Induced orientations on Banach menifolds.

by

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Introduction. The purpose of this note is to prove a theorem about orientations of submanifolds of certain Banach manifolds which includes some results of K.D. Elworthy and A.J. Tromba. Roughly we shall prove that if M, N and Q are Banach manifolds with c-structures and $f: M \to Q$, $g: N \to Q$ are smooth transversal maps one of which is Fredholm and c-structure preserving, then the fibered product $M \times_Q N$ is a manifold with c-structure which is orientable if M, N and Q are orientable. (The terms are explained below). The proof is conceptual and quite standard in the sense that it arises from extensions of standard methods of finite dimensional theory. The arguments will be accordingly brief.

We refer to [1] and [5] for general information on concepts and properties of manifolds used in the sequel.

1. For B a real Banach space, let $L(B)$ be the Banach algebra of bounded linear operators under the norm topology and $GL(B) \subset L(B)$ the multiplicative subgroup of inversible elements. Let $c(B) \subset L(B)$ be the closed ideal of completely continuous operators and $L_c(B)$ and $GL_c(B)$ the subsets of $L(B)$ and $GL(B)$, respectively, of operators of the form $I + T$, $T \in c(B)$.

Then $GL_c(B)$ is a subgroup of $GL(B)$, and it is known that $GL_c(B)$ has precisely two components, cf. [3]. We denote the component containing the identity $SL_c(B)$ and the other $SL_c^-(B)$. Given a Banach manifold M modelled on B a c-structure on M is an admissible atlas $\{\varphi_i, U_i\}$ maximal with respect to the property: For any i,j the differential $d(\varphi_j\varphi_i^{-1})$ at any point lies in $GL_c(B)$. The c-structure is orientable if it admits a subatlas for which the differentials actually lie in $SL_c(B)$. An orientation is a subatlas maximal with respect to this property. Given a c-manifold M a submanifold $M_0 \subset M$ is a c-submanifold if there exist charts $(\varphi_i, U_i) \in c_M$ covering M_0 with $\varphi_i(U_i \cap M_0)$ open in the model space $B_0 \subset B$ of M_0. All finite dimensional submanifolds are clearly c-submanifolds. Other examples are given by the finite codimensional submanifolds. This is less obvious, and we outline the argument. First some preliminaries.

A smooth map $f: M \to N$ between c-manifolds modelled on B is a c-map if for any local representative $\psi_j f \varphi_i^{-1}$ of f the differential $d(\psi_j f \varphi_i^{-1})$ at any point is in $L_c(B)$. Now, given a Banach manifold M modelled on B and a Fredholm map $f: M \to B$ of index 0, there is a unique c-structure on M which makes f a c-map (with respect to the canonical c-structure on B). This fundamental observation is due to Elworthy and Tromba. The proof is short and simple and can be found in [2]. The argument being local the model B can actually be replaced by an arbitrary c-manifold modelled on B. This gives

Theorem (Elworthy - Tromba). Let M, N be Banach manifolds on the same model, and let $f: M \to N$ be a Fredholm map of index 0. Given a c-structure c_N on N, there is a unique c-structure

$c_M = f^*c_N$ on M making f a c-map.

There is a straightforward generalization of this theorem which is useful. Suppose M and N are manifolds modelled on Banach spaces B and C, respectively, and $f: M \to N$ is a Fredholm map of index n with $n \geq 0$, say. Then there is an isomorphism $B \cong C \times \mathbb{R}^n$. Choosing one such we get a Fredholm map of index 0 between manifolds modelled on $C \times \mathbb{R}^n$

$$i \circ f : M \to N \subset N \times \mathbb{R}^n$$

By the theorem if N hence $N \times \mathbb{R}^n$ has a c-structure, this pulls back uniquely to M by $i \circ f$. It is clear that this gives the unique c-structure on M such that f becomes a c-map in the following extended sense: For any local representative $\psi_j f \varphi_i^{-1}$ of f the differential $d(\psi_j f \varphi_i^{-1})$ at any point differs from the projection $pr: C \times \mathbb{R}^n \to C$ by a completely continuous linear map. It is easy to see that the induced c-structure c_M depends on the splitting of B only up to completely continuous perturbations, i.e. if $B \cong' C \times \mathbb{R}^n$ is another isomorphism such that the composite $B \cong' C \times \mathbb{R}^n \cong B$ is in $GL_c(B)$ then the corresponding c'_M equals c_M. If in stead $f: M \to N$ is of index $n \leq 0$, there is isomorphism $C \cong B \times \mathbb{R}^n$ and an analogous statement, the projection $C \times \mathbb{R}^n \to C$ being replaced by the injection $B \to B \times \mathbb{R}^n$. Altogether

<u>Corollary</u>. Let M, N be manifolds modelled on Banach spaces B, C respectively and $f: M \to N$ a Fredholm map of index n. Let c_N be a c-structure on N.

If $n \geq 0$ ($n \leq 0$), then for any splitting $B \cong C \times \mathbb{R}^n$ ($C \cong B \times \mathbb{R}^n$) there is a unique c-structure $c_M = f^*c_N$ on M

making f a c-map. c_M depends on the splitting only up to completely continuous perturbations.

If $N_0 \subset N$ is a finite codimensional submanifold modelled on a closed subspace $C_0 \subset C$, then any c-structure on N induces canonically a c-structure on N_0 since the inclusion map $N_0 \subset N$ is Fredholm and C splits over C_0 canonically up to completely continuous perturbations. This makes N_0 a c-submanifold of N. Conversely, for any c-submanifold N_0 of N the inclusion $N_0 \subset N$ is a c-map, hence the c-structure of N_0 is the one induced from N. We collect these observations in

Lemma. In a manifold with c-structure every finite codimensional submanifold (as well as every finite dimensional submanifold) inherits a unique c-structure which makes it a c-submanifold.

The reader should notice that not every submanifold with c-structure need be a c-submanifold. We shall make use of the lemma later.

2. If M is a c-manifold modelled on B, the tangent bundle τ_M is a bundle with fiber B and structure group $GL_c(B)$. M is orientable if and only if τ_M can be reduced to an $SL_c(B)$-bundle. We shall look at general vector bundles with structure group $GL_c(B)$. Since the Banach space B will vary during the discussion, we omit explicit reference to it and write GL, GL_c, SL_c for the groups in question. A c-bundle is abbreviation for a vector bundle with structure group GL_c. All base spaces are

assumed paracompact with the homotopy type of CW-complexes.

A c-bundle is <u>orientable</u> if it admits a reduction of the structure group to SL_c , which is of index 2 in GL_c . Corresponding to the inclusion $SL_c \subset GL_c$ there is the double covering of classifying spaces $\rho\colon BSL_c \to BGL_c$ ([4] p.44). A c-bundle ξ over X is orientable if and only if a classifying map $f_\xi\colon X \to BGL_c$ lifts to BSL_c

$$\begin{array}{ccc} & \nearrow & BSL_c \\ & & \downarrow \rho \\ X & \longrightarrow & BGL_c \end{array}$$

The associated double covering ξ_2 of ξ is up to isomorphism the pull-back $f_\xi^*\rho$. Clearly ξ is orientable if and only if ξ_2 has a section. If ξ is the bundle $p\colon E \to X$, denote ξ_2 by $p_2\colon E_2 \to X$. The pull-back of ξ by p_2 is a c-bundle over E_2 . We have bundle maps

$$\begin{array}{ccccc} p_2^*E & \to & E & \to & EGL_c \\ \downarrow & & p\downarrow & & \downarrow \\ E_2 & \xrightarrow{p_2} & X & \xrightarrow{f_\xi} & BGL_c \end{array} \qquad \begin{array}{ccc} E_2 & \to & BSL_c \\ p_2\downarrow & & \downarrow \\ X & \xrightarrow{f_\xi} & BGL_c \end{array}$$

Hence $p_2^*\xi$ is classified by $f_\xi \circ p_2$ which lifts to BSL_c , i.e. $p_2^*\xi$ is always orientable. In the case where X is a c-manifold M and ξ is the tangent bundle τ_M , $p_2\tau_M$ is just the tangent bundle of the double covering manifold T_2M over M . Hence for any c-manifold M the associated double covering T_2M is an orientable c-manifold.

Since ξ_2 is a double covering, it is classified as such by a map from X into $\mathbb{RP}^\infty = K(\mathbb{Z}_2,1)$. Let

$$g_\xi : X \to \mathbb{RP}^\infty$$

be a classifying map, so that

$$w(\xi) = [g_\xi] \in [X, \mathbb{RP}^\infty] = H^1(X;\mathbb{Z}_2)$$

It follows that $w(\xi)$ is the only obstruction to orienting ξ . Clearly $w(\xi) = g_\xi^*(w)$, where $w \in H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ is the universal Stiefel - Whitney class.

Next observe that if ξ ξ' are c-bundles, then so are $\xi \times \xi'$ and $\xi \oplus \xi'$ (when base spaces coincide). We need the following

<u>Lemma</u>. Let ξ, ξ' be c-bundles over X . Then

$$w(\xi \oplus \xi') = w(\xi) + w(\xi')$$

The standard proofs from finite dimensional theory involving the total Stiefel - Whitney class of a direct sum of vector bundles cannot be used. In stead we proceed as follows.

Consider two c-bundles ξ, ξ' over X, X', respectively, with associated double coverings ξ_2, ξ_2' . Let ι, ι' be the fiber involutions on E_2, E_2', respectively (same notations as above). Then $\iota \times \iota'$ is a fixed point free involution on $E_2 \times E_2'$ which gives rise to a commutative diagram of covering maps

$$\begin{array}{ccc} E_2 \times E_2' & \xrightarrow{p_2 \times p_2'} & X \times X' \\ \downarrow & \nearrow_{p_2 \hat{\otimes} p_2'} & \\ E_2 \hat{\otimes} E_2' & & \end{array}$$

where $E_2 \otimes E_2' = E_2 \times E_2' / \iota \times \iota'$ and $E_2 \times E_2' \to E_2 \hat{\otimes} E_2'$ is the identification map. The notation $\xi_2 \hat{\otimes} \xi_2' : E_2 \hat{\otimes} E_2' \xrightarrow{p_2 \hat{\otimes} p_2'} X \times X'$ is appropriate because if $\bar{\xi}_2$ and $\bar{\xi}_2'$ denote the associated line bundles to ξ_2 and ξ_2' respectively, then $\bar{\xi}_2 \hat{\otimes} \bar{\xi}_2'$ is the associated line bundle to $\xi_2 \hat{\otimes} \xi_2'$. For the same reason, if $X' = X$ and $\Delta: X \to X \times X$ is the diagonal, we write $\xi_2 \otimes \xi_2'$ for $\Delta^*(\xi_2 \hat{\otimes} \xi_2')$. Note that we have cononical natural isomorphisms $\xi_2 \hat{\otimes} \xi_2' \cong (\xi \times \xi')_2$ and $\xi_2 \otimes \xi_2' \cong (\xi \oplus \xi')_2$.

The operation $\hat{\otimes}$ is functorial. Hence if $g: X \to \mathbb{RP}^\infty$ and $g': X' \to \mathbb{RP}^\infty$ are the classifying maps for ξ_2 and ξ_2', respectively, so that

$$\begin{array}{ccc} E_2 & \to & S^\infty \\ p_2\downarrow & & \downarrow \\ X & \overset{g}{\to} & \mathbb{RP}^2 \end{array} \qquad \begin{array}{ccc} E_2' & \to & S^\infty \\ p_2'\downarrow & & \downarrow \\ X & \overset{g'}{\to} & \mathbb{RP}^\infty \end{array}$$

are pull-backs, then so is

$$\begin{array}{ccc} E_2 \hat{\otimes} E_2' & \to & S^\infty \hat{\otimes} S^\infty \\ \downarrow & & \downarrow \\ X \times X & \overset{g \times g'}{\to} & \mathbb{RP}^\infty \times \mathbb{RP}^\infty \end{array}.$$

Let $\varkappa_2$ be the universal covering $S^\infty \to \mathbb{RP}^\infty$. Then if $h: \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ is a classifying map for $\varkappa_2 \hat{\otimes} \varkappa_2$, $k = h \circ (g \times g') \circ \Delta$ is a classifying map for $\xi_2 \otimes \xi_2'$. By the Kunneth formula it is immediate that $h^*(w) = w \times 1 + 1 \times w$, hence $(g \times g')^* h^*(w) = w(\xi) \times 1 + 1 \times w(\xi')$ and $k^*(w) = w(\xi) + w(\xi')$. In other words $w(\xi \oplus \xi') = w(\xi) + w(\xi')$. This completes the proof of the lemma.

Directly or from the relation with the associated line bundle it is easily seen that $\xi_2 \cong \xi_2'$ if and only if $\xi_2 \otimes \xi_2'$ is trivial, i.e. that $\xi_2 \cong \xi_2'$ if and only if $w(\xi) = w(\xi')$. A map $\varphi: X \to X'$ is said to be <u>orientable with respect to the c-bundles</u>

ξ, ξ' over X, X' if $\varphi^*\xi_2' \cong \xi_2$ or equivalently if $\varphi^*(w(\xi')) = w(\xi)$.

We close this section by some additional remarks concerning the lemma. If $0 \to \xi' \to \xi \to \xi'' \to 0$ is a split exact sequence of c-bundles, then there are vector bundle isomorphisms $\xi \cong \xi' \oplus \xi''$. In general this does not imply that $w(\xi)$ equals $w(\xi') + w(\xi'')$ (in contrast to the finite dimensional situation) since the different c-structures may not be related by the isomorphism. We point out a simple situation where they are related. First, if ξ, ξ' are two c-bundles over X with the same fiber it is clear what we mean by a c-homomorphism $\xi \to \xi'$. More generally, if the fiber of one bundle is product by a finite vector space of the other it is still clear what to mean by a c-homomorphism $\xi \to \xi'$ (cf. the corollary in section 1). Next consider two c-bundles ξ', ξ'' over X and the trivial split exact sequence

$$0 \to \xi \xrightarrow{(I,0)} \xi' \oplus \xi'' \xrightarrow{(0,I)} \xi'' \to 0$$

In this case we have $w(\xi' \oplus \xi'') = w(\xi') + w(\xi'')$ by the lemma. Perturbing the trivial situation slightly yields the rather obvious result:

Given a split exact sequence of c-bundles

$$0 \to \xi' \xrightarrow{(i',j')} \eta' \oplus \eta'' \xrightarrow{(i'',j'')} \xi'' \to 0$$

where $\xi' \xrightarrow{i'} \eta'$ and $\eta'' \xrightarrow{j''} \xi''$ are c-homomorphisms. Then $w(\xi') + w(\xi'') = w(\eta' \oplus \eta'') = w(\eta') + w(\eta'')$.

3. Let M, Q be Banach manifolds and $f\colon M \to Q$ a smooth map. Let c_M and c_Q be c-structures on M and Q , respectively.

Then there are will defined orientation classes $w(\tau_M) \in H^1(M;\mathbb{Z}_2)$ and $w(\tau_Q) \in H^1(Q;\mathbb{Z}_2)$. Write $w(f) = w(\tau_N) - f^*(w(\tau_Q)) = w(\tau_N) + f^*(w(\tau_Q))$. f is <u>orientable</u> with respect to c_M and c_Q if $w(f) = 0$.

Suppose next that N is a Banach manifold and $g: N \to Q$ a Fredholm map which is transversal to f. Let $P = M \times_Q N$ be the fibered product of f and g. Then P is a manifold, and we have a commutative diagram of smooth maps

$$\begin{array}{ccc} M & \xrightarrow{f} & Q \\ {\scriptstyle \bar{g}}\uparrow & & \uparrow{\scriptstyle g} \\ P & \xrightarrow{\bar{f}} & N \end{array}$$

Moreover, if g is Fredholm of index n, then either P is empty or $\bar{g}$ is Fredholm of index n. We may therefore assume that P (when $\neq \emptyset$) and N have c-structures $c_P = \bar{g}^*c_M$ and $c_N = g^*c_Q$ with respect to given splittings of the models (cf. corollary of section 1). We can now state the result we have been heading toward.

<u>Theorem</u>. Let M, N and Q be c-manifolds and $f: M \to Q$, $g: N \to Q$ smooth transversal maps. Let $P = M \times_Q N$ be the pull-back and $\bar{g}: P \to M$, $\bar{f}: P \to N$ the associated maps. Suppose g is Fredholm of index n. Then either P is empty or $\bar{g}$ is Fredholm of index n. Give P a c-structure $c_P = \bar{g}^*c_M$ and assume $c_N = g^*c_Q$ with respect to some splitting of the model. Then if g is orientable, so is $\bar{g}$, and if f is orientable, so is $\bar{f}$.

Before giving the proof we state some corollaries

Corollary 1. Let M, Q be c-manifolds, $Q_o \subset Q$ a submanifold of finite codimension n and $f: M \to Q$ a smooth map transversal to Q_o. Write M_o for $f^{-1}Q_o$ and f_o, g_o, g for the maps $M_o \to Q_o$, $M_o \subset M$, $Q_o \subset Q$. Then M_o is either empty or an n-codimensional submanifold of M, and if g, respectively f, is orientable, so is g_o, respectively f_o.

This results from the theorem and the lemma of section 1. Specializing further we get

Corollary 2. If M, Q and Q_o are orientable, so is M_o.

Corollary 3. If Q is finite dimensional and $q \in Q$ is a regular value of f, then the inclusion $f^{-1}q \subset M$ is always orientable. In partiqular $f^{-1}q$ is orientable whenever M is.

Proof Let Q be an open coordinate neighborhood of q. Applying corollary 1 to the situation

$$\begin{array}{ccc} f^{-1}Q & \to & Q \\ g_o' \cup & & \cup g' \\ f^{-1}q & \overset{f_o}{\to} & q \end{array}$$

we conclude that g_o' is orientable. However $f^{-1}Q$ is open in M and so the inclusion $f^{-1}Q \subset M$ is certainly orientable. It follows that the composite $f^{-1}q \subset f^{-1}Q \subset M$ is orientable.

Corollary 4. Let N, Q be c-manifolds and $g: N \to Q$ a Fredholm map of index p such that $g^*c_Q = c_N$ (with respect to a splitting of the model). Let $Q^o \subset Q$ be an m-dimensional submanifold and write N^o for $g^{-1}Q^o$ and g^o, f^o, f for the maps $N^o \to Q^o$, $N^o \subset N$, $Q^o \subset Q$. Then N^o is either empty or an $(m+p)$-dimensional submanifold of N. If f, respectively g, is orientable so is f^o, respectively g^o.

Proof. That N^o is finite dimensional of dimension $m+p$ is an elementary consequence of the Fredholm property. Since N^o and Q^o are finite dimensional, they have unique c-structures and therefore $c_{N^o} = g^{o*}c_{Q^o}$. Now apply the theorem.

Corollary 5. If N, Q and Q_o are orientable, so is N_o.

Corollaries 2 and 5 and the last part of corollary 3 are due to Elworthy and Tromba, [3]. Their proofs appeal directly to the definition of a orientation (cf. section 1) hence are more elementary but quite computational. We turn to the proof of the theorem of this section.

Since $f\colon M \to Q$ and $g\colon N \to Q$ are transversal, there results a split exact sequence of vector bundles over $P = M \times_Q N$

$$0 \to \tau_P \to \bar{g}^*\tau_M \oplus \bar{f}^*\tau_N \to \bar{f}^*g^*\tau_Q \to 0$$

induced by the maps in the pull-back diagram. This follows easily from the standard transversality theorem in the case where g is the inclusion of a submanifold, cf. [1] p.45 . The general case is deduced from the special by observing that f is transversal to g if and only if $f \times g : M \times N \to Q \times Q$ is transversal to the diagonal inclusion $\Delta \subset Q \times Q$. This yields the split exact sequence above. By the assumptions in the theorem all the bundles are c-bundles such that the maps $\tau_P \to \bar{g}^*\tau_M$ and $\tau_N \to g^*\tau_Q$ induced from $\bar{g}: P \to M$ and $g: N \to Q$ are c-homomorphisms. Therefore also the pull-back $\bar{f}^*\tau_N \to \bar{f}^*g^*\tau_Q$ is a c-homomorphism. Then the final remarks of section 2 apply and shows that

$$w(\tau_P) + w(\bar{f}^*g^*\tau_Q) = w(\bar{g}^*\tau_M) + w(\bar{f}^*\tau_N)$$

in $H^1(P;\mathbb{Z}_2)$ or, since the Stiefel-Whitney class is functorial,

$$w(\tau_P) + \bar{f}^*g^*(w(\tau_Q)) = \bar{g}^*(w(\tau_M)) + \bar{f}^*(w(\tau_N)) \ .$$

Suppose g is orientable. Then $g^*(w(\tau_Q)) = w(\tau_N)$, hence $\bar{g}^*(w(\tau_M)) = w(\tau_P)$ showing that $\bar{g}$ is orientable. Similarly one gets that $\bar{f}$ is orientable if f is orientable (using the fact that $\bar{f}^*g^*(w(\tau_Q)) = \bar{g}^*f^*(w(\tau_Q))$).

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