Involutions and Fredholm Maps

by

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Introduction. Let E be a Banach space and $K : E \to E$ a completely continuous map (i.e. such that the image of a bounded set has compact closure). Assume that K is odd (but not necessarily linear) and let A_r be the set of solutions of the equation $x + K(x) = 0$ at the sphere S_r of radius r from the origin. By a theorem of Granas [4, Theorem 10, p.45], if $I + K$ maps S_r to a proper subspace of E, then A_r is non-empty. The purpose of this article is to initiate a closer study of the solution set A_r in a more general context. Thus, let X be a paracompact Hausdorff space with a fixed point free involution T, and let $\varphi: X \to E$ be a proper equivariant map. We define a numerical invariant called the coindex of φ and estimate the size of $A(f) = \{x \in X | f(Tx) = f(x)\}$ in terms of this invariant, where $f : X \to E$ is any compact perturbation of φ. The methods we use are based on those of Conner and Floyd [1], [2], suitably extended to the infinite dimensional situation. As in [1] the method often covers the more general case where T is replaced by a finite group of homeomorphisms acting freely on X.

The actual computation of coind φ requires in practice

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considerable regularity of the map φ. One case which seems more tractable than others is where X is a differentiable manifold modelled on a Banach space and φ is a Fredholm map. This case gains considerable importance in view of recent development, see e.g. [3]. The most interesting example to have in mind is perhaps the one where φ is derived from a non-linear partial differential operator on a bounded region in R^n, see again [3].

In section 1 we summerize standard properties of the coindex of a space with involution and in section 2 we define the coindex of an equivariant map from a space with involution to a Banach space. In section 3 there is a local computation of the coindex of a Fredholm map. Section 4 deals with the degree of a map from one Banach manifold to another and section 5 relates the degree to the coindex. Section 6 establishes equivariant transversality which is used in section 7 where the global result on the coindex of a Fredholm map is proved.

1. Coindex of a space with involution. Let X be a paracompact Hausdorff space and $T: X \to X$ a fixed point free involution on X. Then $X \to X/T$ is a double covering with a characteristic class $c \in H^1(X/T)$ (Cech cohomology, coefficients $\mathbb{Z}_2$). Define the coindex of (X,T) to be the largest non-vanishing power of c; by abuse of notation

$$\text{coind } X = \sup\{n ; c^n \neq 0\} .$$

In the notation of Conner and Floyd [ ] the coindex map is written $\text{co-ind}_{\mathbb{Z}_2} X$, and the authors observe that it has the following properties:

1.1 (Conner-Floyd) The coindex map assigns to each paracompact Hausdorff space X with a fixed point free involution a non-negative integer or ∞, such that

(Functoriality) If $f: X \to Y$ is an equivariant map between spaces with involutions, then $\operatorname{coind} X \leq \operatorname{coind} Y$.

(Additivity) If A, B are closed invariant subsets of X and $X = A \cup B$, then $\operatorname{coind} X \leq \operatorname{coind} A + \operatorname{coind} B + 1$.

(Continuity) If A is a closed invariant subset of S, then $\operatorname{coind} A = \operatorname{coind} U$ for some closed invariant neighbourhood U of A.

(Dimensionality) $\operatorname{coind} S^n = n$, $n = 0, 1, \ldots$

and such that

(Stability) If X is compact, then $\operatorname{coind} SX = \operatorname{coind} X + 1$.

Here SX means the suspension of X equipped with the fixed point free involution $(x,t) \to (T(x), 1-t)$. It is an easy consequence of the additivity property that quite generally $\operatorname{coind} SX \leq \operatorname{coind} X + 1$.

The properties listed in 1.1 do not characterize the coindex map. In fact a coindex based on the characteristic class with twisted integral coefficients (instead of $\mathbb{Z}_2$-coefficients) satisfies 1.1 as well. And if L is any principal ideal domain, there is a coindex map based on the characteristic class with twisted L-coefficients, having the properties 1.1 with the possible exception of the stability property. We refer to [1] for the details. Until further notice coind will stand for any map satisfying 1.1 except for the stability property. For convenience we also add the definition coind $\emptyset = -1$, and observe that then 1.1 remains true also in the cases where any of the spaces occurring are empty provided $S\emptyset$ is interpreted as S^o. A coindex map is stable if it has the stability property. For an example of a non-stable coindex map of a somewhat different character than those already mentioned, let coind X be the smallest integer n such that there is an equivariant map $X \to S^n$; see again [ ].

The following result in a somewhat different setting is due to Yang [9]:

1.2 (Yang) Let $f: X \to R^n$ be any map and let $A(f) \subset X$ be the set of points x such that $f(x) = f(Tx)$. Then $A(f)$ is a closed invariant subset of X and

$$\text{coind } A(f) \geq \text{coind } X - n$$

Proof. Form the map $\varphi = f - f \circ T : X \to R^n$. Then φ is equivariant (with respect to the standard involution in R^n) and $A(\varphi) = A(f)$. Thus we may as well assume f equivariant. Let U be a closed invariant neighbourhood of $A(f)$ such that coind U = coind A (the continuity property) and V a closed invariant neighbourhood of U such that coind V = coind U . Then $X - \mathring{U}$ and V are closed invariant subsets covering X and so $\text{coind } X \leq \text{coind}(X - \mathring{U}) + \text{coind } V + 1$, by additivity. On the other hand, existence of an equivariant map

$$X - \mathring{U} \xrightarrow{f} R^n - o \to S^{n-1}$$

shows that $\text{coind}(X - \mathring{U}) \leq n - 1$. Thus $\text{coind } X \leq n - 1 + \text{coind } A(f) + 1$.

2. *Coindex of an equivariant map.* In the sequel E denotes a Banach space with its standard involution (one fixed point, the origin). If $\varphi: X \to E$ is any equivariant map, we define the *coindex* of φ by coind $\varphi \geq p$ if for any sufficiently large finite dimensional subspace $F \subset E$ coind $\varphi^{-1}F \geq p + \dim F$.

As an example consider the case where $X = S$, the unit sphere in E , and φ is the inclusion $S \subset E$. Then for any finite dimensional $F \subset E$ $\varphi^{-1}F$ is the unit sphere in F , and so coind $\varphi^{-1}F \geq \dim F - 1$. It follows that coind $\varphi = -1$. Similarly, or φ is the constant map to the origin, then the coindex of φ is ∞ ; and if $X = S_F$, the unit sphere in a finite dimensional subspace $F \subset E$, and φ is the inclusion $S_F \subset E$, then the coindex of φ is $-\infty$. Thus the coindex of a map takes values in the range of all integers with the two extremes $-\infty$ and ∞ included.

A map $K: X \to E$ is *compact* (or *finite dimensional*) if im K lies in a compact (or finite dimensional) subset of E . A map $f: X \to E$ is a *compact perturbation* (or *finite dimensional perturbation*) of φ if $f = \varphi + K$ for some compact (or finite dimensional) map $K: X \to E$.

Remark. A compact perturbation of a proper map is proper.

Our first result is an extension of Yang's theorem 1.2.

2.1 Theorem. Let $\varphi: X \to E$ be a proper equivariant map and $f: X \to E$ a compact perturbation of φ . If im f lies in a k-codimensional subspace of E , then

$$\text{coind } A(f) \geq \text{coind } \varphi + k .$$

Proof. Let $E_k \subset E$ be a k-codimensional subspace containing im f and let $E^k \subset E$ be some complement. Let K be the compact map $f - \varphi$ and assume first that K is finite dimensional, i.e. that $\text{im } K \subset E^m$ for some m-dimensional subspace E^m of E . Finally let E^n be any finite dimensional subspace containing $E^m + E^k$. Then $\varphi^{-1}E^n$ * into $E^n \cap E_k$. Let $f^n: \varphi^{-1}E^n \to E^n \cap E_k$ be the restricted map. Then, by theorem 1.2 $\text{coind } A(f^n) \geq \text{coind } \varphi^{-1}E^n - (n-k)$, since clearly $\dim E^n \cap E_k = n-k$. Since for sufficiently large E^n $\text{coind } \varphi^{-1}E^n - n$ dominates $\text{coind } \varphi$, we get $\text{coind } A(f^n) \geq \text{coind } \varphi + k$. However, $A(f) \subset \varphi^{-1}E^m \subset \varphi^{-1}E^n$ as is easily checked, and so $A(f) = A(f^n)$. This proves the theorem in the case where K is finite dimensional.

In the case of a general compact map K let U be a closed invariant neighbourhood of A(f) such that coind U = coind A(f) . Suppose there is a finite dimensional compact map $K': X \to E$ such that $\text{im } f' \subset E_k$ and $A(f') \subset U$, $f' = \varphi + K'$. Since the inclusion map $A(f') \subset U$ is equivariant, we get $\text{coind } A(f) = \text{coind } U \geq \text{coind } A(f') \geq \text{coind } \varphi + k$, the last inequality by the first part of the proof. We now show that there are such maps K' .

First observe that given $U \supset A(f)$ as above there is an $\epsilon > 0$ such that $\|f(y) - f(Ty)\| \leq \epsilon$ implies $y \in U$. In fact, otherwise we could pick out a sequence of points $y_i \in X - U$ with $\|f(y_i) - f(Ty_i)\| \leq \frac{1}{i}$. However, the map $f - f\ T = 2\varphi + (K-K\ T)$ is a compact perturbation of a proper map and therefore proper. Therefore $\{y_i\}$ would contain a subsequence converging to some point $y_o \in X - A(f)$; which is impossible since by continuity $f(y_o) - f(Ty_o)$ should equal 0 .

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* is a closed invariant subspace of X , and f maps $\varphi^{-1}E^n$

Next, let $\pi: E \to E$ be the projection of E to E^k with kernel E_k. Then $\pi \circ f = \pi \circ \varphi - \pi \circ K$ is the zero map, since $\operatorname{im} f \subset E_k$. Let K_δ be a compact finite dimensional δ-approximation to K (cf. [6]), and form $K' = K_\delta + \pi \circ K - \pi \circ K_\delta$ and $f' = \varphi + K'$. Then K' is a finite dimensional compact map, and $\pi \circ f'$ is zero so that $\operatorname{im} f' \subset E_k$. Now, $K - K' = (1-\pi)(K-K_\delta)$. Therefore, $\|K(x) - K'(x)\| \leq \|1-\pi\| \cdot \delta$ and so $\|f(x) - f'(x)\| \leq \|1-\pi\| \cdot \delta$ for all $x \in X$. Suppose $y \in A(f')$. Then $\|f(y) - f(Ty)\| = \|f(y) - f'(y) - f(Ty) + f'(Ty)\| \leq \|f(y) - f'(y)\| + \|f(Ty) - f'(Ty)\| \leq 2\|1 - \pi\| \cdot \delta$. Hence, for δ sufficiently small $\|f(y) - f(Ty)\| \leq \epsilon$ and so $y \in U$, i.e. $A(f') \subset U$. This completes the proof of Theorem 2.1.

In particular, if we apply Theorem 2.1 to the case where X is S and φ is the inclusion $i: S \subset E$, we find that for any compact map $K: S \to E$ such that $x + K(x)$ lies in E_k (some k-codimensional subspace of E) coind $A(i+K) \geq k-1$. This, of course, implies that cov. dim $A(i+K) \geq k-1$, which is a slightly refined version of the Granas-Borsuk-Ulam theorem, cf. [4].

<u>Remark.</u> The first part of the proof shows that if $\varphi: X \to E$ is any equivariant map (not necessarily proper) then the conclusion of Theorem 2.1 remains true provided f is a finite dimensional (not necessarily compact) perturbation of φ.

A map $\varphi: X \to E$ is <u>finitely bounded</u> if for every finite dimensional subspace $F \subset E$, $\varphi \mid \varphi^{-1}F$ is bounded.

<u>Remark.</u> If φ is proper and finitely bounded, then $\varphi^{-1}F$ is compact when F is finite dimensional. Therefore, if

$\pi: E \to E$ is a linear map with finite dimensional kernel, $\pi \circ \varphi$ is again proper and finitely bounded. Any compact perturbation of a finitely bounded map is finitely bounded.

As an application of theorem 2.1 we give:

<u>2.2 Theorem.</u> Let $\varphi: X \to E$ be any equivariant map. Then the following are equivalent:

(1) $\operatorname{coind} \varphi \geq p$

(2) For every finite dimensional subspace F of E $\operatorname{coind} \varphi^{-1}F \geq p + \dim F$

Moreover, if φ is proper and finitely bounded, then (1) and (2) are each equivalent to

(3) For every finite dimensional subspace F of E and every compact equivariant perturbation f of φ $\operatorname{coind} f^{-1}F \geq p + \dim F$.

<u>Proof.</u> We first show that (1) implies (2). Thus, let $F \subset E$ be an arbitrary finite dimensional subspace and $F' \supset F$ a finite dimensional subspace such that $\operatorname{coind} \varphi^{-1}F' \geq p + \dim F'$. Let $\pi: F' \to F''$ be an epimorphism with kernel F. Then, by 1.2

$$\begin{aligned} \operatorname{coind} A(\pi \circ \varphi | \varphi^{-1}F') &\geq \operatorname{coind} \varphi^{-1}F' - \dim F'' \\ &\geq p + \dim F' - \dim F'' \\ &= p + \dim F . \end{aligned}$$

The conclusion now follows from the fact that $A(\pi \circ \varphi | \varphi^{-1}F')$ equals $\varphi^{-1}F$.

Next we assume that φ is proper and finitely bounded and show that (2) implies (3). Thus, let $F \subset E$ be arbitrary finite dimensional and f equivariant and compactly related to φ (i.e. such that $f - \varphi$ is a compact map). Let $\pi: E \to E$ be a projection with kernel F so that $A(\pi \circ f) = f^{-1}F$. Since $\ker \pi$ is finite dimensional, $\pi \circ \varphi$ is again proper and $\pi \circ f$ is a compact perturbation of $\pi \circ \varphi$. Therefore, since $\operatorname{im} \pi \circ f$ lies in a subspace of E of codimension equal $\dim F$, by theorem 2

$$\operatorname{coind} A(\pi \circ f) \geq \operatorname{coind} \pi \circ \varphi + \dim F .$$

Since $\pi \circ \varphi$ differs from φ by a finite dimensional map $\operatorname{coind} \pi \circ \varphi = \operatorname{coind} \varphi \geq p$ and so

$$\operatorname{coind} f^{-1}F \geq p + \dim F .$$

The implications $(3) \Rightarrow (2) \Rightarrow (1)$ are trivial.

It follows from the definition that the coindex of a map is invariant under finite dimensional equivariant perturbations. For proper finitely bounded maps it is invariant under compact perturbations in virtue of theorem 2.2 so we have the following corollary.

2.3 Corollary. If φ is proper and finitely bounded and f is a compact equivariant perturbation of φ, then $\operatorname{coind} f = \operatorname{coind} \varphi$.

We now show that the coindex can be computed by flags in reasonable cases. Let $\mathcal{F}$ be a directed family of finite dimensional subspaces of E and $\hat{\mathcal{F}}$ the family of all subspaces of E contained in some member of $\mathcal{F}$. Then $\hat{\mathcal{F}}$ is likewise a directed family of finite dimensional subspaces. Associated to $\mathcal{F}$ there is the notion of the coindex of φ with respect to $\mathcal{F}$ based on either of the two equivalent properties:

a) For any sufficiently large $F \in \mathcal{F}$ $\operatorname{coind} \varphi^{-1}F \geq p + \dim F$.

b) For any $F \in \mathcal{F}$ $\operatorname{coind} \varphi^{-1}F \geq p + \dim F$.

The fact that these are equivalent follows from the first part of the proof of theorem 2.2, with F, F' required to be in $\mathcal{F}$. Denote the coindex of φ with respact to $\mathcal{F}$ by $\operatorname{coind}_{\mathcal{F}} \varphi$. Then the following is true

c) $\operatorname{coind}_{\hat{\mathcal{F}}} \varphi = \operatorname{coind}_{\mathcal{F}} \varphi$

<u>Proof</u>. Obviously $\operatorname{coind}_{\hat{\mathcal{F}}} \varphi \leq \operatorname{coind}_{\mathcal{F}} \varphi$

To verify the opposite inequality let p be any integer not exceeding $\operatorname{coind}_{\mathcal{F}} \varphi$. (If $\operatorname{coind} \varphi = -\infty$, there is nothing to show.) We have to check that $\operatorname{coind} \varphi^{-1}F \geq p + \dim F$ for all $F \in \hat{\mathcal{F}}$. But any $F \in \hat{\mathcal{F}}$ is contained in some $F' \in \mathcal{F}$, for which $\operatorname{coind} \varphi^{-1}F' \geq p + \dim F'$. Again the first part of the proof of theorem 2.2 gives the desired inequality.

A <u>flag</u> $\mathcal{F} = \{E^n\}$ in E is a sequence $E^1 \subset E^2 \subset \dots$ of subspaces such that $\dim E^n = n$ and $\cup E^n$ is dense in E . 1)

<u>2.4 Theorem.</u> Let $\varphi : X \to E$ be a proper and bounded equivariant map and $\mathcal{F} = \{E^i\}$, $i = 1,2,\dots$ a flag in E . Then $\operatorname{coind} \varphi = \operatorname{coind}_{\mathcal{F}} \varphi$.

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1) Thus for E to admit flags it must be separable hence second countable.

Proof. Clearly $\operatorname{coind} \varphi \leq \operatorname{coind}_{\mathcal{F}} \varphi = \operatorname{coind}_{\hat{\mathcal{F}}} \varphi$. We show that $\operatorname{coind}_{\hat{\mathcal{F}}} \varphi \leq \operatorname{coind} \varphi$. If $\operatorname{coind}_{\hat{\mathcal{F}}} \varphi$ is not $-\infty$, let p be any integer not exceeding $\operatorname{coind}_{\hat{\mathcal{F}}} \varphi$. Suppose there is a finite dimensional subspace $F_0 \subset E$ such that $\operatorname{coind} \varphi^{-1} F_0 < p + \dim F_0$. Let $U \subset X$ be a closed invariant neighbourhood of $\varphi^{-1} F_0$ such that $\operatorname{coind} U = \operatorname{coind} \varphi^{-1} F_0$. Then $\varphi(X-\operatorname{int}U)$ and F_0 are disjoint closed sets in E , and $\varphi(X-\operatorname{int}U)$ is bounded. Hence there is a distance $> \epsilon > 0$ between F_0 and $\varphi(X-\operatorname{int}U)$. Let $r > 0$ be a bound for φ so that $\varphi X \subset B(r)$ (the ball of radius r). By the definition of a flag there is a finite dimensional space $F_1 \in \hat{\mathcal{F}}$ with $\dim F_1 = \dim F_0$ such that any element in $F_1 \cap B(r)$ is within distance $< \epsilon$ of an element in $F_0 \cap B(r)$ and conversely. Then $\varphi^{-1} F_1 \subset U$. Otherwise $F_1 \cap \varphi(X-\operatorname{int}U)$ would be non-empty, which is impossible since $y \in F_1 \cap \varphi(X-\operatorname{int}U)$ implies $\operatorname{dist}(y, F_0) < \epsilon$ as well as $\operatorname{dist}(y, F_0) > \epsilon$. It follows that $\operatorname{coind} \varphi^{-1} F_1 \leq \operatorname{coind} U = \operatorname{coind} \varphi^{-1} F_0$, and so $\operatorname{coind} \varphi^{-1} F_1 - \dim F_1 \leq \operatorname{coind} \varphi^{-1} F_0 - \dim F_0 < p$ which contradict the assumptions. Hence we must have $\operatorname{coind} \varphi^{-1} F_0 \geq p + \dim F_0$. Since $F_0 \subset E$ was arbitrary finite dimensional, this implies $\operatorname{coind} \varphi \geq p$, which again implies $\operatorname{coind} \varphi \geq \operatorname{coind}_{\hat{\mathcal{F}}} \varphi$.

3. Local coindex of a Fredholm map. Theorem 2.1 poses the problem of computing the coindex of an equivariant map φ into E . In general this is a difficult task, since it requires considerable knowledge about the filtration on X pulled back from E by φ . One case which seems more tractable than others, however, is where X is a differensiable manifold modelled on a Banach space and φ is a Fredholm map, cf. [3]. A Fredholm

map $\varphi: X \to Y$ between Banach manifolds is a smooth map such that $(d\varphi)_x$ has finite dimensional kernel and cokernel at every $x \in X$. The <u>index</u> of φ is $\dim \ker (d\varphi)_x - \dim \operatorname{coker} (d\varphi)_x$, which is independent of x for a connected manifold X.

We start by proving the following local result, which still is true for arbitrary coindex maps.

<u>3.1 Theorem.</u> Let E, F be Banach spaces, $D \subset E$ a symmetric open neighbourhood of the origin in E and $\varphi: D \to F$ an equivariant Fredholm map of Fredholm index $q \geq 0$. Then for any sufficiently small ball B centered at the origin

$$\operatorname{coind} \varphi|(B - o) = q - 1$$

For a stable coindex this is true also if $q < 0$.

<u>Proof.</u> Assume $q = 0$ and let $E_o = \ker d\varphi$ and $F^o = \operatorname{im} d\varphi$ (the differential $d\varphi$ taken at the origin in E). Also let $E^o \subset E$ and $F_o \subset F$ be complementary subspaces to E_o and F^o respectively. Then $d\varphi$ can be condidered a linear map $E_o \oplus E^o \to F_o \oplus F^o$ which is zero on E_o and maps E^o isomorphically to F^o. Let $\Psi: E_o \oplus E^o \to F_o \oplus F^o$ be a linear map which maps E_o isomorphically to F_o and is zero on E^o. Form $\varphi + \psi : D \to F$, where ψ is just the restriction of Ψ to D. Then $\varphi + \psi$ is equivariant, and $d(\varphi+\psi) = d\varphi + \Psi$ is an isomorphism. Hence $\varphi + \psi$ is a local equivariant diffeomorphism around the origin. Now, to compute the coindex of φ close to the origin, consider $(B-o) \cap \varphi^{-1}\{F_o \oplus F'\}$ for finite dimensional $F' \subset F^o$ and a small ball B around $o \in E$. Then $\varphi^{-1}\{F_o \oplus F'\} = (\varphi+\psi)^{-1}\{F_o \oplus F'\}$ and so $\varphi + \psi$ establishes an equivariant homeomorphism $(B-o) \cap \varphi^{-1}\{F_o \oplus F'\} \to (\varphi+\psi)(B-o) \cap (F_o \oplus F')$.

It follows that these two sets have the same coindex. Furthermore, $(\varphi+\psi)(B)$ is a neighbourhood of the origin $o \in F$ and so contains a small ball B'. This gives equivariant inclusions $(B'-o) \cap (F_o \oplus F') \subset (\varphi+\psi)(B-o) \cap (F_o \oplus F') \subset F_o \oplus F' - o$ showing that the coindex of $(\varphi+\psi)(B-o) \cap (F_o \oplus F')$ is precisely $\dim F_o \oplus F' - 1$. Therefore $\operatorname{coind} \varphi|B - o \geq -1$. Since clearly -1 is the greatest lower bound for $\operatorname{coind} (B-o) \cap \varphi^{-1}\{F_o \oplus F\} - \dim F_o \oplus F'$ as F' runs through the finite dimensional subspaces of F^o, the coindex of $\varphi|B - o$ is in fact precisely -1. This proves the result in the case where $q = 0$. If $q > 0$, replace φ by the composite map

$$D \xrightarrow{\varphi} F \xrightarrow{i} F \oplus \mathbb{R}^q$$

which is then Fredholm of index 0. Applying the special case just proved gives $\operatorname{coind} i \circ \varphi|B - o = -1$ for B a small ball. Thus, for sufficiently large $F'' = F' \oplus \mathbb{R}^q \subset F^o \oplus \mathbb{R}^q$ $\operatorname{coind} (B-o) \cap (i \circ \varphi)^{-1}\{F_o \oplus F''\} - \dim F_o \oplus F''$ equals -1. But $(i \circ \varphi)^{-1}\{F_o \oplus F''\} = \varphi^{-1}\{F_o \oplus F'\}$. It follows that $\operatorname{coind} (B-o) \cap \varphi^{-1}\{F_o \oplus F'\} - \dim F_o \oplus F'$ equals $q - 1$ for F' large, or equivalently that $\operatorname{coind} \varphi|B - o = q - 1$. Finally suppose that $q < 0$. In this case replace φ by the composite map

$$D \times \mathbb{R}^{-q} \xrightarrow{pr} D \xrightarrow{\varphi} F$$

which is then Fredholm of index 0. Again by the first part of the proof we find $\operatorname{coind} (B''-o) \cap (\varphi \circ pr)^{-1}\{F_o \oplus F'\} - \dim F_o \oplus F' = -1$ where $B'' \subset D \times \mathbb{R}^{-q}$ is a small ball of the form $B \times B'$ around o in $D \times \mathbb{R}^{-q}$. Suspending $(B-o) \cap \varphi^{-1}\{F_o \oplus F'\}$ $-q$ times we get

$$S^{-q}(B- \ ) \cap S^{-q}\varphi^{-1}\{F_o \oplus F'\} \subset (B''-o) \cap (\varphi \circ pr)^{-1}\{F_o \oplus F'\} \to$$
$$S^{-q}(B-o) \cap S^{-q}\varphi^{-1}\{F_o \oplus F'\}$$

where the maps are equivariant. Therefore, if the coindex map is stable, coind $(B''-o) \cap (\varphi \circ pr)^{-1}\{F_o \oplus F'\}$ = coind $(B-o) \cap \varphi^{-1}\{F_o \oplus F'\} - q$ or coind $(B-o) \cap \varphi^{-1}\{F_o \oplus F'\} - \dim F_o \oplus F' = q-1$. This again implies coind $\varphi|B-o = q-1$.

In section 7 we give a considerable improvement of theorem 3.1. However, in doing so it is necessary to restrict attention to cohomology coindices and smooth separable Banach spaces (i.e. separable Banach spaces with smooth partitions of unity).

4. The degree of a map. We turn to the definition and properties of the degree of a map. Since equivariance is irrelevant in this case, we may conveniently forget about the involution T on X . For a more complete discussion we refer to [3].

Let $L(E)$ be the Banach algebra of bounded linear operators on E and $GL(E)$ the multiplicative subgroup of invertible elements. Let $c(E)$ be the completely continuous operators and $L_c(E)$ and $GL_c(E)$ the subsets of $L(E)$ and $GL(E)$, respectively, of operators of the form $I + T$, $T \in c(E)$. Then $GL_c(E)$ is a subgroup of $GL(E)$. It is known that $GL_c(E)$ has two components. We denote the component containing the identity $SL_c(E)$ and the other $SL_c^-(E)$. Given a Banach manifold M a c-structure on M is an admissible atlas $\{\varphi_i, U_i\}$ maximal with respect to the property: For any i,j the differential $d(\varphi_j\varphi_i^{-1})$ at any point lies in $GL_c(E)$. The c-structure is orientable if it admits a subatlas for which the differentials actually lie in $SL_c(E)$. An orientation is a subaltas maximal with respect to this property. Observe that any finite dimensional manifold has a unique c-structure and that orientability in this case has its usual meaning. A smooth map $f: M \to N$ between c-manifolds (i.e. manifolds with distinguished c-structures) is a c-map if for any local representative $\psi_j f \varphi_i^{-1}$ of f the differential $d(\psi_j f \varphi_i^{-1})$ at any point is in $L_c(E)$. This implies that f is Fredholm of index 0. Suppose f is a proper c-map between oriented manifolds M, N with N connected. Then the oriended degree of f is defined:

By the Smale-Sard theorem f has a regular value y in N. Then $f^{-1}\{y\} \subset M$ consists of a finite number of points. Count these with their proper signs; this gives the degree,

$$\deg f = \sum_{x \in f^{-1}\{y\}} \operatorname{sgn} df_x .$$

The sign (of f) at $x \in f^{-1}\{y\}$ is determined as follows: Take any local representative $\psi_j f \varphi_i^{-1}$ around x. The derivative $d(\psi_j f \varphi_i^{-1})$ at $\varphi_i(x)$ is then in $GL_c(E)$ since x is a regular point. Define $\operatorname{sgn} df_x$ to be 1 if $d(\psi_j f \varphi_i^{-1})$ is in $SL_c(E)$ and -1 otherwise. (The value does not depend on the choice of local representative.) This definition of degree obviously extends the finite dimensional one, cf. [5].

Suppose now that $N = E$ with its canonical c-structure and that $f: M \to E$ is just Fredholm of index 0. Then, by a result of Elworthy and Tromba [3], there is a unique c-structure $c_f = \{\varphi_i, U_i\}$ on M making f a c-map. We will say that f is orientable if c_f is orientable. Then, if f is proper, the degree of f is defined, and it can be shown that up to sign it is a proper Fredholm homotopy invariant. In particular the parity of the oriented degree of a proper Fredholm map $f: M \to E$ of index 0 is defined and invariant under proper Fredholm homotopies. It is easy to see that this invariant is precisely the degree mod 2 of f as defined by Smale, [7].

Given $f: M \to E$ as above we next turn to the computation of $\deg f$ by homological methods. But first we need a corollary of a result of Elworthy and Tromba. We briefly indicate the proof.

4.1 Lemma. Let $f: M \to E$ be a Fredholm map of index 0, transverse to $E^n \subset E$. If f is orientable, so is $M^n = f^{-1}E^n$.

Proof. M^n is an n-dimensional regular submanifold of M with a normal bundle which can be realized as a tubular neighbourhood in M. This implies that M^n can be covered by local coordinate neighbourhoods of M (trivial parts of the tubular neighbourhood), each of which is nicely diffeomorphic to open product sets $U^n \times U'$ in E. In these trivializations the local images of M^n are the slices $U^n \times 0$, and the local representatives of f take the form

$$(x,y) \to (x'(x,y),\ y'(y))$$

where $y': E' \to E'$ is a linear operator on a complement of E^n. The reader may check that these trivializations restrict to an orientable atlas on M^n.

Remark. An actual orientation of c_f on M restricts to an orientation on M^n, such that if φ'_i, φ'_j are restrictions of charts φ_i, φ_j on M to M^n, then $d(\varphi'_j\,\varphi'^{-1}_i)$ is in $SL(E^n)$ if and only if $d(\varphi_j\varphi_i^{-1})$ is in $SL_c(E)$, the differentials taken at any point in the domain of $\varphi'_j\varphi'^{-1}_i$.

Remark. The considerations above hold under more general circumstances. In particular we later use the simple generalisation of lemma 4.1 where E is replaced by an open subset $N \subset E$.

Again consider an orientable proper Fredholm map $f: M \to E$ which is transversal to $E^n \subset E$, with M^n and $f^n: M^n \to E^n$ as above. Let $y \in E^n$ be a regular value for f^n. Then y is a regular value for f and $f^{-1}\{y\} = (f^n)^{-1}\{y\}$. Choose an orientation for M (with respect to c_f). Then M^n inherits an orientation, and $\operatorname{sgn} df^n_x = \operatorname{sgn} df_x$ for all $s \in f^{-1}\{y\}$, by the first remark above. Thus $\deg f = \deg f^n$. However, $\deg f^n$ can be computed by well known homological methods: Let $\gamma^n \in H^n_c(E^n)$ be a generator (Cech cohomology with compact supports, coefficients $\mathbb{Z}$). Then $\deg f$ is up to sign the value on γ^n of the composite homomorphism

$$H^n_c(E^n) \xrightarrow{f^{n*}} H^n_c(M^n) \cong H_o(M^n) \xrightarrow{\varepsilon} \mathbb{Z}$$

In particular we can choose γ^n such that the homological degree comes out with the right sign.

If $E^m \subset E^n$ are two finite dimensional subspaces of E to which f is transverse we get a diagram

$$\begin{array}{ccccccc}
H^n_c(E^n) & \xrightarrow{f^{n*}} & H^n_c(M^n) & \cong & H_o(M^n) & \longrightarrow & \mathbb{Z} \\
\cong\uparrow & & \uparrow & & \uparrow & & \| \\
H^m_c(E^m) & \xrightarrow{f^{m*}} & H^m_c(M^m) & \cong & H_o(M^m) & \longrightarrow & \mathbb{Z}
\end{array}$$

where $H^m_c(E^m) \to H^n_c(E^n)$ is the suspension or the Thom isomorphism of the normal bundle of E^m in E^n, and $H^m_c(M^m) \to H^n_c(M^n)$ is the composite of the Thom isomorphism $H^m_c(M^m) \to H^n_c(U^n)$ and the transfer $H^n_c(U^n) \to H^n_c(M^n)$, U^n being an open tubular neighbourhood of M^m in M^n. This diagram commutes when $H^m_c(E^m) \to H^n_c(E^n)$ is the particular Thom map which sends γ^m to γ^n.

Similarly, if f is transversal to an ascending sequence $\{E^n\}$ in E we get an infinite commutative ladder of groups and homomorphisms, each stage of which computes the degree of f.

Suppose next that in fact a countable collection $\{E^n\}$ is picked out at random in E and that f is not necessarily transversal to $\{E^n\}$. Let $\{E^n\}$ be a sequence of complements in E to the members of $\{E^n\}$ such that we have short exact sequences

$$0 \to E^n \xrightarrow{i_n} E \xrightarrow{j_n} E_n \to 0$$

The composites

$$M \xrightarrow{f} E \xrightarrow{j_n} E_n$$

are σ-proper Fredholm maps. Therefore their regular value sets V_n are residual by the Sard-Smale theorem. It follows that the sets $j_n^{-1}V_n$ are residual, and therefore so is their intersection V'. If $y \in V'$ then $j_n(y)$ is a regular value of $j_n \circ f$, and so the origin $o \in E_n$ is a regular value of $j_n \circ (f-y)$. Then the translate $f-y$ is transverse to E^n for all n. Hence $f-ty$ is a smooth compact finite-dimensional homotopy from f to $g = f-y$ with $g \pitchfork \{E^n\}$. In particular $\deg f = \deg g$. Now define $M^n = g^{-1}E^n$ for $n = 1,2,\ldots,$ and we may apply the discussion above with g,g^n substituted for f,f^n. Observe also that we may choose $\|y\|$ as small as we want, so that $\|f-(f-ty)\|$ is small throughout the homotopy.

Finally let V be a closed symmetric neighbourhood of the origin in E and $f\colon (V,\mathrm{bd}V) \to (E,E-o)$ with f proper and bounded and Fredholm in $V-\mathrm{bd}V$. Then $f\,\mathrm{bd}V$ is closed and hence bounded away from $o \in F$. Therefore, if D is a small open ball around o in F, $M = f^{-1}D$ is an open subset in $V-\mathrm{bd}V$ and

$f_D: M \to D$ is a proper Fredholm map between oriented manifolds. Then the degree of f_D is well defined and obviously independent of the particular choice of D. By definition this is the degree of $f: (V, bdV) \to (E, E-0)$. If $\{E^n\}$ is a flag in E, we may suppose that f is transversal to $\{E^n\}$ on the interior of V, otherwise f can be deformed into such a map by a small compact homotopy $(V, bdV) \times I \to (E, E-0)$, and it is easy to check that the degree stays fixed under such a deformation. According to our ealier set-up we can now get the degree homologically from the composites

$$H_c^n(D^n) \to H_c^n(M^n) \cong H_o(M^n) \to \mathbb{Z}$$

On the other hand we have the commutative diagram (using earlier notations and stting $B^n = V^n \cap bdV$)

$$\begin{array}{ccccccc}
H_c^n(D^n) & \xrightarrow{f^{n*}} & H_c^n(M^n) & \cong & H_o(M^n) & \to & \mathbb{Z} \\
\cong \downarrow & & \downarrow & & \downarrow & & \| \\
H_c^n(E^n) & & H_c^n(V^n-B^n) & \cong & H_o(V^n-B^n) & \to & \mathbb{Z} \\
\cong \downarrow & & \cong \downarrow & & \| & & \| \\
H^n(E^n, E^n-0) & \xrightarrow{f^{n*}} & H^n(V^n, B^n) & \cong & H_o(V^n-B^n) & \to & \mathbb{Z}
\end{array}$$

Thus we may equally well compute the degree from the composite map

$$H^n(E^n, E^n-0) \xrightarrow{f^{n*}} H^n(V^n, B^n) \cong H_o(V^n-B^n) \to \mathbb{Z}$$

5. Degree and cohomology coindex. We relate the degree to the cohomology coindex for finite dimensional spaces. Throughout this section coindex stands for the coindex based on the $\mathbb{Z}_2$-characteristic cohomology map. By a manifold here and in the sequal we mean a separable metrizable space which carries a

smooth manifold structure. Relative manifolds are similarly defined. The extra topological condition is for convenience. It can be avoided, at least at the expence of introducing conditions on the maps occuring.

First we make some general remarks. Consider again the space X with the fixed point free involution T and let $p: X \to X_T = X/T$ be the covering map defined by T. Associated to this double covering is a local system of groups on X_T: the stalk at any point $x' \in X_T$ is $\mathbb{Z}$, and the action of $\pi(X_T,x')$ on $\mathbb{Z}$ is given by the representation $\pi(X_T,x') \to \mathrm{Aut}\,(\mathbb{Z}) = \mathbb{Z}_2$ which is simply the canonical projection

$$\pi(X_T,x') \to \pi(X_T,x')/p_{\#}\pi(X,x) , \qquad x \in p^{-1}\{x'\} .$$

This is the local orientation system of the covering $X \to X_T$. We shall denote it $\mathbb{Z}_T$. Observe that the pull-back of $\mathbb{Z}_T$ to X is the trivial system $\mathbb{Z}$ (up to equivalence).

If X_T is path connected, there can be at most two non-equivalent local systems with stalk $\mathbb{Z}$ on X_T. It follows that (in any case) local systems with stalk $\mathbb{Z}$ are self dual under the tensor pairing: tensor product of a local system with itself yields the trivial local system. Now introduce the notation

$$G_1 = G_3 = G_5 = \dots = \mathbb{Z}$$
$$G_2 = G_4 = G_6 = \dots = \mathbb{Z}_T$$

Then G_n is a local system on X_T for $n \geq 1$ and $\mathbb{Z}_T \otimes G_n = G_{n+1}$ for all n. Next observe that if X is S^n with the antipodal action, then G_n is precisely the local orientation system for the <u>manifold</u> $X_T = P^n$ for every n (cf. [8] 6A3 on p. 357) so that $H^n(P^n;G_n) \cong \mathbb{Z}$ and $H^n(P^n;G_{n+1}) = \mathbb{Z}_2$ by Poincaré duality [1]. Furthermore there is the following exact portion of the Smith-Gysin sequence (with coefficients G_n) of the double covering $p: S^n \to P^n$

$$0 \to H^n(P^n;G_n) \xrightarrow{p^*} H^n(S^n;\mathbb{Z}) \to H^n(P^n;G_{n+1}) \to 0 .$$

Therefore p^* is always multiplication by 2 .

<u>5.1 Theorem.</u> Let M be a compact orientable manifold of dimension n with a fixed point free involution T and $\varphi: M \to S^n$ an equivariant map of odd degree. Then $\operatorname{coind} M = n$.

<u>Proof.</u> Let M_T be the quotient manifold M/T . There is a commutative square

$$\begin{array}{ccc} H^n(S^n;\mathbb{Z}) & \xrightarrow{\varphi^*} & H^n(M;\mathbb{Z}) \\ p^* \uparrow & & \uparrow p^* \\ H^n(P^n;G_n) & \xrightarrow{\varphi_T^*} & H^n(M_T;G_n) \end{array}$$

Let $\gamma \in H^n(P^n;G_n)$ and $g \in H^n(S^n;\mathbb{Z})$ be generators such that $p^*\gamma = 2g$ and let $c = \varphi_T{}^*\gamma$. Choose an orientation of M and let $[M] \in H_n(M;\mathbb{Z})$ be the corresponding fundamental homology class. Then $\varphi_*[M]$ is an odd multiple of $g_* \in H_n(S^n;\mathbb{Z})$ (the dual generator of g) since the degree of φ is odd and

1) If Y is a path connected space, G a local system on Y with stalk $\mathfrak{G}$, and $\sigma: \pi(Y,y) \to \operatorname{Aut}(\mathfrak{G})$ the action of $\pi(Y,y)$ on $\mathfrak{G}$ at a point y , then $H_0(Y;G) \cong \mathfrak{G}/(g-\sigma(x)g)$, $g \in \mathfrak{G}$, $x \in \pi(Y,y)$.

$$\begin{aligned} \langle p^*c,[M]\rangle &= \langle \varphi^* p^* \gamma,[M]\rangle \\ &= \langle p^*\gamma, \varphi_*[M]\rangle \\ &= \langle 2g,(2k+1)g_*\rangle \\ &= 2(2k+1) \end{aligned}$$

If M_T' is any component of M_T, let $M' = p^{-1}M_T'$. Again by Poincaré duality $H^n(M_T';G_n) \approx \mathbb{Z}$ if G_n is the orientation system of M_T' and $H^n(M_T';G_n) \approx \mathbb{Z}_2$ if G_n is not the orientation system of M_T'. In the latter case $(p|M')^*c = 0$ since $c|M_T'$ is of finite order. Hence there must exist components M_T' for which G_n is the orientation system. For such a component the map $p^*: H^n(M_T';G_n) \to H^n(M';\mathbb{Z})$ sends a generator to a class whose value on $[M']$ is ± 2. Therefore

$$c|M_T' \equiv 0 \pmod 2$$

if and only if

$$\langle p^*(c|M_T'),[M']\rangle \equiv 0 \pmod 4$$

Since $\langle p^*c,[M]\rangle = \Sigma\langle p^*(c|M_T'),[M']\rangle$, the sum $\Sigma\langle p^*(c|M_T'),[M']\rangle$ is not zero mod 4. Therefore, for some component M_T' $c|M_T' \not\equiv 0 \pmod 2$. Hence $c \not\equiv 0 \pmod 2$. Finally, if $c_T \in H^1(M_T;\mathbb{Z}_2)$ is the characteristic class of the covering $M \to M_T$, then $c_T^n \in H^n(M_T;\mathbb{Z}_2)$ is the reduction mod 2 of c, hence $c_T^n \neq 0$. It follows that coind $M \geq n$. This completes the proof of the theorem.

<u>5.2 Corollary</u>. Let (X,A) be a compact orientable smooth relative manifold of dimension n with a smooth involution [2] which is fixed point free on A. Let

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2) Mapping A to A, of course.

$\varphi: (X,A) \to (\mathbb{R}^n, \mathbb{R}^n - o)$ be an equivariant map of odd degree with respect to the origin $o \in \mathbb{R}^n$. Then coind $A = n-1$.

Proof. Let $K = \varphi^{-1}\{0\}$. Then K contains the fixed points under the involution and K is bounded away from A. By the continuity property there is a closed invariant neighbourhood U of A disjoint from K such that coind U = coind A. Let $Y = X - K$ and $Y_T = Y/T$, $A_T = A/T$, where T is the involution. Then (Y_T, A_T) is a smooth relative manifold and $U_T = U/T$ is a closed neighbourhood of A_T. Let $N_T \subset Y_T$ be an n-dimensional manifold with boundary $\partial N_T = M_T$ such that $N_T \subset Y_T - A_T$ and $Y_T - \text{int}\, U_T \subset N_T - M_T$. Then M_T is contained in U_T. Let M be the lift of M_T to $Y \subset X$. Then M is a compact orientable manifold of dimension $n-1$ contained in U and so T is fixed point free on M. Consider the equivariant map

$$M \overset{\varphi}{\to} \mathbb{R}^n - o \to S^{n-1}$$

The degree of this map is clearly equal to the degree with respect to the origin of $\varphi: (X,A) \to (\mathbb{R}^n, \mathbb{R}^n - o)$, hence it is odd. Now apply theorem 5.1 to get coind $M = n-1$. Since $M \subset U$, coind $M \leq$ coind U = coind A. Thus coind $A \geq n-1$. But clearly also coind $A \leq$ coind $\mathbb{R}^n - 0 = n-1$. This completes the proof of the corollary.

6. Equivariant transversality. In this section we prove a transversality theorem for equivariant map.

A manifold V is said to be smoothly normal if given disjoint closed sets $A, B \subset V$ there is a smooth function $\eta: V \to \mathbb{R}$ such that:

(1) $\eta(x) \in I$ for all $x \in V$

(2) $\eta(x) = 0$ for $x \in A$

(3) $\eta(x) = 1$ for $x \in B$

(4) $\eta(x) = 0$ implies all partial derivatives of all orders of η vanish at x .

Any manifold modelled on a separable Banach space with smooth partitions of unity is smoothly normal.

We first prove the following local result.

<u>6.1 Lemma.</u> Let V be a smoothly normal manifold with closed subsets A,B . Let E be a Banach space and $\{E^n\}$ a countable [3] collection of finite dimensional subspaces, and let $\varphi: V \to E$ be a Fredholm map which is transversal to $\{E^n\}$ on some neighbourhood of A . Given $\epsilon > 0$ and a closed neighbourhood N_B of B there is a smooth homotopy

$$H: V \times I \to E$$

such that

(1) $H(x,0) = \varphi(x)$ for $x \in V$.

(2) $\|H(x,t) - \varphi(x)\| < \epsilon$ for all $x \in V$, $t \in I$.

(3) There is a one-dimensional space $E_1 \subset E$ such that $H(x,t) - \varphi(x) \in E_1$ for all $x \in V$, $t \in I$.

(4) There is a neighbourhood N_A of A such that $H(x,t) = \varphi(x)$ for $x \in N_A$, $t \in I$.

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3) The cases of principal interest are when $\{E^n\}$ is a finite collection (e.g. with one member) or a flag.

(5) $H(x,t) = \varphi(x)$ for $x \in V - N_B$, $t \in I$.

(6) $H(\cdot,1)$ is transversal to $\{F^n\}$ on some neighbourhood of B .

Proof. Let U be an open neighbourhood of A such that φ is transversal to $\{E^n\}$ on U, and let M be a closed neighborhood of B contained in $\mathrm{int}N_B$. Then $A \cup (V - \mathrm{int}N_B)$ is a closed set disjoint from the closed set $M - U$. Let N be a closed neighbourhood of $A \cup (V-\mathrm{int}N_B)$ disjoint from $M - U$. Since V is smoothly normal there is a smooth map $\eta: V \to \mathbb{R}$ such that

(1) $\eta(x) \in I$ for all $x \in V$.

(2) $\eta(x) = 0$ for $x \in N$.

(3) $\eta(x) = 1$ for $x \in M - U$.

(4) $\eta(x) = 0$ implies all partial derivatives of η vanish at x .

Then $\frac{\varphi}{\epsilon\eta} : [V - \eta^{-1}(0)] \to E$ is a Fredholm map so that, by Smale's theorem [7, theorem 1.3], there is $y \in F$ with $\|y\| < 1$ such that $\frac{\varphi}{\epsilon\eta} + y$ is transversal to $\{E^n\}$ on $V - \eta^{-1}(0)$. Then

$$H(x,t) = \varphi(x) + t\,\epsilon\eta(x)y$$

is a homotopy satisfying (1), (2), (3) trivially. For (4) we observe that N will do as N_A in (4). For (5) we have that $N \supset (V-\mathrm{int}N_B) \supset V - N_B$ so $H(x,t) = \varphi(x)$ for $x \in V - N_B$. For (6) we have that $H(\cdot,1) = \varphi + \epsilon\eta y$ is transversal to $\{E^n\}$ on $V - \eta^{-1}(0)$. Also it is transversal to $\{E^n\}$ on $U \cap \eta^{-1}(0)$. Since $M \cap \eta^{-1}(0) \subset U \cap \eta^{-1}(0)$, it follows that $H(\cdot,1)$ is transversal to $\{E^n\}$ on M and this is a neighbourhood of B . Hence,

(6) holds and the proof is complete.

Now we prove the following global result.

6.2 Theorem. Let T be an involution on a smoothly normal manifold X and let K be the set of fixed points of T. Let E be a Banach space and $\{E^n\}$ a countable collection of finite dimensional subspaces, and suppose $\varphi: X \to E$ is an equivariant Fredholm map which is transversal to $\{E^n\}$ on a neighbourhood of K. Then there is a smooth homotopy

$$H: X \times I \to E$$

such that:

(1) For any $t \in I$, $H(\cdot,t): X \to E$ is an equivariant Fredholm map.

(2) There is a compact subset $C \subset E$ such that
$$H(x,t) - \varphi(x) \in C \quad \text{for all } x \in X, \quad t \in I.$$

(3) There is a neighbourhood N of K such that
$$H(x,t) = \varphi(x) \quad \text{for } x \in N, \quad t \in I.$$

(4) $H(\cdot,1)$ is transversal to $\{E^n\}$ on all of X.

Proof. Let W be a neighbourhood of K on which φ is transverse regular to $\{E^n\}$ and choose a neighbourhood W' of K with $\overline{W}' \subset W$. Let $\{U_i, V_i\}$ be a countable collection of open subsets of X such that:

(a) $\bigcup U_i \cup \bigcup TU_i = X - K$

(b) V_i is disjoint from TV_i

(c) $\overline{U}_i \subset V_i$

By induction on i we construct a sequence of homotopies $H_i: X \times I \to F$ for $i = 1,2,\ldots$ such that:

(d) $H_1(\cdot,0) = \varphi$

(e) $H_{i+1}(\cdot,0) = H_i(\cdot,1)$ for $i \geq 1$

(f) There is an element $y_i \in F$ with $\|y_i\| < \frac{1}{2^i}$ such that $H_i(x,t) - H_i(x,0)$ is in the closed interval joining $-y_i$ to y_i.

(g) $H_i(\cdot,t): X \to F$ is an equivariant map.

(h) $H_i(x,t) = H_i(x,0)$ on some neighbourhood of $[\overline{W}' - K] \cup \overline{U}_1 \cup\ldots\cup \overline{U}_{i-1} \cup T\overline{U}_1 \cup\ldots\cup T\overline{U}_{i-1}$

(i) $H_i(\cdot,1)$ is transversal to $\{F^n\}$ on some neighbourhood of $\overline{U}_i \cup T\overline{U}_i$.

Assuming H_j defined for $j < i$ where $i \geq 1$ let $\varphi_{i-1} = H_{i-1}(\cdot,1)$ (or $\varphi_0 = \varphi$ in case $i = 1$). Then φ_{i-1} is transversal to $\{F^n\}$ on some neighbourhood of $[\overline{W}' - K] \cup \overline{U}_1 \cup\ldots\cup \overline{U}_{i-1} \cup T\overline{U}_1 \cup\ldots\cup T\overline{U}_{i-1}$. Let $A_i = ([\overline{W}' - K] \cup \overline{U}_1 \cup\ldots\cup \overline{U}_{i-1}) \cap V_i$ and $B_i = \overline{U}_i$. Applying the local lemma 6.1 to $\varphi_{i-1}|V_i$ with A_i, B_i closed sets in V_i with $\varepsilon = \frac{1}{2^i}$ and with N_{B_i} any closed neighbourhood of B_i contained in V_i we obtain a homotopy

$$J_i: V_i \times I \to F$$

such that:

(j) $J_i(x,0) = \varphi_{i-1}(x)$ for $x \in V_i$

(h) There is $y_i \in F$ with $\|y_i\| < \frac{1}{2^i}$ such that $J_i(x,t) - \varphi_{i-1}(x)$ is in the interval from $-y_i$ to y_i.

(l) $J_i(x,t) = \varphi_{i-1}(x)$ in some neighbourhood of A_i

(m) $J_i(x,t) = \varphi_{i-1}(x)$ for $x \in V_i - N_{B_i}$

(n) $J_i(\cdot,1)$ is transversal to $\{F^n\}$ on B_i.

Define $J_i^!: TV_i \times I \to F$ so that $J_i^!(x,t) = J_i(Tx,t)$. By (m) we can extend J_i and $J_i^!$ to a homotopy

$$H_i: X \times I \to F$$

such that $H_i | V_i \times I = J_i$, $H_i | TV_i \times I = J_i^!$, and $H_i(x,t) = \varphi_{i-1}(x)$ for $x \in X - (V_i \cup TV_i)$. Then H_i has the properties (d) - (i) inclusive.

With the H_i defined we define $H: X \times I \to F$ by the formula

$$H(x,t) = H_i\left(x, \frac{t-(1-\frac{1}{i})}{\frac{1}{i} - \frac{1}{i+1}}\right), \quad 1 - \frac{1}{i} \leq t \leq 1 - \frac{1}{i+1}$$

$$H(x,1) = H_i(x,1), \quad x \in U_i \cup TU_i \cup K$$

Then H has properties (1) and (4). It also has property (3) because $H(x,t) = \varphi(x)$ for $x \in W'$, $t \in I$. To show H has property (2) let C be the set of sums of $[-y_1,y_1] + [-y_2,y_2] + \ldots$. This is compact because $\|y_i\| < \frac{1}{2^i}$. Then

$$H(x,t) - \varphi(x) \in C \quad \text{for all } x \in X, \quad t \in I,$$

completing the proof.

7. Global coindex of a Fredholm map. We assume E is a separable Banach space admitting smooth partitions of unity and coindex is the coindex based on $\mathbb{Z}_2$-characteristic cohomology class.

7.1 Theorem. Let V be a closed symmetric neighbourhood of the origin in E and $\varphi: (V, bdV) \to (E, E-o)$ a proper equivariant Fredholm map of Fredholm index 0. Suppose φ is bounded and orientable of odd degree relative to the origin. Then

$$\operatorname{coind} \varphi | bdV = -1 .$$

Proof. First observe that since $\varphi bdV \subset E - 0$, it follows that $\operatorname{coind} \varphi | bdV \leq -1$. Thus it suffices to verify the opposite inequality. Next, since φ is Fredholm of index 0 at the origin, there is a finite dimensional map $\psi: V \to E$ with support in $\operatorname{int} V$ such that $\varphi + \psi$ is a local diffeqmorphism around the origin besides being proper equivariant and Fredholm of index 0 (cf. first part of the proof of theorem 3.1). Since the degree only depends on the values of the map at bdV, $\varphi + \psi$ also has odd degree with respect ot the origin in E, and since the coindex is invariant under finite dimensional perturbations, $\operatorname{coind} (\varphi+\psi)|bdV = \operatorname{coind} \varphi|bdV$. Thus we may as well work with $\varphi + \psi$, or what comes to the same, we may as well assume that φ is a local diffeomorphism at the origin.

Next let $\{E^n\}$ be a flag in E., Since φ is a local diffeomorphism, φ is transversal to $\{E^n\}$ in a neighbourhood around the origin in E. By theorem 6.2 there is a map $\varphi': (V, bdV) \to (E, E-o)$ smooth on $\operatorname{int} V$ and transversal to $\{E^n\}$, which is homotopic to φ through smooth equivariant compact perturbations of φ. In particular φ' is proper orientable equivariant and Fredholm of index 0 and has odd degree. Moreover, by corollary 2.3

coind $\varphi'|bdV$ = coind $\varphi|bdV$. Again we may as well continue with φ' instead of φ , or equivalently, we may suppose that φ is transversal to $\{E^n\}$ on int V . Next, let $V^n = \varphi^{-1}F^n$, $B^n = bdV \cap V^n$. Then the (V^n,B^n) are coherently orientable compact invariant relative manifolds of dimension n ; compact since the $\varphi|V^n: V^n \to F^n$ are both proper and bounded and coherently orientable by the remark following lemma 4.1. At this point we shall use the fact that both the degree and the coindex are computable by means of the flag $\{E^n\}$, i.e. in terms of the filtration $\{V^n,B^n\}$ on V,bdV . For the degree this means the following: There is a commutative diagram

$$\begin{array}{ccccccc} H^{n+1}(E^{n+1},E^{n+1}-0) & \xrightarrow{\varphi^*} & H^{n+1}(V^{n+1},B^{n+1}) & \cong & H_0(V^{n+1}-B^{n+1}) & \to & \mathbb{Z} \\ \uparrow \cong & & \uparrow & & \uparrow & & \uparrow \mathrm{id.} \\ H^n(E^n,E^n-0) & \xrightarrow{\varphi^*} & H^n(V^n,B^n) & \cong & H_0(V^n-B^n) & \to & \mathbb{Z} \end{array}$$

where the two first vertical maps are transfers induced by the respective normal structures, and the third vertical map is induced by the inclusion. The unspecified horizontal maps are duality isomorphisms and augmentations. Thus the unique generators of the groups $H^n(E^n,E^n-0)$ are all mapped to the same element of $\mathbb{Z}$ by the composite horizontal maps. This element is the degree of φ with respect to $o \in E$ (cf. section 4). By assumption it is odd. Similarly coind $\varphi|bdV$ is computable in terms of the filtration B^n; i.e. coind $\varphi|bdV \geq -1$ iff coind $B^n \geq n-1$ for all n (cf. theorem 2.4).

The result now follows from corollary 5.2 which applies to the relative manifold (V^n,B^n) .

Remark. According to Elworthy and Tromba [ ] the map φ is always orientable and of odd degree if $\widetilde{K}O(V)$ is the trivial group, e.g. if V is contractible.

Remark. The proof of theorem 7.1 applies without change to the more general case where (V, bdV) is replaced by a relative manifold (X, A) with involution modelled on a smooth Banach space E, except for the first part where φ has to be modified (smoothly, equivariantly, ...) so as to be transversal to the flag $\{E^n\}$ on a neighbourhood of the fixed point set C. Since C must be compact, this can probably always be done. The proof fully covers the case where C is empty or contains one point.

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