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# Stability of Upwind Mixed Discretizations of Convection Diffusion Equations

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**Abstract**

In this thesis we make use of a complex of exponentially upwinded differential forms to study a mixed discretization of convection diffusion equations. Interpolation operators and smoothed projections with various properties are constructed, using ideas from finite element exterior calculus and finite element systems. Several continuity and infsup conditions for the mixed formulations are proven. We identify possible candidates for *natural norms* of the problem and give a full analysis of a 1-dimensional discretization in the regime of vanishing viscosity using these norms, with stability proven up to logarithmic terms in the viscosity.

# Contents

# Chapter 1

# Introduction

In this thesis we shall study exponentially upwinded and downwinded mixed discretizations of the convection diffusion equations

$$\begin{aligned} \nabla \cdot (\alpha \nabla p - \beta p) &= f \\ p|_{\partial\Omega} &= 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} -\alpha \Delta p + \beta \cdot \nabla p &= f \\ p|_{\partial\Omega} &= 0 \end{aligned} \tag{1.2}$$

where the convection $\beta \in \mathbb{R}^d$, the viscosity $\alpha \in \mathbb{R}$, $f \in L^2$ and $p$ is the unknown function. We will study these discretizations for $\Omega$ a convex domain, but some of our proofs will make use of interpolators and smoothed projections that can be constructed when $\Omega$ is any polygonal Lipschitz domain. Note that (1.1) and (1.2) are essentially the same equation since $\nabla \cdot (\beta p) = \beta \cdot \nabla p$ for $\beta$ constant. We will start by investigating the case where $\alpha \sim |\beta|$ before moving on to the more difficult case $\alpha << |\beta|$. This last case turns the above equations into singularly perturbed problems and we get the creation of boundary layers in the solutions, which causes many discretization schemes to become unstable. A lot of work has been done on finding stabilised methods for these kinds of problems, and probably the most famous is the Streamline Upwind Petrov-Galerkin method (SUPG). See [15] for a detailed overview of these methods. A description of mixed formulations in general can be found in [3]. We will follow the ideas of [6], [7] and [9] in using an upwind discretization with test functions that are piecewise exponential. Christiansen *et al.* proves stability for the Petrov-Galerkin formulation of a convection diffusion equation with test functions of this type in [6], however stability of mixed formulations still remains open. This idea is related to methods of exponential fitting [5],[21] and is contained in the frameworks of finite element exterior calculus (FEEC) [1] and finite element systems (FES) [7].

In the next section we define the notation used throughout this thesis, and give a very general overview of differential forms which we shall need later. In the last section of this chapter we present our reading of the beginning of [2] to discuss the wellposedness of so-called generalized saddle-point problems, of which the mixed formulations of (1.1) and (1.2) are examples of. Here, we find that finite dimensional approximations to generalized saddle-point problems are stable if and only if several infsup-conditions are satisfied. In Chapter 2 we look at interpolators onto two different upwind complexes, one of which is defined in [7], and investigate when they are bounded uniformly in the small parameter $\alpha$. These interpolators are then used in Chapter 3 to show the existence of smoothed projections onto our upwind spaces, using constructions based on those found in [1], [8] and [7]. Lastly we use these interpolators and projections in Chapter 4 to show infsup-conditions for our upwind mixed discretizations.

## 1.1 Notation and Preliminaries

### Notation and Preliminaries

Throughout this thesis $\Omega$ will denote an open, bounded, connected subset of $\mathbb{R}^d$ with polygonal Lipschitz boundary. In Chapter 4 we will further assume it is a convex domain, or equivalently, a rectangular domain. $\mathcal{T}$ will be a partition of $\Omega$ into a finite set of $d$-cubes, called a cubic or rectangular mesh. For any $n$-dimensional cube $T \in \mathcal{T}_h$, $\Delta_j(T)$ will denote the set of $j$-dimensional subcubes of $T$, where $j \leq n$. We consider a family of partitions $\{\mathcal{T}_h\}$ indexed by a discretization parameter

$$h = \max_{T \in \mathcal{T}_h} \{h_T\}, \qquad h_T = \operatorname{diam} T.$$

We will assume for this family of partitions, that there exist a constant $C_{\text{mesh}} > 0$ independent of $\mathcal{T}_h$, called the mesh regularity constant, such that

$$h_T^d \leq C_{\text{mesh}} |T|, \qquad T \in \mathcal{T}_h, \tag{1.3}$$

where $|T|$ denotes the volume of $T$. This is called shape-regularity. If we replace $h_T$ with $h$ in (1.3) we get the additional assumption of quasi-uniformity, i.e. a uniform bound on $h/h_T$. For $T \in \mathcal{T}_h$ let

$$\mathcal{T}_h(T) = \{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$$

denote its macroelement and $T^*$ the corresponding domain. The notation $f \preceq g$ means that $f \leq Cg$, for a constant $C > 0$. This notation will only be used when the constant is independent of the parameters of interest (typically $\alpha$ and $h$).

The space of smooth differential $k$-forms on $\Omega$ will be denoted as $\Lambda^k(\Omega)$, when it is obvious from context we drop $\Omega$ and just write $\Lambda^k$. A continuous differential form will be denoted as $C\Lambda^k(\Omega)$. For a complete introduction to differential forms we refer to [20], however we shall note the most important properties here. In all dimensions $d$, the space $\Lambda^0$ is just the space of smooth functions $\mathbb{R}^d \to \mathbb{R}$, and the space $\Lambda^1$ can be described using the dual basis of vectors as follows. If $v : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field, it is of the form

$$v(x) = g_1(x)e_1 + ... + g_d(x)e_d,$$

where $\{e_1, ..., e_d\}$ is the standard basis for $\mathbb{R}^d$ and $g_i$ is a smooth function for all $i$. Letting $\{\mathrm{d}x_1, ..., \mathrm{d}x_d\}$ be the dual basis to $\{e_1, ..., e_d\}$, given by $\mathrm{d}x_i(e_j) = \delta_{ij}$, then $\omega \in \Lambda^1$ can be written as

$$\omega_x = f_1(x)\,\mathrm{d}x_1 + ... + f_d(x)\,\mathrm{d}x_d,$$

for smooth functions $f_i$. For $\omega \in \Lambda^k$ and $\eta \in \Lambda^j$, the wedge product $\omega \wedge \nu \in \Lambda^{k+j}$ is defined by

$$(\omega \wedge \eta)_x(v_1, ..., v_{k+j}) = \sum_{\sigma} (\mathrm{sign}\sigma) \omega_x(v_{\sigma(1)}, ..., v_{\sigma(k)}) \eta_x(v_{\sigma(k+1)}, ..., v_{\sigma(k+j)})$$

where the sum is over all permutations $\sigma$ of $\{1, ..., k+j\}$ such that $\sigma(1) < ... < \sigma(k)$ and $\sigma(k+1) < ... < \sigma(k+j)$. Observe that $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$ and $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. Any $\omega \in \Lambda^k$ then has a unique representation of the form

$$\omega_x = \sum_{\sigma \in \Sigma(k,d)} f_\sigma(x) \, \mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(k)}, \tag{1.4}$$

where $f_i$ is a smooth function, and $\Sigma(k, d)$ is the set of increasing maps $\{1, ..., k\} \to \{1, ..., d\}$. For $\omega \in \Lambda^k$ a $k$-form given by (1.4), the exterior derivative $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$ is defined as

$$\mathrm{d}\omega = \sum_{\sigma} \sum_{i=1}^{d} \frac{\partial f_\sigma}{\partial x_i} \, \mathrm{d}x_i \wedge \mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(k)}.$$

It has the property that $\mathrm{d} \circ \mathrm{d} = 0$, and satisfies the Leibniz rule

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \nu + (-1)^k \omega \wedge \mathrm{d}\eta, \qquad \omega \in \Lambda^k, \eta \in \Lambda^j.$$

In 3 dimensions $\mathrm{d} : \Lambda^0 \to \Lambda^1$ corresponds to the gradient operator $\nabla$ for vector fields, $\mathrm{d} : \Lambda^1 \to \Lambda^2$ corresponds to the curl operator $\nabla \times$, and $\mathrm{d} : \Lambda^2 \to \Lambda^3$ corresponds to the divergence operator $\nabla \cdot$. Since the space $\Lambda^k$ of differential k-forms has $\binom{d}{k} = \binom{d}{d-k}$ basis elements, we can define the Hodge star operator $\star : \Lambda^k \to \Lambda^{d-k}$ by

$$< \star(\omega), \eta > \, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_d = \omega \wedge \eta,$$

where $< \cdot, \cdot >$ is the usual $L^2$ inner product. Note that $\star(\mathrm{d}x^i)$ and $\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge ... \wedge \mathrm{d}x^d$ are equal up to a sign. For $\phi$ a smooth map from $\Omega \subset \mathbb{R}^d$ to $\Omega' \subset \mathbb{R}^n$, the pullback $\phi^* : \Lambda^k(\Omega') \to \Lambda^k(\Omega)$ is given by

$$(\phi^*\omega)_x(v_1, ..., v_k) = \omega_{\phi(x)}(D\phi_x(v_1), ..., D\phi_x(v_k)),$$

where the linear map $D\phi_x : \mathbb{R}^d \to \mathbb{R}^n$ is the derivative of $\phi$ at $x$. The pullback commutes with the exterior derivative,

$$\phi^* \circ \mathrm{d} = \mathrm{d} \circ \phi^*,$$

and it distributes over the wedge product,

$$\phi^*(\omega \wedge \eta) = \phi^*\omega \wedge \phi^*\eta.$$

Let $L^p\Lambda^k(\Omega)$ denote the space of differential forms such that the functions $f_i$ in (1.4) are in $L^p(\Omega)$ and let $H^1\Lambda^k(\Omega)$ denote the space of differential forms such that the functions $f_i$ in (1.4) are in $H^1(\Omega)$. The Sobolev space of differential $k$-forms $W^p_{\mathrm{d}}\Lambda^k(\Omega)$ is then defined by

$$W^p_{\mathrm{d}}\Lambda^k(\Omega) = \{\omega \in L^p\Lambda^k(\Omega) : \mathrm{d}\omega \in L^p\Lambda^{k+1}(\Omega)\},$$

where $\mathrm{d}\omega$ is defined in a suitable weak sense, e.g. such that the integration by parts identity

$$\int_\Omega \mathrm{d}\omega \wedge \eta = (-1)^{k-1} \int_\Omega \omega \wedge \mathrm{d}\eta$$

holds for all compactly supported $\eta \in \Lambda^{d-k-1}$. For $p = 2$, $W^p_{\mathrm{d}}\Lambda^k(\Omega)$ is denoted as $H\Lambda^k(\Omega)$. Again, we will drop $\Omega$ when it is obvious from context. Note that $W^p_{\mathrm{d}}\Lambda^{d-1}$ corresponds to the space of vector fields $L^p_{\mathrm{div}}$ given by

$$L^p_{\text{div}}(\Omega) = \{u \in L^p : \nabla \cdot u \in L^p\}$$

where $\nabla \cdot u$ denotes the weak divergence of $u$. For $p = 2$ we will denote $L^p_{\text{div}}$ as $H_{\text{div}}$. In 3 dimensions the differential 2-form corresponding to the vector field $u = (u_1, u_2, u_3)$ can be written as [1]

$$u_1 \star (\,\mathrm{d}x) + u_2 \star (\,\mathrm{d}y) + u_3 \star (\,\mathrm{d}z) = u_3 \,\mathrm{d}x \wedge \,\mathrm{d}y - u_2 \,\mathrm{d}x \wedge \,\mathrm{d}z + u_1 \,\mathrm{d}y \wedge \,\mathrm{d}z.$$

In Chapter 2 and 3 we discuss interpolators and smoothed projections for differential forms, while in Chapter 4 we use them on vector fields in $L^p_{\text{div}}$. This is a slight abuse of notation, since we do not make it explicit that a transformation similar to the above has been used.

## 1.2 Generalized Saddle-Point Problems

The mixed formulations of the convection diffusion equations are examples of what's usually referred to as generalized saddle-point problems (see e.g. [14] and [10]). We need to prove several continuity and infsup conditions to ensure such problems are wellposed. We therefore start with an abstract presentation of such variational problems following [2]. For $X_i$, $Y_i$ $(i = 1, 2)$ real reflexive Banach spaces and $a : X_2 \times X_1 \to \mathbb{R}$, $b_i : X_i \times Y_i \to \mathbb{R}$ $(i = 1, 2)$ continuous bilinear forms, we consider the following abstract variational problem. Find $(u, p) \in X_2 \times Y_1$ such that

$$\begin{aligned} a(u, v) + b_1(p, v) &= < g, v > \qquad && \forall v \in X_1, \\ b_2(q, u) &= < f, q > \qquad && \forall q \in Y_2. \end{aligned} \tag{1.5}$$

It is wellposed when the following conditions hold [2]. First there must exist constants $C_0, C_i > 0$ such that we have continuity conditions

$$\begin{aligned} |a(u, v)| &\leq C_0 \,||u||_{X_2}||v||_{X_1} \\ |b_i(p, v)| &\leq C_i \,||p||_{Y_i}||v||_{X_i} \end{aligned} \tag{1.6}$$

for $i = 1, 2$. Secondly, the $b_i$'s must satisfy the infsup conditions

$$\sup_{v \in X_i} \frac{b_i(p,v)}{||v||_{X_i}} \geq B_i ||p||_{Y_i}, \qquad \forall p \in Y_i, \tag{1.7}$$

for some constants $B_i > 0$. Lastly, letting $K_i = \{u \in X_i : b_i(q,u) = 0 \quad \forall q \in Y_i\}$, we need $a$ to either satisfy the infsup conditions

$$\begin{aligned} \sup_{v \in K_1} \frac{a(u,v)}{||v||_{X_1}} &\geq A_1 ||u||_{X_2}, \qquad && \forall u \in K_2, \\ \sup_{u \in K_2} a(u,v) &> 0, && \forall v \in K_1 \setminus \{0\}, \end{aligned} \tag{1.8}$$

or

$$\begin{aligned} \sup_{u \in K_2} \frac{a(u,v)}{||u||_{X_2}} &\geq A_2 ||v||_{X_1}, \qquad && \forall v \in K_1, \\ \sup_{v \in K_1} a(u,v) &> 0, && \forall u \in K_2 \setminus \{0\}, \end{aligned} \tag{1.9}$$

where $A_1.A_2 > 0$ are constants. According to Remark 2.1 in [2] the above two conditions (1.8) and (1.9) are equivalent. The following proposition is important.

**Proposition 1.** *Conditions* (1.6), (1.7) *and* (1.8) *(or* (1.6), (1.7) *and* (1.9)*) are equivalent to existence and uniqueness of a solution to* (1.5).

This is Theorem 2.1 of of [2] and the proof is found there.

**The finite-dimensional approximation.** Let $Y_i^h \subset Y_i$ and $X_i^h \subset X_i$ be finite dimensional subspaces. To ensure stability and convergence of a finite-dimensional approximation to (1.5), defined as find $u \in X_2^h$ and $p \in Y_1^h$ such that

$$\begin{aligned} a(u,v) + b_1(p,v) &= < g, v > \qquad && \forall v \in X_1^h \\ b_2(q,u) &= < f, q > && \forall q \in Y_2^h, \end{aligned} \tag{1.10}$$

we need the analogous discrete infsup conditions to (1.7), (1.8) and (1.9) to be satisfied as well. In other words there must exist a constant $\hat{B}_i > 0$ independent of $h$ such that

$$\sup_{v\in X_i^h} \frac{b_i(p,v)}{||v||_{X_i}} \geq \hat{B}_i ||p||_{Y_i}, \qquad \forall p \in Y_i^h, \tag{1.11}$$

for $i = 1, 2$. Furthermore, letting $K_i^h = \{u \in X_i^h : b_i(q,u) = 0 \quad \forall q \in Y_i^h\}$, we need the bilinear form $a$ to either satisfy the infsup conditions

$$\begin{aligned} &\sup_{v\in K_1^h} \frac{a(u,v)}{||v||_{X_1}} \geq \hat{A}_1 ||u||_{X_2}, &&\forall u \in K_2^h, \\ &\sup_{u\in K_2^h} a(u,v) > 0, &&\forall v \in K_1 \setminus \{0\}, \end{aligned} \tag{1.12}$$

or

$$\begin{aligned} &\sup_{u\in K_2^h} \frac{a(u,v)}{||u||_{X_2}} \geq \hat{A}_2 ||v||_{X_1}, &&\forall v \in K_1^h, \\ &\sup_{v\in K_1^h} a(u,v) > 0, &&\forall u \in K_2 \setminus \{0\}, \end{aligned} \tag{1.13}$$

for constants $\hat{A}_1, \hat{A}_2 > 0$ independent of $h$. The error estimates can be found in [2]. Note that they depend on the above continuity and infsup-constants.

# Chapter 2

# Interpolators

We are primarily interested in finding $L^2$ stable projections onto exponentially upwinded $H_{\text{div}}$-elements, but we will look at projections onto slightly more general spaces. First we must study interpolators onto these spaces. Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal Lipschitz domain and $\mathcal{T}_h$ a rectangular grid on $\Omega$. Furthermore, let $\beta \in \mathbb{R}^d$ be a constant vector field and $\alpha << |\beta|$ a small parameter. An important assumption in [6] was the flow being aligned with the mesh, i.e. $\beta = (\beta_1, 0, ..., 0)$, and upwinding was only done in that direction. The test functions used were, in 2 dimensions, piecewise of the form

$$a_1 + a_2 e^{\frac{-\beta_1 x}{\alpha}} + a_3 y + a_4 e^{\frac{-\beta_1 x}{\alpha}} y, \tag{2.1}$$

and similarly for higher dimensions. The corresponding upwinding in all directions to (2.1) would in 2 dimensions belong to the differential complex

$$\begin{aligned}
\Lambda_h^0[\alpha] &= \{\omega : \omega|_T = a_1^T + a_2^T e^{\frac{-\beta_1 x}{\alpha}} + a_3^T e^{\frac{-\beta_2 y}{\alpha}} + a_4^T e^{\frac{-\beta_1 x}{\alpha}} e^{\frac{-\beta_2 y}{\alpha}}\}, \\
\Lambda_h^1[\alpha] &= \{\omega : \omega|_T = e^{\frac{-\beta_1 x}{\alpha}} (a_2^T + b_2^T e^{\frac{-\beta_2 y}{\alpha}})\, \mathrm{d}x + e^{\frac{-\beta_2 y}{\alpha}} (a_1^T + b_1^T e^{\frac{-\beta_1 x}{\alpha}})\, \mathrm{d}y\}, \\
\Lambda_h^2[\alpha] &= \{\omega : \omega|_T = a^T e^{\frac{-\beta_1 x}{\alpha}} e^{\frac{-\beta_2 y}{\alpha}}\, \mathrm{d}x \wedge \mathrm{d}y\},
\end{aligned} \tag{2.2}$$

which has the property that

$$\Lambda_h^0[\alpha] \xrightarrow{\mathrm{d}} \Lambda_h^1[\alpha] \xrightarrow{\mathrm{d}} \Lambda_h^2[\alpha].$$

In example 5.31 of [7] we can find the defining relations for this complex (and the corresponding higher dimensional ones) as

$$\begin{aligned} \mathrm{d}_\alpha^* \,\mathrm{d}\omega &= 0, \\ \mathrm{d}_\alpha^* \omega &= 0, \end{aligned}$$

where $\mathrm{d}_\alpha$ is defined by

$$\mathrm{d}_\alpha \omega = \alpha \,\mathrm{d}\omega + \beta \wedge \omega, \tag{2.3}$$

with the 1-form $\beta = \beta_1 \,\mathrm{d}x_1 + ... + \beta_d \,\mathrm{d}x_d$ and $\mathrm{d}^*$ denoting the formal adjoint of $\mathrm{d}$. We shall not use test functions from this complex in our upwinding scheme, however, since there are two problems with these function spaces for small $\alpha$. First, we would like interpolators onto our upwind spaces to be have bounded $L^2$ norm uniformly in $\alpha$ when $\alpha \to 0$. This is not the case however, as we shall see from the following argument. The usual degrees of freedom for an interpolator $\mathrm{I}$ onto a finite element space is given by

$$\int_e \mathrm{I}\omega = \int_e \mathrm{Tr}_e \omega \qquad e \in \Delta_k(\mathcal{T}_h)$$

in the cases $k \in \{d-1, d\}$. Looking at the last space $\Lambda_h^2[\alpha]$ in (2.2) we observe that interpolating $\omega \in L^p$ onto this space on a rectangle $T$ is given by the integral

$$\int_T \omega = \int_T \mathrm{I}(\omega) = \int_T a e^{-\frac{\beta_1 x + \beta_2 y}{\alpha}} \,\mathrm{d}x \,\mathrm{d}y.$$

Assuming $\int_T \omega = 1$ and $T = [0,1]^2$ to simplify, we observe that the constant is determined by

$$a = 1/(\alpha^2\beta_1\beta_2[1-e^{-\frac{\beta_1}{\alpha}}][1-e^{-\frac{\beta_2}{\alpha}}]).$$

Calculating the $L^p$ norm of $\mathrm{I}(w)$ on $T$ we get

$$\begin{aligned}||\mathrm{I}(\omega)||_{L^p} &= \Big(\int_0^1\int_0^1 a^p e^{-p\frac{\beta_1 x+\beta_2 y}{\alpha}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{(1/p)},\\ &= a\alpha^{2/p}\Big(\frac{\beta_1\beta_2}{p^2}[1-e^{-\frac{p\beta_1}{\alpha}}][1-e^{-\frac{p\beta_2}{\alpha}}]\Big)^{(1/p)} \sim \alpha^{2(1/p-1)}.\end{aligned}$$

Hence, $||\mathrm{I}(\omega)||_{L^p} \to \infty$ when $\alpha \to 0$ for any $1 < p < \infty$ and we can not get a bound $||\mathrm{I}(\omega)||_{L^p} \leq C||\omega||$ for any norm $||\cdot||$ on $\omega$ where the constant $C > 0$ is independent of $\alpha$. Specifically, $L^2$ boundedness uniformly in $\alpha$ is not possible. Since interpolating $\omega$ (with well-defined traces) onto the space $\Lambda_h^1[\alpha]$ also involves an integration of $e^{\frac{-\beta x}{\alpha}}$ we will get the same problem for this space.

The second problem with interpolation onto the complex (2.2) is that we would prefer the existence of convergence estimates that are independent of $\alpha$ when $h \to 0$. This is only possible for the 0-forms in (2.2) since the others will, for fixed constants, converge to $0\,\mathrm{d}x + 0\,\mathrm{d}y$ or $0\,\mathrm{d}x\,\mathrm{d}y$ when $\alpha \to 0$. The same will happen in all dimensions $d$ for $k$-forms with $k > 0$. Since the choice of constants depend on $\alpha$ we can not expect convergence estimates independent of $\alpha$. We will therefore in the case of $(L^2, H_{\mathrm{div}})$ elements study a new complex defined by the relations

$$\begin{aligned}\mathrm{d}^*\,\mathrm{d}_\alpha\omega &= 0,\\ \mathrm{d}^*\omega &= 0,\end{aligned}\tag{2.4}$$

which in two dimensions is the complex

$$\begin{aligned}\Lambda_h^0[\alpha] &= \{\omega : \omega|_T = a_1^T + a_2^T e^{\frac{-\beta_1 x}{\alpha}} + a_3^T e^{\frac{-\beta_2 y}{\alpha}} + a_4^T e^{\frac{-\beta_1 x}{\alpha}} e^{\frac{-\beta_2 y}{\alpha}}\},\\ \Lambda_h^1[\alpha] &= \{\omega : \omega|_T = (a_2^T + b_2^T e^{\frac{-\beta_2 y}{\alpha}})\,\mathrm{d}x + (a_1^T + b_1^T e^{\frac{-\beta_1 x}{\alpha}})\,\mathrm{d}y\},\\ \Lambda_h^2[\alpha] &= \{\omega : \omega|_T = a^T\,\mathrm{d}x \wedge \mathrm{d}y\}.\end{aligned}$$

It has the property that

$$\Lambda_h^0[\alpha] \xrightarrow{\mathrm{d}_\alpha} \Lambda_h^1[\alpha] \xrightarrow{\mathrm{d}_\alpha} \Lambda_h^2[\alpha],$$

and we observe that the exterior derivative $\mathrm{d}$ has been replaced by the operator $\mathrm{d}_\alpha$.

**Proposition 2.** *For* $\mathrm{d}_\alpha$ *defined as in* (2.3) *we have* $\mathrm{d}_\alpha^2 = \mathrm{d}_\alpha \circ \mathrm{d}_\alpha = 0$.

*Proof.* Let a differential k-form $\omega$ be given. We then have

$$\begin{aligned}\mathrm{d}_\alpha(\mathrm{d}_\alpha\omega) &= \mathrm{d}_\alpha(\alpha\,\mathrm{d}\omega + \beta\wedge\omega) = \alpha\,\mathrm{d}(\alpha\,\mathrm{d}\omega + \beta\wedge\omega) + \beta\wedge(\alpha\,\mathrm{d}\omega + \beta\wedge\omega)\\ &= \alpha\,\mathrm{d}(\beta\wedge\omega) + \beta\wedge(\alpha\,\mathrm{d}\omega) = -\alpha\beta\wedge(\mathrm{d}\omega) + \beta\wedge(\alpha\,\mathrm{d}\omega) = 0\end{aligned}$$

where we have used that $\mathrm{d}^2 = \mathrm{d}\circ\mathrm{d} = 0$, $\beta\wedge\beta = 0$ and $\mathrm{d}(\beta\wedge\omega) = -\beta\wedge(\mathrm{d}\omega)$ for $\beta$ a constant 1-form. □

Observe how the complexes defined by (2.4) must always contain the constants, so it's possible to have convergence estimates independent of $\alpha$, as we prove in the next chapter. Also note the similarity of the above 1-forms to the usual lowest order Raviart-Thomas elements, being piecewise

$$(a_1 + b_1 x)\,\mathrm{d}y + (a_2 + b_2 y)\,\mathrm{d}x. \tag{2.5}$$

We will look at the complex (2.4) first with the usual degrees of freedom, and then modified with degrees of freedom such that the interpolators commutes with $\mathrm{d}_\alpha$.

## 2.1 Standard degrees of freedom

For a k-form $\omega$ the standard degrees of freedom for the interpolators $\mathrm{I}_h^\alpha$ onto the complex (2.4) is given by

$$\int_e \mathrm{I}_h^\alpha\omega = \int_e \mathrm{Tr}_e\omega \qquad e\in\Delta_k(\mathcal{T}_h)$$

in the cases $k \in \{d-1, d\}$. An important property for the $d-1$-forms in dimension $d$ in this complex is that they are of the form $\sum_i f_i(x_i, \alpha) \star(\,\mathrm{d}x_i)$, where $f_i(x_i, \alpha) = a_i + b_i e^{\frac{-\beta_i x_i}{\alpha}}$. We therefore look at the slightly more general space $\Lambda_h[\alpha] = \Lambda_h^{d-1}[\alpha]$ of $d-1$-forms defined by

$$\Lambda_h[\alpha] = \{\omega \in W_{\mathrm{d}}^p \Lambda^{d-1}(\Omega) : \quad \omega|_T = \sum_i f_i(x_i, \alpha) \star(\,\mathrm{d}x_i), T \in \mathcal{T}_h\}, \tag{2.6}$$

Recall that $\star(\,\mathrm{d}x_i)$ and $\mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge ... \wedge \mathrm{d}x_d$ are equal up to a sign. So, for upwinding in all directions we have

$$f_i(x_i, \alpha) = a_i + b_i e^{\frac{-\beta_i x_i}{\alpha}} \qquad a_i, b_i \in \mathbb{R}, \tag{2.7}$$

and for upwinding in the direction of the flow, $\beta = (\beta_1, 0, .., 0)$, we have

$$\begin{aligned} f_1(x_1, \alpha) &= a_1 + b_1 e^{\frac{-\beta_1 x_1}{\alpha}} && a_1, b_1 \in \mathbb{R}, \\ f_i(x_i, \alpha) &= a_i + b_i x_i && a_i, b_i \in \mathbb{R} \quad i = 2, ..., d. \end{aligned} \tag{2.8}$$

The space $H^s$ has a well-defined trace operator for $s > 1/2$ and for $\omega \in H^s\Lambda^k(\Omega)$ we have $\mathrm{Tr}_e\omega \in H^{s-1/2}(\partial\Omega)$ [19]. This gives us the following important proposition.

**Proposition 3.** *The interpolators* $\mathrm{I}_h^\alpha : H^1\Lambda^{d-1} \to \Lambda_h[\alpha]$ *for functions* $f_i$ *of the form* (2.7) *have the property that*

$$||\mathrm{I}_h^\alpha(\omega)||_{L^2} \leq C||\omega||_{H^1}$$

*for a constant* $C > 0$ *independent of* $h$ *and* $\alpha$.

*Proof.* Let $\omega \in H^1\Lambda^{d-1}$ be given and let $f_i$ be an exponential of the form (2.7). Since the trace operator is well-defined and we are using a rectangular mesh, the degrees of freedom for the interpolator $\mathrm{I}_h^\alpha$ on the rectangle $T$ are given by integrals of the form

$$\int_e \mathrm{Tr}_e \omega = \int_e f_i(x_i, \alpha) \star (\,\mathrm{d}x_i) = f_i(x_i, \alpha) \int_e \star(\,\mathrm{d}x_i)$$

for $i = 1, ..., d$ and $e \in \Delta_{d-1}(T)$. We observe that this is equivalent to keeping the value of $f_i(x_i, \alpha)$ fixed on both "ends" of the rectangle $T$ in the direction $i$. The function $f_i$ of the form (2.7) can therefore not blow up when $\alpha \to 0$. Hence, $||\mathrm{I}_h^\alpha \omega||_{L^2}$ is bounded for any given $\omega$, and so $||\mathrm{I}_h^\alpha \omega||_{L^2} \le C||\omega||_{H^1}$, where $C > 0$ is a constant independent of $\alpha$. □

**Proposition 4.** *The interpolators* $\mathrm{I}_h^\alpha : H^1\Lambda^{d-1} \to \Lambda_h[\alpha]$ *for functions* $f_i$ *of the form* (2.7) *have the convergence estimates*

$$||\omega - \mathrm{I}_h^\alpha(\omega)||_{L^2} \le C\frac{h}{\alpha}||\omega||_{H^1}$$

*for a constant* $C > 0$ *independent of* $h$ *and* $\alpha$.

*Proof.* Let $\omega \in H^1\Lambda^{d-1}$ be given, and let $\mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)$ denote the usual space of piecewise constant forms. We then have on the rectangle $T$

$$\begin{aligned} ||\omega - \mathrm{I}_h^\alpha \omega||_{L^2\Lambda^{d-1}(T)} &= \inf_{\eta \in \mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)} ||\omega - \eta - \mathrm{I}_h^\alpha(\omega - \eta)||_{L^2\Lambda^{d-1}(T)} \\ &= \inf_{\eta \in \mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)} ||(I - \mathrm{I}_h^\alpha)(\omega - \eta)||_{L^2\Lambda^{d-1}(T)}, \\ &\le C \inf_{\eta \in \mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)} ||\omega - \eta||_{H^1\Lambda^{d-1}(T)} \\ &\le C(T)|\omega|_{H^1}, \end{aligned}$$

where the last inequality follows from the Bramble-Hilbert lemma or Clément interpolation (see e.g [4]). It then follows from a standard scaling argument that $C(T) = C'\frac{h}{\alpha}$ for a constant $C' > 0$ independent of $h$ and $\alpha$.

□

**Corollary 5.** *The interpolators* $\mathrm{I}_h^\alpha : H^1\Lambda^{d-1} \to \Lambda_h[\alpha]$ *for functions* $f_i$ *of the form* (2.7) *have the bound*

$$(||\mathrm{I}_h^\alpha \omega||_{\mathrm{d}}) \leq C||\omega||_{H^1}$$

*for a constant* $C > 0$ *indepedent of* $h$*, where* $||\cdot||_{\mathrm{d}} = ||\cdot||_{L^2} + ||\,\mathrm{d}\cdot||_{L^2}$*.*

*Proof.* Given $\omega \in H^1\Lambda^{d-1}$, let $\omega_h \in \Lambda_h[\alpha]$ solve

$$< \mathrm{d}\omega_h, \mathrm{d}w > + < \omega_h, w > = < l, w > \qquad w \in \Lambda_h[\alpha],$$

where $l$ is defined by

$$< l, w > := < \mathrm{d}\omega, \mathrm{d}w > + < \omega, w > .$$

Clearly $||\omega_h||_{\mathrm{d}} \leq ||\omega||_{H^1}$. Then it follows from an inverse inequality [4] that

$$\begin{aligned} |\mathrm{I}_h^\alpha \omega|_{\mathrm{d}} &\leq |\mathrm{I}_h^\alpha \omega - \omega_h|_{\mathrm{d}} + |\omega_h|_{\mathrm{d}} \preceq h^{-1}||\mathrm{I}_h^\alpha(\omega - \omega_h)||_{L^2} + ||\omega||_{H^1}, \\ &\preceq |\omega|_{H^1} + ||\omega||_{H^1} \preceq ||\omega||_{H^1}. \end{aligned}$$

□

Next we look at $\mathrm{d} \circ \mathrm{I}_h^\alpha$. Observe that we have the property

$$\int_T \mathrm{d}(\mathrm{I}_T^\alpha \omega) = \int_{\partial T} \mathrm{I}_T^\alpha \omega = \int_{\partial T} \mathrm{Tr}_{\partial T}\omega = \int_T \mathrm{d}\omega, \tag{2.9}$$

for $\mathrm{I}_h^\alpha$ on all spaces $\Lambda_h[\alpha]$. Letting $\Lambda_h[\alpha]$ be of the form (2.7) we note that the image $\mathrm{dI}_h^\alpha(\Lambda_h[\alpha])$ is a subset of

$$\mathrm{dI}_h^\alpha(\Lambda_h[\alpha]) \subset \{\omega \in L^2 : \omega_T = \sum_i b_i e^{\frac{-\beta_i x_i}{\alpha}} : b_i \in \mathbb{R}, T \in \mathcal{T}_h\},$$

which locally has dimension $d$, instead of dimension 1 as we would expect from differential complexes in finite element. This is a reason why the complex (2.4) must use the operator $\mathrm{d}_\alpha$ instead of $\mathrm{d}$.

## 2.2 Modified degrees of freedom

The problem with the above interpolators is that they do not commute with the exterior derivative $\mathrm{d}$ nor any other such operator. This is a useful property to have, and we shall therefore modify the degrees of freedom for our interpolators such that the new interpolators $\mathrm{J}_h^\alpha$ commutes with $\mathrm{d}_\alpha$. First we define the space of k-forms $W^p_{\mathrm{d}_\alpha}\Lambda^k$, in dimension $d$, by

$$W^p_{\mathrm{d}_\alpha}\Lambda^k = \{\omega \in L^p\Lambda^k :\, \mathrm{d}_\alpha\omega \in L^p\Lambda^{k+1}\},$$

with the norm

$$||\omega||_{p,\,\mathrm{d}_\alpha} = ||\omega||_{L^p} + ||\,\mathrm{d}_\alpha\omega||_{L^p}.$$

Then for the complex (2.4), we choose degrees of freedom on the modified interpolators $\mathrm{J}_h^\alpha$ such that

$$\int_e e^{\frac{\beta\cdot x}{\alpha}}\mathrm{J}_h^\alpha(\omega) = \int_e e^{\frac{\beta\cdot x}{\alpha}}\omega \qquad e\in\Delta_k(\mathcal{T}_h) \tag{2.10}$$

for any $k\in\{d-1,d\}$ and $\omega\in W^p_{\mathrm{d}_\alpha}\Lambda^k$ with well-defined traces.

**Proposition 6.** *The interpolators* $\mathrm{J}_h^\alpha$ *defined above commutes with* $\mathrm{d}_\alpha$.

*Proof.* To prove $\mathrm{d}_\alpha\circ\mathrm{J}_h^\alpha = \mathrm{J}_h^\alpha\circ\mathrm{d}_\alpha$, it is enough to show that $\mathrm{J}_h^\alpha\omega = 0$ implies $\mathrm{J}_h^\alpha(\,\mathrm{d}_\alpha\omega) = 0$ since $\mathrm{J}_h^\alpha$ is a projection. So, assuming $\mathrm{J}_h^\alpha\omega = 0$, we have on the rectangle $T$

$$\begin{aligned} 0 = \int_{\partial T} e^{\frac{\beta\cdot x}{\alpha}}\omega &= \int_T \mathrm{d}(e^{\frac{\beta\cdot x}{\alpha}}\omega) = \int_T (e^{\frac{\beta\cdot x}{\alpha}}\,\mathrm{d}\omega + \,\mathrm{d}(e^{\frac{\beta\cdot x}{\alpha}})\wedge\omega),\\ &= \alpha^{-1}\int_T e^{\frac{\beta\cdot x}{\alpha}}(\alpha\,\mathrm{d}\omega + \beta\wedge\omega) = \alpha^{-1}\int_T e^{\frac{\beta\cdot x}{\alpha}}\mathrm{J}_h^\alpha(\,\mathrm{d}_\alpha\omega), \end{aligned}$$

which is what we want. □

Unfortunately these modified interpolators do not necessarily give a bound on $||\mathrm{J}_h^\alpha \omega||_{L^p}$ uniformly in $\alpha$. We only have the next proposition.

**Proposition 7.** *Let $k \in \{d-1, d\}$. Then the interpolators $\mathrm{J}_h^\alpha$ onto the spaces $\Lambda_h^d[\alpha]$ in (2.4), defined by the degrees of freedom (2.10), are $L^p$ stable uniformly in $\alpha$ in the case $p = \infty$.*

*Proof.* First note that traces are always defined for $\omega \in L^\infty$. Looking at the $d$-forms we recall that they are piecewise constant, and so for $\omega \in W^p_{\mathrm{d}_\alpha}\Lambda^d$ we have $\mathrm{J}_h^\alpha \omega|_T = a_T$ on a rectangle $T$. Hence

$$\begin{aligned} \int_T a e^{\frac{\beta \cdot x}{\alpha}} &= \int e^{\frac{\beta \cdot x}{\alpha}} \omega, \\ a_T &= \frac{\int e^{\frac{\beta \cdot x}{\alpha}} \omega}{\int e^{\frac{\beta \cdot x}{\alpha}}}. \end{aligned}$$

The constant $a_T$ will not blow up when $\alpha \to 0$, since by Hölder's inequality we have

$$\frac{|\int e^{\frac{\beta \cdot x}{\alpha}} \omega|}{\int e^{\frac{\beta \cdot x}{\alpha}}} \leq ||\omega||_{L^\infty}.$$

For $\omega \in W^p_{\mathrm{d}_\alpha}\Lambda^{d-1}$ recall that we can write $\mathrm{J}_h^\alpha \omega|_T = \sum_i f_i(x_i, \alpha) \star (\,\mathrm{d}x_i)$ for $f_i$ of the form (2.7). It follows that for an edge $e \in \Delta_{d-1}(T)$ we have

$$\begin{aligned} \int_e e^{\frac{\beta \cdot x}{\alpha}} \mathrm{Tr}_e \omega = \int_e e^{\frac{\beta \cdot x}{\alpha}} f_i(x_i, \alpha) \star (\,\mathrm{d}x_i) = f_i(x_i, \alpha) \int_e e^{\frac{\beta \cdot x}{\alpha}} \star (\,\mathrm{d}x_i), \\ f_i(x_i, \alpha) = \frac{\int_e e^{\frac{\beta \cdot x}{\alpha}} \mathrm{Tr}_e \omega}{\int_e e^{\frac{\beta \cdot x}{\alpha}} \star (\,\mathrm{d}x_i)}, \end{aligned}$$

and the result follows from Hölder's inequality and an argument similar to Proposition 3. □

# Chapter 3

# Smoothed Projections

We continue from the previous chapter and prove that our interpolators can be used to construct smoothed projections onto our exponentially upwinded space from the complex defined by (2.4). This construction will be used on both the interpolators $\mathrm{I}_h^\alpha$ with standard degrees of freedom in the case $k = d-1$ and the interpolators $\mathrm{J}_h^\alpha$ with modified degrees of freedom in the cases $k \in \{d-1, d\}$. The benefits of using $\mathrm{I}_h^\alpha$ is that the resulting projections $\mathrm{P}_h^\alpha$ will be $L^2$ stable uniformly in $\alpha$, the mesh does not need to be quasi-uniform and it can be used with an extension operator, the drawback is that we get no bound on $\mathrm{d}\mathrm{P}_h^\alpha$ or $\mathrm{d}_\alpha \mathrm{P}_h^\alpha$. The benefit of using $\mathrm{J}_h^\alpha$ is that they commute with $\mathrm{d}_\alpha$ and so we can get a bound on $\mathrm{d}_\alpha \Pi_h^\alpha$ for the resulting smoothed projections $\Pi_h^\alpha$. Unfortunately, this bound will in general be dependent on $\alpha$. The arguments in this section closely follows those found in [1], [7] and [8] for (non-upwinded) piecewise polynomial elements.

**The smoothing operator.** In both cases we will make use of a smoothing operator constructed as follows. First we need for $x \in \mathbb{R}^d$, $y \in \mathbb{B}^d = \{y \in \mathbb{R}^d : |y| \leq 1\}$ and $0 < \epsilon \leq \epsilon_0$, a function $\Phi_y : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\Phi_y(x) = x + \epsilon g_h(x) y,$$

where $g_h$ both satisfies $g_h(x) \simeq h_T$ for $x \in T$ and is of sufficient regularity for our needs (it will be further explained in each case). Let $\Phi_y^*$ denote the pullback of $\Phi_y$, then the smoothing operator $R_h^\epsilon$ on a differential form $\omega$ is defined as

$$(R_h^\epsilon \omega)_x = \int_{\mathbb{B}} \rho(y)(\Phi_y^* E\omega)_x \, \mathrm{d}y,$$

where $\rho \in C_0^\infty(\mathbb{B}^d)$ is some mollifier function satisfying $0 \leq \rho(y) \leq 1$ and $\int_{\mathbb{B}^d} \rho(y)\,\mathrm{d}y = 1$. In the first construction $E$ will be an extension operator explained below while in the second it will just be the identity. $E$ will have the property of commuting with the exterior derivative $\mathrm{d}$, and since the same holds true for pullbacks, it will hold for $R_h^\epsilon$ as well. In other words we have $\mathrm{d} \circ R_h^\epsilon = R_h^\epsilon \circ \mathrm{d}$.

## 3.1 Standard Degrees of Freedom

Let $g_h : \Omega \to \mathbb{R}^+$ be the piecewise affine function determined by setting, on vertices

$$g_h(x) = \frac{1}{|\mathcal{T}_h(x)|} \sum_{T \in \mathcal{T}_h(x)} h_T, \quad x \in \Delta_0(\mathcal{T}_h)$$

where $h_T = \mathrm{diam}(T)$ for $T \in \mathcal{T}_h$, $\mathcal{T}_h(x) = \{T | T \in \mathcal{T}_h, x \in T\}$ and $|\mathcal{T}_h(x)|$ is the cardinality of $\mathcal{T}_h(x)$. The functions $g_h$ are uniformly Lipschitz continuous, with Lipschitz constants $L_{\text{mesh}}$ depending on the shape-regularity constant $C_{\text{mesh}}$.

**The extension operator.** Using the smoothing operator on/near the boundary is troublesome. We therefore need an extension operator $E : H\Lambda^k(\Omega) \to H\Lambda^k(\tilde{\Omega})$ where $\tilde{\Omega} \supset \bar{\Omega}$. The construction of such an extension operator can be found in part 4.1 of [8], but intuitively it can be thought of as a reflection with respect to the boundary. It is defined as a pullback and it has the properties that $\mathrm{d} \circ E = E \circ \mathrm{d}$, $E \in \mathcal{L}(H\Lambda^k(\Omega), H\Lambda^k(\tilde{\Omega}))$ and there exist an $\epsilon_0 > 0$ such that $\mathbb{B}_\epsilon(x) \subset \tilde{\Omega}$ for any $x \in \Omega$ and $0 < \epsilon \leq \epsilon_0$.

Using this extension operator we find that the smoothing operator $R_h^\epsilon$ is a map from $L^2\Lambda^k(\Omega)$ to $C\Lambda^k(\Omega)$ for all $0 \leq k \leq d$.

**Scaling.** For $T \in \mathcal{T}_h$ recall from the introduction that $\mathcal{T}_h(T)$ denotes its macroelement and $T^*$ the corresponding domain. If $T \cap \partial\Omega \neq \emptyset$ we extend $T^*$ to also include

$$\{x \in \tilde{\Omega} \setminus \Omega : \mathrm{dist}(x, T) \leq h_T\}.$$

Let $F$ be a map from $T \in \mathcal{T}_h$ onto a reference simplex $\hat{T}$ given by $F(x) = (x - x_0)/h_T$, where $x_0$ is the first vertex of $T$. Define $\hat{T}^* = F(T^*)$. The operator

$$\hat{R}_h^\epsilon = F^{*-1} R_h^\epsilon F^* : L^2\Lambda^k(\hat{T}^*) \to L^2\Lambda^k(\hat{T})$$

is then the smoothing operator in the space of scaled variables, satisfying

$$(\hat{R}_h^\epsilon \omega)_x = \int_{B_1} \rho(y)(\hat{\Phi}_y^* \hat{E}\omega)_x \,\mathrm{d}y.$$

Here, $\hat{E} = F^{*-1} E F^*$ is the scaled extension operator, $\hat{\Phi}_y : \hat{T} \to \hat{T}^*$ satisfies

$$\hat{\Phi}_y(x) = x + \epsilon \hat{g}_h(x) y$$

and $\hat{g}_h(x) = h_T^{-1} g_h(F^{-1}x)$ is the scaled mesh function. Clearly the matrices $D\hat{\Phi}_y$ have the property that

$$|D\hat{\Phi}_y - I| \leq \epsilon L_h \tag{3.1}$$

on $\hat{T}$, where $L_h$ is the Lipschitz constant of $\hat{g}_h$.

**Lemma 8.** *For each $\epsilon \in (0, \epsilon_0]$ there is a constant $c(\epsilon)$, independent of $T \in \mathcal{T}_h$ and $h$, such that*

$$||\hat{R}_h^\epsilon||_{\mathcal{L}(L^2\Lambda^k(\hat{T}^*), C\Lambda^k(\hat{T}))} \leq c(\epsilon)$$

This is Lemma 4.2 in [8] and the proof is found there.

For $\mathrm{I}_T^\alpha$ the interpolator onto $\Lambda_h[\alpha]$ of the form (2.7) described in Chapter 2, we have the following important lemma.

**Lemma 9.** *Let $\omega \in \Lambda^{d-1}(\hat{T}^*)$ with $w|_{\hat{T}}$ of the form (2.6) for $\hat{T}' \in \mathcal{T}(\hat{T})$. Then there is a constant $C > 0$, independent of $T \in \mathcal{T}_h$, $h$, $\epsilon$ and $\alpha$, such that*

$$||\mathrm{I}^{\alpha}_{\hat{T}}(I - \hat{R}^{\epsilon}_h)\omega||_{L^2\Lambda^{d-1}(\hat{T})} \leq C\epsilon \sum_{\hat{T}' \in \mathcal{T}(\hat{T})} ||\omega||_{L^\infty(\hat{T}')}$$

*for $\epsilon$ small enough.*

*Proof.* First note that $\mathrm{I}^{\alpha}_{\hat{T}}\,\omega$ is determined by integrals of the form $\int_e \mathrm{Tr}_e\omega$ on subrectangles $e \in \Delta_{d-1}(\hat{T})$. We decompose the edges $e$ into $e_\epsilon$ and $e \smallsetminus e_\epsilon$, where

$$e_\epsilon = \{x \in e : \mathrm{dist}(x, \partial e) \geq C\epsilon\}.$$

For $v_1, .., v_{d-1}$ unit tangent vectors to $e$ we have

$$\begin{aligned}
&|(\omega)_x(v_1, ..., v_{d-1}) - (\omega)_z(D\Phi_y v_1, ..., D\Phi_y v_{d-1})| \\
&\leq |(\omega)_x(v, ..., v_{d-1}) - (\omega)_z(v, ..., v_{d-1})| + |(\omega)_z(v, ..., v_{d-1}) - (\omega)_z(D\Phi_y v_1, ..., D\Phi_y v_{d-1})| \\
&\leq |D\Phi_y - I| \sum_{\hat{T}' \in \mathcal{T}(\hat{T})} ||\omega||_{L^\infty(\hat{T}')} \\
&\leq C\epsilon \sum_{\hat{T}' \in \mathcal{T}(\hat{T})} ||\omega||_{L^\infty(\hat{T}')},
\end{aligned}$$

since $(\mathrm{Tr}_e\omega)_x = (\mathrm{Tr}_e\omega)_z$ for all edges $e$ when $\omega$ is of the form (2.6).

Hence,

$$\begin{aligned}
|\int_{e_\epsilon} (I - \hat{R}^{\epsilon}_h)\omega| &= |\int_{\mathbb{B}} \rho(y) \int_{e_\epsilon} (\omega - (\Phi_y)^*\omega)\,\mathrm{d}y| \\
&\leq C\epsilon \sum_{\hat{T}' \in \mathcal{T}(\hat{T})} ||\omega||_{L^\infty(\hat{T}')}.
\end{aligned}$$

Lastly, we observe that

$$|\int_{e \smallsetminus e_\epsilon} (I - \hat{R}^{\epsilon}_h)\omega| \leq |\int_{e \smallsetminus e_\epsilon} \omega| + |\int_{e \smallsetminus e_\epsilon} \hat{R}^{\epsilon}_h\omega| \leq c\epsilon||\omega||_{L^\infty\Lambda^{d-1}(\hat{T}^*)},$$

and the result follows. $\square$

**The smoothed projection.** The interpolation operators $\mathrm{I}_h^\alpha$ and the smoothing operators $R_h^\epsilon$ will be used to define the projection operators $\mathrm{P}_h^\alpha$ onto the upwinded finite element spaces $\Lambda_h[\alpha]$.

**Proposition 10.** *For each $\epsilon \in (0, \epsilon_0]$ there exist a constant $c(\epsilon)$ such that*

$$||\mathrm{I}_h^\alpha R_h^\epsilon||_{\mathcal{L}(L^2\Lambda^{d-1}(\Omega), L^2\Lambda^{d-1}(\Omega))} \leq c(\epsilon)$$

*for all $h$.*

*Proof.* Shape regularity of $\mathcal{T}_h$ implies bounded overlap, so

$$\sum_{T \in \mathcal{T}_h} ||\omega||_{L^2\Lambda^{d-1}(T^*)} \leq c||\omega||_{L^2\Lambda^{d-1}(\Omega)}.$$

Hence it suffices to show that

$$||\mathrm{I}_h^\alpha R_h^\epsilon||_{\mathcal{L}(L^2\Lambda^{d-1}(T^*), L^2\Lambda^{d-1}(T))} \leq c(\epsilon).$$

Using the scaling map $F$ defined above we observe that

$$\begin{aligned} ||\mathrm{I}_h^\alpha R_h^\epsilon||_{\mathcal{L}(L^2\Lambda^{d-1}(T^*), L^2\Lambda^{d-1}(T))} &= ||F^{*-1}\mathrm{I}_T^\alpha R_h^\epsilon F^*||_{\mathcal{L}(L^2\Lambda^{d-1}(\hat{T}^*), L^2\Lambda^{d-1}(\hat{T}))} \\ &= ||\mathrm{I}_{\hat{T}}^\alpha \hat{R}_h^\epsilon||_{\mathcal{L}(L^2\Lambda^{d-1}(\hat{T}^*), L^2\Lambda^{d-1}(\hat{T}))}, \end{aligned}$$

so since $\mathrm{I}_h^\alpha$ is uniformly bounded in $\mathcal{L}(C\Lambda^{d-1}(\hat{T}), L^2\Lambda^{d-1}(\hat{T}))$, the result follows from Lemma 8.

□

For a fixed $\epsilon$ the operators $\mathrm{I}_h^\alpha R_h^\epsilon$ are uniformly bounded maps, with respect to $h$ and $\alpha$, from $L^2\Lambda^{d-1}(\Omega)$ onto $\Lambda_h[\alpha]$ in the $L^2$ norm. However, they are not projections since they are not invariant on $\Lambda_h[\alpha]$. The next proposition fixes this.

**Proposition 11.** *There exist a constant $C > 0$, independent of $\epsilon$, $h$ and $\alpha$, such that*

$$||I - \mathrm{I}_h^\alpha R_h^\epsilon|_{\Lambda^{d-1}}||_{\mathcal{L}(L^2\Lambda_h[\alpha], L^2\Lambda_h[\alpha])} \leq C\epsilon$$

*for* $\epsilon \in (0, \epsilon_0]$.

*Proof.* Because of Proposition 10 and scaling it is enough to show that

$$||I - \mathrm{I}_{T^*}^\alpha \hat{R}_h^\epsilon|||_{\mathcal{L}(L^2\Lambda_h[\alpha](\hat{T}*), L^2\Lambda_h[\alpha](\hat{T}))} \leq C\epsilon.$$

For $\omega \in \Lambda_h[\alpha](\hat{T}^*)$ we have $\mathrm{I}_{\hat{T}}^\alpha = w|_{\hat{T}}$, so from the compactness of the macroelements we know that

$$\sum_{\hat{T}' \in \mathcal{T}(\hat{T})} ||\omega||_{L^\infty\Lambda^{d-1}(T')} \leq c||\omega||_{L^2\Lambda_h[\alpha](\hat{T}^*)}.$$

Hence the bound follows from Lemma 9. □

By choosing $\epsilon \in (0, \epsilon_1]$, where $\epsilon_1 < \epsilon_0$ it follows from Proposition 11 that $\mathrm{I}_h^\alpha R_h^\epsilon|_{\Lambda^{d-1}}$ is invertible, with the inverse $\mathrm{Q}_h^\epsilon$ satisfying

$$||\mathrm{Q}_h^\epsilon||_{\mathcal{L}(L^2\Lambda_h[\alpha], L^2\Lambda_h[\alpha])} \leq 2.$$

We now fix $\epsilon \in (0, \epsilon_1]$. The smoothed interpolation operator $\mathrm{P}_h^\alpha$ is then defined by

$$\mathrm{P}_h^\alpha = \mathrm{Q}_h^\epsilon \mathrm{I}_h^\alpha R_h^\epsilon.$$

Putting it all together we get the following result.

**Corollary 12.** *The projections* $\mathrm{P}_h^\alpha$ *defined above satisfies*

$$||\mathrm{P}_h^\alpha \omega||_{L^2\Lambda^{d-1}} \preceq ||\omega||_{L^2\Lambda^{d-1}},$$

*for a constant independent of* $h$ *and* $\alpha$. *Furthermore, for all* $\omega \in L^2\Lambda^{d-1}(\Omega)$, $\mathrm{P}_h^\alpha \omega \to \omega$ *in* $L^2$ *as* $h \to 0$.

**Proposition 13.** *For $\Lambda_h[\alpha]$ of the form* (2.7) *we have the convergence estimate*

$$||\omega - \mathrm{I}_h^\alpha \omega||_{L^2} \leq ch^s ||\omega||_{H^s} \qquad \omega \in H^s,$$

*for* $0 < s \leq 1$.

This estimate follows from the corresponding estimate for projections onto piecewise constants. More formally we have the following slight variation of the proof of Theorem 5.6 in [1].

*Proof.* Letting $\mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)$ be the usual space of piecewise constant forms, we have

$$\begin{aligned} ||\omega - \mathrm{P}_h^\alpha \omega|| &= \inf_{\mu \in \Lambda_h[\alpha]} ||(I - \mathrm{P}_h^\alpha)(\omega - \mu)|| \preceq \inf_{\mu \in \Lambda_h[\alpha]} ||\omega - \mu|| \\ &\preceq \inf_{\mu \in \mathcal{P}_0\Lambda^{d-1}(\mathcal{T}_h)} ||\omega - \mu|| \preceq h^s ||\omega||_{H^s}, \end{aligned}$$

where the last inequality is a well-known result and follows from the Bramble-Hilbert lemma or Clément interpolation (see e.g [4]). □

*Remark:* This estimate is optimal in the sense that any "improved" version would necessarily depend on $\alpha$, since all $\omega \in \Lambda_h[\alpha]$ of the form (2.7) converges to piecewise constant forms when $\alpha \to 0$.

## 3.2 Modified Degrees of Freedom

In this section we assume the mesh $\mathcal{T}_h$ is quasi-uniform in addition to being shape-regular. We can therefore choose $g_h(x) = h$ as was done in [1]. Observe that for $\mathrm{d}_\alpha$ defined in (2.3) we have

$$
\begin{aligned}
(R_h^\epsilon \,\mathrm{d}_\alpha \omega)_x &= \int_{\mathbb{B}} \rho(y)(\Phi_y^* \,\mathrm{d}_\alpha \omega)_x \,\mathrm{d}y = \int_{\mathbb{B}} \rho(y)(\Phi_y^*(\alpha \,\mathrm{d}\omega + \beta \wedge \omega))_x \,\mathrm{d}y, \\
&= \int_{\mathbb{B}} \rho(y)(\alpha \,\mathrm{d}(\Phi_y^* \omega) + \Phi_y^*(\beta) \wedge \Phi_y^*(\omega))_x \,\mathrm{d}y, \\
&= \int_{\mathbb{B}} \rho(y)(\alpha \,\mathrm{d}(\Phi_y^* \omega) + \beta \wedge \Phi_y^*(\omega))_x \,\mathrm{d}y, \\
&= \alpha \,\mathrm{d}(\int_{\mathbb{B}} \rho(y)(\Phi_y^* \omega)_x \,\mathrm{d}y + \beta \wedge (\int_{\mathbb{B}} \rho(y)\Phi_y^* \omega)_x \,\mathrm{d}y = (\,\mathrm{d}_\alpha R_h^\epsilon \omega)_x,
\end{aligned}
$$

where we have used that $\Phi_y^*(\beta) = \beta$ when $g_h$ is a constant function and $\beta$ is a constant form. If we wanted to use the above construction again for these interpolators $\mathrm{J}_h^\alpha$, the main difference would be in Lemma 9. However, we shall use a different construction here. Let the mollifier function $\rho$ have the property that

$$
\int_{\mathbb{B}^d} \rho(y) f(y) \,\mathrm{d}y = f(0) \qquad \forall f \in E \otimes P_d,
$$

where $E = \{f(y) = a + be^{-\frac{\beta}{\alpha}\epsilon h y} : a, b \in \mathbb{R}\}$ and $P_d$ is the set of polynomials of degree at most $d$. Then we can use most of the theory of [7] section 5.3 with only minor modifications. Letting $\nabla$ denote the usual gradient of a tensor field, we have the following important results.

**Lemma 14.** *Let $k \in \{d-1, d\}$ and $\omega \in \Lambda_h^k[\alpha]$ from the complex* (2.4)*, then $R_h^\epsilon \omega = \omega$ and $\nabla R_h^\epsilon \omega = \nabla \omega$.*

*Proof.* The case $k = d$ follows from [7] since $\Lambda_h^k[\alpha]$ is then the space of piecewise constant $d$-forms. Assume $k = d-1$. First observe that $g_h$ constant implies

$$
D_x \Phi_y(x) v = v,
$$

and so

$$
\begin{aligned}
(\Phi_y^* \omega)_x(v_1, ..., v_k) &= \omega_{\Phi_y(x)}(D_x \Phi_y(x) v_1, ..., D_x \Phi_y(x) v_k), \\
&= \sum_i (a_i^T + b_i^T e^{-\frac{\beta}{\alpha}(x+\epsilon h y)}) \star (\,\mathrm{d}x_i)(v_1, ..., v_k)
\end{aligned} \tag{3.2}
$$

is in the space $E \otimes P_{d-1}$ as a function of y. Its value at $y = 0$ is $\omega_x(v_1, ..., v_k)$, hence

$$(R_h^\epsilon \omega)_x(v_1, ..., v_k) = \int_{\mathbb{B}} \rho(y)(\Phi_y^* \omega)_x(v_1, ..., v_k)\,\mathrm{d}y = \omega_x(v_1, ..., v_k),$$

and we have $R_h^\epsilon \omega = \omega$. For $\nabla R_h^\epsilon \omega = \nabla \omega$ we can use the same argument on expression (5.18) in [7] for $(D_x \Phi_y^* \omega)_x(v_1, ..., v_d)$.

□

For $T \in \mathcal{T}_h$ recall that $T^*$ denotes its macroelement. Let $T_\epsilon^* = \{\mathbb{B}(x, \epsilon h) : x \in T\}$.

**Lemma 15.** *For a fixed $\epsilon > 0$ and any $T \in \mathcal{T}_h$ of maximal dimension we have the estimates*

$$||\nabla R_h^\epsilon \omega||_{L^2(T)} \preceq ||\nabla \omega||_{L^2(T_\epsilon^*)}.$$

*and*

$$||\omega - R_h^\epsilon \omega||_{L^2(T)} \preceq h||\nabla \omega||_{L^2(T_\epsilon^*)} \tag{3.3}$$

This is proposition 5.58 in [7] and the proof is essentially the same since our $R_h^\epsilon$ also preserves constants. Note that we can use $h$ in (3.3) because of quasi-uniformity. The above lemma together with the properties of $\mathrm{J}_h^\alpha$ gives us the proposition below by a standard scaling argument, as long as $\epsilon$ is chosen small enough such that $T_\epsilon^* \subset T^*$ for all $T \in \mathcal{T}_h$.

**Proposition 16.** *For $k \in \{d-1, d\}$ and $\omega \in L^2\Lambda^k$ defined on $T_\epsilon^*$ we have the estimates*

$$||\mathrm{J}_h^\alpha R_h^\epsilon \omega||_{L^2(T)} \leq C||\omega||_{L^2(T_\epsilon^*)},$$
$$||\omega - \mathrm{J}_h^\alpha R_h^\epsilon \omega||_{L^2(T)} \leq C'h||\nabla \omega||_{L^2(T_\epsilon^*)},$$

for constants $C, C' > 0$ independent of $h$. Lastly, choosing $\epsilon$ so small that $||(I - \mathrm{J}_h^\alpha R_h^\epsilon)|_{\Lambda_h^k[\alpha]}|| \leq \frac{1}{2}$, then $\mathrm{J}_h^\alpha R_h^\epsilon|_{\Lambda_h^k[\alpha]}$ is invertible with norm less than 2. Defining $\Pi_h^\alpha$ by

$$\Pi_h^\alpha = (\mathrm{I}_h^\alpha R_h^\epsilon|_{\Lambda_h^k[\alpha]})^{-1}\mathrm{I}_h^\alpha R_h^\epsilon,$$

we get the next proposition.

**Corollary 17.** *For $k \in \{d-1, d\}$, the projections $\Pi_h^\alpha$ defined above satisfies*

$$||\Pi_h^\alpha \omega||_{L^2\Lambda^k} \leq C_1||\omega||_{L^2\Lambda^k},$$
$$||\,\mathrm{d}_\alpha \Pi_h^\alpha \omega||_{L^2\Lambda^k} \leq C_2||\,\mathrm{d}_\alpha\, \omega||_{L^2\Lambda^k}$$

*for constants $C_1, C_2 > 0$ independent of $h$. Furthermore, for all $\omega \in L^2\Lambda^k(\Omega)$, $\Pi_h^\alpha\omega \to \omega$ in $L^2$ as $h \to 0$.*

We have the convergence estimate

$$||\omega - \Pi_h^\alpha \omega||_{L^2} \leq Ch^s||\omega||_{H^s} \qquad \omega \in H^s, \tag{3.4}$$

for $0 \leq s \leq 1$. This estimate follows from the essentially the same proof as in Proposition 13. The main difference is that the constant $C > 0$ can depend on $\alpha$.

*Remark:* Note the missing extension operator in this construction, and without it these projections can only be used in the case of periodic boundary conditions. The reason we lack an extension operator $E$ in this case is that we would need it to both preserve the space $\Lambda_h^k[\alpha]$ in some sense, e.g. for $\omega \in \Lambda_h^k[\alpha](\Omega)$ we would need $E\omega \in \Lambda_h^k[\alpha](\tilde{\Omega})$ where $\tilde{\Omega} \supset \bar{\Omega}$ and have the property that $E(\beta) = \beta$ for $\beta$ a constant 1-form. Constructing such an operator is difficult.

### Original Complex

We can construct smoothed projections for the interpolators of our original complex (2.2) as well. Since we do not use this complex, we will not go through this construction. The smoothed projections onto this complex would have the property of commuting with $\mathrm{d}$.

# Chapter 4

# Convection Diffusion

## 4.1 Upwind $(L^2, H_{div})$-Formulation for $\alpha \sim |\beta|$

We start by assuming $\alpha \sim |\beta|$ and show the continuous and discrete infsup-conditions in this case. Our main assumption in this chapter is that $\Omega$ is a convex domain. We will further assume $\Omega \subset \mathbb{R}^2$, but our proofs will also work for $\Omega \subset \mathbb{R}^3$ with only minor modifications. Let $u = -\alpha\nabla p + \beta p$ then a $(L^2, H_{div})$ mixed formulation of (1.1) is find $p \in L^2$ and $u \in H_{\mathrm{div}}$ such that

$$
\begin{aligned}
\int_\Omega u \cdot v \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega (\alpha\nabla \cdot v + \beta \cdot v) p \,\mathrm{d}x\,\mathrm{d}y = 0 \qquad\qquad & \forall v \in H_{\mathrm{div}} \\
\int_\Omega (\nabla \cdot u) q \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f q \,\mathrm{d}x\,\mathrm{d}y \quad & \forall q \in L^2.
\end{aligned}
\tag{4.1}
$$

Define

$$
\begin{aligned}
a(u, v) &= \int_\Omega u \cdot v \,\mathrm{d}x\,\mathrm{d}y, \\
b_1(p, v) &= \int_\Omega (\alpha\nabla \cdot v + \beta \cdot v) p \,\mathrm{d}x\,\mathrm{d}y, \\
b_2(q, u) &= \int_\Omega (\nabla \cdot u) q \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{4.2}
$$

then the mixed formulation can be written as

$$\begin{aligned} a(u,v)+b_1(p,v) &= 0 && \forall v\in H_{\text{div}} \\ b_2(q,u) &= < f,q > && \forall q\in L^2. \end{aligned} \tag{4.3}$$

We observe that the continuity conditions on the bilinear forms $a, b_1$ and $b_2$ with $u, v \in H_{\text{div}}$ and $p, q \in L^2$ follows from Hölder's inequality

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |u\cdot v|\,\mathrm{d}x\,\mathrm{d}y \le ||u||_{L^2}||v||_{L^2} \le ||u||_{H_{\text{div}}}||v||_{H_{\text{div}}}, \\ |b_1(p,v)| &\le \int_\Omega |\alpha p\nabla\cdot v|\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |p\beta\cdot v|\,\mathrm{d}x \le C_1||p||_{L^2}(||\nabla\cdot v||_{L^2}+||v||_{L^2}) \\ &\le C_1||p||_{L^2}||v||_{H_{\text{div}}} \\ |b_2(q,u)| &\le \int_\Omega |(\nabla\cdot u)q|\,\mathrm{d}x\,\mathrm{d}y \le ||\nabla\cdot u||_{L^2}||q||_{L^2} \le ||u||_{H_{\text{div}}}||q||_{L^2}, \end{aligned} \tag{4.4}$$

where the constant $C_1 > 0$ depends on both $\alpha$ and $\beta$.

**Proposition 18.** *Problem* (4.1) *is wellposed with the norms $H_{\text{div}}$ and $L^2$.*

*Proof.* This follows from Proposition 1 and Lemma 19, Lemma 21 and Lemma 23 below. □

The infsup constants will in general depend on $\alpha$ and $\beta$ when using the standard norms on $H_{\text{div}}$ and $L^2$, but that is not a problem in this case since we have assumed $\alpha, |\beta| \sim 1$. Let $\mathcal{T}_h$ be a rectangular mesh of $\Omega$ and define $X_1^h, X_2^h \subset H_{\text{div}}$ and $Y^h \subset L^2$ by

$$\begin{aligned} X_1^h &= \{v\in H_{\text{div}} : v|_T = (a_T + b_T e^{\frac{-\beta_1 x}{\alpha}}, c_T + d_T e^{\frac{-\beta_2 y}{\alpha}}),\, T\in\Delta_d(\mathcal{T}_h)\} \\ X_2^h &= \{u\in H_{\text{div}} : u|_T = (a_T + b_T x, c_T + d_T y)\, T\in\Delta_d(\mathcal{T}_h)\} \\ Y^h &= \{q\in L^2 : q|_T = a_T,\, T\in\Delta_d(\mathcal{T}_h)\} \end{aligned} \tag{4.5}$$

Note that $\nabla\cdot X_2^h = Y^h$ and $(X_2^h, Y^h)$ is the standard lowest order Raviart-Thomas elements, while $X_1^h$ is an exponentially upwinded test space corresponding to $X_2^h$. The upwinded mixed discretization of (1.1) is then find $u \in X_2^h$ and $p \in Y^h$ such that

$$\begin{aligned} a(u,v) + b_1(p,v) &= 0 & \forall v \in X_1^h \\ b_2(q,u) &= < f, q > & \forall q \in Y^h. \end{aligned}$$

We begin with the infsup conditions on $b_1$.

**Lemma 19.** *There exists a constant $B_1 > 0$ depending on $\alpha$ and $\beta$ such that the continuous infsup condition*

$$\sup_{v \in H^1} \frac{b_1(p,v)}{||v||_{H^1}} \geq B_1 ||p||_{L^2}, \qquad \forall p \in L^2.$$

*is satisfied.*

*Proof.* Given $p \in L^2$, let $\phi \in H^1$ be defined by $\nabla\phi = v$ and choose the optimal test function $v$ such that $\phi$ is the weak solution to $\alpha\Delta\phi + \beta \cdot \nabla\phi = p$ with $\phi|_{\partial\Omega} = 0$. Formally we have

$$-\int_\Omega \alpha \nabla\phi \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \beta \cdot \nabla\phi\psi \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega p\psi \, \mathrm{d}x \, \mathrm{d}y, \tag{4.6}$$

for all $\psi \in H_0^1$. Standard Sobolev theory [11] gives us the estimates

$$||v||_{L^2} = ||\nabla f||_{L^2} \leq C||p||_{L^2},$$

for a constant $C > 0$ that can in general depend on $\alpha$. Elliptic regularity and $p \in L^2$ implies $\phi \in H^2$, and so

$$-\int_\Omega \nabla\phi \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \Delta\phi\psi \, \mathrm{d}x \, \mathrm{d}y$$

for all $\psi \in H_0^1$. Choose $\psi_n \in H_0^1$ such that $\psi_n \to p$ in $L^2$, then we have

$$\begin{aligned} b_1(p,v) &= \int_\Omega \alpha p \Delta\phi \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega p\beta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_n \int_\Omega \alpha\psi_n\Delta\phi\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \psi_n\beta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_n \int_\Omega p\psi_n\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |p|^2\,\mathrm{d}x\,\mathrm{d}y = ||p||^2_{L^2}, \end{aligned}$$

and the result follows, since equation (4.6) then gives us an $\alpha$-dependent bound on $||v||_{H^1}$. □

*Remark:* In the case $a << |\beta|$, we could try choosing Dirichlet conditions $\phi = 0$ on the inflow boundary and the Neumann conditions $\phi \cdot n = 0$ on the outflow boundary in the above argument to avoid boundary layers in optimal test function $v$.

Note that $\phi \in H^2$ implies the optimal test function $v \in (H^1)^d$ which is an important ingredient in the next proof.

**Proposition 20.** *There exists a constant $\hat{B}_1 > 0$ independent of $h$ such that the discrete infsup condition* (1.11) *($i = 1$), with the spaces $X_1^h$ and $Y_1^h = Y^h$ defined as in* (4.5)*, is satisfied with the $L^2$ and $H_{\text{div}}$ norms.*

*Proof.* Let $\mathrm{I}_h^\alpha : H_{\text{div}} \to X_1^h$ be the interpolators onto the complex (2.4) described in Chapter 2. Recall that they satisfy a convergence estimate and we have $||\mathrm{I}_h^\alpha v||_{H_{\text{div}}} \le C||v||_{H^1}$, for some constant $C > 0$ independent of $h$. We begin by noting that for $p \in L^2$ and $v \in H^1$ its optimal test function found in Lemma 19, we have

$$\begin{aligned} |b_1(p, v - \mathrm{I}_h^\alpha v)| &= |\sum_T \int_T \alpha p_T \nabla\cdot(v - \mathrm{I}_h^\alpha v)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega p\beta\cdot(v - \mathrm{I}_h^\alpha v)|\,\mathrm{d}x\,\mathrm{d}y \\ &\le |\sum_T \alpha p_T \int_{\partial T} (v - \mathrm{I}_h^\alpha v)\cdot n\,\mathrm{d}S| + \int_\Omega |p\beta\cdot(v - \mathrm{I}_h^\alpha v)|\,\mathrm{d}x\,\mathrm{d}y \\ &\le |\beta|\,||p||_{L^2}||v - \mathrm{I}_h^\alpha v||_{L^2} \\ &\preceq h||p||_{L^2}||v||_{H^{1/2-\epsilon}} \xrightarrow[h\to 0]{} 0. \end{aligned}$$

Here, we have used that $\int_{\partial T}(v - \mathrm{I}_h^\alpha v)\cdot n\,\mathrm{d}S = 0$ for $v \in H^{1/2+\epsilon}$ by the degrees of freedom for $\mathrm{I}_h^\alpha$. Hence, we can find a constant $C > 0$ such that

$$|b_1(p, v - \mathrm{I}_h^\alpha v)| \leq C|b_1(p, \mathrm{I}_h^\alpha v)| \tag{4.7}$$

for all $h$ smaller than some $h_0$. Let $h$ be smaller than $h_0$ and let $p \in Y^h \subset L^2$. From the previous lemma we know that

$$B_1||p||_{L^2} \leq \sup_{v\in H^1} \frac{b_1(p,v)}{||v||_{H^1}} = \sup_{v\in H^1} \frac{b_1(p, v - \mathrm{I}_h^\alpha v) + b_1(p, \mathrm{I}_h^\alpha v)}{||v||_{H^1}}$$

Using property (4.7) we have

$$\begin{aligned} B_1||p||_{L^2} &\preceq \sup_{v\in H^1} \frac{b_1(p, \mathrm{I}_h^\alpha v)}{||v||_{H^1}} \preceq \sup_{v\in H^1} \frac{b_1(p, \mathrm{I}_h^\alpha v)}{||\mathrm{I}_h^\alpha v||_{H_{\mathrm{div}}}} \\ &\preceq \sup_{v\in X_1^h} \frac{b_1(p,v)}{||v||_{H_{\mathrm{div}}}}. \end{aligned}$$

$\square$

The discrete infsup condition on $b_2(q,u) = \int_\Omega (\nabla \cdot u) q \,\mathrm{d}x\,\mathrm{d}y$ for Raviart-Thomas elements is a well-known result. We give it here for completeness sake.

**Lemma 21.** *There exists a constant $B_2 > 0$ depending on $\alpha$ and $\beta$ such that the continuous infsup condition*

$$\sup_{u\in H_{\mathrm{div}}} \frac{b_2(q,u)}{||u||_{H_{\mathrm{div}}}} \geq B_2||q||_{L^2}, \qquad \forall q \in L^2.$$

*is satisfied.*

*Proof.* Let $q \in L^2$ be given. Let the optimal trial function $u \in H_{\mathrm{div}}$ be of the form $u = \nabla f$ for some $f \in H_0^1$, such that $f$ is the weak solution to

$$\Delta f = \nabla \cdot u = q.$$

It follows from standard Sobolev theory that $||u||_{L^2} = ||\nabla f||_{L^2} \leq C||q||_{L^2}$ for some constant $C > 0$. We then have

$$\begin{aligned}
\sup_{u \in H_{\mathrm{div}}} \frac{b_2(q,u)}{||u||_{H_{\mathrm{div}}}} &= \sup_{u \in H_{\mathrm{div}}} \frac{\int_\Omega \nabla \cdot uq \,\mathrm{d}x\,\mathrm{d}y}{||u||_{L^2} + ||\nabla \cdot u||_{L^2}}, \\
&= \sup_{u \in H_{\mathrm{div}}} \frac{||q||_{L^2}^2}{||u||_{L^2} + ||q||_{L^2}} \geq B_2||q||_{L^2}.
\end{aligned}$$

□

**Proposition 22.** *There exists a constant $\hat{B}_2 > 0$ independent of $h$ such that the discrete infsup condition* (1.11) *(where $i = 2$), with the spaces $X_2^h$ and $Y_2^h = Y_h$ are defined as in* (4.5)*, is satisfied.*

*Proof.* Let $p \in Y_h \subset L^2$ be given. Let $\Pi_h : H_{\mathrm{div}} \to X_h^2$ be the smoothed projection found in [8] or [7], it has the properties that $||\Pi_h u||_{H_{\mathrm{div}}} \preceq ||u||_{H_{\mathrm{div}}}$ and

$$\int_T \nabla \cdot \Pi_h u \,\mathrm{d}x\,\mathrm{d}y = \int_T \nabla \cdot u \,\mathrm{d}x\,\mathrm{d}y \tag{4.8}$$

on each rectangle $T$. Then from the previous lemma we have

$$\begin{aligned}
B_2||q||_{L^2} \leq \sup_{u \in H_{\mathrm{div}}} \frac{b_2(q,u)}{||u||_{H_{\mathrm{div}}}} &= \sup_{u \in H_{\mathrm{div}}} \frac{b_2(q, u - \Pi u) + b_2(q, \Pi u)}{||u||_{H_{\mathrm{div}}}}, \\
= \sup_{u \in H_{\mathrm{div}}} \frac{b_2(q, \Pi u)}{||u||_{H_{\mathrm{div}}}} &\preceq \sup_{u \in H_{\mathrm{div}}} \frac{b_2(q, \Pi u)}{||\Pi u||_{H_{\mathrm{div}}}}, \\
\preceq \sup_{u \in X_2^h} \frac{b_2(q,u)}{||u||_{H_{\mathrm{div}}}}&,
\end{aligned}$$

where we have used property (4.8) to find that

$$b_2(q, u - \Pi_h u) = \sum_T q_T \int_T (\nabla \cdot u - \nabla \cdot \Pi_h u) \,\mathrm{d}x\,\mathrm{d}y = 0.$$

□

Next we look at the continuous infsup condition (1.8) for $a$.

**Lemma 23.** *There exist a constant $A > 0$ such that the continuous infsup condition*

$$\sup_{v \in K_1} \frac{a(u,v)}{||v||_{H_{\text{div}}}} \geq A||u||_{H_{\text{div}}}, \qquad \forall u \in K_2,$$

*is true for the spaces*

$$K_1 = \{v \in H_{\text{div}} : \alpha\nabla \cdot v + \beta \cdot v = 0\},$$
$$K_2 = \{u \in H_{\text{div}} : \nabla \cdot u = 0\}.$$

*Proof.* Let $u \in K_2$ be given, since it is divergence free there must exist a $\phi \in H^1$ such that $u = \nabla \times \phi$, where $\nabla\times : \mathbb{R} \to \mathbb{R}^2$ is the usual curl-operator from one to two dimensions. For any test function $v \in K_1$ we know from standard Helmholtz decomposition or Hodge decomposition that there exist functions $f, F \in H^1$ such that

$$v = \nabla f + \nabla \times F.$$

The optimal test function $v$ is then chosen such that $\nabla \times F = u$ and $f|_{\partial\Omega} = 0$. From the definition of $K_1$ we have that $f$ must satisfy the equation

$$\alpha\Delta f + \beta \cdot \nabla f = -\beta \cdot \nabla \times F = -\beta \cdot u \tag{4.9}$$

and so $v$ is uniquely determined for every $u$. It also follows from standard Sobolev theory that $||\nabla f|| \leq C||u||_{L^2}$. Elliptic regularity gives $f \in H^2$ and from (4.9) we have

$$||\nabla \cdot v||_{L^2} = ||\Delta f||_{L^2} \leq C'||u||_{L^2},$$

and so

$$||v||_{H_{\text{div}}} = ||\nabla f + \nabla \times F||_{L^2} + ||\Delta f||_{L^2} \leq C''||u||_{L^2}.$$

The constants $C, C', C'' > 0$ above will in general depend on $\alpha$ and $\beta$. Using integration by parts on $a$ we observe that

$$a(u,v) = \int_\Omega u \cdot (\nabla f + \nabla \times F)\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega \nabla \cdot uf\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |u|^2\,\mathrm{d}x\,\mathrm{d}y = ||u||^2_{L^2},$$

and so the result follows. □

*Remark:* The other infsup condition in (1.8)

$$\sup_{u\in K_2} a(u,v) > 0, \qquad\qquad \forall v \in K_1 \setminus \{0\},$$

is trivial, since for a given $v$, just use the Helmholtz decomposition on $v$ and choose $u = \nabla \times F$.

Lastly we need to show the discrete infsup condition

$$\sup_{v\in K_1^h} \frac{a(u,v)}{||v||_{H_{\mathrm{div}}}} \geq a||u||_{H_{\mathrm{div}}}, \quad \forall u \in K_2^h,$$

for the spaces

$$\begin{aligned} K_1^h &= \{v \in X_1^h : \int_T (\alpha\nabla \cdot v|_T + \beta \cdot v_T)\,\mathrm{d}x\,\mathrm{d}y = 0 \qquad \forall T \in \Delta_d(\mathcal{T}_h)\}, \\ &= \{v \in X_1^h : \alpha\nabla \cdot v + \beta \cdot v = 0\}, \\ K_2^h &= \{u \in X_2^h : \nabla \cdot u = 0\}. \end{aligned} \tag{4.10}$$

*Remark:* We have used that $(\alpha\nabla + \beta) \cdot (X_1^h)$ and $\nabla \cdot (X_2^h)$ are both subsets of the piecewise constants in the above description of the spaces $K_i^h$.

Unfortunately we have been unable to prove this. However, we do have a weaker result after we have modified the norms in the next section.

## 4.2 Natural norms

Later we will assume that $\alpha$ is a very small parameter. Since the convergence estimates for the above discretization are bounded by the inverse of the above infsup-constants, i.e. "constants" of order $1/\alpha$, we will need new convergence estimates that are independent of $\alpha$. This leads to what's referred to as a *natural norm* [16], which in our case is an $\alpha$ and $\beta$ dependent norm such that the continuity and infsup-constants are independent of these parameters. We only need our constants to be independent of $\alpha$, however, so any $\beta$ dependencies can in principle be ignored. We can also ignore logarithmic terms in $\alpha$, since $|\log(\alpha)|$ will be small enough for almost all applications. Note the following result

**Proposition 24.** *An almost natural norm of the bilinear form*

$$c(\phi, \psi) = \int_\Omega \alpha \nabla\phi \cdot \nabla\psi + \beta \cdot \nabla\phi\psi$$

*is, up to logarithmic terms in $\alpha$, given by*

$$|||\phi|||^2 = \alpha|\phi|^2_{H^1} + ||\beta \cdot \nabla\phi||^2_{(C_0,C_1)_{1/2,2}},$$

*where $||\cdot||_{(C_0,C_1)_{1/2,2}}$ is an interpolation norm [19] between $C_0 = H^{-1}$ and $C_1 = \beta \cdot \nabla(H^1_0)$.*

See [16], [17] and [18] for details. Observe that the second term is essentially a non-standard $H^{1/2}$ norm. We will not use this result, but looking at this norm and the properties of our complex (2.4), we suspect the natural norms for our problem could be of the form

$$\begin{aligned} ||u||_w &= ||u||_{H^{1/2}_w} + ||\,\mathrm{d}u||_{L^2}, \\ ||v||_{w,\mathrm{d}_\alpha} &= ||v||_{H^{-1/2}_w} + ||\,\mathrm{d}_\alpha v||_{L^2}, \end{aligned} \tag{4.11}$$

where $||\cdot||_{H^{1/2}_w}$ is one of the many possible $H^{1/2}$-norms. Analogous to the $H^{1/2}_w$-norm used in [6], it has to be weak enough to allow discontinuities, since tangential components of $u$ are not necessarily continuous. $H^{-1/2}_w$ is then the dual space of this unknown space $H_w^{1/2}$. Note that we have slightly abused notation in (4.11) by using the exterior derivative on a vector field. This is done to indicate the symmetric nature of these norms. We observe that the continuity conditions on the bilinear forms $a$ and $b_1$ with these new norms follows from Hölder's inequality,

$$
\begin{aligned}
&|a(u,v)| = |<u,v>| \leq ||u||_{H_w^{\frac{1}{2}}} ||v||_{H_w^{-\frac{1}{2}}} \leq ||u||_w ||v||_{w,\,\mathrm{d}_\alpha}, \\
&|b_1(p,v)| \leq \int_\Omega |(\alpha\nabla\cdot v + \beta\cdot v)p| \,\mathrm{d}x \leq ||\,\mathrm{d}_\alpha v||_{L^2} ||p||_{L^2} \leq ||v||_{w,\,\mathrm{d}_\alpha} ||p||_{L^2}, \\
&|b_2(q,u)| \leq \int_\Omega |(\nabla\cdot u)q| \,\mathrm{d}x \leq ||u||_w ||q||_{L^2}.
\end{aligned}
$$

where $< \cdot, \cdot >$ denotes the dual pairing between $H_w^{1/2}$ and $H_w^{-1/2}$. Assuming these norms actually are natural for our problem, i.e. that the continuous infsup condition (1.9) for $a$ is true with them, we can finally prove a weaker version of our discrete infsup condition (1.13). Under the additional assumption that we have some extra regularity on the optimal trial function $u$. This may be a too strong assumption, but it is natural to expect that once we have some regularity on an optimal trial function, we can get a little more by for example elliptic regularity.

**Proposition 25.** *Assuming the continuous infsup condition* (1.9) *on $a$ is true with the norms* (4.11)*, and furthermore that the optimal trial function $u$ is actually in $H^1$, then the discrete infsup condition* (1.13)

$$
\sup_{u\in K_2^h} \frac{a(u,v)}{||u||_w} \geq \hat{A}||v||_{w,\,\mathrm{d}_\alpha}, \quad \forall v \in K_1^h, \tag{4.12}
$$

*is true for the spaces* (4.10).

*Proof.* Let $v \in K_1^h \subset K_1$ be given and let $u \in K_2$ be its corresponding optimal trial function. Again, let $\Pi_h : H_{\mathrm{div}} \to X_2^h$ be the smoothed projections found in [8] or [7] for piecewise linear elements. We then have

$$
\begin{aligned}
a(u - \Pi_h u, v) &= \int_\Omega (u - \Pi_h)\cdot v \,\mathrm{d}x\,\mathrm{d}y \leq ||u - \Pi_h u||_{H_w^{1/2}} ||v||_{H_w^{-1/2}} \\
&\preceq h^{1/2} |u|_{H^1} ||v||_{H_w^{-1/2}} \xrightarrow[h\to 0]{} 0,
\end{aligned}
$$

and so the result would follow from a similar argument as in Proposition 20. $\square$

*Remark:* If we modify our discretization by taking $p$ from the space

$$Y_1^h = \{p \in L^2 : p|_T = p_T e^{\frac{\beta_1 x + \beta_2 y}{\alpha}}, \quad T \in \Delta_d(\mathcal{T}_h)\},$$

and using the norm (4.11) on $v$, we can use our smoothed projections $\Pi_h^\alpha$ to find a simpler proof of the discrete infsup condition on $b_1$, assuming we find an extension operator for it.

*Proof.* Let $p \in Y_1^h \subset L^2$ and $v \in H_{\text{div}}$ its optimal test function be given. Then we have

$$\begin{aligned} b_1(p, v - \Pi_h^\alpha v) &= \int_\Omega p \, \mathrm{d}_\alpha(v - \Pi_h^\alpha v) = \int_\Omega p( \mathrm{d}_\alpha v - \Pi_h^\alpha \mathrm{d}_\alpha v), \\ &= \sum_T \int_T p_T e^{\frac{\beta_1 x + \beta_2 y}{\alpha}} ( \mathrm{d}_\alpha v_T - \Pi_h^\alpha \mathrm{d}_\alpha v_T) = 0, \end{aligned}$$

by the degrees of freedom for our projections $\Pi_h^\alpha$. It then follows from Lemma 19 that

$$\begin{aligned} ||p||_{L^2} &\preceq \sup_{v \in H_{\text{div}}} \frac{b_1(p,v)}{||v||_{L^2} + || \mathrm{d}_\alpha v||_{L^2}} = \sup_{v \in H_{\text{div}}} \frac{b_1(p,v)}{||v||_{L^2} + || \mathrm{d}_\alpha v||_{L^2}} \\ &\preceq \sup_{v \in H_{\text{div}}} \frac{b_1(p, \Pi_h^\alpha v)}{||\Pi_h^\alpha v||_{L^2} + || \mathrm{d}_\alpha \Pi_h^\alpha v||_{L^2}} \\ &\preceq \sup_{v \in X_1^h} \frac{b_1(p,v)}{||v||_{L^2} + || \mathrm{d}_\alpha v||_{L^2}}. \end{aligned}$$

$\square$

## 4.3 Upwind ($L^2, H_{div}$)-Formulation for $\alpha << |\beta|$

**Stability in 1D**

We investigate the case $\alpha << |\beta|$ in dimension 1, and show the infsup conditions with constants independent of $\alpha$ using the norms (4.11). In one dimension our equation (1.1) becomes

$$(\alpha p_x - \beta p)_x = f \qquad p|_{\partial\Omega} = 0,$$

and to simplify we set $\Omega = (0,1)$. A simple rescaling can turn it into

$$(\alpha p_x - p)_x = f \qquad p|_{\partial\Omega} = 0$$

so we can without loss of generality assume $\beta = 1$. Setting $u = \alpha p_x - p$, the mixed formulation is then find $p \in L^2(\Omega)$ and $u \in H^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega uv \,\mathrm{d}x + \int_\Omega p(\alpha v_x + v)\,\mathrm{d}x &= 0 && \forall v \in H^1(\Omega) \\ \int_\Omega u_x q \,\mathrm{d}x &= \int_\Omega fq \,\mathrm{d}x && \forall q \in L^2(\Omega). \end{aligned}$$

Since $H_{\mathrm{div}} = H^1$ in dimension 1 the natural norms of this problem becomes much easier to handle. We can use the $H^1$ norm on $u$, and for $v$ use the norm $||\cdot||_\alpha$ defined by

$$||v||_\alpha = ||v||_{H^{-1/2}} + ||\alpha v_x + v||_{L^2}. \tag{4.13}$$

We observe that the continuity conditions on the bilinear forms $a$, $b_1$ and $b_2$ with these new norms also follows from Hölder's inequality.

$$\begin{aligned} |a(u,v)| &= |\int_\Omega uv\,\mathrm{d}x| \le ||u||_{H^1/2}||v||_{H^{-1/2}} \le ||u||_{H^1}||v||_\alpha, \\ |b_1(p,v)| &\le \int_\Omega |(\alpha v_x + v)p|\,\mathrm{d}x \le ||\alpha v_x + v||_{L^2}||p||_{L^2} \le ||v||_\alpha ||p||_{L^2}, \\ |b_2(q,u)| &\le \int_\Omega |u_x q|\,\mathrm{d}x \le ||u||_{H^1}||q||_{L^2}. \end{aligned}$$

Since this is in one dimension the mesh is just a partition of the unit interval, and so $\mathcal{T}_h = \{T_1, ..., T_n\}$ where $n = [1/h]$. The spaces for the finite dimensional approximation becomes

$$\begin{aligned} X_1^h &= \{v \in H^1 : v|_T = a_T + b_T e^{-\frac{x}{\alpha}},\, T \in \mathcal{T}_h\}, \\ X_2^h &= \{u \in H^1 : u|_T = a_T + b_T x,\, T \in \mathcal{T}_h\}, \\ Y^h &= \{q \in L^2 : q|_T = a_T,\, T \in \mathcal{T}_h\}. \end{aligned} \tag{4.14}$$

**Lemma 26.** *The continuous infsup condition on $b_1(p, v) = \int_\Omega (\alpha v_x + v) p \,\mathrm{d}x$ with the norm $||\cdot||_\alpha$ on $v$ and the $L^2$ norm on $u$ is satisfied.*

*Proof.* Let $v$ be the solution to

$$\alpha v_x + v = p, \qquad v(0) = 0,$$

then

$$v(x) = \int_0^x \frac{1}{\alpha} e^{\frac{(s-x)}{\alpha}} p(s) \,\mathrm{d}s. \tag{4.15}$$

Using the function $G_\alpha$ from [6] defined by

$$G_\alpha(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \tag{4.16}$$

we observe that (4.15) can be written as

$$v = G_\alpha * p. \tag{4.17}$$

Since $||G_\alpha||_{L^1} \leq 1$ and $p \in L^2$ we have by Young's inequality [19] that $||v||_{L^2} \leq ||p||_{L^2}$. Hence, $||v||_\alpha \preceq ||v||_{L^2} + ||\alpha v_x + v||_{L^2} \preceq ||p||_{L^2}$ and

$$\frac{b_1(p, v)}{||v||_\alpha} \succeq \frac{||p||^2_{L^2}}{||p||_{L^2}} \succeq ||p||_{L^2},$$

where the constant is independent of $\alpha$. $\square$

**Proposition 27.** *The discrete infsup condition on $b_1(p, v) = \int_\Omega (\alpha v_x + v) p \, \mathrm{d}x$ with the norm $|| \cdot ||_\alpha$ on $v$ and the $L^2$ norm on $p$ is satisfied.*

*Proof.* Let $p \in Y_1^h \subset L^2$ be given. Define $v$ as the solution to

$$\alpha v_x + v = p \qquad v(0) = 0,$$

then it is a piecewise exponential function in the space $X_1^h$, since $\mathrm{d}_\alpha(X_1^h) = Y^h$. As in the above lemma $v$ can be written as $v = G_\alpha * p$, for $G_\alpha$ defined in (4.16), and so by the same argument $||v||_\alpha \preceq ||p||_{L^2}$. The result follows. $\square$

**Lemma 28.** *The continuous infsup condition on $b_2(q, u) = \int_\Omega q u_x \, \mathrm{d}x$ is satisfied with the $H^1$ norm on $u$ and the $L^2$ norm on $q$.*

*Proof.* Let $q \in L^2$ be given and let $u$ solve

$$v_x = q \qquad u(0) = 0.$$

Then by Poincaré's inequality we have $||u||_{L^2} \preceq ||u_x||_{L^2} \preceq ||q||_{L^2}$, and so

$$\frac{b_2(q, u)}{||u||_{H^1}} = \frac{||q||^2_{L^2}}{||u||_{H^1}} \geq B_1 \frac{||q||^2_{L^2}}{||q||_{L^2}} = B_1 ||q||_{L^2}.$$

$\square$

**Proposition 29.** *The discrete infsup condition on $b_2(q, u) = \int_\Omega q u_x \, \mathrm{d}x$ is satisfied with the $H^1$ norm on $u$ and the $L^2$ norm on $q$.*

*Proof.* Let $q \in Y^h \subset L^2$ be given and let $u$ solve

$$u_x = q \qquad u(0) = 0.$$

Then $u$ is a piecewise linear function in $X_1^h$, and the result follows from the same argument as in the above lemma. $\square$

**Lemma 30.** *The continuous infsup condition* (1.9) *on* $a(u,v) = \int_\Omega uv\,\mathrm{d}x$ *is satisfied, up to logarithmic terms in* $\alpha$, *for* $u \in K_2$ *with the* $H^1$*-norm and* $v \in K_1$ *with the norm* $||\cdot||_\alpha$.

*Proof.* Observe that $u \in K_2$ means $u_x = 0$ which implies $u$ is of the form

$$u = c$$

for $c$ constant. On the other hand $v \in K_1$ means $\alpha v_x + v = 0$ which implies

$$v = ke^{-\frac{x}{\alpha}},$$

for $k$ constant. We can choose $k = 1$ and by a simple calculation we have

$$a(u,v) = \int_\Omega uv\,\mathrm{d}x = \int_0^1 ce^{-\frac{x}{\alpha}}\,\mathrm{d}x = c\alpha(1 - e^{-\frac{1}{\alpha}}).$$

To estimate $||v||_\alpha = ||v||_{H^{-1/2}}$ we use the fact that an equivalent norm for $H^{-1/2}$ is in dimension 1 given by the square-root (of the absolute value) of the integral

$$\int_0^1 \int_0^1 \log|x-y| v(x)v(y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 \log|x-y| e^{-\frac{x+y}{\alpha}}\,\mathrm{d}x\,\mathrm{d}y. \quad (4.18)$$

Using substitutions $\frac{x}{\alpha} \to x$ and $\frac{y}{\alpha} \to y$ the integral (4.18) turns into

$$\begin{aligned}\alpha^2 \int_0^{1/\alpha} \int_0^{1/\alpha} \log|\alpha(x-y)| e^{-x-y}\,\mathrm{d}x\,\mathrm{d}y = {} & \alpha^2 \int_0^{1/\alpha} \int_0^{1/\alpha} \log|\alpha| e^{-x-y}\,\mathrm{d}x\,\mathrm{d}y, \\ & + \alpha^2 \int_0^{1/\alpha} \int_0^{1/\alpha} \log|x-y| e^{-x-y}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \quad (4.19)$$

and we immediately see that

$$\alpha^2 \int_0^{1/\alpha} \int_0^{1/\alpha} \log|\alpha| e^{-x-y} \, \mathrm{d}x \, \mathrm{d}y = \alpha^2 \log(\alpha)[1 - e^{-\frac{1}{\alpha}}]^2.$$

As for the remaining term in (4.19) we use the substitutions $u = x + y$, $v = x - y$ and observe

$$\begin{aligned} \alpha^2 \int_0^{1/\alpha} \int_0^{1/\alpha} \log|(x-y)| e^{-x-y} \, \mathrm{d}x \, \mathrm{d}y = \alpha^2 & \int_0^{1/\alpha} \int_{-u}^{u} \log|v| e^{-u} 2 \, \mathrm{d}v \, \mathrm{d}u, \\ & + \alpha^2 \int_{1/\alpha}^{2/\alpha} \int_{u-2/\alpha}^{2/\alpha - u} \log|v| e^{-u} 2 \, \mathrm{d}v \, \mathrm{d}u, \end{aligned} \tag{4.20}$$

where we have used that the determinant of the Jacobian is 2. continuing with the first integral in (4.20) we get

$$\alpha^2 \int_0^{1/\alpha} \int_{-u}^{u} \log|v| e^{-u} 2 \, \mathrm{d}v \, \mathrm{d}u = 2\alpha^2 \int_0^{1/\alpha} 2u(\log(u) - 1) e^{-u} \, \mathrm{d}u = \alpha^2 O(1),$$

since

$$\int_0^{1/\alpha} u(\log(u) - 1) e^{-u} \, \mathrm{d}u \xrightarrow[\alpha \to 0]{} -\gamma,$$

where $\gamma$ is the Euler–Mascheroni constant. Next we look at the second integral in (4.20)

$$\begin{aligned} \alpha^2 \int_{1/\alpha}^{2/\alpha} \int_{u-2/\alpha}^{2/\alpha - u} \log|v| e^{-u} 2 \, \mathrm{d}v \, \mathrm{d}u &= 2\alpha^2 \int_{1/\alpha}^{2/\alpha} 2(\frac{2}{\alpha} - u)(\log(\frac{2}{\alpha} - u) - 1) e^{-u} \, \mathrm{d}u, \\ &= 4\alpha^2 e^{-\frac{2}{\alpha}} \int_0^{1/\alpha} w(\log(w) - 1) e^{w} \, \mathrm{d}w, \end{aligned} \tag{4.21}$$

where we have used the substitution $w = \frac{2}{\alpha} - u$. This last integral (4.21) is $O(\alpha^2)$ since it follows, by for example Hölder's inequality, that

$$e^{-\frac{2}{\alpha}} \int_0^{1/\alpha} w(\log(w) - 1)e^w \, \mathrm{d}w \xrightarrow[\alpha \to 0]{} 0.$$

All in all we have

$$||v||^2_{H^{-1/2}} \leq C(\alpha^2 \log(\alpha) + \alpha^2),$$

for a constant $C > 0$ independent of $\alpha$. We conclude that

$$||v||_\alpha \preceq \alpha\sqrt{\log(\alpha)},$$

and

$$\frac{a(u,v)}{||v||_\alpha} \succeq \frac{c\alpha}{\alpha\sqrt{\log(\alpha)}} \succeq \log(\alpha)^{-1/2}||u||_{H^1},$$

which is what we want.

□

**Proposition 31.** *The discrete infsup condition* (1.13) *on* $a(u,v) = \int_\Omega uv \, \mathrm{d}x$ *is satisfied, up to logarithmic terms in* $\alpha$, *for* $u \in K_2^h$ *with the* $H^1$*-norm and* $v \in K_1^h$ *with the norm* $|| \cdot ||_\alpha$.

*Proof.* Observe that $u \in K_2^h$ means $(u_i)_x = 0$ on each subinterval $T_i$, and so $u_i$ is of the form $u_i = c_i$ for $c_i$ constant. Continuity between intervals, however, requires these constants are all equal, i.e. that $c_i = c$ for all $i$. Similarly for $v \in K_1^h$ we have $\alpha(v_i)_x + (v_i) = 0$ which implies

$$v_i = k_i e^{-\frac{x}{\alpha}},$$

for $k_i$ constant. Again continuity between intervals requires $k_i = k$ for all $i$. Hence, this infsup condition is the same as in Lemma 30.

□

Putting it all together we get the following result.

**Corollary 32.** *The exponentially upwinded discretization for the one dimensional problem described above is stable up to logarithmic terms in $\alpha$.*

## 4.4 Downwind $(L^2, H_{div})$-Formulations

Let $u = \nabla p$ then a $(L^2, H_{div})$ mixed formulation of (1.2) is find $p \in L^2$ and $u \in H_{\mathrm{div}}$ such that

$$\begin{aligned} \int_\Omega u \cdot v \,\mathrm{d}x + \int_\Omega p \nabla \cdot v \,\mathrm{d}x\,\mathrm{d}y &= 0 && \forall v \in H_{\mathrm{div}} \\ \int_\Omega (\beta \cdot u - \alpha \nabla \cdot u) q \,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega f q \,\mathrm{d}x && \forall q \in L^2. \end{aligned}$$

Define

$$\begin{aligned} a(u, v) &= \int_\Omega u \cdot v \,\mathrm{d}x\,\mathrm{d}y, \\ b_1(p, v) &= \int_\Omega p \nabla \cdot v \,\mathrm{d}x\,\mathrm{d}y, \\ b_2(q, u) &= \int_\Omega (\beta \cdot u - \alpha \nabla \cdot u) q \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.22}$$

then this mixed formulation can be written in the same form as (4.3). As above, let $\mathcal{T}_h$ be our rectangular mesh. This time we define $X_1^h, X_2^h \subset H_{\mathrm{div}}$ and $Y^h \subset L^2$ as

$$\begin{aligned} &X_1^h = \{u \in H_{\mathrm{div}} : u|_T = (a_T + b_T x, c_T + d_T y)\, T \in \Delta_d(\mathcal{T}_h)\} \\ &X_2^h = \{v \in H_{\mathrm{div}} : u|_T = (a_T + b_T e^{\frac{\beta_1 x}{\alpha}}, c_T + d_T e^{\frac{\beta_2 y}{\alpha}}),\, T \in \Delta_d(\mathcal{T}_h)\} \\ &Y^h = \{q \in L^2 : q|_T = a_T,\, T \in \Delta_d(\mathcal{T}_h)\}. \end{aligned} \tag{4.23}$$

Note that the only difference between (4.23) and (4.5) is that $X_1^h$ and $X_2^h$ have been interchanged (up to a sign). The proofs for the continuity and infsup-conditions are essentially the same as above, since we have just switched the roles of $u$ and $v$ (note the similarity of $b_i$ in (4.2) and $b_{1-i}$ in (4.22)). We just need to choose the other infsup condition for $a(u, v)$. The sign difference in $X_2^h$ is caused by the sign difference in $b_2$. The key difference in this case is that the new error norm on $u$ will be old norm on $v$, but this won't give us a better convergence estimate since our exponentially upwind/downwind spaces have worse convergence properties than the regular piecewise linear spaces.

# Chapter 5

# Concluding Remarks

We have shown that a complex of differential forms which are piecewise exponential can be used to construct an upwind mixed discretization of convection diffusion equations. Several continuity and infsup-conditions have been proven for this discretization, but only a weak version of the discrete infsup-condition on $a$ in the case $\alpha \sim |\beta|$ was found. Possible candidates for the *natural norms* of our problem have been identified and we have used these norms to prove stability of our one dimensional discretization, up to logarithmic terms in $\alpha$, in the case $\alpha << |\beta|$. Finding out whether these norms are natural in higher dimensions as well, and determine exactly which $H^{1/2}$ norm is correct, still remains.

While we have made use of the interpolators and smoothed projections constructed for our complex (2.4) in our proofs, they lack many desired properties. Since we now have good norms for our problem, it makes it easier to see which properties are the most essential for our smoothed projections and which properties we can spare. The projection $\mathrm{P}_h^\alpha$ constructed in Chapter 3 is $L^2$ stable uniformly in $\alpha$ while the smoothed projection $\Pi_h^\alpha$ commutes with $\mathrm{d}_\alpha$. To prove a discrete infsup condition using a smoothed projection we would prefer one that satisfies at least both of these properties. This is difficult, but since we only need $H_w^{-1/2}$ stability uniformly in $\alpha$ it is possible that our projections $\Pi_h^\alpha$ can be shown to satisfy this. $\Pi_h^\alpha$ also lacks an extension operator, and both smoothed projections should be constructed for all $k$-forms in the complex (2.4).

**Thoughts on stability in 2D in the case** $\alpha << |\beta|$

1. Using the norms (4.11) the main difficulty (after finding out exactly which

$H^{1/2}$-norm is correct) lies in proving the infsup conditions on $a(u, v)$ and $b_1(p, v)$. For the discrete infsup-condition on $b_1$ we can, for a given $p \in Y^h$, define the optimal test function $v \in X_1^h$ by

$$\mathrm{d}_\alpha v = p, \tag{5.1}$$

but then we can't use our current smoothed projections to get an $\alpha$-independent bound on $v$. We would have to look for some other argument. Recalling definition (4.5) of $X_1^h$ and $Y_h$ we observe that (5.1) turns into

$$a_1^T \beta_1 + a_2^T \beta_2 = a^T$$

on each rectangle $T$, where $v|_T = (a_1^T + b_1^T e^{\frac{-\beta_1 x}{\alpha}}, a_2^T + b_2^T e^{\frac{-\beta_2 y}{\alpha}})$ and $p|_T = a^T$. It then looks intuitively true that we can get such a bound $v$ since the only role of the $b_i$'s will be to enforce continuity between the normal components of $v$. We might therefore expect the term $b_1^T e^{\frac{-\beta_1 x}{\alpha}}$ to be bounded either by $a_1^T$ itself or the constant $a_1^{T'}$ in the "next" rectangle $T'$.

As for the continuous infsup condition on $a$, if we wished to use a similar argument as in Lemma 23 we would need a bound of the form $||\nabla f||_{H_w^{-1/2}} \preceq ||u||_{H_w^{1/2}}$ for the function $f$ defined there. Looking at Proposition 24 and equation (4.9) we can get the bound $||\beta \cdot \nabla f||_{H^{-1/2}} \preceq ||\beta \cdot u||_{L^2}$, which is almost, but not quite what we want.

2. Some of our interpolators were stable uniformly in $\alpha$ when using the $L^1$ or $L^\infty$ norms. Using spaces other than the usual $L^2$ Hilbert spaces is not common for finite element methods, but it has been done in e.g. [13]. A problem with using these norms is that neither $L^1$ or $L^\infty$ are reflexive Banach spaces, a requirement in Proposition 1. This could perhaps be fixed by looking at the space of $L^1$ trial functions with continuous test function $C$, since $L^1$ is dense in the dual space of $C$. Optimal test function would then be sign$(u)$ in the case $u \in L^1$ and the Dirac delta in the case $u \in C$. Finite element spaces are not suited for using the test function sign$(u)$ since a linear/exponential/monotone function can switch signs within an element, something that would require the sign function to be discontinuous inside a rectangle $T$. On the other hand, our piecewise exponential spaces could perhaps be used to approximate the Dirac delta, at least in the limit $\alpha \to 0$.

# Appendix A

# Appendix

## A.1 Elliptic Regularity

**Proposition 33.** *Let $\Omega$ be a convex domain. Then for each $f \in L^2(\Omega)$ there exist a unique $p \in H^2(\Omega)$ such that $p$ is the solution of*

$$\begin{aligned} Lp &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega \end{aligned}$$

*when L is an elliptic operator.*

This is Theorem 3.2.1.2 in [12] and the proof is found there.

## A.2 Upwind $(H^1, L^2)$-Formulation

Let $u = \nabla p$ then a $(H^1, L^2)$ mixed formulation of (1.2) is find $p \in H_0^1$ and $u \in L^2$ such that

$$\begin{aligned} \int_\Omega u \cdot v \,\mathrm{d}x - \int_\Omega \nabla p \cdot v \,\mathrm{d}x \,\mathrm{d}y &= 0 && \forall v \in L^2 \\ \int_\Omega (\alpha u \cdot \nabla q + \beta \cdot uq) \,\mathrm{d}x \,\mathrm{d}y &= \int_\Omega f q \,\mathrm{d}x \,\mathrm{d}y && \forall q \in H_0^1. \end{aligned} \tag{A.1}$$

Define

$$\begin{aligned} a(u,v) &= \int_\Omega u\cdot v \,\mathrm{d}x\,\mathrm{d}y, \\ b_1(p,v) &= -\int_\Omega \nabla p\cdot v\,\mathrm{d}x\,\mathrm{d}y, \\ b_2(q,u) &= \int_\Omega (\alpha u\cdot\nabla q + \beta\cdot uq)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{A.2}$$

then the mixed formulation can be written as

$$\begin{aligned} a(u,v)+b_1(p,v) &= 0 \qquad && \forall v\in L^2 \\ b_2(q,u) &= <f,q> && \forall q\in H_0^1. \end{aligned} \tag{A.3}$$

First observe that $||\cdot||_\alpha$ defined by

$$||q||_\alpha = ||\,\mathrm{d}_\alpha q||_{L^2} = ||\alpha\nabla q+\beta q||_{L^2}$$

is actually a norm on $H_0^1$, since $\alpha\nabla q+\beta q = 0$ and $q|_{\partial\Omega}=0$ implies $q=0$. Using this norm we observe that the continuity conditions follows from Hölder's inequality

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |u\cdot v|\,\mathrm{d}x \le ||u||_{L^2}||v||_{L^2}, \\ |b_1(p,v)| &\le \int_\Omega |\nabla p\cdot v|\,\mathrm{d}x \le ||p||_{H^1}||v||_{L^2} \\ |b_2(q,u)| &\le \int_\Omega |(\alpha\nabla q+\beta q)\cdot u)|\,\mathrm{d}x\,\mathrm{d}y \le ||q||_\alpha||u||_{L^2}. \end{aligned} \tag{A.4}$$

Define spaces

$$\begin{aligned} &Y_1^h = \{p\in H_0^1 :\, p|_T = a_T + b_T x + c_T y + d_T xy,\, T\in\mathcal{T}_h\} \\ &Y_2^h = \{q\in H_0^1 :\, q|_T = q|_T = a_T + b_T e^{\frac{-\beta_1 x}{\alpha}} + c_T e^{\frac{-\beta_2 y}{\alpha}} + d_T e^{\frac{-\beta_2 y}{\alpha}} e^{\frac{-\beta_1 x}{\alpha}},\, T\in\mathcal{T}_h\} \\ &X_1^h = \{v\in L^2 :\, v|_T = (a_T + b_T y, c_T + d_T x)\, T\in\mathcal{T}_h\} \\ &X_2^h = \{u\in L^2 :\, u|_T = (a_T + b_T e^{\frac{-\beta_1 y}{\alpha}}, c_T + d_T e^{\frac{-\beta_2 x}{\alpha}}),\, T\in\mathcal{T}_h\} \end{aligned} \tag{A.5}$$

Note that $Y_1^h$ and $X_1^h$ are the usual piecewise polynomial spaces for this problem, satisfying $\nabla(Y_1^h) = X_1^h$, while $Y_2^h$ and $X_2^h$ are the exponentially upwind spaces corresponding to the complex (2.4) of differential forms, satisfying $\alpha\nabla + \beta)(Y_2^h) = X_2^h$. The infsup condition on $b_1$ for the spaces $X_1^h$ and $Y_1^h$ defined above is a well-known result.

**Proposition 34.** *There exists a constant $\hat{B}_1 > 0$ independent of $h$ such that the discrete infsup condition* (1.11) *($i = 1$), with the spaces $X_1^h$ and $Y_1^h$ defined as in* (A.5)*, is satisfied with the $H^1$ norm on $p$ and the $L^2$ norm on $v$.*

*Proof.* Let $p \in Y_1^h \subset H_0^1$ be given. Since $\nabla(Y_1^h) = X_1^h$ we choose the optimal test function $v \in X_1^h$ such that $v = -\nabla p$. Then we have

$$\frac{b_1(p, v)}{||v||_{L^2}} = \frac{-\int_\Omega v \cdot \nabla p \,\mathrm{d}x\,\mathrm{d}y}{||\nabla p||_{L^2}} = |p|_{H^1},$$

and the result follows from Poincaré's inequality. □

Next, we look at the discrete infsup condition on $b_2$.

**Proposition 35.** *There exists a constant $\hat{B}_2 > 0$ independent of $h$ such that the discrete infsup condition* (1.11) *($i = 2$), with the spaces $X_2^h$ and $Y_2^h$ defined as in* (A.5)*, is satisfied with the norm $||\cdot||_\alpha$ on $q$ and the $L^2$ norm on $u$.*

*Proof.* Let $q \in Y_2^h \subset H_0^1$ be given. Since $(\alpha\nabla + \beta)(Y_2^h) = X_2^h$ we choose the optimal trial function $u \in X_2^h$ such that $u = \alpha\nabla q + \beta q$. Then we have

$$\frac{b_2(q, u)}{||u||_{L^2}} = \frac{\int_\Omega |\alpha\nabla q + \beta q|^2 \,\mathrm{d}x\,\mathrm{d}y}{||u||_{L^2}} = \frac{||q||_\alpha^2}{||q||_\alpha} = ||q||_\alpha.$$

□

*Remark:* The continuous infsup conditions have essentially the same proofs.

While we lack proofs of the infsup conditions on $a$ for this problem, note that stability in 1D follows from the above arguments since the continuous and discrete infsup condition on $a$ would then be trivial, as $v \in K_i, K_i^h$ would then imply $v = 0$.

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