \\ \rho_{1,\Delta+1}^{v,v} \\ \rho_{1,\Delta+1}^{c,c} \\ \rho_{1,\Delta+1}^{v,c} \\ \rho_{2,\Delta+1}^{c,v} \\ \rho_{1,\Delta+2}^{c,v} \\ \vdots \end{pmatrix} \quad (2.43)$$

L_Δ is a matrix whose elements are given by the coefficients from equations (2.36)-(2.39) for a given Δ . We will show that the only solution to equation (2.42) is the trivial one, that $\mathbf{v}_\Delta = \mathbf{0}$.

If we start at the beginning, where $n = 0$ and $n' = \Delta$, we have the first four rows of L:

$$\begin{pmatrix} ig\sqrt{\Delta} & (-\kappa\frac{\Delta}{2}-P) & \gamma & 0 & 0 & \kappa\sqrt{\Delta+1} & 0 & 0 & 0 \\ 0 & P & (-\kappa\frac{\Delta}{2}-\gamma) & -ig & ig\sqrt{\Delta+1} & 0 & \kappa\sqrt{\Delta+1} & 0 & 0 \\ 0 & 0 & -ig & -(\frac{\kappa}{2}(i+1)+\frac{\gamma}{2}+\frac{P}{2}) & 0 & ig\sqrt{\Delta+1} & 0 & \kappa\sqrt{2(\Delta+1)} & 0 \\ 0 & 0 & ig\sqrt{\Delta+1} & 0 & -(\frac{\kappa}{2}(i+1)+\frac{\gamma}{2}+\frac{P}{2}) & -ig & 0 & 0 & \kappa\sqrt{\Delta+2} \end{pmatrix} \quad (2.44)$$

Technically this is only a part of the first four rows, since the vector \mathbf{v}_Δ is of infinite length. However the set of equations (2.36)-(2.39) for $n = 0$ only concern the nine elements explicitly written in equations (2.43). This is more or less the general case. Equations (2.36)-(2.39), for any given n , only concern the elements

$$\begin{pmatrix} \rho_{n,\Delta-1}^{v,c} \\ \rho_{n-1,n+\Delta}^{c,v} \\ \rho_{n,n+\Delta}^{v,v} \\ \rho_{n,n+\Delta}^{c,c} \\ \rho_{n,n+\Delta}^{v,c} \\ \rho_{n+1,n+\Delta}^{c,v} \\ \rho_{n,n+\Delta+1}^{c,v} \\ \rho_{n+1,n+\Delta+1}^{v,v} \\ \rho_{n+1,n+\Delta+1}^{c,c} \\ \rho_{n+2,n+\Delta+1}^{v,c} \\ \rho_{n+1,n+\Delta+2}^{c,v} \end{pmatrix} \quad (2.45)$$

For $n = 0$ the element $\rho_{n-1,\Delta}^{c,v}$ does not exist, giving the start of the vector \mathbf{v}_Δ in equation (2.43).

We see that the matrix in equation (2.44) can be row reduced to a upper triangular matrix, by adding the third row times $\sqrt{\Delta+1}$ to the fourth row. This can always be done, assuming that no parameters are zero.

We need so show that we can reduce the entire matrix L_Δ to a upper triangular matrix. We will now write the matrix rows which constitutes the equations (2.36)-(2.39) for a given n . Only the columns which have non-zero elements, in other words those multiplied with the elements given in vector (2.45), will be written out. We will also write, for the same columns, the same equations (2.36)-(2.39) for $n + 1$. These four next rows are of the same form as for n , but moved four places to the right. We will also do this for the four next equations for $n + 2$. Since we only write those columns which have non-zero elements for the equations for n , those columns multiplied with the elements written above, we do not write out the entirety of the equations for $n + 1$ and $n + 2$, since they also involve other elements. However, those columns we do write will give ut the diagonal passing through the equations for n , $n + 1$ and $n + 2$. The part of the matrix L_Δ we consider is then:

$$\begin{pmatrix} g\sqrt{n'} - ig\sqrt{n} (-\kappa \frac{n+n'}{2} - P) & \gamma & 0 & 0 & B_{n,n'} & 0 & 0 & 0 \\ 0 & 0 & P & (-\kappa \frac{n+n'}{2} - \gamma) & -ig\sqrt{n+1} & ig\sqrt{n'+1} & 0 & B_{n,n'} & 0 & 0 \\ 0 & 0 & 0 & -ig\sqrt{n+1} & A_{n,n'} & 0 & ig\sqrt{n'+1} & 0 & B_{n+1,n'} & 0 \\ 0 & 0 & 0 & ig\sqrt{n'+1} & 0 & A_{n,n'} & -ig\sqrt{n+1} & 0 & 0 & B_{n,n'+1} \\ 0 & 0 & 0 & 0 & ig\sqrt{n'+1} & -ig\sqrt{n+1} & -\kappa \frac{(n+n'+2)}{2} - P & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P & -\kappa \frac{n+n'}{2} - P & -ig\sqrt{n+2} & ig\sqrt{n'+2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ig\sqrt{n+2} & A_{n+1,n'+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig\sqrt{n'+2} & 0 & A_{n+1,n'+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig\sqrt{n'+2} & -ig\sqrt{n+2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.46)$$

where we, for the sake of space, have introduced:

$$A_{n,n'} = -\left(\frac{\kappa}{2}(n + n' + 1) + \frac{\gamma}{2} + \frac{P}{2}\right) \quad (2.47)$$

and

$$B_{n,n'} = \kappa \sqrt{(n+1)(n'+1)} \quad (2.48)$$

Comparing equation (2.46) with equation (2.50) we see that the diagonal in (2.46) passes from element (1, 2) through to element (9, 10).

We see that this part of the matrix L_Δ can quite easily be transformed into an upper triangular matrix. We see that if we add the third row times $\frac{\sqrt{n'+1}}{\sqrt{n+1}}$ to the fourth row, we eliminate the element (4, 4), which lies under the diagonal. This also makes the element (4, 5) non-zero, which we want

seeing as this element is a diagonal element. This element is now equal to $A_{n,n'} \frac{\sqrt{n'+1}}{\sqrt{n+1}}$. We must now eliminate the element (5, 5) below the diagonal. This done by subtracting the fourth row multiplied by $\frac{ig\sqrt{n+1}}{A_{n,n'}}$. This, most importantly, does not eliminate element (5, 6), which becomes $-2ig\sqrt{n+1}$. The next rows which have elements left of the diagonal are the eight and the ninth row. However, this is just a repeat of what we have just done, barring increasing the value of n by one. The eight and ninth row is after all the same equations as row four and five. The fourth row is equation (2.39) for n while the fifth row is equation (2.37) for $n+1$. On the other hand the eight row is equation (2.39) for $n+1$ while the ninth row is equation (2.37) for $n+2$.

We see that we can reduce any general slice of the matrix L_Δ to an upper triangular matrix. We also see that the eliminations of variables being done one place in the matrix, like for the fourth and the fifth row in equations (2.46), does not influence our elimination of the elements below the diagonal in the eight and the ninth row. From this we draw the conclusion that the matrix L_Δ , through any number of rows, can be transformed into an upper triangular matrix. The matrix L_Δ is of infinite size, but we can still transform it into a upper triangular matrix through any number of rows. If the matrix L_Δ can be reduced to a upper triangular matrix, equation (2.42) has only one solution, namely that $\mathbf{v}_\Delta = \mathbf{0}$.

One aspect which should be mentioned is that the equations (2.36)-(2.39) also held true for the diagonal system, if $n' = n$ where $\Delta = 0$. Have we showed that, contrary to our earlier results and the laws of probability, the diagonal elements will also tend to zero? We have of course not showed this. One of our earlier assumption was that no parameter of the matrix L_Δ was zero, and this included Δ . When $\Delta = 0$, as for the diagonal system, things are quite different. The element $\rho_{n,\Delta-1}^{v,c}$ does not longer exist. This will lead to a different diagonal, and in addition we will have some different cancellation than earlier caused by $\Delta = 0$. We will now write the five times four sub-matrix in the upper left of the matrix L_0 . Those five rows are the only rows containing non-zero elements for the first four columns. The four columns are those multiplied by the elements

$$\begin{pmatrix} \rho_{0,0}^{v,v} \\ \rho_{0,0}^{c,c} \\ \rho_{1,0}^{v,c} \\ \rho_{0,1}^{c,v} \end{pmatrix} \quad (2.49)$$

The sub-matrix is then:

$$\begin{pmatrix} -P & \gamma & 0 & 0 \\ P & -\gamma & -ig & ig \\ 0 & -ig & A_{0,0} & 0 \\ 0 & ig & 0 & A_{0,0} \\ 0 & 0 & ig & -ig \end{pmatrix} \quad (2.50)$$

If we row-reduce this we get the matrix:

$$\begin{pmatrix} -P & \gamma & 0 & 0 \\ 0 & ig & 0 & (\frac{\kappa}{2} + \frac{\gamma}{2} + \frac{P}{2}) \\ 0 & 0 & -(\frac{\kappa}{2} + \frac{\gamma}{2} + \frac{P}{2}) & (\frac{\kappa}{2} + \frac{\gamma}{2} + \frac{P}{2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.51)$$

We see that we can not transform L_0 into an upper triangular matrix. There is an equivalence between being able to transform a matrix to an upper triangular matrix and that the only solution to the homogeneous matrix equation being $\mathbf{0}$ [9]. Irrespective of how we can transform the rest of the matrix L_0 we can not transform the full matrix into an upper triangular matrix. Therefore the steady state solutions of the diagonal set of elements are not zero, which we already knew. The diagonal case is thus very different from the non-diagonals. The procedure for showing that the steady state solutions for the elements of the non-diagonal sets are zero does not work for the diagonal system.

We will now return to the non-diagonal sets of elements. We have showed that the density matrix can be separated into an infinite number of sets of elements, where each set's time development depends only on that same set. That means, as mentioned, that if the system starts in a quantum state with only elements in the diagonal set, it will continue to be so for all time. An easy example of this is to start the system in its ground state $|v, 0\rangle$. The system will then decrease its energy afterwards by the energy pumped into the system, proportional to the pump rate P . However, all the elements in the non-diagonal sets will forever remain zero.

As we saw in the last section it is the non-diagonal elements of the photon density matrix ² which gives rise to the phase information of the system. If

²All of which are found as sums of elements in the sets of non-diagonal elements, per equation (1.54)

the only non-zero elements are the ones of the diagonal set of elements, we will have no phase distinctiveness, i.e. the probability distribution in phase space will be rotational invariant. Even if we do know the diagonal elements and from them the radial distribution, the angular distribution will be uniform. This is in contrast to a coherent state, in which the probability distribution is closely centered in both direction. A coherent state is after all the quantum mechanical analogue to a classical monochromatic wave, where both the energy and the phase is known. However, what we have shown in this section is that even if we start the system in for instance a coherent state, where the elements of the non-diagonal sets are non-zero, we will still lose this phase information. We don't know how long it takes, but eventually the angular distribution will become less and less sharp until it becomes completely uniform and all phase information is lost.

When we introduced the coherent states we saw that in a harmonic oscillator a coherent state remained a coherent state as time passed. What we have seen here is that outside forces, represented by P, κ, γ , disturb the system and eventually destroy valuable information about the phase of the system. This is avoided in the radial direction, where the parameter balances each other out.

Chapter 3

The time dependent solution of the two-level system

In the last chapter we found the steady-state solution of the two-level system. The diagonal elements, which gives us the photon profile, were found numerically to have a non-zero solution, if P is not set to zero of course. We also saw analytical that the non-diagonal elements tend to zero. Consequently the system, independent of initial values or of parameters, will lose all phase-information. Unanswered is how the system develops from any initial state to the steady state found. While the off-diagonal elements tends to zero, the time-scale of this process was not found in the previous chapter. A system that starts out with all non-diagonal elements set to zeros will remain so for all time. But if we imagine a system that starts out with non-zeros off-diagonal elements, like a coherent state, the system may retain its phase information for a long time. There are several ways to solve the time-dependent problem given by the Liouville-von Neumann equation. In this chapter we will solve the Liouville-von Neumann equation as a coupled set of ordinary differential equations. This will be achieved using the Runge-Kutta 4 method, the workhorse of numerical differential equation solving.

3.1 The Runge-Kutta 4 method

The Runge-Kutta 4 method is widely used for solving differential equation numerically. The Runge-Kutta 4 method is a compromise between accuracy and numerical speed. The elementary Euler method, while being superior

with respect to speed and pedagogic, is not very accurate. On the other side, other methods like higher order Runge-Kutta methods are more accurate than the fourth order Runge-Kutta, but require far more numerical operations and are therefore slower. The Runge-Kutta 4 method are quite accurate while not being order of magnitudes slower than the Euler method.

The Runge-Kutta 4 method is as follows[7]: The derivative of a function y is given by the function $f(y, t)$, which is a function of the current value y and the time t . We are at the k 'th step, where $y = y_k$ and $t = t_k$. To calculate the next step, separated by a time h , we first calculate the following:

$$\begin{aligned}
 k_1 &= hf(y_k, t_k) \\
 k_2 &= hf\left(y_k + \frac{k_1}{2}, t_k + \frac{h}{2}\right) \\
 k_3 &= hf\left(y_k + \frac{k_2}{2}, t_k + \frac{h}{2}\right) \\
 k_4 &= hf(y_k + k_3, t_k + h)
 \end{aligned} \tag{3.1}$$

The next step is then found by:

$$y_{k+1} = y_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.2}$$

All these k 's are just different approximations to the average derivative between the real values $y(t_k)$ and $y(t_{k+1})$.

Our derivative is given by equation (2.36) to (2.39). This also holds true for the diagonal system, where $n' = n$. If we consider N photons we have $4N$ elements for each set of equations. We store these elements in a vector v . We consider four elements in this vector, $v[j]$, $v[j+1]$, $v[j+2]$ and $v[j+3]$, corresponding to the values of $\rho_{n,n'}^{v,v}$, $\rho_{n,n'}^{c,c}$, $\rho_{n+1,n'}^{v,c}$ and $\rho_{n,n'+1}^{c,v}$. Their derivative,

stored in the vector d , is then found by:

$$\begin{aligned}
d[i] &= -ig(\sqrt{n}v[i-1] - \sqrt{n'}v[n-2]) + \kappa\sqrt{n+1}\sqrt{n'+1} * v[i+4] \\
&\quad - \left(\frac{1}{2}\kappa(n+n') + P\right)v[i] + \gamma v[i+1]; \\
d[i+1] &= -ig(\sqrt{n+1}v[i+2] - \sqrt{n'+1}v[i+3]) + \kappa\sqrt{n+1}\sqrt{n'+1}v[i+5] \\
&\quad - \left(\frac{1}{2}\kappa(n+n') + \gamma\right)v[i+1] + Pv[i]; \\
d[i+2] &= -ig(\sqrt{n+1}v[i+1] - \sqrt{n'+1}v[i+4]) + \kappa\sqrt{n+2}\sqrt{n'+1}v[i+6] \\
&\quad + \left(\frac{1}{2}\kappa(-n-n'-1) - \frac{1}{2}\gamma - \frac{1}{2}P\right)v[i+2]; \\
d[i+3] &= -ig(\sqrt{n+1}v[i+4] - \sqrt{n'+1}v[i+1]) + \kappa\sqrt{n+1}\sqrt{n'+2}v[i+7] \\
&\quad + \left(\frac{1}{2}\kappa(-n-n'-1) - \frac{1}{2}\gamma - \frac{1}{2}P\right)v[i+3];
\end{aligned} \tag{3.3}$$

This is implemented in appendix B. We have checked the long term limit of the photon expectation value. The solution converges, and it converges (for the same parameters) to the same value as the program we used in the last chapter.

3.2 Time development of expectation value

In figure 3.1 and 3.2 we see the time development for the photon expectation value and the value of g_2 , where we start the system in a coherent state $|z| = 2$ and its steady state photon expectation value $\langle n \rangle$ is around 40. We see that the photon expectation value and g_2 hits their respective steady state value around the same time and, as we will see, quite quickly. That the photon expectation value and g_2 hits their steady state value around the same time indicates that the energy and the energy fluctuation becomes stable at the same time.

In figure 3.3 we how photon expectation values increase from their initial state around $\langle n \rangle = 4$, up until their final state around $\langle n \rangle = 40$. We see that this process happens more slowly for higher values of γ , and correspondingly lower values of κ .

The explanation may be something like this: The energy is pumped into the system by exciting the atom from its energy ground level to its excited

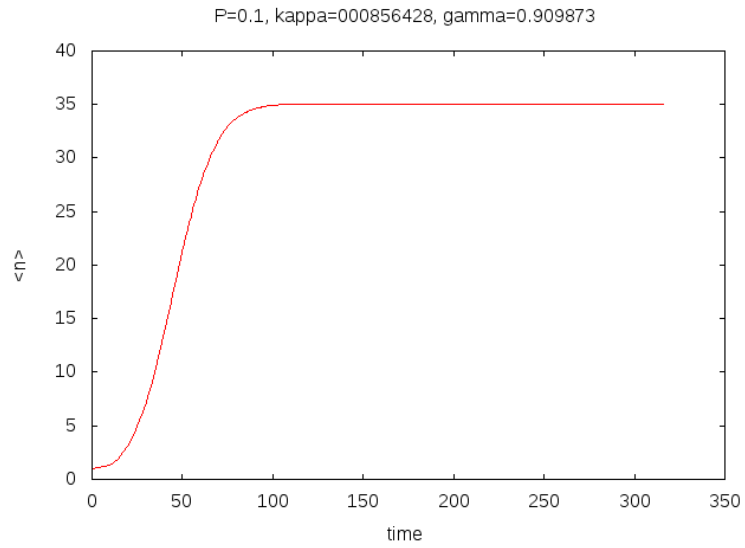
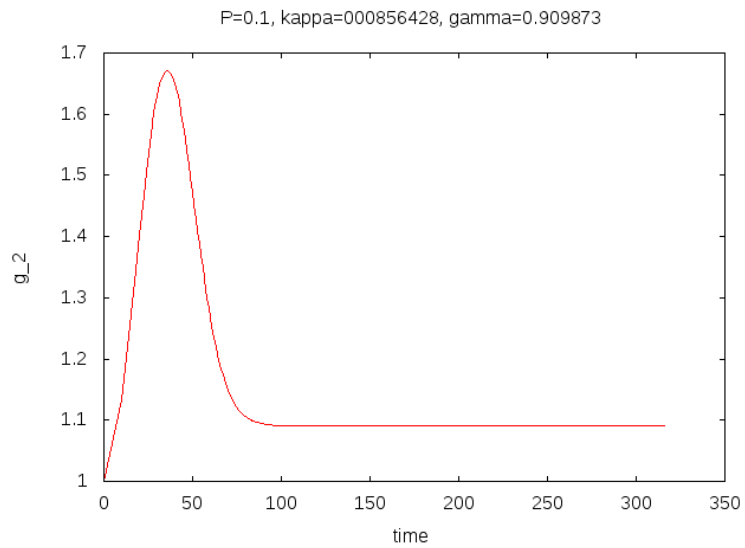


Figure 3.1: The time development of the expectation value

Figure 3.2: The time development of the $g_2(0)$

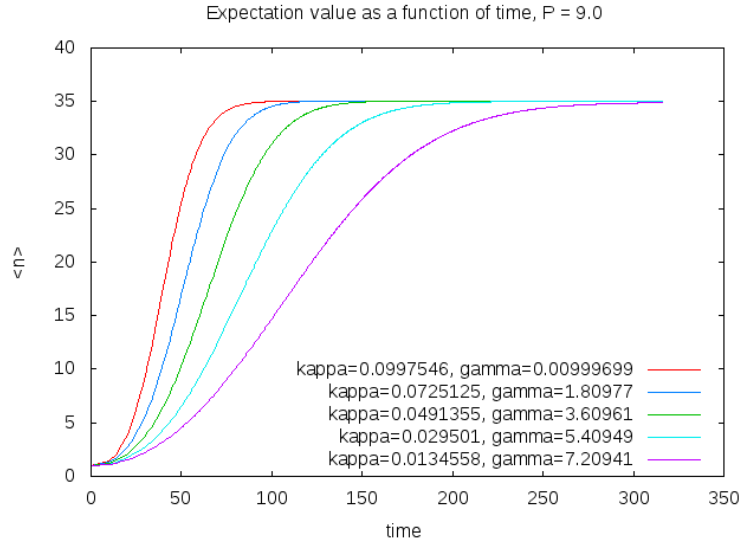


Figure 3.3: The time development of the photon expectation number $\langle n \rangle$. The photon number and the pump rate P is fixed, while κ and γ is varied.

level. From there the excited state may send out a cavity photon, or it can emit a photon which is not a cavity photon with a rate γ , in which case the energy leaves the system without any practical use, i.e. the energy does not becomes laser light. The cavity photons can be reabsorbed by the atom again, or escape the cavity as laser light with rate κ . When we increase γ we must decrease κ if we are to keep the final photon number $\langle n \rangle$ fixed. In the steady state case these two changes neutralizes each other, but in the time dependent case the systems with the higher values of γ takes longer time to reach the steady state. An increased γ means that more energy will "leak" from the atom part of the system out into the environment. Since all the energy that will be stored as photons in the cavity must be pumped into the system the γ parameter acts as a delay when it comes to building up a high photon number in the cavity.

3.3 Time development in phase space

In figure 3.4 to figure 3.8 we see the time development in phase space. We start with a coherent state where the probability is closely centered around a

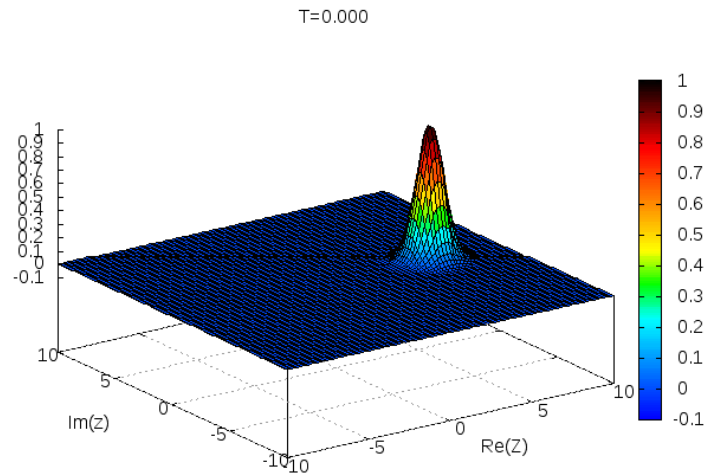
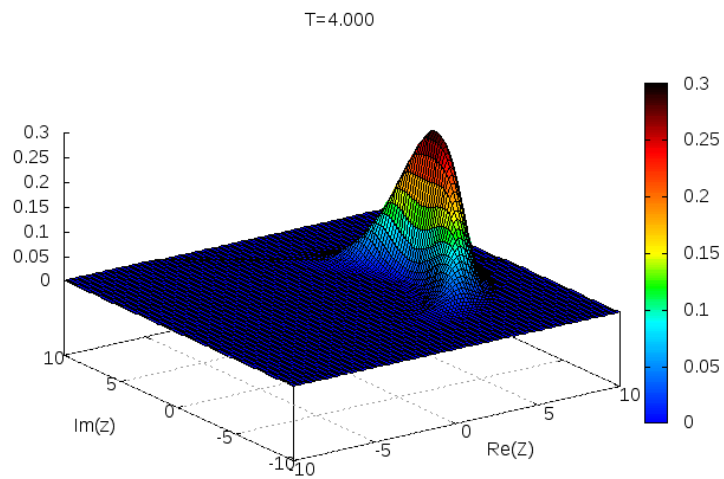
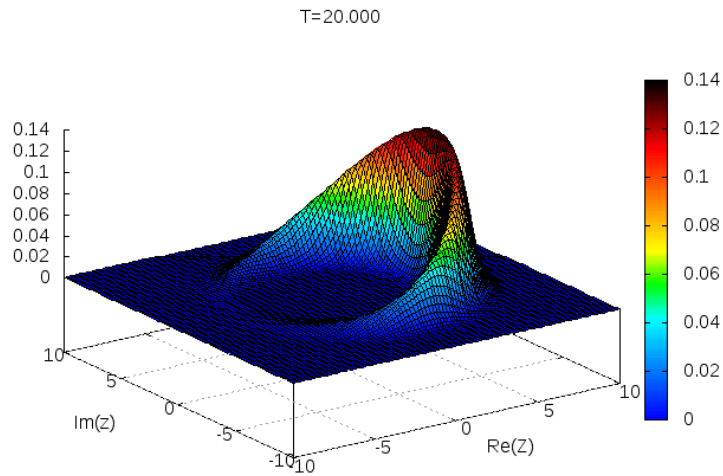
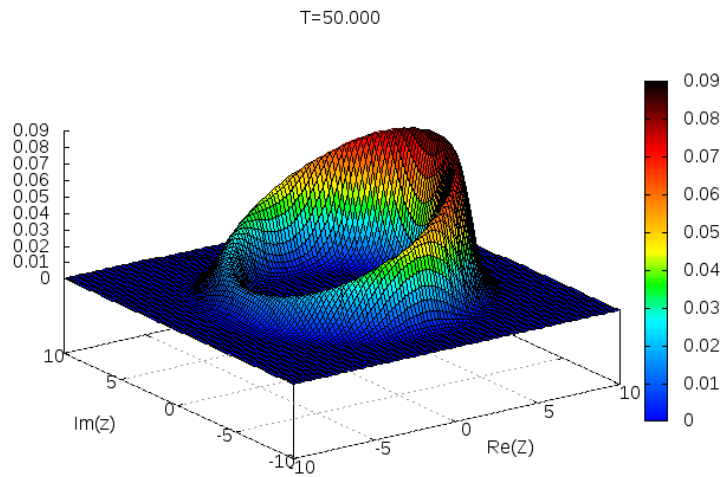


Figure 3.4: The probability distribution in phase space, initial state

Figure 3.5: The probability distribution in phase space, $t=400$

Figure 3.6: The probability distribution in phase space, $t=2000$ Figure 3.7: The probability distribution in phase space, $t=5000$

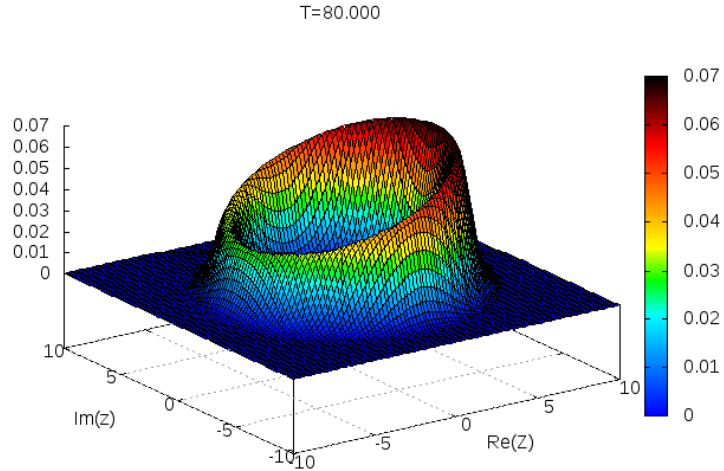


Figure 3.8: The probability distribution in phase space, $t=8000$

point. Then it starts spreading out in both direction. After a while it becomes a ring-formed structure, which eventually becomes uniform at which point all phase information is lost.

In the last section we saw long time it takes before the expectation value of the photon number reaches its steady state solution. In this section we see that the phase information is lost at a slower rate. The radial distribution reaches its equilibrium faster than the angular probability distribution.

3.4 Angular probability distribution

In the last section we saw how the probability distribution in phase space developed, from a quite sharp coherent state to a state where all phase information was lost. We have also seen that this process takes far less time than the time it takes for the photon statistics to find it's equilibrium. To study this dephasing process we will in this section simplify the probability distribution and only consider the angular probability distribution. The angular probability distribution is of course found by integrating over the radial component. If we start the system in a coherent state z , such that $|z| = \sqrt{\langle n \rangle}$, the probability flux in the radial direction will be small. It will therefore be

meaningful to only study the angular probability distribution.

3.4.1 Standard laser

In a standard laser the dephasing is a diffusion process. The diffusion constant D is given by [14]:

$$D == \frac{\Gamma_{sp} + \Gamma_{cav}}{8\langle n \rangle} \quad (3.4)$$

where Γ_{sp} is the emission into the photon mode and Γ_{cav} is the probability a photon has of escaping the cavity.

3.4.2 A Diffusion model for angular probability distribution for the two level system

Initial state

We start the system in a coherent state, z , where the photon expectation value is equal to the steady state solution. The probability of the system having the value z' in phase space is then $P(z') = e^{-|z'-z|^2}$. For the sake of simplicity we place the coherent start position on the real axis, $z = |z| = r_0$. We write z' as $z' = re^{i\phi}$. The angular probability distribution is given with

$$P(\phi) = \int_0^\infty e^{-|z'-z|^2} dr = \int_0^\infty e^{-|re^{i\phi}-r_0|^2} dr \quad (3.5)$$

Using the law of cosines we get

$$P(\phi) = \int_0^\infty e^{-(r^2-2r_0 \cos(\phi)r+r_0^2)} dr \quad (3.6)$$

Solving this integral is not trivial, but can be found in references[Wolfram alpha]. The solution is:

$$P(\phi) = \frac{1}{2}e^{-r_0^2}(\sqrt{\pi}r_0 \cos \phi e^{r_0^2 \cos^2 \phi} \operatorname{erfc}(-r_0 \cos \phi) - 1) \quad (3.7)$$

This is not a very elegant solution, but we can use it to check the numerical integration. The numerical integration is in practice quite simple. The integral $\int_0^\infty P(r, \phi) dr$ is approximated to the sum $P(\phi) = \sum_{i=0}^N P(i\Delta r, \phi) \Delta r$, where N must be large enough that $N\Delta r$ is a good approximation to

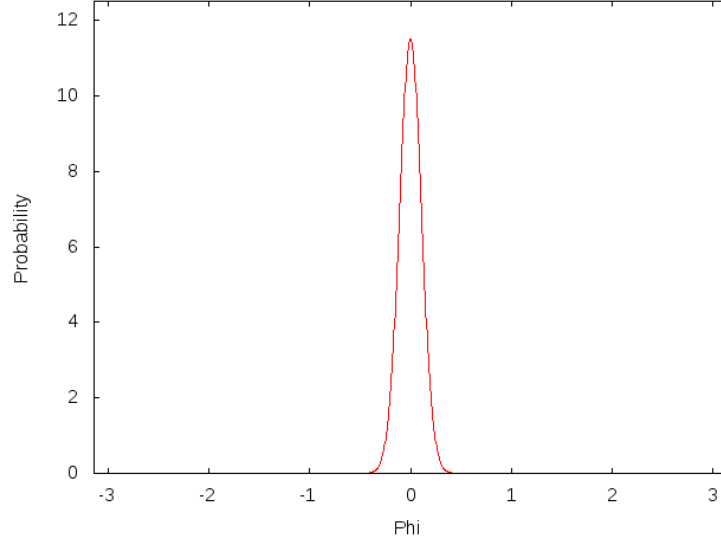


Figure 3.9: Analytical angular distribution for a coherent state $z = 6.5$

infinity, in practice we set $N\Delta r > |z| + 1$. Figure 3.9 shows the angular probability distribution of a coherent state, given by equation (3.7). We note that as to be expected from a coherent state the distribution is strongly centered around the origin. We also see that the distribution resembles a Gaussian distribution, which will be exploited later.

Seing as the exact solution of the initial (coherent) angular probability distribution is not very easy to work with, we can make a very reasonable approximation. If $|z| \gg 1$, the initial probability distribution will be far away from the origin, and concentrated one one side of it. If we start in a coherent state where z is real, positive and fulfills $z \gg 1$ the probability distribution will be close to zero for all positions z' in phase space, if the real part of z' is negative. We can therefore approximate the integral in equation (3.6) with the following:

$$P(\phi) = \int_{-\infty}^{\infty} e^{-(r^2 - 2r_0 \cos(\phi)r + r_0^2)} dr \quad (3.8)$$

This integral is found to be

$$P(\Phi) = r_0 \cos \phi \sqrt{\pi} e^{r_0^2(\cos^2 \phi - 1)} \quad (3.9)$$

If we start with a coherent state $|z\rangle$, where z is real and $z \gg 1$, the angular probability distribution is non-zero only for small values of ϕ . We can then use Taylor expansions to find that the angular probability distribution can be approximated with a Gaussian distribution.

$$P(\phi) = r_0 \left(1 - \frac{1}{2}\phi^2 + \dots\right) \sqrt{\pi} e^{-r_0^2 \sin^2 \phi} \approx r_0 \sqrt{\pi} e^{-r_0^2 \phi^2} \quad (3.10)$$

This approximation to the angular probability is quite nicer than the exact expression given in equation (3.7). And as we see in figure 3.10 the Gaussian approximation seems quite similar to the exact distribution as seen in figure 3.9. In figure 3.11 the absolute difference between the Gaussian approximation and the exact distribution is shown.

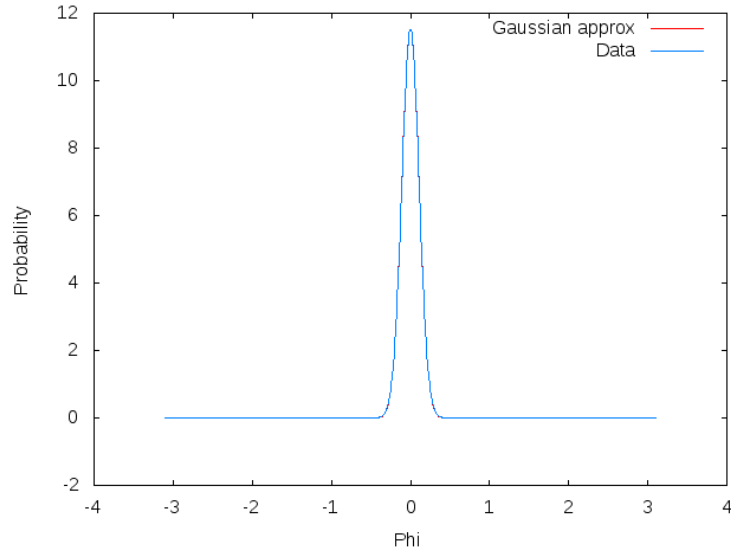


Figure 3.10: Gaussian approximation to Angular Distribution for a coherent state, $z = 6.5$

It seems like this Gaussian approximation is a good one.

Short term development

Now we know how the initial angular probability distribution is, that it can be approximated to a Gaussian distribution. Now the question is how the probability distribution develops in time.

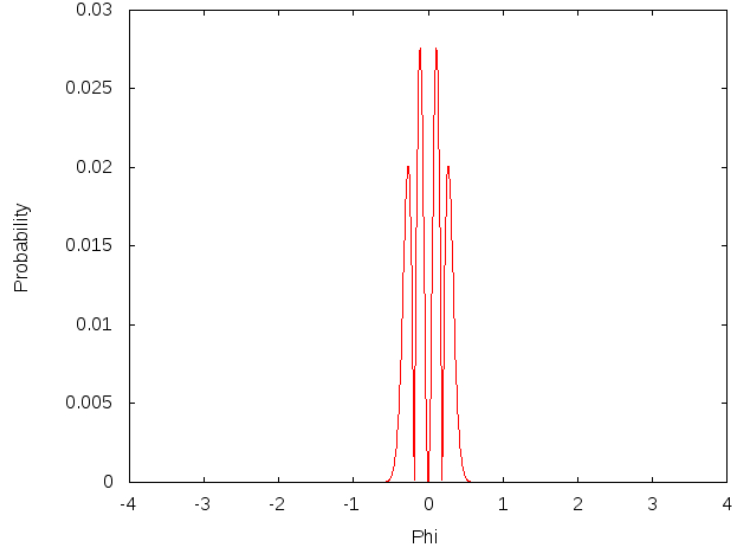


Figure 3.11: Absolute difference between the Gaussian approximation in figure 3.10 and the exact solution in figure 3.9

If the time development of our one-atomic laser follows the same dephasing process as for a normal laser, the dephasing will have the time development as a diffusion process. That means that it would have followed equation (3.11)

$$\frac{\partial P}{\partial t} = D\nabla^2 P \quad (3.11)$$

We saw earlier that the angular probability distribution could be approximated with a Gaussian distribution. A Gaussian distribution remain Gaussian at all later times if it develops accordingly to equation (3.11), in the form of:

$$P(\phi, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{\phi^2}{4Dt}} \quad (3.12)$$

D is the diffusion constant, the defining parameter in a diffusion process.

If the time development follows a diffusion model the angular probability distribution will follow equation (3.12). If we then find the numerically integrated angular probability distribution at two points in time, one at the start and one after some time Δt has passed, we can find the diffusion constant. The diffusion constant will be found by testing for different values of D , and

choosing the one which gives the most accurate description of the angular probability distribution at time Δt .

We will now study the time development of the angular probability distribution and see how well it matches up with a diffusion model. For the following example we use the parameters $P = 12, \kappa = 0.1, \gamma = 0.0, g = 1.0$. Figure 3.12 shows how the angular probability distribution develops in time,

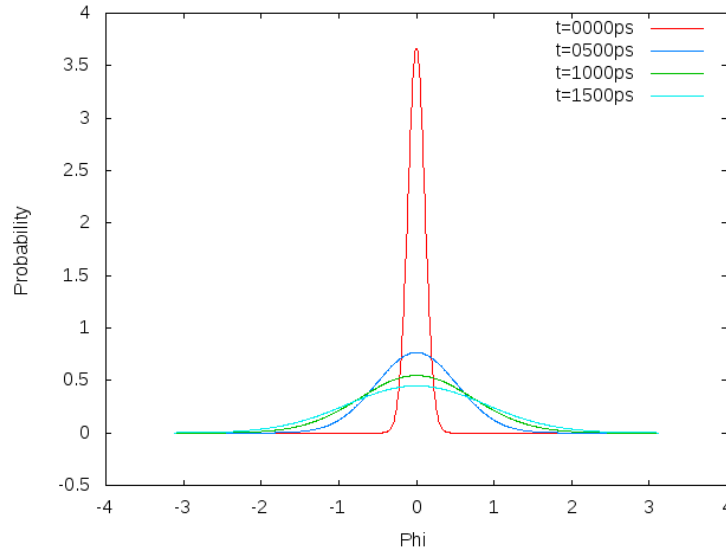


Figure 3.12: Time development of the angular probability distribution, in steps of 500. The parameters are $P = 12, \kappa = 0.1, \gamma = 0.0$.

from a sharply peaked coherent state into a more diffuse state. We now use the two first time points to find the best fitting diffusion constant. For these parameters we find that the best fit is for $D = 0.0001295$.

Figure 3.13 shows the angular probability distribution for a diffusion model after $t = 2000$, where we start the system in the Gaussian state closest to the angular probability distribution of the coherent state $z = 6.5$. If the diffusion model is an accurate model, this distribution should closely resemble the angular probability distribution we get by numerical integration of the time developed coherent state. Figure 3.14 is the difference between these two distribution, the diffusion model and the time developed coherent state. We see that the the the model fits very well where ϕ are close to the origin, where the probability density is highest. We see that the model is

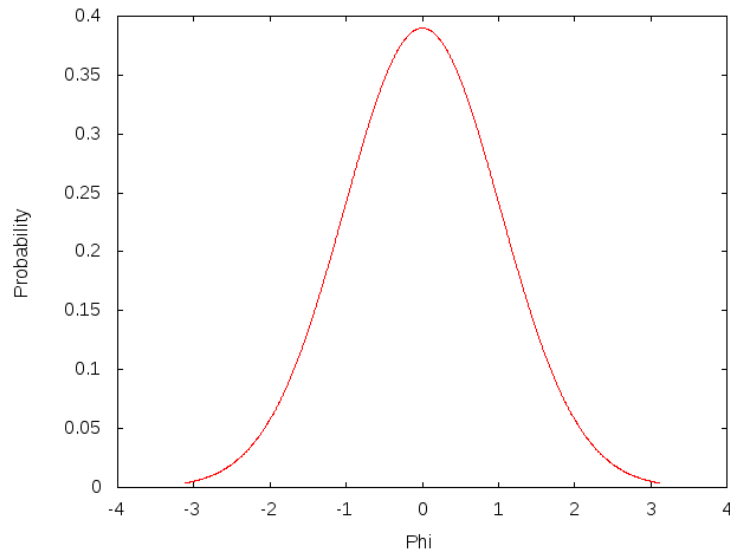


Figure 3.13: Angular probability distribution, Diffusion model, at $t=2000g$. The parameters are $P = 12$, $\kappa = 0.1$, $\gamma = 0.0$.

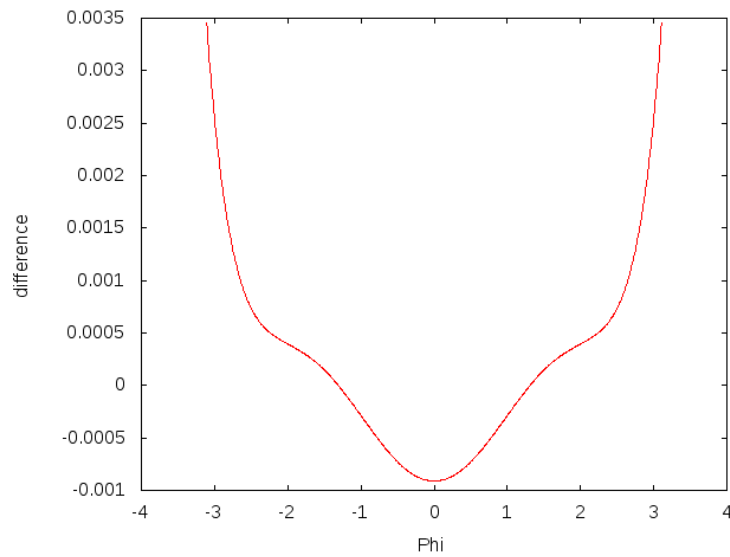


Figure 3.14: Difference between the numerical integration and the diffusion model, for the same point in time and for the same parameters as for figure 3.13

somewhat less accurate for $\phi \rightarrow \pm\pi$. The reason for this will be discussed in the next section.

Long term development

The increasing inaccuracy when $\phi \rightarrow \pm\pi$ gets more pronounced when we step forward in time. This is shown in figure 3.15, which show that the actual angular probability distribution is larger than the Gaussian diffusion model at the edges.

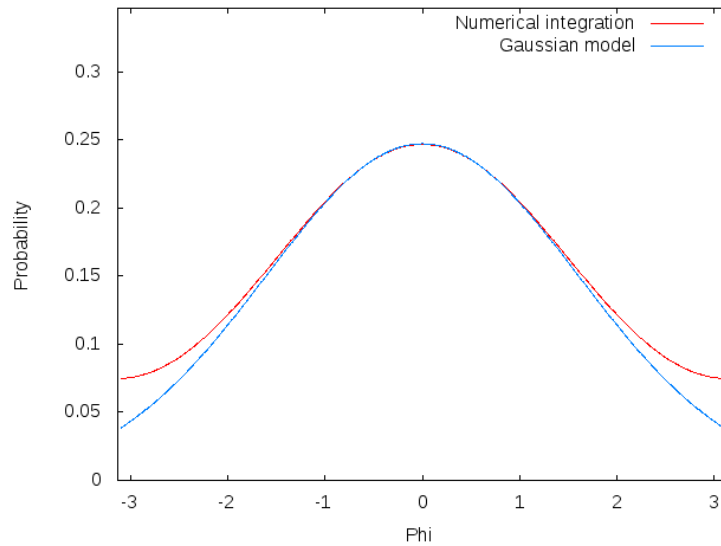


Figure 3.15: Gaussian diffusion model and the numerical integration of the time developed coherent state, after 5000

An other deficit with the Gaussian diffusion model is that the Gaussian diffusion model tends to 0 when t tends to ∞ . This is not remotely close to the actual situation, where we have seen that the angular probability distribution becomes uniform but nonzero. Probability is, after all, conserved. We have $P(\phi) \rightarrow \frac{1}{2\pi}$ when $t \rightarrow \infty$, seeing as $\int_{-\pi}^{\pi} P(\phi)d\phi = 1$.

The reason for the inaccuracy of the Gaussian diffusion model as shown in figure 3.15 can be seen as a consequence of the fact that the probability diffuses clockwise and anti-clockwise. The probability flux clockwise goes toward $\phi = -\pi$, while the anti-clockwise probability flux diffuses towards $\phi = \pi$. However, these two point are one and the same. The probability

density at $\phi = \pi$ therefore increases faster than if the variable ϕ had been linear, as seen in figure 3.15.

Can we include this into the Gaussian diffusion model? We achieve the desired overlap effect to explain 3.15 if we add two Gaussian distribution equal to the one we already have, only centered around the fictional points $\phi = \pm 2\pi$. This also makes sense, seeing as $\pm 2\pi$ represents the same point as 0. In fact, ϕ equal to any multiples of 2π represent the same point as $\phi = 0$. The more complete Gaussian diffusion model then include an infinity of Gaussian distributions, centered around all multiples of 2π .

This also solves the problem when $t \rightarrow \infty$. In the original Gaussian diffusion model an finite amount of probability is spread thin over an infinite space. Now an infinite amount of probability that is spread over the same area, with a full probability of one centered around each multiple of 2π .

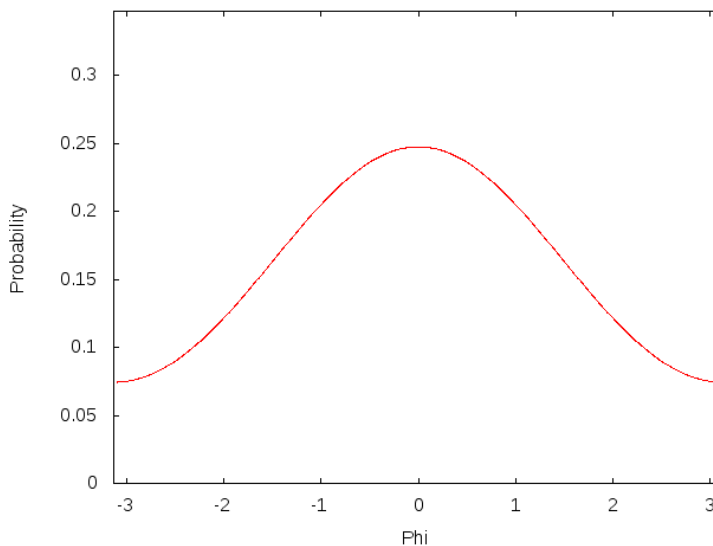


Figure 3.16: Periodically Gaussian model after 5000, same situation as in figure 3.15

Figure 3.16 shows this new, improved periodical Gaussian diffusion model. This seem to resemble the actual angular probability distribution as shown in figure 3.15. In figure 3.17 the difference in percent between the actual distribution and the periodically Gaussian diffusion model is shown, where we see that the the periodical model fits quite well.

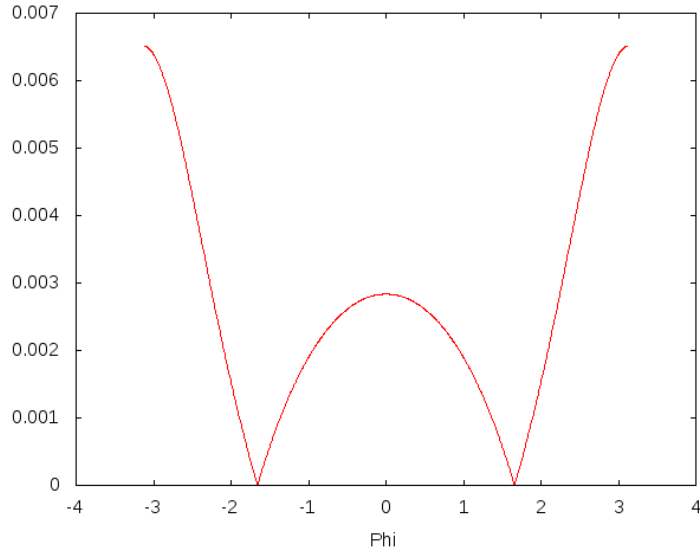


Figure 3.17: The difference in percent between the actual angular probability distribution and the periodically Gaussian diffusion model, for $t=5000$

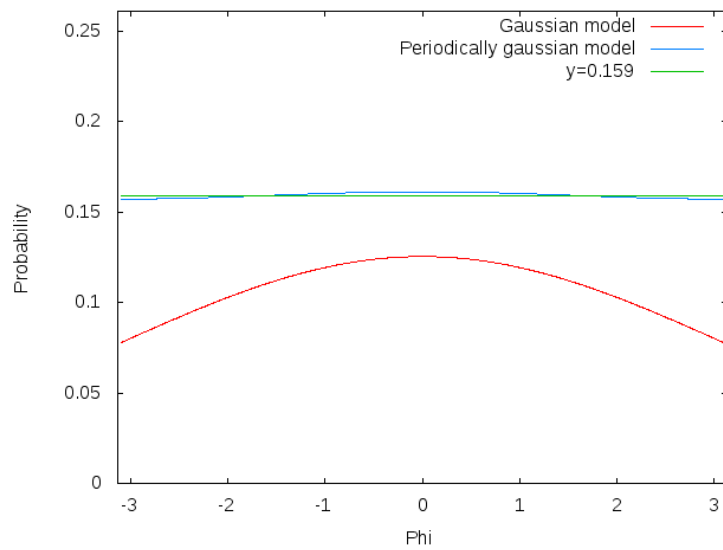


Figure 3.18: The Gaussian and periodically Gaussian diffusion model, for $t=19500$

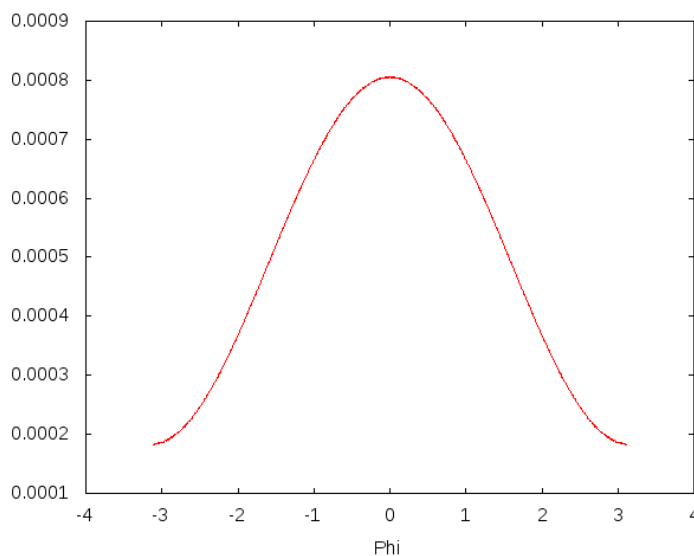


Figure 3.19: The difference in percent between the actual angular probability distribution and the periodically Gaussian diffusion model, for $t=19500$

In figure 3.18 and figure 3.19 we have stepped further forward in time, at $t=19500$ ps after we started with a coherent state. Figure 3.18 shown the by now large difference between the two models, the Gaussian and the periodically Gaussian diffusion model. We see that the Gaussian diffusion model are continuing to diffuse towards zero, while the periodically Gaussian diffusion model seems to go towards $\frac{1}{2\pi} \approx 0.159$. Figure 3.19 shows the percentage difference between the actual angular probability distribution and the periodically Gaussian diffusion model, which we see is at no point higher than 0.0009%, which must be characterized as a good agreement between the actual distribution and the model.

3.4.3 The Diffusion Parameter

In a standard laser the phase drift in the form of diffusion is said to be due to random disturbances in phase space. This give rise to a random walk in the angular direction. ¹ A random walk becomes a diffusion process when the steps of the walk are small enough.

¹The radial distribution in phase space is in balance as we have seen earlier in this chapter and in the last, while the angular distribution in phase space will as we have seen

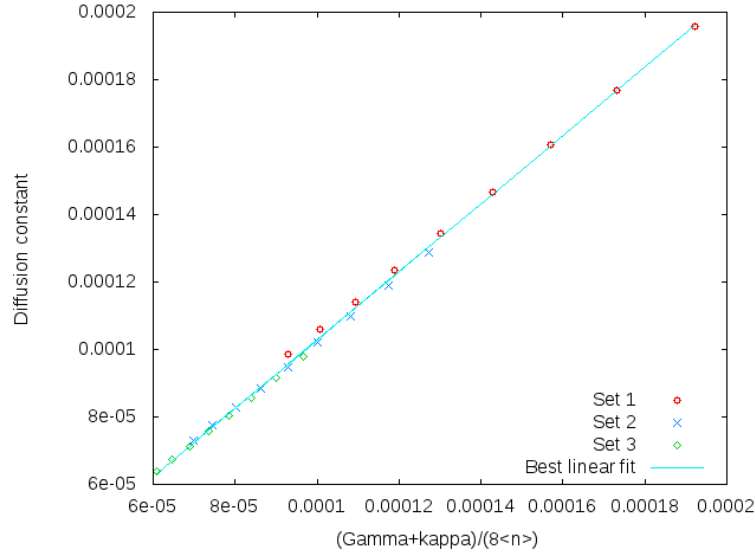


Figure 3.20: The data points are the calculated diffusion constant plotted against the variable $\frac{\Gamma+\kappa}{8\langle n \rangle}$. The line is the best linear fit to the points

Figure 3.20 plot the diffusion constant D against the parameter $\frac{\Gamma+\kappa}{8\langle n \rangle}$, in addition to the best linear approximation through all of the points. The diffusion constant is calculated for three different set of parameters. For each set the parameter P is fixed, while κ and γ is varied so that $\langle n \rangle$ remain fixed. Those fixed parameters are different for each set. There seems to be a linear connection between the diffusion constant D and the parameter $\frac{\Gamma+\kappa}{8\langle n \rangle}$.

This can perhaps be explained by considering the random walk model. In the standard laser the random walk is caused by disturbances to the field. What disturbances have we here? The parameter κ causes photons to escape the photons, which is one disturbance. As mention in the earlier chapter, the parameter Γ can be considered the effective coupling rate between the atom subsystem and the cavity subsystem. The parameter was introduced in an approximation, but we know that the assumption that was made seems correct for a quite high photon number. If the parameter Γ can be considered the effective coupling rate between the atom and the cavity subsystem then that causes another disturbance. The diffusion constant should be a function of the disturbance. From figure 3.20 it looks like the diffusion constant is

diffuse out until it becomes flat.

simply proportional to the sum of κ and Γ .

Chapter 4

Concluding remarks

We have, in this thesis, studied the steady state solution of our one-atom two-level system. We have seen how the simple one-atom two-level system, for certain regions of phase space, acts similar to a laser. That is, we reach a high photon number, and the photon probability distribution has the statistics of a coherent state. For a high photon number to be achieved we have seen that the parameter κ and γ must be restricted. The Poisson statistics are found when the photon number is high, and when $\frac{\partial \langle n \rangle}{\partial t} > 0$.

We have also seen that while we may achieve a photon distribution equal to that of a coherent state, the phase/angular probability distribution will not remain sharp or coherent. While a coherent state remain a coherent state if we only have a harmonic oscillator, our harmonic oscillator, the electromagnetic field, has been coupled with an atom and to a wider, more chaotic universe. We may have proven analytically that eventually all phase information will become irrevocably lost.

We have also shown that the angular/phase probability distribution nearly obey the diffusion equations, which enabled us to find a diffusion constant. That means that even if, with time, all phase information is lost, we can somewhat influence the speed of which the phase information is lost. This diffusion constant seems to be proportional to the sum of Γ , which may be viewed as an effective coupling constant between the atom and the cavity, and κ , which is the probability of a photon escaping the cavity. These two parameters may be viewed as the parameters contribution to the disturbances to the photon part of the system.

An open question is if this can be considered a optimization problem of a sort: To find parameters which maximize the photon number $\langle n \rangle$, while at

the same time minimize the diffusion constant D and so increase the time the system will conserve its phase distinctiveness.

Appendix

Appendix A

C++ program for finding the steady state solution

```
#include <iostream>
#include <fstream>
#include <string>
#include <stdlib.h>
#include <cmath>
#include <armadillo>
using namespace std;
using namespace arma;

vec con_1(int N, double P, double kappa, double gamma, double g);
vec con_2(int N, double P, double kappa, double gamma, double g);
vec con_3(int N, double P, double kappa, double gamma, double g);
vec con_4(int N, double P, double kappa, double gamma, double g);
vec solve_system(double P, double kappa, double gamma, double g);
vec solve_eq(int N, double P, double kappa, double gamma, double g);
double exp_val(vec denvec);

//Functions to construct vectors
//These vectors are just a handy way of handling the coefficients of the equations
vec con_1(int N, double P, double kappa, double gamma, double g){
    vec v;
```

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```

    v.zeros(2*N+1);
    v(0) = P;
    for(int i=1;i<=N;i++){
        v(2*i-1) = -2*g*g/kappa;
        v(2*i) = P;
    }
    return v;
}

vec con_2(int N, double P, double kappa, double gamma, double g){
    vec v;
    v.zeros(2*N+1);
    v(0) = -gamma;
    for(int i=1;i<=N;i++){
        v(2*i-1) = kappa*(i-0.5) + 0.5*gamma -0.5*P + 2*g*g/kappa;
        v(2*i) = -kappa*i-gamma;
    }
    return v;
}

vec con_3(int N, double P, double kappa, double gamma, double g){
    vec v;
    v.zeros(2*N+1);
    v(0) = -kappa;
    for(int i=1;i<=N;i++){
        v(2*i-1) = kappa*(2*i-0.5)+ 1.5*gamma+0.5*P;
        v(2*i) = -kappa*(i+1);
    }
    return v;
}

vec con_4(int N, double P, double kappa, double gamma, double g){
    vec v;
    v.zeros(2*N+1);
    v(0) = 0.0;
    for(int i=1;i<=N;i++){
        v(2*i-1) = -kappa*(i+1);
        v(2*i) = 0.0;
    }
}

```

```

    }
    return v;
}
//Functions for elements in vectors

//Function for solving lin. eq. for N photons
vec solve_eq(int N, double P, double kappa, double gamma, double g){
    vec v1,v2,v3,v4,x;
    v1 = con_1(N,P,kappa,gamma,g);
    v2 = con_2(N,P,kappa,gamma,g);
    v3 = con_3(N,P,kappa,gamma,g);
    v4 = con_4(N,P,kappa,gamma,g);
    int counter = 0;//Number of negative elements
    x.zeros(2*N+6);//Include four extra elements to simplify calculations
    x(2*N+1) = 1.0;//Arbitrary factor, will change due to normalization later. May chan
    //Starts from the highest elements and work our way downwards
    for(int n=2*N;n>=0;n--){

        x(n) = -(1/v1(n))*(v2(n)*x(n+1)+v3(n)*x(n+2)+v4(n)*x(n+4));
        if(x(n) < 0){
            counter += 1;
        }
        x /= abs(sum(x));
        //cout << n << endl;

    }
    if(counter>0){
        cout << "Warning, negative probabilities" << endl;
    }
    return x.subvec(0,2*N+1);
}
//Solves the system for larger and larger values of photons
// until the solutions converge
vec solve_system(double P, double kappa, double gamma, double g){
    vec u;
    int N = 5;
    double P_N = 1.0;
    double eps = pow(10.0,-300);

```

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```
while(P_N > eps){
    // cout << N << endl;
    u.zeros();
    u = solve_eq(N,P,kappa,gamma,g);
    P_N = u(2*N) + u(2*N+1);
    N += 5;
}
//cout << N<< endl;
return u;
}
//Expectation value
double exp_val(vec denvec){
    int length;
    length = denvec.n_elem;
    double evn=0.0;

    for(int i=0;i<length/2;i+=1){
        evn += i*(denvec(2*i) + denvec(2*i+1));
    }
    return evn;
}
//Upper level population
double ulp(vec denvec){
    int length;
    length = denvec.n_elem;
    double nc=0.0;

    for(int i=1;i<length;i+=2){
        nc += denvec(i);
    }
    return nc;
}

int main(){
    //Parameters
    double P=0.9
    double kappa=0.01;
    double gamma=0.02;
```



```
double g=0.1;
//Initiate empty vector
vec denvec;
denvec.zeros();
//Solves the system
denvec = solve_system(P,kappa,gamma,g);
//Write out the expectation value for the photon number
cout << exp_val(denvec) << endl;

return 0;
}
```

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Appendix B

C++ program for finding the time development

```
#include <iostream>
#include <fstream>
#include <string>
#include <stdlib.h>
#include <cmath>
#include <armadillo>
using namespace std;
using namespace arma;
//Declaring functions
cx_vec derivative(cx_vec denvec, double P, double kappa, double gamma, double g);
cx_vec next_step(cx_vec denvec, double P, double kappa,
double gamma, double g,int i);
cx_vec steady_state(double P, double kappa, double gamma, double g);
//double expectation_value_N(int N, double P, double kappa, double gamma, double g);
double exp_val(cx_vec denvec);
double normalization_const(cx_vec dv);

//Declaring constants
const cx_double j = cx_double(0,1);
```

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```

//Find the derivatiooves on any set of elements, stored in denvec
//Works for both the diagonal sets and any non-diagonal
cx_vec derivative(cx_vec denvec,double P, double kappa,
                 double gamma, double g, int i){
    int M;
    M = denvec.n_elem;
    cx_vec D;
    D.set_size(M);
    double m;
    double o;
    if(i==0){//For the diagonal case, the derivatives is a bit different
        for(int n=0;n<M;n+=4){
            m = (double) n;
            m/=4;//Photon number
            o = m+i;//n' = n in the diagonal case

            if(n==0){//Exception handling start
D(n) = + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+4) - (0.5*kappa*(m+o) + P)
            *denvec(n) + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
            + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+5)-(0.5*kappa*(m+o)+gamma)
            *denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1) - sqrt(o+1)*denvec(n+4))
            + kappa*sqrt(m+2)*sqrt(o+1)*denvec(n+6)
            + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+2);
D(n+3) = -j*g*(sqrt(m+1)*denvec(n+4) - sqrt(o+1)*denvec(n+1))
            + kappa*sqrt(m+1)*sqrt(o+2)*denvec(n+7)
            + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);
            }
            else if(n==M-4){//Exception handling end
D(n) = -j*g*(sqrt(m)*denvec(n-1)-sqrt(o)*denvec(n-2))
            - (0.5*kappa*(m+o) + P)*denvec(n) + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
            -(0.5*kappa*(m+o)+gamma)*denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1)) + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)
            *denvec(n+2);
D(n+3) = -j*g*( - sqrt(o+1)*denvec(n+1))
            + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);
            }
        }
    }
}

```

```

    }
    else{
D(n) = -j*g*(sqrt(m)*denvec(n-1)-sqrt(o)*denvec(n-2)) + kappa*sqrt(m+1)
        *sqrt(o+1)*denvec(n+4) - (0.5*kappa*(m+o) + P)*denvec(n)
        + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
        + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+5)-(0.5*kappa*(m+o)+gamma)
        *denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1) - sqrt(o+1)*denvec(n+4))
        + kappa*sqrt(m+2)*sqrt(o+1)*denvec(n+6)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+2);
D(n+3) = -j*g*(sqrt(m+1)*denvec(n+4) - sqrt(o+1)*denvec(n+1))
        + kappa*sqrt(m+1)*sqrt(o+2)*denvec(n+7)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);

    }
}
}
else{//For the non-diagonal sets
    for(int n=1;n<M;n+=4){
        m = (double) n -1;
        m/=4;
        o = m+i;

        if(n==1){
D(0) = j*g*sqrt(o)*denvec(1)+kappa*sqrt(o)*denvec(3)
        -0.5*(kappa*(o-1)+gamma+P)*denvec(0);
D(n) = +j*g*sqrt(o)*denvec(0)+ kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+4)
        - (0.5*kappa*(m+o) + P)*denvec(n) + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
        + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+5)-(0.5*kappa*(m+o)+gamma)
        *denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1) - sqrt(o+1)*denvec(n+4))
        + kappa*sqrt(m+2)*sqrt(o+1)*denvec(n+6)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+2);
D(n+3) = -j*g*(sqrt(m+1)*denvec(n+4) - sqrt(o+1)*denvec(n+1))
        + kappa*sqrt(m+1)*sqrt(o+2)*denvec(n+7)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);

```

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```

    }
    else if(n==M-4){
D(n) = -j*g*(sqrt(m)*denvec(n-1)-sqrt(o)*denvec(n-2))
        - (0.5*kappa*(m+o) + P)*denvec(n) + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
        -(0.5*kappa*(m+o)+gamma)*denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1))
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+2);
D(n+3) = -j*g*( - sqrt(o+1)*denvec(n+1))
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);
    }
    else{
D(n) = -j*g*(sqrt(m)*denvec(n-1)-sqrt(o)*denvec(n-2))
        + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+4)
        - (0.5*kappa*(m+o) + P)*denvec(n) + gamma*denvec(n+1);
D(n+1) = -j*g*(sqrt(m+1)*denvec(n+2) - sqrt(o+1)*denvec(n+3))
        + kappa*sqrt(m+1)*sqrt(o+1)*denvec(n+5)
        -(0.5*kappa*(m+o)+gamma)*denvec(n+1) + P*denvec(n);
D(n+2) = -j*g*(sqrt(m+1)*denvec(n+1) - sqrt(o+1)*denvec(n+4))
        + kappa*sqrt(m+2)*sqrt(o+1)*denvec(n+6)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+2);
D(n+3) = -j*g*(sqrt(m+1)*denvec(n+4) - sqrt(o+1)*denvec(n+1))
        + kappa*sqrt(m+1)*sqrt(o+2)*denvec(n+7)
        + (0.5*kappa*(-m-o-1)-0.5*gamma-0.5*P)*denvec(n+3);
    }
}
}

return D;
}

//Calculates the next step
cx_vec next_step(cx_vec denvec, double P, double kappa,
                double gamma, double g, int i){
double dt = 0.1;

```

```

cx_vec A;
//A = dt*derivative(denvec,P,kappa,gamma,g,i);
//The Runge Kutta 4 method:
cx_vec K1,K2,K3,K4;//Runge Kutta 4
K1 = derivative(denvec,P,kappa,gamma,g,i);
K2 = derivative(denvec+0.5*dt*K1, P,kappa,gamma,g,i);
K3 = derivative(denvec+0.5*dt*K2, P,kappa,gamma,g,i);
K4 = derivative(denvec + dt*K3, P,kappa,gamma,g,i);
A = dt/6.0*(K1+2*K2+2*K3+K4);

return A;
}
//Returns the expectation value when given the diagonal set
double exp_val(cx_vec denvec){

double sum;
vec rdv = real(denvec);
int lenvec;
lenvec = denvec.n_elem;
for(int l=0;l<lenvec;l+=4){
    sum += 1/4*(rdv(l) + rdv(l+1));
}

return sum;
}

double normalization_const(cx_vec denvec){
double sum;
vec rdv = real(denvec);
int lenvec;
lenvec = denvec.n_elem;
for(int k=0;k<lenvec;k+=4){
    sum += rdv(k) + rdv(k+1);
}

return sum;
}

```

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```

}

double factorial(int n){
    double prod=1;

    if(n>0){
        for(int i=n;i>0;i--){
            prod*=i;
        }
    }

    return prod;
}

//The pseudo-probabiliy function <z|rho|z>
double alpha(field<cx_vec>denvecs,double R, double I,int N,int M){
    cx_double sum=0.0;
    for(int n=0;n<N;n++){
        sum += (denvecs(0)(4*n) + denvecs(0)(4*n+1))*exp(-(R*R+I*I))
            *pow(R*R+I*I,n)/factorial(n);
    }

    for(int m=1;m<M;m++){
        for(int n=0;n<N;n++){
            sum += exp(-(R*R+I*I))/(sqrt(factorial(n))
                *sqrt(factorial(n+m)))*2*real((denvecs(m)(4*n+1)
                + denvecs(m)(4*n+2))*pow(R-j*I,n)*pow(R+j*I,n+m));
            //cout << m << " " << n << " " << sum << endl;
        }
    }
}

```



```

//cout << imag(sum) << endl;
//cout << "-----" << endl;
return real(sum);
}

int main()
{
  //Parameters
  double P=0.9;
  double kappa=0.01;
  double gamma=0.02;
  double g=0.1;
  int N=120;//Number of photons
  int M=40;//Number of sets of coupled ODEs.
  field<cx_vec> denvecs(M);
  denvecs(0).zeros(4*N);
  for(int i=1;i<M;i++){
    denvecs(i).zeros(4*N+1);
  }

  //mat denmat_photon;
  cx_double z;
  z = 5.79;

  cx_double ro;
  //Initiate the system to be a coherent state z
  for(int n=0;n<N;n++){
    ro = exp(-abs(z)*abs(z))*pow(z,n)*pow(conj(z),n)/factorial(n);
    denvecs(0)(4*n) = 0.5*ro;
    denvecs(0)(4*n+1) = 0.5*ro;
  }
  for(int m=1;m<M;m++){

```

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```

for(int n=1;n<N;n++){
    ro = exp(-abs(z)*abs(z))*pow(z,n)*pow(conj(z),n+m)
        /(sqrt(factorial(n))*sqrt(factorial(n+m)));
    denvecs(m)(4*n+1) = 0.5*ro;
    denvecs(m)(4*n+2) = 0.5*ro;
}
//cout << sum(abs(denvecs(m))) << endl;
}
//for(int m=0;m<M;m++){
// cout << sum(denvecs(m)) << endl;
//}
//Makes a grid in phase space
double norm=1.0;
double S= 10;
int NR = 100;
int NI = 100;

vec Ivec,Rvec;
Rvec = linspace(-S,S,NR);
Ivec = linspace(-S,S,NI);
ofstream myfile("zspace.txt");
//ofstream myfile("decay.txt");
int T=100;
// myfile << T*2000 << endl;

myfile << T+1 << endl;
myfile << NR << endl;
myfile << NI << endl;

for(int i=0;i<NR;i++){
    myfile << Rvec(i) << endl;
}
for(int i=0;i<NI;i++){
    myfile << Ivec(i) << endl;
}
double A;

```

```

for(int i=0;i<T*1000+1;i++){
    cout << "Time-step: " << i << endl;
    #pragma omp parallel for
    //Steps foreward for all sets
    for(int j=0;j<M;j++){
        denvecs(j) += next_step(denvecs(j),P,kappa,gamma,g,j);
    }
    //Normalize all sets
    norm = normalization_const(denvecs(0));
    #pragma omp parallel for
    for(int j=0;j<M;j++){
        denvecs(j) /= norm;
    }
    //Write out the distribution in phase space
    if(i % 1000==0){
        for(int k=0;k<NR;k++){
            #pragma omp for ordered schedule(dynamic)
            for(int l=0;l<NI;l++){
                // cout << alpha(denvecs,Rvec(k),Ivec(l),N,M) << endl;
                A = alpha(denvecs,Rvec(k),Ivec(l),N,M);
                #pragma omp ordered
                myfile << A << endl;
            }
        }
    }

}

return 0;

}

```


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