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A CONSTRUCTION OF ~~THE~~ INNER FUNCTIONS ON THE UNIT BALL IN C^P

by

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In their recent paper [1], Hakim and Sibony came very close to constructing inner functions on the unit ball in C^p. In fact, only some minor modifications of their arguments are necessary to produce bona fide inner functions. The present paper carries out these modifications. Since we need to point out some information not explicitly stated (but clear from the context) in their paper, and to make the expositon self-contained, we repeat most of the material in their paper. Our lemma 1, for instance, is identical to their Lemme 3, and our application of lemma 1 in the proof of lemma 2 is the same as theirs in the proof of Theoreme 1.

The following notations are used: B is the unit ball in C^p, $B_R=\{z\in C^p: |z|<R\}$. We use the distance

$$\delta(z_1,z_2) = \frac{1}{\sqrt{2}}\|z_1-z_2\| \leqslant \sqrt{2} \quad \text{when} \quad z_1,z_2\in B$$

When $z_1,z_2\in \partial B$, we get

$$\delta^2(z_1,z_2) = \frac{1}{2}\|z_1-z_2\|^2 = 1 - \mathrm{Re}\langle z_1,z_2\rangle$$

When $z\in\partial B$, we let $B(z,r)$ denote the ball in ∂B with δ-radius r. The area of $B(z,r)$ is denoted $A(r)$. An exact formula for $A(r)$ is given in [1]. The following obvious estimate, however, is sufficient for our purposes:

(1) There exist constants C_1 and C_2 such that

$$C_1 r^{2p-1} \leqslant A(r) \leqslant C_2 r^{2p-1}, \quad 0<r\leqslant\sqrt{2}$$

Finally μ is the ordinary area measure on ∂B, normalized such that $\mu(\partial B)=1$. The area $A(r)$ refers to this measure.

<u>Lemma 1</u>. There exist positive numbers ε_0, C and A, depending only on the dimension, such that: If ε, a and R are positive numbers satisfying $0<2\varepsilon<a<1$, $\varepsilon\leqslant\varepsilon_0$ and $R<1$, and f is a continuous function on ∂B, $U\subset\partial B$ an open set such that $\mu(U)=\mu(\bar{U})<1$ with $|f(z)|>a$ for all $z\in U$, then there is an entire function g

and an open set $V \subset \partial B$ such that

(a) $\|f+g\|_\infty \leqslant \max\{\|f\|_\infty, 1\} + 2\varepsilon$

(b) $\|g\|_{B_R} \leqslant \varepsilon$

(c) $|f(z)+g(z)| > a - 2\varepsilon$ on $U \cup V$

(d) $\bar{U} \cap \bar{V} = \emptyset$ and $\mu(V)=\mu(\bar{V}) \geqslant C \arccos a[\frac{\log 1/a}{\log A/\varepsilon}]^{\frac{2p-1}{2}}(1-\mu(U))$

Proof: Let $U^\gamma = \{z \in \partial B;\ \delta(z,U)<\gamma\}$ and $V^\gamma = \partial B \setminus U^\gamma$. U^γ is an open set and $\lim_{\gamma\to 0} \mu(U^\gamma)=\mu(\bar{U})=\mu(U)$, hence there exists γ_1 such that

$$\mu(V^\gamma) = 1 - \mu(U^\gamma) > \tfrac{1}{2}(1-\mu(U)) \quad \text{whenever} \quad \gamma \leqslant \gamma_1$$

Since f is uniformly continuous there exists γ_2 such that $|f(z)-f(z')|<\varepsilon$ when $\delta(z,z')<\gamma_2$. Let $r>0$ be a number such that $r \leqslant \min(\gamma_1,\gamma_2)$. Choose a maximal disjoint family $\{B(z_j,r)\}_{j=1}^{N_r}$ of balls with $z_j \in V^r$. Maximal means that any disjoint family of balls with centers in V^r and radii r will have not more than N_r balls. Since all of these balls lie outside U, (1) gives

(2) $$N_r C_1 r^{2p-1} \leqslant 1 - \mu(U)$$

The balls $B(z_j, 2r)$ must cover V^r, hence (1) gives

(3) $$N_r C_2 2^{2p-1} r^{2p-1} \geqslant \mu(V^r) > \tfrac{1}{2}(1-\mu(U))$$

(2) and (3) together give

(4) $$\frac{C_3}{r^{2p-1}}(1-\mu(U)) \leqslant N_r \leqslant \frac{C_4}{r^{2p-1}}(1-\mu(U))$$

We now seek an estimate on how many points z_j can be at a certain distance from $z \in \partial B$. Let

$$V_k(z) = \{z_j;\ kr \leqslant \delta(z,z_j) < (k+1)r\} \quad k=1,\ldots,[\tfrac{\sqrt{2}}{r}]$$

and $N_k(z)=\text{card } V_k(z)$. If $z_j \in V_k(z)$ then $B(z_j,r)\subset B(z,(k+2)r)$ hence (1) gives

$$N_k(z)C_1 r^{2p-1} \leq C_2(k+2)^{2p-1} r^{2p-1}$$

which implies

(5) $$N_k(z) \leq C_5 k^{2p-1}$$

Let $g(z)=\sum_{j=1}^{N_r} \beta_j e^{-n(1-\langle z,z_j\rangle)}$ where β_j is defined by $\beta_j=0$ if $|f(z_j)|\geq 1$ and $|f(z_j)+\beta_j|=|f(z_j)|+|\beta_j|=1$ otherwise.

We shall show that n and r can be chosen such that g satisfies the lemma. When $z\in\partial B$ we get

$$g(z) = \sum_{j=1}^{N_r} |\beta_j| e^{-n\delta^2(z,z_j)} e^{i\theta_{n,j}(z)} =$$
$$\sum_{k=0}^{[\frac{\sqrt{2}}{r}]} \sum_{z_j\in V_k(z)} |\beta_j| e^{-n\delta^2(z,z_j)} e^{i\theta_{n,j}(z)}$$

There is at most one point in $V_0(z)$. This gives, by (5)

$$|g(z)| \leq 1 + C_5\left[\sum_{k=0}^{[\frac{\sqrt{2}}{r}]} k^{2p-1} e^{-k^2nr^2}\right]$$

Assuming nr^2 is large, we get

$$|g(z)| \leq 1 + C_5 \sum_{k=1}^{\infty} e^{-knr^2} < 1 + 2C_5 e^{-nr^2} := 1 + Ae^{-nr^2}$$

Hence, if we choose n and r such that

(6) $$nr^2 = \log A/\varepsilon$$

we get

(7) $$|g(z)| < 1 + \varepsilon \quad \text{and} \quad |g(z)| < \varepsilon \quad \text{if} \quad z\in\partial B \quad \text{and}$$
$$z \notin B(z_j,r) \quad \text{for any } j \text{ with } \beta_j\neq 0$$

Notice that there exists $\varepsilon_0>0$ such that if $\varepsilon\leq\varepsilon_0$ condition (6) implies that the above calculations hold. We shall also assume ε_0 is small enough to guarantee that $nr^2 \geq 1$. Condition (6) can be satisfied with arbitrarily small r and arbitrarily large n.

(7) implies parts of (a) and (c). Since U is outside all the balls, $|f(z)+g(z)|>a-\varepsilon$ on U. If $z \in \partial B$ and $z \notin B(z_j,r)$ for any j with $\beta_j \neq 0$, we get $|f(z)+g(z)| \leq \|f\|_\infty + \varepsilon$. If $z \in B(z_j,r)$ for some j with $\beta_j \neq 0$, we get

$$(8)\quad |f(z)+g(z)| \leq |f(z)-f(z_j)| + |f(z_j)+\beta_j e^{-n(1-<z,z_j>)}| + \Big|\sum_{k\neq j} \beta_k e^{-n(1-<z,z_k>)}\Big| \leq \varepsilon + 1 + \varepsilon = 1 + 2\varepsilon$$

This proves (a).

Let $W = \bigcup_{j=1}^{N_r} B(z_j,r)$. We shall now determine a certain open subset V of W such that $|f(z)+g(z)|>a-2\varepsilon$ in V and give an estimate on its area. If $|f(z_j)| \geq 1$, then $\beta_j=0$, so (7) gives that $|f(z)+g(z)|>1-2\varepsilon>a-2\varepsilon$ in $B(z_j,r)$, so we can let the entire ball be in V.

Next, we pick out certain subsets of the balls $B(z_j,r)$ with $\beta_j \neq 0$. To do this, introduce the notations $\alpha = |f(z_j)|$, $s = |e^{-n(1-<z,z_j>)}| = e^{-n\delta^2(z,z_j)}$ and $\theta = \arg(e^{-n(1-<z,z_j>)}) = n\,\mathrm{Im}<z,z_j>: = ny$. If $\pi(z)$ is the projection of z on the real tangent space of ∂B at z_j, y is the component of $\pi(z)$ orthogonal to the complex tangent space. Since

$$|f(z)+g(z)| \geq |f(z_j)+\beta_j e^{-n(1-<z,z_j>)}| - |f(z)-f(z_j)| - \Big|\sum_{k\neq j} \beta_k e^{-n(1-<z,z_j>)}\Big| \geq |f(z_j)+\beta_j e^{-n(1-<z,z_j>)}| - 2\varepsilon =$$

$$|\alpha+(1-\alpha)se^{i\theta}| - 2\varepsilon$$

we get that $|f(z)+g(z)|>a-2\varepsilon$ if $|\alpha+(1-\alpha)se^{i\theta}|>a$, hence if

$$\alpha^2 + 2\alpha(1-\alpha)s\cos\theta + (1-\alpha)^2 s^2 \geq a^2$$

This holds if $s>a$ and $\cos\theta>a$. $s>a$ holds if

$$n\delta^2(z,z_j) \leq \log\frac{1}{a}$$

hence in a ball with radius ρ, such that

$$(9) \qquad n\rho^2 = \log\frac{1}{a}$$

(6) and (9) show that we may assume $\rho<r$. The condition $\cos ny>a$. means that we have to pick out certain strips in the ball $B(z_j,\rho)$ An easy geometric argument shows that these strips will have a total area which is at least $\frac{\text{arc cos } a}{2\pi}A(\rho)$. The set V obtained satisfies $\mu(\bar{V})=\mu(V)$ and $\bar{V}\cap\bar{U}=\emptyset$.

We now get by (1),(4),(6) and (9)

$$\mu(V) \geq \frac{\text{arc cos } a}{2\pi}A(\rho) \cdot N_r \geq \frac{C_2}{2\pi}(\text{arc cos } a)\rho^{2p-1}N_r \geq$$

$$\frac{C_1C_3}{2\pi}(\text{arc cos } a)(\frac{\rho}{r})^{2p-1}(1-\mu(U)) =$$

$$\frac{C_1C_3}{2\pi}(\text{arc cos } a)\left[\frac{\log 1/a}{\log A/\varepsilon}\right]^{\frac{2p-1}{2}}(1-\mu(U))$$

This proves (c) and (d). Finally, if $|z|\leq R$, then $\mathrm{Re}(1-\langle z,z_j\rangle)\geq 1-R$. Hence, since $nr^2\geq 1$, (4) gives

$$|g(z)| \leq N_r e^{-n(1-R)} \leq C_4\frac{1}{r^{2p-1}}e^{-n(1-R)} \leq C_4 n^{\frac{2p-1}{2}} e^{-n(1-R)}$$

Choosing n large enough proves (b).

<u>Remark</u>: Lemma 1 holds with $U=\emptyset$, in which case the condition $|f(z)|>a$ on U is empty and $V_r=\partial B$ for all r. It is also clear from the construction that $\|g\|_U<\varepsilon$, a property we shall not need.

<u>Lemma 2</u>: Let f be a continuous function on ∂B with $\|f\|_\infty\leq 1$ and let $\varepsilon>0$, $R<1$. Then there is an entire function h and an open set $U\subset\partial B$ such that

(1) $\|f+h\|_\infty \leq 1 + \varepsilon$

(2) $\|h\|_{B_R} \leq \varepsilon$

(3) $|f(z)+h(z)| > 1 - \varepsilon$ for $z \in U$

(4) $\mu(U) > 1 - \varepsilon$

Proof: Let $a=1 - \frac{1}{2}\varepsilon$ and choose ε_i such that $4\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$. Apply lemma 1 to the data $a, \varepsilon_1, R, f, U=\emptyset$ to produce an entire function h_1 and an open set U_1 such that

(a) $\|f+h_1\|_\infty \leq 1 + 2\varepsilon_1$

(b) $\|h_1\|_{B_R} \leq \varepsilon_1$

(c) $|f(z)+h_1(z)| > a - 2\varepsilon_1$ on U_1

(d) $\sigma_1 = \mu(U_1) \geq C \text{ arc cos } a \left[\frac{\log 1/a}{\log A/\varepsilon_1}\right]^{\frac{2p-1}{2}}$

Suppose entire functions $h_1,\ldots,h_n$ have been chosen, together with open sets $U_1,\ldots,U_n$ such that if $W_i = \bigcup_{k=1}^{i} U_k$, then $\overline{U_{i+1}} \cap \overline{W_i} = \emptyset$ and $\mu(U_i)=\mu(\bar{U}_i)=\sigma_i$. The function h_{n+1} and the open set U_{n+1} is then obtained by applying lemma 1 to the data $a-2\sum_{k=1}^{n} \varepsilon_k, \varepsilon_{n+1}, R, f+(h_1+\ldots+h_n), W_n$. This produces a sequence $\{h_k\}$ of entire functions and a sequence $\{U_k\}$ of disjoint open sets such that

(a) $\|f+\sum_{k=1}^{n} h_k\| \leq 1 + 2\sum_{k=1}^{n} \varepsilon_k < 1 + \varepsilon$

(b) $\|\sum_{k=1}^{n} h_k\|_{B_R} \leq \sum_{k=1}^{n} \|h_k\|_{B_R} \leq \sum_{k=1}^{n} \varepsilon_k < \varepsilon$

(c) $|f(z)+\sum_{k=1}^{n} h_k(z)| > a -2\sum_{k=1}^{n} \varepsilon_k > a - \frac{1}{2}\varepsilon = 1 - \varepsilon$ on W_n

(d) $$\sigma_n = \mu(U_n) \geq C \arccos\left(a-2\sum_{k=1}^{n-1}\varepsilon_k\right)\left[\frac{\log 1/a - 2\sum_{k=1}^{n-1}\varepsilon_k}{\log A/\varepsilon_n}\right]^{\frac{2p-1}{2}}\left(1-\sum_{k=1}^{n-1}\sigma_k\right)$$

$$\geq C \arccos a\left[\frac{\log 1/a}{\log A/\varepsilon_n}\right]^{\frac{2p-1}{2}}\left(1-\sum_{k=1}^{n-1}\sigma_k\right)$$

If $\sum_{k=1}^{\infty}\sigma_k<1$, (d) shows that there is a constant C_6 such that

$$\sigma_n \geq C_6 \frac{1}{[\log A/\varepsilon_n]^{\frac{2p-1}{2}}}$$

This is clearly impossible if $\sum_{n=1}^{\infty}\frac{1}{[\log A/\varepsilon_n]^{\frac{2p-1}{2}}} = +\infty$, which can be achieved by

$$\varepsilon_n = A\tau^{n^{\left(\frac{2}{2p-1}\right)}}$$

for some small τ. Hence we may assume that $\sum_{k=1}^{\infty}\sigma_k=1$, so for n sufficiently large, $U=W_n$, we get

$$\mu(U) = \sum_{k=1}^{n}\sigma_k > 1 - \varepsilon$$

which is (4) in the lemma. Letting $h=\sum_{k=1}^{n}h_k$, (1),(2) and (3) are just (a),(b) and (c)

<u>Remark</u>: We shall apply lemma 2 repeatedly with the hypothesis $\|f\|_\infty<a$ for some a, in which case the conclusions of the lemma hold with 1 replaced by a in (1) and (3). We shall refer to f,a,ε,R as data for the lemma.

<u>Theorem</u>: There exist inner functions in B

Proof: Let $\{a_i\},\{\varepsilon_i\}$ be sequences such that a_i increases strictly to 1, $a_i+\varepsilon_i<a_{i+1}$ and $\sum_{i=1}^{\infty}\varepsilon_i<\frac{1}{2}$. Apply lemma 2 to the data $f_0=0$, $a_1,\varepsilon_1,R_1=\frac{1}{2}$ to get an entire function f_1 and an open set $U_1 \subset \partial B$ such that

(1) $\qquad \|f_1\|_\infty \leq a_1 + \varepsilon_1 \leq a_2$

(2) $\qquad \|f_1\|_{B_{R_1}} \leq \varepsilon_1$

(3) $\qquad |f_1(z)| > a_1 - \varepsilon_1$ for $z \in U_1$

(4) $\qquad \mu(U_1) > 1 - \varepsilon_1$

Since f is continuous, there exists an R_2, such that $R_1 < R_2 < 1$ and such that

(5) $\qquad |f_1(R_1 z)| > a_1 - 2\varepsilon_1$ for $z \in U_1$

Suppose that we have inductively found entire functions $f_1, \ldots, f_n$, open sets $U_1, \ldots U_n$ and real numbers $R_1, \ldots, R_{n+1}$, such that, if we let $h_n = \sum_{i=1}^{n} f_i$, then

(1) $\qquad \|h_n\|_\infty \leq a_{n+1}$

(2) $\qquad \|f_i\|_{B_{R_i}} \leq \varepsilon_i$, $R_i < R_{i+1} < 1$ for $i=1,\ldots,n$

(3) $\qquad |h_n(z)| > a_n - \varepsilon_n$ for $z \in U_n$

(4) $\qquad \mu(U_n) > 1 - \varepsilon_n$

(5) $\qquad |h_n(R_{n+1} z)| > a_n - 2\varepsilon_n$ for $z \in U_n$

We then apply lemma 2 to the data $h_n, a_{n+1}, \varepsilon_{n+1}$ and R_{n+1} to produce a new function f_{n+1} and an open set U_{n+1}. Properties (1) to (4) follow immediately for h_{n+1}, and (5) is just a consequence of its continuity. We assume $\lim R_n = 1$. By (2),

$$h = \lim h_n = \sum_{i=1}^{\infty} f_i$$

exists and satisfies $\|h\|_{B_{\frac{1}{2}}} < \frac{1}{2}$. By (1), $\|h\|_\infty \leq 1$. Let

$$V_j = \bigcap_{n \geq j} U_n$$

Then $V_j \subset V_{j+1}$ and by (4) $\mu(V_j) \geq 1 - \sum_{n \geq j} \varepsilon_n$, hence $\lim \mu(V_j) = 1$ and $U := \bigcup_{j=1}^{\infty} V_j$ has full measure. If $z \in U$, then there exists j such that $z \in U_n$ for all $n \geq j$. For such n (2) and (5) imply

$$|h(R_{n+1}z)| > a_n - 2\varepsilon_n - \sum_{k>n} \varepsilon_k \to 1 \quad \text{when} \quad n \to \infty$$

Hence, if $\lim_{t \to 1} h(tz)$ exists, which it does almost everywhere in U, its absolute value must be 1. This concludes the proof of the theorem.

Remark: We can, without any additional effort, prove a more general version of this theorem. To do this, replace the number 1 by a strictly positive, continuous function H on ∂B. Hence, in lemma 1, we now assume that $|f(z)| > aH(z)$ on U. We can carry out exactly the same construction, assuming that r is small enough to guarantee that $|H(z) - H(z')| < \varepsilon$ when $\delta(z, z') < r$. This will just add one ε to our inequalities. β_j is now defined by $\beta_j = 0$ if $|f(z_j)| \geq H(z_j)$ and $|f(z_j) + \beta_j| = |f(z_j)| + |\beta_j| = H(z_j)$ otherwise. This time $|f(z) + g(z)| > aH(z) - 3\varepsilon$ in a ball with $\beta_j \neq 0$ if $|\alpha + (H(z_j) - \alpha)se^{i\theta}| > aH(z_j)$, which is also satisfied if $s \geq a$ and $\cos\theta \geq a$. Hence, the conclusion holds with (b) and (d) unchanged and

(a) $\quad |f(z) + g(z)| \leq \max\{|f(z)|, H(z)\} + 3\varepsilon$

(c) $\quad |f(z) + g(z)| > aH(z) - 3\varepsilon$ on $U \cup V$

From this lemma 2 can be immediately generalized. The assumption is now that $|f(z)| \leq H(z)$ and the conclusion holds with (2) and (4) unchanged and

(1) $\quad |f(z) + h(z)| \leq H(z) + \varepsilon$

(3) $\quad |f(z) + h(z)| > H(z) - \varepsilon$ for $z \in U$

Finally, the theorem also generalizes immediately. The sequences $\{a_i\}$ and $\{\varepsilon_i\}$ must now be chosen such that $a_i H(z)+\varepsilon_i < a_{i+1} H(z)$ for all z and R_{n+1} must be chosen such that

$$a_n H(z) + 2\varepsilon_n > h_n(R_{n+1} z) > a_n H(z) - 2\varepsilon_n \quad \text{for} \quad z \in U_n$$

which is clearly possible by uniform continuity. This proves:

<u>Theorem</u>: Let H be a strictly positive, continuous function on ∂B. Then there exists $f \in H^\infty(B)$ such that $|\lim_{t\to 1} f(tz)| = H(z)$ almost everywhere on ∂B.

REFERENCES

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