Transformation Groups on Complex Stiefel manifolds.

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0. Introduction.

In the theory of transformation groups a most fundamental, but in general quite difficult problem, is the classification of the possible orbit structures for actions of a compact Lie group G on a given space X. The well known P.A. Smith theory (as generalized by Borel, Conner, and others) gives beautiful results when X is of the simplest topological type (e.g. acyclic, cohomology sphere, cohomology projective space) and G is a torus or a p-torus. Moreover, when G is a classical group, restriction of the action to the maximal torus of G combined with structural splitting theorems on the characteristic class level for torus actions, result in nice regularity theorems for classical group actions on spaces of such simple topological type ([H 1]).

It is our assertion that the time is ripe for applying more sophisticated methods now available in algebraic topology and equivariant cohomology theory in a more serious study of transformation groups on certain spaces of more complicated topological types. The most natural spaces to consider are various homogeneous spaces, which accomodate a rich variety of natural actions. In this paper we give the full proof for one starting theorem in the field of large transformation groups on homogeneous spaces. Our main result is:

Theorem 1.

Let $X = W_{n,k}$ be the complex Stiefel manifold of $(n-k)$-frames in complex n-space $\mathbb{C}^n$, $k > \frac{1}{2}n$, and let $G = SU(n)$. Then any non-trivial, smooth action of G on X is conjugate

to the linear action.
(The "linear action" is the transitive action on $W_{n,k}$ induced from the standard linear action of G on $\mathbb{C}^n$).

If $k = n-1$, X is the sphere S^{2n-1}, and the result is well known (it is also an easy consequence of the geometric weight system for the restriction of the action to the maximal torus of G, ([H 1])). Fore more complicated spaces X, there is in general not much hope of obtaining such complete structural information on the cohomology of torus actions; hence it is to be expected that one must combine the partial cohomological information available with strong use of subtler topological constructions. The rather involved proof of Theorem 1 bears this expectation out for the case under study.

In section 1 we use the explicit classification of homogeneous spaces of SU(n) whose first Pontrjagin classes vanish and "local characteristic class theory" for the G-space X to study the possible orbit types for the action. It turns out that a few possibilities for principal orbit types, notably SU(n)/Sp(r) and SU(n)/SO(r) cannot be eliminated solely by local characteristic class theory, and we clear up those cases in section 2. It is worth to note that in the dimension range $k > \frac{1}{2}n+1$ the desired reduction for the above two cases is an application of the result of Allday-Halperin on the torus rank of a space. In the limiting case $\frac{1}{2}n < k \leq \frac{1}{2}n+1$, however, a more delicate method, involving the equivariant cohomology of the embedding of a minimal orbit in X with respect to the action of different subtori of G is called for. In section 3 we rely on the (global) cohomology theory of torus and p-torus actions to conclude that all isotropy groups are connected. A consequence is that the orbit projection is a fibration. In section 4 we proceed to a more detailed study of the orbit projection. An application of Steenrod squares is sufficient to reach our final conclusion under certain strong restrictions on n and k, ([H 1]) , this result may be somewhat improved by applying reduced p-powers. This is in a sense dual to the use of cohomology operations in the section problem for standard fibrations of complex Stiefel manifolds. The elimination of the limiting cases $SU(n)/SU(n-1)$ and $SU(n)/SU(k+1)$ as possible principal orbit types, depend, however, on higher order cohomology operations; they are obtained by reducing to known results on the fibre homotopy types of complex Stiefel manifolds.

We note that most of the methods of this paper are also relevant for other homogeneous spaces. Clearly they yield much information for Stiefel manifolds also outside the dimension restriction $k > \frac{1}{2}n$. This restriction is used, however, in the proof of Theorem 1; the striking simplicity of this result and the wide dimension range still covered, justifies it at present.

With some modifications (real Stiefel manifolds are products of spheres in special dimensions), similar results can be proved for real and quaternionic Stiefel manifolds. We have chosen to work out the details of the complex case here, in particular the orbit exclusion problem of section 2 appears only for that case.

Notations.

We let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, denote the integers, rationals, reals, complex numbers, respectively, and k any one of these rings. Let V be a k-module, then $\Lambda_k(V)$ denotes the graded Grassman k-algebra spanned by V, and $\Lambda_k^p(V)$ its subspace of (grading) degree p.

The natural representations of the classical groups $SU(r)$, $SO(r)$, $Sp(r)$ are denoted by μ_r, ρ_r, ν_r respectively. Inclusions between these, such as $SU(r) \subset SU(n)$, $SO(r) \subset SU(n)$, $Sp(r) \subset SU(2r)$, etc. always refer to standard inclusions.

If the cohomology algebra $H^*(X;k)$ is isomorphic to $H^*(Y;k)$, we denote this by $X \sim_k Y$.

Let G be a compact transformation group on the space X. Then $X_G = E_G \times_G X \to B_G$ is the bundle associated to a universal G bundle $E_G \to B_G$ by G's action on X.

By abuse of language we call the identity component G_x^o of an isotropy group G_x the "connected isotropy group av x"; correspondingly G/G_x^o represents the "connected orbit type".

1. Local Characteristic Class Theory.

Let X be the complex Stiefel manifold of $(n-k)$-frames in $\mathbb{C}^n$, then $X = SU(n)/SU(k)$ as a homogeneous space, and $X \sim_{\mathbb{Z}} S^{2k+1} \times \ldots \times S^{2n-1}$. X is stably parallellizable (parallellizable for $k < n-1$), hence all its Pontrjagin- and Stiefel-Whitney classes vanish. Let $G = SU(n)$ act smoothly on X. If $k = n-1$, $X = S^{2n-1}$, and it is known that any non-trivial G-action must be transitive. Thus, for the remainder of this paper we assume, without loss of generality, that $\frac{1}{2}n < k < n-1$ and $n \geq 5$. The above observation allows us to apply the computations of Pontrjagin classes of homogeneous spaces of ([H2]), combined with conditions on characteristic classes determined by the equivariant embedding of the orbit into X, to exclude most homogeneous spaces of G as possible orbits.

We recall:

Let G/H be a homogeneous space of G and let T be a maximal torus of H. Then $\pi^* : H^*(G/H;\mathbb{Q}) \to H^*(G/T;\mathbb{Q})$ is injective, and $p^* : H^*(G/T,\mathbb{Q}) \to H^*(G_T;\mathbb{Q})$ induced from the projection $p : G_T \to G/T$ is an isomorphism. Here $G_T = E_G \times_T G$, and $j : G_T \to B_T$ may be considered the fibre bundle associated to the universal T-bundle $E_G \to E_G/T = B_T$ by T's action on G by left translation. There is an obvious map α from the representation ring of H to the (equivariant) KO-group of G/H. The following splitting principle for homogeneous vector bundles over homogeneous spaces is the basic setting of Borel-Hirzebruch ([BHI]):

Let Φ be a real representation of H with weight system $\Omega(\Phi)$ and let $\alpha(\Phi)$ be the associated G-vector bundle over G/H. Then $p^*(\pi^*(P(\alpha(\Phi)))) = j^*(\underset{w\in\Omega(\Phi)}{\pi}(1+w))$, where P is the total rational Pontrjagin class and each weight w in $H^1(T;\mathbb{Q})$ is identified by transgression with an element of $H^2(B_T;\mathbb{Q})$. In particular, the tangent bundle $\tau(G/H) = \alpha(Ad_G/H - Ad_H)$; since $\alpha(Ad_G/H)$ is trivial, we have $\tau(G/H) = -\alpha(Ad_H)$ in $\widetilde{KO}(G/H)$; hence $p^*(\pi^*(P'(\tau(G/H)))) = j^*(\underset{w\in\Delta(H)}{\pi}(1+w)) = j^*(\underset{w\in\Delta^+(H)}{\pi}(1-w^2))$,

where $\Delta(H)$ and $\Delta^+(H)$ is the root system and a positive root system of H respectively, and P' is the total dual rational

Pontrjagin class. Let PH^k be the homogeneous part of degree $2k$ in $\prod_{w\in\Delta^+(H)}(1-w^2) = 1 - PH^2 + PH^4 - \ldots$. Then $P_i(G/H) = 0$, $i = 1,2,3$ if and only if PH^2, PH^4 and PH^6 are zero mod $\ker j^*$, where $\ker j^*$ is the ideal generated by the elementary symmetric functions in the weights of the complex n-dimensional representation ψ defined by the embedding of H in $G = SU(n)$. An explicit computation is now possible, and gives the following classification: ([H 2]), Theorem 1):

Theorem 2.

Let $\psi : H \subset SU(n)$ be a compact, connected Lie group with a given almost faithful, complex representation ψ. If $P_k(SU(n)/\psi H)=0$ for $k = 1,2,3$, then the possibilities for all such pairs (H,ψ) modulo trivial representations are given by the following list:

(i) H is any subtorus

(ii) H is semisimple and $\psi = Ad_H$

(iii) $H = SU(r)\times \tilde{H}$, $n|30$ and $\psi = \mu_r \otimes \mu_r + Ad_{\tilde{H}}$

(iv) $H =$ a) $SU(r)$ with $\psi = \mu_r$ or $2\mu_r$,

b) $SO(r)$ with $\psi = \rho_r$, $\dim_{\mathbb{C}}\rho_r = r$

c) $Sp(r)$ with $\psi = \nu_r$, $\dim_{\mathbb{C}}\nu_r = 2r$

d) G_2 with $\psi = \varphi_1$ or $2\varphi_1$, $\dim_{\mathbb{C}}\varphi_1 = 7$

(v) $H = Sp(1)^{\ell}$, $\ell \geq 1$, $\psi = k\cdot(\nu_1^{(1)} + \nu_1^{(2)} + \ldots + \nu_1^{(\ell)})$, $k = 1,2,4$.

(vi) $H =$ a) $SU(3)\times SU(3)$ with $\psi = k(\mu_3+\mu_3') + \ell(\bar{\mu}_3+\bar{\mu}_3')$, $k+\ell = 1$ or 2.

b) $G_2 \times G_2$ with $\psi = \varphi_1 + \varphi_1'$ or $2(\varphi_1+\varphi_1')$.

(vii) $H =$ a) $SU(r)$, $r = 3,4,5$ with $\psi = \mu_r + \bar{\mu}_r$.

b) $SU(3)$ with $\psi = k\mu_3 + \ell\bar{\mu}_3$, $k+\ell = 3,6$.

c) $Sp(2)$ with $\psi = \nu_2 + \Lambda^2\nu_2$

d) $Spin(8)$ with $\psi = \Delta^+ + \Delta^-$.

Now, let G act smoothly on a manifold M, then any orbit G/G_x embeds in M with homogeneous normal bundle associated to the slice representation Φ_x of G_x; $i : G/G_x \to M$, $i^*(\tau(M)) = \tau(G/G_x) + \alpha(\Phi_x) = -\alpha(Ad_{G_x}) + \alpha(\Phi_x)$ in $\widetilde{KO}(G/G_x)$. Evaluation of this equation at the characteristic class level provides strong restrictions on the possibilities of orbit types and slice representations. In particular, if G_x is a principal isotropy subgroup H, Φ_x is trivial. Hence, for M stably parallellizable, all Pointrjagin classes of the principal orbit type G/H must vanish. Since $G/H^o \to G/H$ is a finite covering, this implies that all Pontrjagin classes of G/H^o must also vanish. Consequently the connected principal isotropy subgroup type must be given by one of the subgroups of $G = SU(n)$ listed in Theorem 2.

For actions with a given principal isotropy subgroup type (H), the same equation applied locally at an arbitrary orbit type G/G_x gives strong limitations on the possible pairs (G_x, Φ_x), espesially when combining with the fact that the principal orbit type of the representation Φ_x must be G_x/H. We quote the following results from [H 2] :

Let $G = SU(n)$ act smoothly on a manifold M, and let the principal isotropy subgroup type be (H).

Theorem 3.

If $P_1(M) = 0$ and $H^o = SU(r) \subset SU(n)$, $r \geq 3$, then all connected isotropy subgroups G_x^o are also of the type $SU(\ell) \subset SU(n)$, $\ell \geq r$.

Theorem 4.

If $P_1(M) = 0$ and $H^o = Sp(r) \subset SU(n)$, $r \geq 2$, then all connected isotropy subgroups G_x^o are also of the type $Sp(\ell) \subset SU(n)$, $\ell \geq r$.

Theorem 5.

If $P_1(M) = 0$ and $H^o = SO(r) \subset SU(n)$, $r \geq 5$, then all connected isotropy subgroups G_x^o are also of the type $SO(\ell)$, $\ell \geq r$.

The first main step in the analysis of the action of G on $X = SU(n)/SU(k)$, is given by the following theorem:

Theorem 6.

Let $G = SU(n)$ act smoothly on $X = SU(n)/SU(k)$, with $\frac{1}{2}n < k < n-1$. Then the connected principal isotropy group H^o is of the type $SU(r) \subset SU(n)$, $k \leq r \leq n$.

By Theorem 3 we then have the following:

Corollary: All connected isotropy subgroups are of the type $SU(\ell) \subset SU(n)$, $r \leq \ell \leq n$.

The proof of Theorem 6 is the main subject of sections 1 and 2. We have to eliminate all other possibilities for connected principal isotropy group in Theorem 2 than (iv) with $H^o = SU(r)$, $\psi = \mu_r$. Here $\dim X = n^2 - k^2$, hence $\dim H^o > k^2 - 1 \geq \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$ unless $H = SU(k)$. Elimination of cases (i), (ii), (iii), (v), (vii) follow by dimension arguments. Here (i) and (ii) are straightforward, for (iii) we have: $\dim \psi = r^2 + \dim \tilde{H} \leq n$, i.e. $\dim H^o = r^2 - 1 + \dim \tilde{H} \leq n-1 \leq \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$ for $n \geq 5$, contradicting the above estimate. In (v) $\dim_{\mathbb{C}} \nu_1 = 2$, hence $2\ell \leq n$ and $\dim H^o = 3\ell \leq \frac{3}{2}n \leq \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$ for $n \geq 5$. In (vii) a) $\psi = \mu_r + \bar{\mu}_r$ implies $2r \leq n$, $\dim H^o = r^2 - 1 \leq \frac{1}{4}n + \frac{1}{2}n - \frac{3}{4}$. Recalling that $n \geq 5$, (vii) b) is clearly impossible. In (vii) c) we have $\dim Sp(2) = 10 \leq \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$ for $n \geq 6$, while for $n = 5$ we cannot accomodate the representation $\psi = \nu_2 + \Lambda^2\nu_2$. For (vii) d) $\psi = \Delta^+ + \Delta^-$ (half spin representations) implies $n \geq 16$, hence $\dim H^o = 28 < \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$.

In (iv) d) we have $n \geq 7$, hence $\dim H^o = 14 < \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$. In (iv) a) the possibility $\psi = 2\mu_r$ is ruled out in the same way as (vii) a). In (vi) b) $n \geq 14$, hence $\dim H^o = 28 < \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$.

It remains only to rule out the cases (iv) b) and c) together with the special case (vi) a). The method of local characteristic classes will not suffice here (although some cases, as $SU(n)/SO(r)$ with r odd may be ruled out by an analogous argument with Stiefel-Whitney classes). For example, since $Sp(r)$ is totally non-homologous to zero in $SU(n)$, $n \geq 2r$, it follows that all characteristic classes of $SU(n)/Sp(r)$ vanish ([BH III]). Hence more specialized methods are required here, these are dealt with in the next section.

2. Exclusion of Orbit Types.

In this section $G = SU(n)$ operates smoothly on $X = SU(n)/SU(k)$, $\frac{1}{2}n < k < n-1$, with H^o as principal isotropy group. We start by eliminating the case (vi) a) of Theorem 2.

Since $\dim H^o = \dim(SU(3) \times SU(3)) = 16$, we have $k^2 < 17$, i.e. $k \leq 4$; since $\dim_{\mathbb{C}} \psi = 6$ or 12 it follows that $n \geq 6$; hence the only possibilities are $n = 6$, $k = 4$ and $n = 7$, $k = 4$.

Proposition 1. The cases $X = SU(6)/SU(4)$, $H^o = SU(3) \times SU(3)$ and $X = SU(7)/(SU(4)$, $H^o = SU(3) \times SU(3)$ cannot occur.

Proof: The principal orbit would have codimension one, hence the only possibilities for the path-connected, compact orbit space X/G are S^1 (corresponding to one or no singular orbits) or a closed interval (corresponding to two singular orbits). Let T^{n-1} be the maximal torus of $G = SU(n)$ consisting of diagonal matrices $(\exp 2\pi i\,\theta_1, \dots, \exp 2\pi i\,\theta_n)$, $\theta_1 + \dots + \theta_n = 0$, $n = 6$ or $n = 7$.

Lemma 1. The fixed point set of T^{n-1} is empty.

Proof: Since $\mathrm{rk}\, H = 4 < n-1 = \mathrm{rk}\, T^{n-1}$, there are obviously no fixed points on principal orbits. If G/K is a singular orbit containing fixed points, it follows that K is of maximal rank in G and hence the Euler characteristic $\chi(G/K) = \chi(F(T^{n-1};G/K)) > 0$. Since there are at most two singular orbits, this contradicts $\chi(F(T^{n-1};X)) = \chi(X) = 0$; this proves the lemma.

Now consider the case $n = 6$. Let $T = \{(\exp 2\pi i\, \theta_1, \ldots, \exp 2\pi i\, \theta_6) \mid \theta_1 + \theta_2 + \theta_3 = \theta_4 + \theta_5 + \theta_6 = 0\}$ be the standard maximal torus of $SU(3) \times SU(3)$; then T has fixed points, and is then by definition a geometric weight of the T^5-action. Denote T by $(\theta_1+\theta_2+\theta_3)^{\perp}$. By Weyl group invariance of the geometric weight system for the T^5-action on X it follows that $(\theta_{\sigma(1)} + \theta_{\sigma(2)} + \theta_{\sigma(3)})^{\perp}$ is a geometric weight for all $\sigma \in S_6$ (the Weyl group of $SU(6)$). In the Leray-Serre spectral sequence for the fibration $X_{T^5} = E_{T^5} \times_{T^5} X \to B_{T^5}$ the generators x_9 and x_{11} of $H^*(X;Q)$ are transgressive, by the lemma their transgressions a_{10} and a_{12} cannot both vanish. By restriction all ten different weight vectors $\theta_{\sigma(1)} + \theta_{\sigma(2)} + \theta_{\sigma(3)}$ (as elements of $H^2(B_{T^5},Q)$) must then divide a_{10} and a_{12} (Corollary 1, p. 45 in [H 1]). This is a contradiction, since both a_{10} and a_{12} have dimension less than 20.

In the remaining case $n = 7$ we have $H^*(X;Q) = \Lambda_Q(x_9, x_{11}, x_{13})$. Obviously $T^5 = \{\exp 2\pi i\, \theta_1, \ldots, \exp 2\pi i\, \theta_6, 1);\ \theta_1 + \theta_2 + \ldots + \theta_6 = 0\}$ has no fixed points on principal orbits for the $SU(7)$-action on X. By the lemma a singular isotropy group K cannot have rank 6, assume it has rank 5 with $SU(3) \times SU(3) \subseteq K \subset SU(7)$. Recalling that the slice representation of K has $SU(3) \times SU(3)$ as principal orbit type, it follows quickly that the only possibility is $K^o = S(U(3)\times U(3))$. In this case x_9, x_{11} and x_{13} are again transgressive in the spectral sequence of $X_{T^6} \to B_{T^6}$ for dimension reasons, hence a non-empty fixed point set of any torus in T^6 must be a cohomology product of three odd spheres. However, $F(T^5,G/K)$ has dimension one; it follows that $F(T^5;X)$ is empty. The same argument as in the case $n=6$ applied to the T^5-action, now gives a contradiction, since we again have: $\dim \tau(x_{13}) = 14 < 20$.

q.e.d.

Proposition 2. Let $G = SU(n)$ operate smoothly on $X = SU(n)/SU(k)$, $\frac{1}{2}n+1 < k < n-1$, with connected principal isotropy group of type (H^o). Then $H^o = Sp(\ell)$ and $H^o = SO(\ell)$ are not possible.

Proof: Assume $H^o = Sp(\ell) \subset SU(n)$. We may assume $\ell \geq 2$, by Theorem 4 all other connected isotropy groups are of the type $Sp(t)$, $\ell \leq t \leq \frac{1}{2}n$. Let T^{n-1} be the standard maximal torus of G, and let $Sp(r)$ be the maximal connected isotropy type; then the maximal connected isotropy type of the T^{n-1}-action on X is of type T, where T is the standard maximal r-torus of $Sp(r)$. The minimal model of X is $\Lambda_{\mathbb{Q}}(x_{2k+1},\ldots,x_{2n-1})$ with $\deg x_j = j$, the homotopy Euler characteristic is $k-n$, so the torus rank is $n-k$; hence there must be subtori of T^{n-1} of corank $n-k$ with fixed points ([AH]). Hence $n-1-r \leq n-k < \frac{1}{2}n-1$, i.e. $2r > n$, which contradicts $Sp(r) \subset SU(n)$.

Similarly, if $H^o = SO(\ell)$, we have $\dim H^o = \frac{1}{2}\ell(\ell-1) > \dim SU(3) = 8$, i.e. $\ell \geq 5$, and we may apply Theorem 5 to conclude that all connected isotropy subgroups are standardly embedded $SO(t)$, $\ell \leq t \leq n$. Let $SO(s)$ of rank r be the maximal connected isotropy type, by the above argument we conclude that $2r > n$. Since s is $2r$

or $2r+1$, we have: $s \geq 2r > n$, which contradicts $SO(s) \subset SU(n)$.

q.e.d.

Thus, for most dimensions the desired elimination of (iv) b) and c) of Theorem 2 is a simple consequence of the torus rank theorem. The more complicated limit cases $\frac{1}{2}n < k \leq \frac{1}{2}n+1$ remain; we give the details of the argument for one of those cases and mention the necessary modifications for the others.

Theorem 7. Let $G = SU(n)$ act smoothly on $X = SU(n)/SU(k)$, $\frac{1}{2}n < k \leq \frac{1}{2}n+1$. Then the connected principel isotropy group H^o cannot be of type $Sp(\ell) \subset SU(n)$ or $SO(\ell) \subset SU(n)$.

Proof: If n is even, X is of the type $SU(2r)/SU(r+1)$, if n is odd, $X = SU(2r+1)/SU(r+1)$. We now consider $X = SU(2r)/SU(r+1) \sim S^{2r+3} \times S^{2r+5} \times \ldots \times S^{4r-1}$. Let $H^o = Sp(\ell) \subset SU(2r)$. By the proof of Proposition 2 we have $G_x^o = Sp(r) \subset SU(2r)$ for some point x, so there is an $SU(2r)$-equivariant map p: $Y = SU(2r)/Sp(r) \to G/G_x \xrightarrow{i} X$, where i is inclusion of the orbit through x. Let $T^{2r-1} = \{(\exp 2\pi i\, \theta_1, \ldots, \exp 2\pi i\, \theta_{2r}); \theta_1 + \ldots + \theta_{2r} = 0\}$ be the standard maximal torus of $SU(2r)$. We have $Y \sim_{\mathbb{Z}} S^5 \times S^9 \times \ldots \times S^{4r-3}$, i.e. Y is a cohomology product of $r-1$ odd spheres. The action of T^{2r-1} on Y is by left translations; its invariants are easily computable: The fibration $Y_{SU(2r)} \to B_{SU(2r)}$ is equivalent to $B_{Sp(r)} \to B_{SU(2r)}$; hence the transgressions of the generators of $H^*(Y;Q)$ may be identified with the odd universal Chern classes $c_3, c_5, \ldots, c_{2r-1}$, i.e. with odd elementary symmetric polynomials in $\{\theta_1, \ldots, \theta_{2r}\}$. Let π be the corresponding fibration $Y_{T^{2r-1}} \to B_{T^{2r-1}}$, then $\ker \pi^* = \langle c_3, \ldots, c_{2r-1} \rangle$, the ideal spanned by the odd universal Chern classes; its variety in the Lie algebra of T^{2r-1} consists of all (r-1)-codimensional linear subspaces defined by equations of the form:

$\theta_{\sigma(1)} + \theta_{\sigma(2)} = \cdots = \theta_{\sigma(2r-1)} + \theta_{\sigma(2r)} = 0$, where σ is in the symmetric group S_{2r} (the Weyl group W_G of SU(2r)). By Theorem IV.6 in [H 1] the corresponding corank $r-1$ subtori of T^{2r-1} are precisely the maximal subtori with fixed points in Y. Here the identity permutation corresponds to the standard maximal torus T^r of $Sp(r) \subset SU(2r)$, and the others are its $\frac{(2r)!}{r!\,2^r} = 3.5 \ldots (2r-1)$ W_G- conjugates in T^{2r-1}. Let η_i be the restriction of θ_{2i-1} to T^r, $i = 1, \ldots, r$. The complimentary root system of Sp(r) in SU(2r) is $\{(\eta_i - \eta_j);\ i \neq j\} \cup \{\pm(\eta_i + \eta_j);\ i < j\}$, it is then easy to see that the isotropy representation of Sp(r) on Y is a real form of $\Lambda^2 \nu_r - \theta$, and the fixed point set of T^r, $F(T^r;Y) = F_Y = T^{2r-1}/T^r$ is an (r-1)-torus. (Here θ is the one-dimensional trivial representation).

Proposition 3. Let $G = SU(2r)$ act smoothly on $X = SU(2r)/SU(r+1)$ with $H^o = Sp(k)$, $k \geq 2$. Then $r = 2^\ell - 1$ for a positive integer ℓ.

Proof: Consider now the T^{2r-1}-action on $X \sim S^{2r+3} \times S^{2r+5} \times \ldots \times S^{4r-1}$. The generators of $H^*(X;\mathbb{Q})$ are transgressive in the fibration $X_{T^{2r-1}} \to B_{T^{2r-1}}$, this time by dimension. Since all connected isotropy groups of the SU(2r)-action are of the type Sp(t) with $t \leq r$, no subtorus of T^{2r-1} of corank less than $r-1$ has fixed points. This is precisely the situation dealt with by Theorem VII.7 of [H 1]. The point is now that the corank (r-1) subtori with fixed points are the same $3.5 \ldots (2r-1)$ subtori which we have already computed for Y; since if $x \in F(T)$ for a corank (r-1) subtorus T, then G_x^o is conjugate to $Sp(r) \subset SU(2r)$. Since those subtori are all W_G-conjugate, their fixed point sets are all diffeomorphic to $F_X = F(T^r;X)$, which is a cohomology product of $r-1$ odd spheres. Theorem VII 7 gives: $e(X) = (2r+4)(2r+6)\ldots 4r = 2^{r-1}(r+2)(r+3)\ldots 2r = 3.5\ldots(2r-1)e(F_X)$. Hence $e_r = 3.5\ldots(2r-1)$ divides $f_r = 2^{r-1}(r+2)(r+3)\ldots 2r$. Proposition 3, which is already a strong indication for Theorem 7, now follows from the next lemma.

Lemma 2. $e_r = 3.5\ldots(2r-1)$ divides $f_r = 2^{r-1}(r+2)(r+3)\ldots 2r$ if and only if r is of the form $2^\ell-1$. The quotient is then $2^{2^{\ell+1}-\ell-3}$.

Proof: Here $\dfrac{f_{r+k}}{e_{r+k}} = 2^{2k}\,\dfrac{r+1}{r+k+1}\,\dfrac{f_r}{e_r}$; hence, when the lemma is true for $r = 2^\ell-1$, it cannot hold again until $r+k+1 = 2^{\ell+1}$, etc.

Proposition 4. The equivariant map $p : Y \to X$ induces a non-trivial homomorphism $p^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$.

Proof: We prove that p^* is non-trivial in degree $4r-3$. The observation that X and Y have the same set of distinguished corank $(r-1)$ subtori of T^{2r-1} with fixed points implies that the radical of the ideal in $H^*(B_{T^{2r-1}};\mathbb{Q})$ spanned by the transgressions of the generators $x_{2r+3}, x_{2r+5}, \ldots, x_{4r-1}$ in $H^*(X;\mathbb{Q})$ must again be $\langle c_3, c_5, \ldots, c_{2r-1}\rangle$, (by Theorem IV.6 in [H 1], this radical is again the ideal of the variety spanned by the Lie algebras of those subtori). This is possible only if the transgression $\tau(x_{4r-3}) = c_{2r-1} = \tau(y_{4r-3})$ (modulo lower universal Chern classes). Here p induces a bundle homomoprhism from $Y_{T^{2r-1}} \to B_{T^{2r-1}}$ to $X_{T^{2r-1}} \to B_{T^{2r-1}}$ and a corresponding homomorphism of spectral sequences. On the E_2-level this is defined by p^*; since

$\tau(x_{4r-3}) = c_{2r-1}$, i.e. is not generated by lower Chern classes; it is clear that $p^*(x_{4r-3})$ cannot be zero in $H^{4r-3}(Y;\mathbb{Q})$.

Corollary. The restriction q of p to $F_Y : q : F_Y \to F_X$ induces a non-trivial homomorphism $q^* : H^*(F_X;\mathbb{Q}) \to H^*(F_Y;\mathbb{Q})$.

Proof: We consider the restriction to the T^r-action; then X and Y are both totally non-homologous to zero in the fibrations $X_{T^r} \to B_{T^r}$ and $Y_{T^r} \to B_{T^r}$ respectively. Hence $H^*(X_{T^r},\mathbb{Q})$ and $H^*(Y_{T^r};\mathbb{Q})$ are both free $H^*(B_{T^r};\mathbb{Q})$-modules, with $H^*(X;\mathbb{Q}) = H^*(X_{T^r};\mathbb{Q}) \otimes_{H^*(B_{T^r};\mathbb{Q})} \mathbb{Q}$ and $H^*(Y;\mathbb{Q}) = H^*(Y_{T^r};\mathbb{Q}) \otimes_{H^*(B_{T^r};\mathbb{Q})} \mathbb{Q}$ (the $H^*(B_{T^r};\mathbb{Q})$-module structure on $\mathbb{Q}$ is defined by augmentation). Let $\bar{p}$ be the induced bundle homomorphism from $Y_{T^r} \to B_{T^r}$ to $X_{T^r} \to B_{T^r}$, then $\bar{p}^* : H^*(X_{T^r};\mathbb{Q}) \to H^*(Y_{T^r};\mathbb{Q})$ is compatible with $p^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$. Hence it follows from Proposition 4 that $\bar{p}^*$ remains non-trivial after localizing at the zero ideal of $H^*(B_{T^r};\mathbb{Q})$. By the basic localization theorem of equivariant cohomology (see [H 1], p.45), we obtain then $q^*: H^*(F_X;\mathbb{Q}) \otimes_{H^*(B_{T^r};\mathbb{Q})} R_o \to H^*(F_Y;\mathbb{Q}) \otimes_{H^*(B_{T^r};\mathbb{Q})} R_o$, where R_o is the quotient field of $H^*(B_{T^r};\mathbb{Q})$. It follows that $q^* : H^*(F_X;\mathbb{Q}) \to H^*(F_Y;\mathbb{Q})$ is non-trivial.

q.e.d.

The Weyl group W of $Sp(r)$ operates on F_X and F_Y; and q^* is an W-homomorphism. Here W is the subgroup of W_G which keeps T^r invariant; i.e. the set of $2^r r!$ permutations of $\{\theta_1,\dots,\theta_{2r}\}$ keeping the set of pairs $\{\theta_1,\theta_2\},\dots,\{\theta_{2r-1},\theta_{2r}\}$ invariant; or, equivalently, all permutations and sign changes of $\{\eta_1,\dots,\eta_r\}$. Let W_s be the normal subgroup of W consisting of all sign changes of $\{\eta_1,\dots,\eta_r\}$, then $W/W_s = S_r$, the symmetric group on $\{\eta_1,\dots,\eta_r\}$. Let E be the standard $(r-1)$-dimensional irreducible representation of S_r with Young diagram corresponding to the partition $(r-1,1)$. The corresponding representation of W with kernel W_s is also denoted by E.

Proposition 5. As a graded W-algebra $H^*(F_Y,\mathbb{Q})$ is isomorphic to $\Lambda_{\mathbb{Q}}(E)$. (The elements of E have degree 1).

Proof: W operates by automorphisms on $H^*(F_Y,\mathbb{Q}) = \Lambda_{\mathbb{Q}}(H^1(F_Y,\mathbb{Q}))$, so we only have to show that the W-module $H^1(F_Y;\mathbb{Q})$ is isomorphic to E. Let $t = (\exp(2\pi i\,\theta_1),\dots,\exp(2\pi i\,\theta_{2r}))$, $(\theta_1+\dots+\theta_{2r}=0)$ be in T^{2r-1}, then $t \equiv (\exp(2\pi i(\theta_1+\theta_2)),1,\exp(2\pi i(\theta_3+\theta_4)),1,\dots,\exp(2\pi i(\theta_{2r-1}+\theta_{2r})),1)$ modulo T^r, i.e. $z_1,\dots,z_r$ with $z_i = \theta_{2i-1}+\theta_{2i}$ are homogeneous coordinates for $T^{2r-1}/T^r = F_Y$. Here elements of W_s, corresponding to permutations of the type $(\theta_{2r-1},\theta_{2i})$ act trivially, and W/W_s acts by permutations of $\{z_1,\dots,z_r\}$. We have: $H^1(F_Y;\mathbb{Q}) = \{\sum_{i=1}^{r}\alpha_i z_i;\ \alpha_i\in\mathbb{Q},\sum_{i=1}^{n}\alpha_i = 0\}$. The representation of S_r induced on this vector space by the action through permutations of $\{z_1,\dots,z_r\}$ is precisely the standard irreducible representation of S_r. q.e.d.

Corollary. $H^*(F_Y;\mathbb{Q})$ is an irreducible W-module in each dimension.

Proof: By proposition 5 this is true in dimension 1. The corollary follows once we confirm that $\Lambda^p E$ is the irreducible S_r-module with Young diagram corresponding to the partition $(r-p,1,\dots,1)$. For lack of a reference and for later use, we note how this can be seen by computing characters. Let $E_1 = E\oplus\theta$, where θ is the trivial one-dimensional representation. Then $\Lambda^p E_1 = \Lambda^p E\oplus\Lambda^{p-1}E$. The character of $\Lambda^p E_1$ evaluated at a permutation α with s_i cycles of length p_i, $i=1,\dots,q$, $s_1p_1+\dots+s_qp_q=r$, is easily seen to be the p-th elementary symmetric function in the roots of the polynomial $(\lambda^{p_1}-1)^{s_1}\ldots(\lambda^{p_q}-1)^{s_q}$. Collecting the λ^{n-p}-terms from this product is easily seen to correspond to some of the permissible decompositions of the Young diagram of $(r-p,1,\ldots,1)$ in the Murnaghan-Nakayama rule for computing the value of the character of the corresponding representation on α. To show that the difference is accounted for by the term $\Lambda^{p-1}E$ is an easy combinatorial exercise.

Proof of Theorem 7:

Let $X = SU(2r)/SU(r+1)$ and assume $H^o = Sp(\ell) \subset SU(2r)$. Then $H^*(F_X;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_1,\ldots,u_{r-1})$, with $\deg u_i = d_i > 0$. Let u_j be the smallest possible dimension such that $q^*(u_j)$ is non-zero in $H^*(F_Y;\mathbb{Q})$. By the Corollary to Proposition 5 it follows that $H^{d_k}(F_Y;\mathbb{Q})$ is in the image of q^*. Since $\dim H^{d_k}(F_Y;\mathbb{Q}) > r-1$ unless $d_k = 1$, $r-2$, $r-1$, no other values are possible. If $d_k = 1$, it follows that $H^1(F_Y;\mathbb{Q}) \subset \operatorname{im} q^*$, i.e. $d_1 = \ldots = d_{r-1} = 1$ and $H^*(F_X;\mathbb{Q}) \cong H^*(F_Y;\mathbb{Q})$, which contradicts the fact that $e(X) \neq e(Y)$, (by the proof of Proposition 3). We may assume $r = 2^{\ell}-1$ by Proposition 3. Since d_k is odd, $d_k = r-1$ is impossible. If $d_k = r-2$, we have $\dim H^{d_k}(F_Y;\mathbb{Q}) = r-1$, and $d_1 = \ldots = d_{r-1} = r-2$, i.e. $F_X \sim S^{r-2}\times\ldots\times S^{r-2}$. Hence $e(F_X) = (r-1)^{r-1} = 2^{r-1}(2^{\ell-1}-1)^{r-1}$ and $e(X) = 3.5\ldots(2r-1)e(F_X) = 2^{r-1}(r+2)\ldots(2r)$. If $\ell > 2$, $e(F_X)$ is then not a power of 2, contradicting lemma 2, if $\ell = 2$, $(r=3)$, $e(F_X) = 4$ which also contradicts lemma 2.

This finishes the proof of Theorem 7 for the case $X = SU(2r)/SU(r+1)$, $H^o = Sp(t)$.

There are some modifications of the above argument in the case $X = SU(2r+1)/SU(r+1)$, $H^o = Sp(t)$. We have

$p : Y = SU(2r+1)/Sp(r) \to X$, and there are now $3.5\ldots(2r+1)$ distinguished corank r subtori of the standard maximal torus T^{2r} of $SU(2r+1)$. Lemma 2 and Proposition 3 applies as before (the extra factor $2r+1$ cancels against the extra sphere dimension). In Proposition 4 however, we now observe that $p^*(x_{4r-3})$, $p^*(x_{4r-1})$ and $p^*(x_{4r-3} \cup x_{4r-1})$ are all non-zero in $H^*(Y;\mathbb{Q})$. Now $F_Y = T^{2r}/T^r$ is an r-torus; the representation of W on $H^1(F_Y;\mathbb{Q})$ is isomorphic to the full permutation representation E_1, and $H^*(F_Y;\mathbb{Q})$ is isomorphic to $\Lambda_{\mathbb{Q}}(E_1)$ as an W-algebra, i.e. $H^p(F_Y,\mathbb{Q}) \cong \Lambda^p E \oplus \Lambda^{p-1}E$. By the above version of Proposition 4 there must now be a generator $u_1 \in H^{d_1}(F_X;\mathbb{Q})$ such that $q^*(u_1)$ is not in any 1-dimensional submodule of $H^*(F_Y;\mathbb{Q})$; it follows as before that $\dim u_1 = \ldots = \dim u_{r-1} = 1$ or $r-2$ for generators $u_1,\ldots,u_{r-1}$. Then $e(F_X) = 2^{r-1}(d_r+1) = 2^{2^{\ell+1}-\ell-2}$ by Lemma 2, where $r = 2^\ell - 1$, i.e. $d_r = 2^{2^\ell-\ell} - 1 > 2^{\ell+2} - 3 = 4r+1$ for $\ell > 2$, which is impossible, $4r+1$ being the largest dimension of the generators for $H^*(X;\mathbb{Q})$. For $\ell = 2$ we have $X = SU(7)/SU(4) \sim S^9 \times S^{11} \times S^{13}$ and $F_X \sim S^1 \times S^1 \times S^3$. Let $G_x^o = Sp(3)$, then the slice at x has dimension 6 and it follows that the slice representation of $Sp(3)$ is trivial. Hence $Sp(3)$ is the connected principal isotropy subgroup type, and the orbit space has dimension 6. Since the fixed point set of $T^3 \subseteq Sp(3)$ has dimension 3 on each fibre $SU(2r)/Sp(3)$, the dimension of F_X would be 9. This contradicts $F_X \sim S^1 \times S^1 \times S^3$.

In the second case $\dim u_1 = r-2 = 2^\ell - 3$. Lemma 2 gives $e(F_X) = (r-1)^{r-1}(d_r+1) = (2^\ell-2)^{r-1}(d_r+1) = 2^{2^{\ell+1}-\ell-2}$, which is impossible for $\ell \neq 2$. For $\ell = 2$ we have $r-2 = 1$, which is the case ruled out above.

Finally, for $H^o = SO(t)$ there are the following cases:

a) $X = SU(2r+1)/SU(r+1)$ with $H^o = SO(t)$ and $G_x^o = SO(2r+1)$ for some x. Then $Y = SU(2r+1)/SO(2r+1) \sim_{\mathbb{Q}} S^5 \times S^9 \times \ldots \times S^{4r+1}$ $\sim_{\mathbb{Q}} SU(2r+1)/Sp(r)$. The maximal torus T^r of $Sp(r)$ is also a maximal torus of $SO(2r)$ and $SO(2r+1)$, $F_Y = T^{2r}/T^r$, and the Weyl groups $W_{Sp(r)} = W_{SO(2r+1)}$. In rational cohomology there is no difference from the previous Sp-case, so the above proof applies.

b) $X = SU(2r)/SU(r+1)$ with $G_x^o = SO(2r)$ for some point x. Then $Y = SU(2r)/SO(2r) \sim_{\mathbb{Q}} S^5 \times S^9 \times \ldots \times S^{4r-3} \times S^{2r}$; i.e. Y is the cohomology product of one even and $r-1$ odd spheres, the homotopy Euler characteristic of Y is $-r+1$. Here the Weyl group $W_{SO(2r)}$ is generated by all permutations and an even number of sign changes of $\{\eta_1, \ldots, \eta_r\}$, $W_{Sp(r)}/W_{SO(2r)} = \mathbb{Z}_2$. Let w in $W_{Sp(r)}$ represent the non-trivial element of $W_{Sp(r)}/W_{SO(2r)}$, then $F_Y = F(T_r;Y) = F_1 \cup w F_1$, where $F_1 = T^{2r-1}/T^r$. The $W_{SO(2r)}$-module $H^*(F_1, \mathbb{Q})$ is isomorphic to $\Lambda_{\mathbb{Q}}(E)$ as before, and translation by w induces an $W_{SO(2r)}$-algebra equivalence from $H^*(w F_1;\mathbb{Q})$ to $H^*(F_1;\mathbb{Q})$. Proposition 3, Lemma 2, and Proposition 4 are as before. Proposition 5 is modified to $H^*(F_Y;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(E) \oplus \Lambda_{\mathbb{Q}}(E)$ as an S_r-module. The rest of the proof goes as the case $X = SU(2r)/SU(r+1)$, $G_x^o = Sp(r)$.

c) $X = SU(2r+1)/SU(r+1)$ with $G_x^o = SO(2r)$ as the connected isotropy group type of maximal dimension. Then $Y = SU(2r+1)/SO(2r)$ $\sim_{\mathbb{Q}} S^5 \times S^9 \times \ldots \times S^{4r+1} \times S^{2r}$. This goes as the previous case with $F_1 = T^{2r}/T^r$, $H^*(F_Y;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(E_1) \oplus \Lambda_{\mathbb{Q}}(E_1)$, and we compare with the case $G_x^o = Sp(r)$. In odd degrees there are now two one-dimensional $W_{SO(2r)}$-modules in $H^1(F_Y;\mathbb{Q})$ and in $H^r(F_Y;\mathbb{Q})$. Since

$p^*(x_{4r+1}) \cup p^*(x_{4r-1})$ is non-zero in $H^*(Y;\mathbb{Q})$, it follows as before that the image of q^* must contain an $(r-1)$-dimensional $W_{SO(2r)}$-submodule of $H^*(F_Y;\mathbb{Q})$. The rest of the proof follows the $G_x^o = Sp(r)$ case, with the following modification for $\ell = 2$, $r = 3$: $\dim SU(7)/SU(4) = \dim SU(7)/SO(6) = 33$, hence $G_x^o = SO(6)$ is impossible.

Theorem 2, the estimates at the end of Section 1, Proposition 1, Proposition 2, and Theorem 7 now prove Theorem 6.

3. Reduction of the Orbit Projection to a Fibration.

In this section X is any simply connected, closed, differentiable manifold with $X \sim_{\mathbb{Z}} S^{2k+1} \times S^{2k+3} \times \ldots \times S^{2n-1}$, $\frac{1}{2}n \leq k < n$.

Theorem 8. Let X be as above and let $G = SU(n)$ act smoothly on X. If all connected isotropy groups are of type $SU(\ell) \subset SU(n)$, then all isotropy groups are in fact connected. Moreover, only one orbit type occurs, and the orbit projection is a fibration of X with $SU(n)/SU(r)$ as fibre, $k \leq r \leq n$.

The following lemma is essential for the proof of Theorem 8.

Lemma 3. If $SU(m)$ acts smoothly on $X \sim_{\mathbb{Z}} S^{2k+1} \times S^{2k+3} \times \ldots \times S^{2n-1}$ with non-empty fixed point set and all connected isotropy subgroups of type $SU(\ell) \subset SU(m)$, $\ell \geq 2$, then all isotropy subgroups are connected.

Proof: Let $x \in X$ be a fixed point. Then the isotropy representation of $SU(m)$ at x has connected principal isotropy subgroup of type $SU(r) \subset SU(m)$, $r \geq 2$. The classification of linear $SU(m)$-actions with non-trivial principal isotropy group is well known; we refer to [H 1, p.83] to conclude that for $r > 2$, the isotropy representation at x must be the underlying real representation of $(m-r)\mu_m$ modulo trivial representations. For $r = 2$ the only other possibilities are: (a) $m = 3$ with isotropy representation $[\Lambda^2\mu_3]_{\mathbb{R}} = [\mu_3]_{\mathbb{R}}$ (the contragradient representation of μ_3 is $\Lambda^2\mu_3$), (b) $m = 4$ with isotropy representation $[\mu_4]_{\mathbb{R}} + \varphi$, where φ is a real form of $\Lambda^2\mu_4$. (All equations modulo trivial representations). The principal isotropy subgroup in (b) is of type $SU(2)$, however, the principal isotropy subgroup $Sp(2)$ of φ occurs as a non-principal isotropy subgroup in $[\mu_4]_{\mathbb{R}} + \varphi$, hence (b) cannot occur under the conditions of Lemma 2. By local linearity it now follows that all isotropy groups in a neighbourhood of x are of type $SU(\ell)$, $\ell \geq 2$. Suppose that G_y is a disconnected isotropy subgroup; by conjugation we may assume $G_y^o = SU(\ell) \subseteq G_y$. Here G_y/G_y^o is finite, and we may choose an element z in $G_y \setminus G_y^o$ such that $z^p \in G_y^o$ for a prime p (it is actually easy to choose z such that $z^p = e$). Let K be the subgroup generated by G_y^o and z, then $K/G_y^o \cong \mathbb{Z}_p$. Let V be a subspace of $\mathbb{C}^m$ such that $K \subseteq SU(V)$, but K is not contained in $SU(W)$ for any subspace W of $\mathbb{C}^n$ with $\dim W < \dim V = m'$. Let T and T' be maximal tori of G_y^o and $SU(V)$ respectively, with $T \subseteq T'$. By considering the representation of T defined by the inclusion of T in $G_y^o = SU(\ell)$, it is easy to see that T cannot be maximal torus in $SU(W) \subseteq SU(V)$ for any other subspace W of V than $\mathbb{C}^\ell$. By the conditions of the lemma it is now clear that $F(G_y^o;X) = F(T;X)_* = Z_1$, similarly $F(SU(V),X) = F(T';X)_* = Z_2$. By the dimension restriction $k \geq \frac{1}{2}n$ the generators of $H^*(X,\mathbb{Z})$ are transgressive in the Serre spectral sequence of the fibre bundle $X_T \to B_T$. From the existence of fixed points it follows that the transgressions of those generators are all zero, and Z_1 is again of the integral cohomology of a product of n-k odd spheres. In particular it is connected, similarly for Z_2. Now K is in the normalizer of G_y^o, hence $K/G_y^o = \mathbb{Z}_p$ acts on $Z_1 = F(G_y^o, X)$. Obviously T' also acts on $Z_1 = F(T;X)$ with fixed point set Z_2. By the known orbit structure around x and the choice of V it follows that $F(SU(V);X)$ has full dimension in $F(\mathbb{Z}_p)$ locally around $x \in Z_2 \subset Z_1$. Hence Z_2 must be a connected component of $F(\mathbb{Z}_p;Z_1)$. Since $y \in F(\mathbb{Z}_p;Z_1) \setminus Z_2$, it follows that $F(\mathbb{Z}_p;Z_1)$ has more than one connected component, hence $\dim H^*(F(\mathbb{Z}_p;Z_1);\mathbb{Z}_p) > \dim H^*(Z_2,\mathbb{Z}_p) = 2^{n-k}$ $= \dim H^*(Z_1;\mathbb{Z}_p)$, in contradiction to a well known theorem of Borel.

Remark: The proof of Lemma 3 is essentially given in Theorem VII, 2' of [H 1]. The argument may be applied to k-multiaxial actions in more general situations than the one considered here.

Proof of Theorem 8: We reduce the first part of the theorem to Lemma 3 as follows: Let $H^o = SU(m)$ be a connected isotropy subgroup of $G = SU(n)$ of maximal rank. Then $(H^o)_x = H^o \cap G_x$ for $x \in X$. Let $G_x^o = SU(V)$ with $\dim V = \ell$. If $\ell = k$, the orbit G/G_x is of full dimension in X, i.e. it is all of X, and G_x must be connected. Thus we may assume $k < \ell \leq m$. Then $H^o \cap G_x^o = SU(m) \cap SU(V) = SU(((\mathbb{C}^m)^\perp + V^\perp)^\perp)$; hence $(H_x^o)^o = H^o \cap G_x^o$ $= SU(W)$ with $\dim W \geq n - (n\text{-}m\text{+}n\text{-}\ell) = m + \ell - n \geq 2k + 2\text{-}n \geq 2$. It follows that the action of H^o on X satsifies the conditions of Lemma 3. It is then sufficient to prove that if G_y is disconnected for some $y \in X$, then some isotropy subgroup of the

H^o-action is also disconnected. Let now G_y be disconnected and let $e_1,\ldots,e_n$ be the standard basis of $\mathbb{C}^n = L(e_1,\ldots,e_n)$; i.e. $H^o = SU(L(e_1,\ldots,e_n))$. By conjugation we may assume that $G_y^o = SU(L(e_{n-\ell+1},\ldots,e_n))$, with $\ell > k$. Since G_y normalizes G_y^o, we have $G_y \subset S(U(L(e_1,\ldots,e_{n-\ell}) \times U(L(e_{n-\ell+1},\ldots,e_n)))$. Let $g = (g_1,g_2)$ be in $G_y - G_y^o$ with $g_1 \in U(L(e_1,\ldots,e_{n-\ell}))$, $g_2 \in U(L(e_{n-\ell+1},\ldots,e_n))$, and let g_3 be defined by $g_3(e_{n-\ell+1}) = (\det g_2)e_{n-\ell+1}$, $g_3(e_i) = e_i$ for $n-\ell+1 < i \leq n$. Then $(1, g_3 g_2^{-1}) \in G_y^o$, hence $(g_1,g_3) \in G_y - G_y^o$. Since $n-\ell+1 \leq n-k \leq \frac{1}{2}n < m$, we also have $(g_1,g_3) \in H^o = SU(L(e_1,\ldots,e_m))$. So $(g_1,g_3) \in H_y^o$, but $(g_1,g_3) \notin (H_y^o)^o \subseteq G_y^o$, hence H_y^o is disconnected in contradiction to Lemma 3. This finishes the proof of the first part of Theorem 8.

Our next observation is that X is a multiaxial (regular) $SU(n)$-manifold. The only additional requirement to check is that the slice representation of an isotropy subgroup $SU(\ell)$ is always a multiple of the standard representation modulo trivial representations. This is obvious for $k \geq 2$, since, for the non-transitive case, the principal isotropy subgroup of the slice representation would then again be of the type $SU(r)$ with $r \geq 3$. The case $(n,k) = (2,1)$ has either trivial or transitive G-action. For a multiaxial G-manifold it is known that the orbit space X/G is a topological manifold with boundary (modelled on the space of positive semi-definite Hermitian matrices, and not in general a differentiable manifold with boundary [D]). Let $SU(r)$ be a principal isotropy subgroup. If $r = k$, the action is transitive, and Theorem 8 is trivial. If $r > k$, we have: $\dim X/G = \dim X - \dim SU(n)/SU(r) = r^2 - k^2$. Let $\pi : X \to X/G$ be the orbit projection, then the singular orbits project down to the boundary points of X/G. The fibers of π are of the type $SU(n)/SU(\ell)$ with $\ell \geq r$; hence $H^i(\pi^{-1}(\bar{y});\mathbb{Z}) = 0$ for $i = 1,\ldots,2r$ for all $\bar{y} \in X/G$. From the Vietoris-Begle mapping theorem it now follows that $\pi^* : H^j(X/G;\mathbb{Z}) \to H^j(X;\mathbb{Z})$ is an isomorphism for $0 \leq j \leq 2r$. Choose cohomology classes $\bar{x}_{2j+1} \in H^{2j+1}(X/G;\mathbb{Z})$ such that $x_{2j+1} = \pi^*(\bar{x}_{2j+1})$ form part of the generators of $H^*(X;\mathbb{Z})$, $k \leq j \leq r-1$. Then $\pi^*(\bar{x}_{2k+1} \cup \ldots \cup \bar{x}_{2r-1}) = x_{2k+1} \cup \ldots \cup x_{2r-1}$ is non-zero in $H^{r^2-k^2}(X;\mathbb{Z})$; hence $0 \neq \bar{x}_{2k+1} \cup \ldots \cup \bar{x}_{2r-1}$ in $H^{r^2-k^2}(X/G;\mathbb{Z})$. Then the cohomology group of X/G is non-zero in the top dimension; hence the boundary of X/G must be empty, and there are no singular orbits.

Remark: Let $X = S^{2k+1} \times \ldots \times S^{2r-1} \times SU(n)/SU(r)$ and let $G = SU(n)$ act by left translations on the last factor and trivially on the others. This example shows that any orbit type $SU(n)/SU(r)$ with $k \leq r \leq n$ can occur in Theorem 8.

4. Cohomology Operations and the Reduction to Linear Action.

In this section we let $G = SU(n)$ act smoothly on $X = W_{n,k} = SU(n)/SU(k)$ with $\frac{1}{2}n < k < n-1$. Applying Theorem 6, its corollary, and Theorem 8 it follows that there is only one orbit type $SU(n)/SU(r)$; with $k \leq r \leq n$. It is then clear that the only unsettled part of Theorem 1 is to prove that for $X = W_{n,k}$ this is only possible with $r = k$ or $r = n$, i.e. the transitive or the trivial actions, respectively. In view of the last remark of Section 3, it is obvious that this can be proved only by applying more subtle topological methods which detect the difference between X and $S^{2k+1}\times\ldots\times S^{2n-1}$. The most obvious example of such cohomology operators are Steenrod squares, which distinguish those spaces for $k<n-1$. It is therefore interesting to observe how much information Steenrod squares yield for the G-space X; we prove that they can always be applied to eliminate the orbit type $SU(n)/SU(r)$ with $k+1<r<n-1$. The method has been used in [HS] for the study of a related problem. Although this result can be somewhat strengthened by applying reduced p-powers, the elimination of the remaining limit cases $SU(n)/SU(r)$ with $r=k+1$ or $n-1$ in general requires the deeper knowledge on the fibre homotopy type of Stiefel manifolds obtained by secondary cohomology operations.

Let $\mathbb{Z}_2^n = O(1)\times\ldots\times O(1)\subset O(n)$ be the standard maximal 2-torus of $O(n)$; the inclusions $\mathbb{Z}_2^n\subset O(n)\subset U(n)$ induce standard fibrations of classifying spaces: $B_{\mathbb{Z}_2^n}\to B_{O(n)}$ and $B_{O(n)}\to B_{U(n)}$, and induced homomorphisms : $H^*(B_{U(n)};\mathbb{Z}_2)\to H^*(B_{O(n)};\mathbb{Z}_2)\to H^*(B_{\mathbb{Z}_2^n};\mathbb{Z}_2)$ $=\mathbb{Z}_2[t_1,\ldots,t_n]$, where $t_i\in H^1(B_{\mathbb{Z}_2^n};\mathbb{Z}_2)$ may be identified with the $\mathbb{Z}_2$-linear functional on $\mathbb{Z}_2^n$ defined by the i-th coordinate. Then $H^*(BO(n);Z_2)=Z_2[w_1,\ldots,w_n]$, where the i-th universal Stiefel-Whitney class w_i is identified with the i-th symmetric polynomial $\sigma_i(t_1,\ldots,t_n)$, and $H^*(B_{U(n)};\mathbb{Z}_2)=\mathbb{Z}_2[c_1,\ldots,c_n]$ where $c_i = w_i^2 = \sigma_i(t_1^2,\ldots,t_n^2)$ (mod 2). It follows that $H^*(B_{SU(n)};\mathbb{Z}_2) = \mathbb{Z}_2[c_2,c_3,\ldots,c_n]$.

Proposition 6. a) The Steenrod square operations in $H^*(B_{SU(n)};\mathbb{Z}_2)$ are given by $Sq^{2i+1}c_j = 0$ for i,j and $Sq^{2i}c_j =$ $\sum_{a=o}^{i}\binom{j-i+a-1}{a}c_{i-a}c_{j+a}$ for $i\leq j$.

b) $H^*(X;\mathbb{Z}_2) \cong \Lambda_{\mathbb{Z}_2}(x_{2k+1},\ldots,x_{2n-1})$ with $\deg x_i = i$, and $Sq^{2i}(x_{2j+1}) = \binom{j}{i}x_{2j+2i+1}$ for $i \leq j$, $j+i \leq n-1$, and zero otherwise. Here $\binom{j}{i}$ is the mod 2 binomial coefficient, and $x_{2k+1},\ldots,x_{2n-1}$ is a simple, universally transgressive system of generators for $H^*(X;\mathbb{Z}_2)$.

The formula in a) follows from the Cartan formula for Steenrod squares and a computation of certain symmetric functions, this is done in [B 3] for the real case $B_{SO(n)}$; the same type of computation works here. The transgression maps a universally transgressive generator of dimension $2\ell+1$ into $H^{2\ell+2}(B_{SU(n)},\mathbb{Z}_2)/D^{2\ell+2} = Q^{2\ell+2}$, where $D^{2\ell+2}$ is the subspace generated by decomposable elements in $H^{2\ell+2}(B_{SU(n)},:\mathbb{Z}_2)$. Steenrod squares take decomposable elements into decomposable elements, so there are well defined "Steenrod Squares" Sq^i: $Q^{2\ell+2} \to Q^{2\ell+2+i}$, and in this sense transgression commutes with Steenrod squares. With this observation it is then easy to see that only one entry from the sum in a) survives modulo decomposable elements to give b).

Let $\pi : X \to X/G$ be the orbit projection. It follows from the proof of Theorem 8 that $\pi^* : H^j(X/G;\mathbb{Z}_2) \to H^j(X;\mathbb{Z}_2)$ is an isomorphism for $0 \leq j \leq r$. Let $\bar{x}_{2k+1},\ldots,\bar{x}_{2r-1}$ be in $H^*(X/G;\mathbb{Z}_2)$ with $\pi^*(\bar{x}_j) = x_j$ for $j = 2k+1, 2k+3,\ldots,2r-1$.

Theorem 9. When $G = SU(n)$ acts smoothly on $X = SU(n)/SU(k)$ with $k > \frac{1}{2}n$, the orbit type $SU(n)/SU(r)$ with $k+1 < r < n-1$ cannot occur.

Proof: Assume that the orbit type is $SU(n)/SU(r)$ with $k+1 < r < n-1$. By Proposition 6 we have:

$Sq^2 x_{2r-1} = (r-1)x_{2r+1}$, $Sq^4 x_{2r-1} = \frac{1}{2}(r-1)(r-2)x_{2r+3}$, $Sq^4 x_{2r-3} = \frac{1}{2}(r-2)(r-3)x_{2r+1}$. If r is even, $r-1 \neq 0 \pmod 2$, if $r = 4j+1$, $\frac{1}{2}(r-2)(r-3) \neq 0 \bmod 2$, and if $r = 4j+3$, $\frac{1}{2}(r-1)(r-2) \neq 0 \bmod 2$. Hence $Sq^2(\bar{x}_{2r-1}) = \bar{x}_{2r+1}$, $Sq^4(\bar{x}_{2r-3}) = \bar{x}_{2r+1}$, or $Sq^4(\bar{x}_{2r-1}) = \bar{x}_{2r+3}$ for those cases respectively, where $\pi^*(\bar{x}_{2r+1}) = x_{2r+1}$ or $\pi^*(\bar{x}_{2r+3}) = x_{2r+3}$. Since $\pi^*(\bar{x}_{2k+1} \cup \bar{x}_{2k+3} \cup \ldots \cup \bar{x}_{2r-1} \cup \bar{x}_{2r+i}) = x_{2k+1} \cup x_{2k+3} \cup \ldots \cup x_{2r-1} \cup x_{2r+i}$ which is non-zero in $H^{r^2-k^2+2r+i}(X;\mathbb{Z}_2)$ with $i = 1$ or 3, respectively, we have $H^{r^2-k^2+2r+i}(X/G;\mathbb{Z}_2) \neq 0$ for $i = 1$ or 3, contradicting $\dim X/G = r^2-k^2$. q.e.d.

Remark: It is furthermore clear that for k odd, we have $Sq^2(x_{2k+1}) = x_{2k+3} \pmod 2$, hence $r = k+1$ is not possible in this case by the same argument; similarly, for n odd, $Sq^2(x_{2n-3}) = x_{2n-1}$ so $r = n-1$ is impossible. By applying reduced p-powers, better results are available. For example, from the computations in [B S] one can deduce for the reduced 3-power that $P_3^1(x_{2k+1}) = (k+3)x_{2j+5}$. Combining this with the above results for $Sq^2(x_{2k+1})$ and $Sq^4(x_{2k+1})$, it follows easily that for $k < n-2$, $r = k+1$ is impossible unless k is divisible by 12, similarly, $r = n-1$ is impossible unless n is divisible by 12. This is analogous to the situation for the section problem for complex Stiefel manifolds before higher cohomology operations were introduced into this problem (see [B S], where the same divisibility condition by 12 appears). Although such operations have not had significant applications to transformation group theory so far, it is reasonable to expect them to play a decisive role for settling certain types of problems. Here we apply the stronger results on fibre homotopy types of Stiefel manifolds which can thus be obtained to finally settle the remaining part of Theorem 1 for the general case.

Proposition 7. Let π be the orbit fibration from X to X/G with fiber $F = SU(n)/SU(r)$, $k \leq r \leq n$. Then X/G is homotopy equivalent to $SU(r)/SU(k)$.

Proof: In the spectral sequence of π we have again that all generators of $H^*(F;\mathbb{Z})$ are transgressive for dimension reasons. It follows easily that all transgressions are zero, and consequently that $E_2 = E_\infty$ and $H^*(X;\mathbb{Z}) \cong H^*(X/G;\mathbb{Z}) \otimes H^*(F;\mathbb{Z})$ as a module; hence $H^*(X/G;\mathbb{Z}) = \Lambda_{\mathbb{Z}}(\bar{x}_{2k+1},\ldots,\bar{x}_{2r-1})$. From the homotopy sequence of π it follows that X/G is simply connected. Consider the inclusion $i : K = SU(r)/SU(k) \to SU(n)/SU(k) = X$. Then $y_{2(k+j)-1} = i^*(x_{2(k+j)-1})$, $j = 1,\ldots,r-k$ form a system of generators for $H^*(K;\mathbb{Z})$, with $(\pi \circ i)^*(\bar{x}_{2(k+j)-1}) = y_{2(k+j)-1}$ for $j = 1,\ldots,r-k$. Hence the map $\pi \circ i$ induces an isomorphism in cohomology and is a homotopy equivalence by the Whitehead theorem.

q.e.d.

Theorem 10. When $G = SU(n)$ acts smoothly on $X = SU(n)/SU(k)$ with $k > \frac{1}{2}n$, the orbit type $SU(n)/SU(r)$ with $r = k+1$ or $r = n-1$ cannot occur.

Proof: Let i be the inclusion of the fibre $F = SU(n)/SU(r)$ in the orbit fibration $\pi : X \to X/G$. We now compare this with the standard fibration $p: X = SU(n)/SU(k) \to SU(n)/SU(r) = Y$ with fibre $P = SU(r)/SU(k)$. We have the commutative square:

$$\begin{array}{ccc} X & \xrightarrow{(p,\pi)} & Y \times X/G \\ \pi \downarrow & & \downarrow p_2 \\ X/G & \xrightarrow{id} & X/G \end{array}$$

where id is the identity map and p_2 is projection on the second factor. Now (p,π) is a fibre map from the orbit fibration to the trivial fibration p_2. Let $z_{2r+1},\dots,z_{2n-1}$ be a simple system of universally transgressive generators for $H^*(Y;\mathbb{Z})$ with $p^*(z_{2(r+j)-1}) = x_{2(r+j)-1}$, $j = 1,\dots,(n-r)$. Then it follows from the proof of Proposition 7 that $(i\circ p)^*(z_{2(r+j)-1}) = y_{2(r+j)-1}$, $(i\circ p)$ is an isomorphism in cohomology and a homotopy equivalence by the Whitehead theorem. Hence the restriction of (p,π) to a fibre is a homotopy equivalence, and by a theorem of Dold ([Do], (p,π) is a fibre homotopy equivalence from π to p_2 In particular $X = SU(n)/SU(k)$ is homotopy equivalent to $SU(n)/SU(r)\times SU(r)/SU(k)$, i.e. the standard fibration p is decomposable. For $r = n-1$ we have $Y = S^{2n-1}$ and for $r = k+1$ we have $P = S^{2k+1}$; i.e. the base space or the fibre is a sphere. By corollaries 4.5 and 4.8 in [J], it would then follow that the standard fibration p of X would be fibre homotopically trivial, which is known to be false (e.g. [J], p. 154). q.e.d.

By Theorem 9 and 10 together with the results of the earlier sections, it follows that if $G = SU(n)$ acts smoothly on $X = SU(n)/SU(k)$ with $k > \frac{1}{2}n$, there is one orbit type $SU(n)/SU(r)$ with $r = k$ or n, corresponding to the linear or the trivial action respectively. This completes the proof of our main Theorem 1.

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