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Symmetric states of infinite tensor products of C*-algebras

by

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1. Introduction.

Infinite tensor products of C^*-algebras, and ever more specially of the complex 2×2 matrices, have been of great importance in operator theory. For example, the perhaps most fruitful technique for constructing different types of factors, has been to take weak closures of infinite tensor products in different representations. In addition, some of the C^*-algebras of main interest, those of the commutation and the anti-commutation relations, are closely related to infinite tensor products of C^*-algebras. Adding to these applications the possibility of applying the theory, when all factors in the infinite tensor product are abelian C^*-algebras, to measure theory on product spaces, we see that the theory of infinite tensor products of C^*-algebras may have great potential importance.

In the present Seminar report we shall study the infinite tensor product $\mathfrak{A}$ of a C^*-algebra $\mathfrak{B}$ with itself, viz. $\mathfrak{A} = \overset{*}{\otimes} \mathfrak{B}_i$, where $\mathfrak{B}_i = \mathfrak{B}$, $i = 1,2,\ldots$, and then show how information on the C^*-algebra $\mathfrak{A}$ leads to both new and known results on the different subjects mentioned in the preceding paragraph. Our main technique will be that of asymptotically abelian C^*-algebras as developed in [13] and [14], where the associated group of automorphisms is the one of finite permutations of the factors of $\mathfrak{A}$.

We recall some relevant terminology in operator theory. A <u>von Neumann</u> algebra is a weakly closed complex algebra $\mathfrak{A}$ of operators on a Hilbert space $\mathcal{H}$ such that $A \in \mathfrak{A}$ implies $A^* \in \mathfrak{A}$. Its <u>commutant</u> $\mathfrak{A}'$ is the von Neumann algebra consisting of all operators A' on $\mathcal{H}$ such that $A'A = AA'$ for all $A \in \mathfrak{A}$. A C^*-algebra $\mathfrak{A}$ is a von Neumann algebra if and only if $\mathfrak{A} = \mathfrak{A}''$. A von Neumann algebra $\mathfrak{A}$ is a <u>factor</u> if $\mathfrak{A} \cap \mathfrak{A}' = \mathbb{C}I$, where I is the identity operator on $\mathcal{H}$. By a <u>trace</u> on $\mathfrak{A}$ we mean an additive map tr of the positive operators $\mathfrak{A}^+$ in $\mathfrak{A}$ into $\mathbb{R}^+ \cup \{+\infty\}$ such that $\mathrm{tr}(A) = \mathrm{tr}(UAU^{-1})$ whenever $A \in \mathfrak{A}^+$ and U is a unitary operator in $\mathfrak{A}$. Let $\mathcal{P}$ denote the set of projections in $\mathfrak{A}$. If $\mathfrak{A}$ is a factor there exists up to a scalar multiple a unique normal trace of $\mathfrak{A}$, where normal roughly means that it is order-continuous (cf. Daniell integrals). If the trace is sufficiently normalized we say $\mathfrak{A}$ is of <u>type</u> I_n, $1 \leq n \leq \infty$ if $\mathrm{tr}(\mathcal{P}) = \{0,1,2,\ldots,n\}$, of <u>type</u> II_1 if $\mathrm{tr}(\mathcal{P}) = [0,1]$, of <u>type</u> II_∞ if $\mathrm{tr}(\mathcal{P}) = [0,\infty]$, of <u>type</u> III if $\mathrm{tr}(\mathcal{P}) = \{0,+\infty\}$.

If $\mathfrak{A}$ is a C^*-algebra with identity I, then a

<u>state</u> of $\mathfrak{A}$ is a positive linear functional ρ of $\mathfrak{A}$ such that $\rho(I) = 1$. Then there is a Hilbert space $\mathcal{H}_\rho$, a $*$representation π_ρ of $\mathfrak{A}$ on $\mathcal{H}_\rho$, and a unit vector x_ρ in $\mathcal{H}_\rho$ cyclic under $\pi_\rho(\mathfrak{A})$ (so that vectors of the form $\pi_\rho(A)x_\rho$ are dense in $\mathcal{H}_\rho$) such that $\rho = \omega_{x_\rho} \circ \pi_\rho$, where ω_{x_ρ} denotes the state $A \to (Ax_\rho, x_\rho)$. ρ is a <u>factor state</u> if $\pi_\rho(\mathfrak{A})''$ is a factor, of <u>type</u> X if $\pi_\rho(\mathfrak{A})''$ is a factor of type X, $X = I_n$, $1 \leq n \leq \infty$, II_1, II_∞, III.

We recall from [7] the definition of $\overset{*}{\otimes}\mathcal{B}_i$. Let $\mathcal{B}_i$, $i = 1,2,\ldots$, be a C^*-algebra with identity $I = I_i$. For each integer n let $\underset{n}{\otimes}\mathcal{B}_i$ denote the algebraic tensor product of $\mathcal{B}_1,\ldots,\mathcal{B}_n$. For each $\mathcal{B}_i$ let π_i be a faithful $*$-representation of $\mathcal{B}_i$ on a Hilbert space $\mathcal{H}_i$. Then their tensor product $\underset{n}{\otimes}\pi_i$ is a representation of $\underset{n}{\otimes}\mathcal{B}_i$ on the Hilbert space tensor product $\underset{n}{\overset{h}{\otimes}}\mathcal{H}_i$ of $\mathcal{H}_1,\ldots,\mathcal{H}_n$. In this way a norm $\| \ \|_*$ is defined on $\underset{n}{\otimes}\mathcal{B}_i$. Let $\underset{n}{\overset{*}{\otimes}}\mathcal{B}_i$ denote the C^*-algebra which is the completion of $\underset{n}{\otimes}\mathcal{B}_i$ in this norm. If $m \geq n$ denote by φ_{nm} the canonical imbedding

$$\varphi_{nm} : \underset{n}{\overset{*}{\otimes}}\mathcal{B}_i \to \underset{m}{\overset{*}{\otimes}}\mathcal{B}_i$$

which carries $\underset{n}{\otimes} A_i$ onto $(\underset{n}{\otimes} A_i) \otimes (\underset{i=n+1}{\overset{m}{\otimes}} I_i)$. Then $\overset{*}{\otimes}\mathcal{B}_i$ is defined as the inductive limit of this inductive system of C^*-algebras. If $\mathcal{B}_i = \mathcal{B}$ for each i, $\mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$, and ρ is a state of $\mathcal{B}$, we denote by $\overset{*}{\otimes}\rho$ (or just $\otimes\rho$) the state $\overset{*}{\otimes}\rho_i$ of $\mathfrak{A}$, where $\rho_i = \rho$

for each i. This state is the unique state of $\mathfrak{A}$ such that if $A_i \in \mathcal{B}_i$ and $A_i = I_i$ for all but a finite number of indices, then

$$(\overset{*}{\otimes}\rho)(\otimes A_i) = \prod_i \rho(A_i).$$

If $\mathfrak{A}$ is a C^*-algebra with identity and G a group, we say $\mathfrak{A}$ is asymptotically abelian with respect to G if there is a representation $g \to \tau_g$ of G as $*$-automorphisms of $\mathfrak{A}$ and a sequence g_n in G such that if $A, B \in \mathfrak{A}$ then

$$\lim_{n\to\infty} \| [\tau_{g_n}(A), B] \| = 0,$$

where $[\ ,\ ]$ denotes the Lie commutator. We refer the reader to [13] and [14] for references to other definitions of asymptotically abelian C^*-algebras. A state ρ of $\mathfrak{A}$ is G-invariant if $\rho \circ \tau_g = \rho$ for all $g \in G$. Then there is a unitary representation $g \to U_\rho(g)$ of G on $\mathcal{H}_\rho$ such that

$$U_\rho(g)\pi_\rho(A)U_\rho(g)^{-1} = \pi_\rho(\tau_g A)$$

for all $A \in \mathfrak{A}$, and $U_\rho(g)x_\rho = x_\rho$ for all $g \in G$. Denote by $I(\mathfrak{A})$ the G-invariant states of $\mathfrak{A}$. Denote by $\mathcal{C}(\rho)$ the center of the von Neumann algebra $\pi_\rho(\mathfrak{A})''$, and by $\mathcal{B}(\rho)$ the von Neumann subalgebra of $\mathcal{C}(\rho)$ of operators $A \in \mathcal{C}(\rho)$ such that $U_\rho(g)AU_\rho(g)^{-1} = A$ for all $g \in G$.

Denote by $\mathcal{U}(\rho)$ the group of all $U_\rho(g)$, $g \in G$. By [13] there exists a unique normal G-invariant positive linear map Φ_ρ of $\pi_\rho(\mathfrak{A})''$ onto $\mathcal{B}(\rho)$, such that $\Phi_\rho | \mathcal{B}(\rho)$ is the identity map. Furthermore $I(\mathfrak{A})$ is a simplex. If ρ is G-invariant ρ is said to be <u>strongly clustering</u> if

$$\lim_{n \to \infty} \rho(\tau g_n(A)B) = \rho(A)\rho(B)$$

whenever $A, B \in \mathfrak{A}$. A G-invariant state ρ is then strongly clustering if and only if ρ is an extreme point of $I(\mathfrak{A})$ and $\omega_{x_\rho}(\pi_\rho(A))I = \Phi_\rho(\pi_\rho(A)) = \text{weak} \lim_{n \to \infty} U_\rho(g_n)\,\pi_\rho(A)U_\rho(g_n)^{-1}$ for all self-adjoint $A \in \mathfrak{A}$ [13, Thm.5.4]. If ρ is a G-invariant factor state then ρ is strongly clustering [13, Cor.5.5]. Furthermore [14, Thm.3.1], ρ is of type III if and only if $\omega_{x_\rho} | \pi_\rho(\mathfrak{A})'$ is not a trace, ρ is of type II_1, or I_n, $n < \infty$, if and only if ρ is a trace, and ρ is of type I_∞ or II_∞ if and only if $\omega_{x_\rho} | \pi_\rho(\mathfrak{A})'$ is a trace, but ρ is not a trace.

We shall not include complete proofs in this Seminar report, as they will appear elsewhere. Only rough indications will usually be given. For the theory of C^*-algebras and von Neumann algebras we refer the reader to the two books of Dixmier [3] and [4]. For the theory of infinite tensor products of C^*-algebras the reader is refered to the paper of Guichardet [7].

2. Symmetric states. Let $\mathcal{B}$ be a C^*-algebra with identity. Let $\mathcal{B}_i = \mathcal{B}$, $i = 1,2,\ldots,$ and let $\mathcal{O}\!\mathcal{l} = \overset{*}{\otimes}\mathcal{B}_i$. Let G denote the group of finite permutations of the positive integers N, i.e. an element $g \in G$ is a one-to-one map of N onto itself which leaves all but a finite number of integers fixed. Then g defines a $*$-automorphism, also denoted by g, of $\mathcal{O}\!\mathcal{l}$ by

$$g(\sum \otimes A_i) = \sum \otimes A_{g(i)},$$

where $A_i = I$ for all but a finite number of indices. Following the terminology of Hewitt and Savage [9] we say a state ρ of $\mathcal{O}\!\mathcal{l}$ is symmetric if ρ is G-invariant, i.e. if $\rho \circ g = \rho$ for all $g \in G$. For each integer n we denote by g_n the permutation

$$1) \qquad g_n(k) = \begin{cases} 2^{n-1}+k & \text{if } 1 \leq k \leq 2^{n-1} \\ k \quad -2^{n-1} & \text{if } 2^{n-1} < k \leq 2^n \\ k & \text{if } 2^n < k \end{cases}$$

Lemma 2.1. With the notation introduced above let $A, B \in \mathcal{O}\!\mathcal{l}$. Then

$$\lim_n \| [g_n(A), B] \| = 0,$$

hence $\mathcal{O}\!\mathcal{l}$ is asymptotically abelian with respect to G.

Proof. In order to show the techniques we give a complete proof. We may assume $\|A\| \leq 1$, $\|B\| \leq 1$. Let $\varepsilon > 0$ be given. Then we can choose a finite integer m and operators $A', B' \in \mathfrak{A}$ such that $A' = \sum_j \otimes_i A_{ij}$, $B' = \sum_k \otimes_l B_{kl}$, where $A_{ij} = I$ for $i,j \geq m$, and $B_{kl} = I$ for $k,l \geq m$, and such that $\|A-A'\| < \varepsilon/4$, $\|B-B'\| < \varepsilon/4$. Choose n so that $2^{n-1} \geq m$. Then $[g_n(A'), B'] = 0$, hence

$$\begin{aligned}
\|[g_n(A),B]\| &= \|(g_n(A)-g_n(A'))B + g_n(A')(B-B') + (B'-B)g_n(A') \\
&\quad + B(g_n(A')-g_n(A))\| \\
&\leq \|g_n(A)-g_n(A')\| \|B\| + \|g_n(A')\| \|B-B'\| \\
&\quad + \|B'-B\| \|g_n(A')\| + \|B\| \|g_n(A')-g_n(A)\| \\
&\leq \varepsilon ,
\end{aligned}$$

since g_n is an isometry.

This lemma makes the results in [13] and [14] applicable to $\mathfrak{A}$ and G. It is immediate that if ρ is a state of $\mathfrak{B}$ then the product state $\overset{*}{\otimes}\rho$ on $\mathfrak{A}$ is symmetric. Our first result describes the type of $\overset{*}{\otimes}\rho$ in terms of ρ if ρ is a factor state.

Theorem 2.2. Let $\mathfrak{B}_i = \mathfrak{B}$ be a C^*-algebra with identity. Let $\mathfrak{A} = \overset{*}{\otimes}\mathfrak{B}_i$, and let ρ be a factor state of $\mathfrak{B}$. Then $\overset{*}{\otimes}\rho$ is a factor state of $\mathfrak{A}$. Moreover

1) $\overset{*}{\otimes}\rho$ is of type I_1 if and only if ρ is a homomorphism.
2) $\overset{*}{\otimes}\rho$ is of type I_∞ if and only if ρ is pure and not a homomorphism.

3) $\overset{*}{\otimes}\rho$ is of type II_1 if and only if ρ is a trace and not a homomorphism.

4) $\overset{*}{\otimes}\rho$ if of type II_∞ if and only if $\omega_{x_\rho} \mid \pi_\rho(\mathcal{B})'$ is a trace, and ρ is neither pure nor a trace.

5) $\overset{*}{\otimes}\rho$ is of type III if and only if $\omega_{x_\rho} \mid \pi_\rho(\mathcal{B})'$ is not a trace.

In the proof one has to make use of the theory of infinite tensor products in order to show that if $\omega_{x_\rho} \mid \pi_\rho(\mathcal{B})'$ is a trace, then $\otimes\rho$ is either of type I or II. Having this the rest of the proof is an easy consequence of [14, Thm.3.1], which is referred to in the introduction.

It should be noted that $\otimes\rho$ is never of type I_p, $1 < p < \infty$.

An important special case occurs when $\mathcal{B}_i = M_n$ - the complex $n \times n$ matrices. If ρ is a state of M_n then there exist n orthogonal unit vectors $y_1, \dots, y_n$ in $\mathbb{C}^n$ and real numbers λ_j, $0 \leq \lambda_j \leq 1$, $j = 1, \dots, n$, such that $\sum \lambda_j = 1$, and $\rho = \sum \lambda_j \omega_{y_j}$. Moreover, we can choose the numbering so that $0 \leq \lambda_n \leq \dots \leq \lambda_1 \leq 1$. Even though the y_j's are not necessarily unique, the numbers λ_j are. They are called the <u>eigenvalues</u> for ρ. Then Theorem 2.2 has the following immediate

<u>Corollary 2.3</u>. Let $\mathcal{B}_i = M_n$, $i = 1,2,\dots, n \geq 2$. Let $\mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$. Let ρ be a state of M_n with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. Then $\overset{*}{\otimes}\rho$ is a factor state of $\mathfrak{A}$, and

1) $\overset{*}{\otimes}\rho$ is of type I_∞ if and only if $\lambda_1 = 1$.
2) $\overset{*}{\otimes}\rho$ is of type II_∞ if and only if $\lambda_1 = \ldots = \lambda_k = \frac{1}{k}$, $1 < k < n$.
3) $\overset{*}{\otimes}\rho$ is of type II_1 if and only if $\lambda_1 = \ldots = \lambda_n = \frac{1}{n}$.
4) $\overset{*}{\otimes}\rho$ is of type III if and only if for some j, $0 < \lambda_{j+1} < \lambda_j < 1$.

With $n = 2$ this result has been shown by Glimm [6, pp.587-589] by quite different methods. We remark that all type I_∞ (resp. II_∞, II_1) factors obtained in this way are isomorphic. However, when $n = 2$, Powers [11, Thm.4.8] has shown that the type III factors obtained are all mutually non isomorphic.

We now investigate the simplex $I(\mathfrak{A})$ of symmetric states, and use the notation from the introduction. Let ρ be a symmetric state. If $A \in \mathfrak{A}$ then by a simple argument, see [13, Lem.5.3], every weak limit point D of the sequence

$$\{U_\rho(g_n)\pi_\rho(A)U_\rho(g_n)^{-1}\}$$

belongs to the center $\mathcal{C}(\rho)$ of $\pi_\rho(\mathfrak{A})''$. Furthermore, if D happens to be in $\mathcal{B}(\rho)$ then $D = \Phi_\rho(\pi_\rho(A))$. With g_n as in 1) this happens, hence

$$\Phi_\rho(\pi_\rho(A)) = \text{weak limit } U_\rho(g_n)\pi_\rho(A)U_\rho(g_n)^{-1},$$

for all $A \in \mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$. Using this and [13, Thm.5.4] mentioned in the introduction, we see that every extremal symmetric state is strongly clustering. With the aid of this and similar techniques we have,

<u>Theorem 2.4</u>. Let $\mathcal{B}_i = \mathcal{B}$ be a C^*-algebra with identity, $i = 1,2,\ldots$. Let $\mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$, and let ρ be a symmetric state. Then the following three conditions are equivalent.

1) ρ is a product state $\overset{*}{\otimes}\omega$ with ω a state of $\mathcal{B}$.
2) ρ is an extremal symmetric state.
3) ρ is strongly clustering.

It is straightforward to show that the map $\rho \to \overset{*}{\otimes}\rho$ of the state space of $\mathcal{B}$ into $I(\mathfrak{A})$ is a homeomorphisme into. Hence an application of Theorem 2.4 gives

<u>Theorem 2.5</u>. Let $\mathcal{B}_i = \mathcal{B}$ be a C^*-algebra with identity, $i = 1,2,\ldots$. Let $\mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$. Let $\partial I(\mathfrak{A})$ denote the extreme boundary of the simplex $I(\mathfrak{A})$ of symmetric states of $\mathfrak{A}$. Then the map $\rho \to \overset{*}{\otimes}\rho$ is a homeomorphism of the state space of $\mathcal{B}$ onto $\partial I(\mathfrak{A})$. Hence $\partial I(\mathfrak{A})$ is a closed set.

A <u>face</u> of the simplex $I(\mathfrak{A})$ is a convex subset F such that if $\omega \in F$, $\omega' \in I(\mathfrak{A})$, and for some real $\lambda > 0$, $\omega' \leq \lambda\omega$, then $\omega' \in F$.

Lemma 2.6. With the notation in Theorem 2.5, if X denotes any one of I, II_1, II , III, let $I(\mathfrak{A})_X = \{\rho \in I(\mathfrak{A}) : \pi_\rho(\mathfrak{A})''$ is a von Neumann algebra of type $X\}$. Then $I(\mathfrak{A})_X$ is a face of $I(\mathfrak{A})$.

The proof follows from [13, Thm.3.1], from which it follows that if $\rho' \leq \lambda\rho$, and $\rho', \rho \in I(\mathfrak{A})$ then there is a unique positive operator $B \in \mathfrak{B}(\rho)$ such that $\rho' = \omega_{Bx_\rho} \circ \pi_\rho$. Since $\partial I(\mathfrak{A})$ is closed, by a theorem of Alfsen [1, Thm.1], the closure $\overline{I(\mathfrak{A})_X}$ of $I(\mathfrak{A})_X$ is a face of $I(\mathfrak{A})$. Now all weakly continuous state of a factor are factor states. Since they are also w^*-dense in the state space, the following result now follows from Theorem 2.2.

Theorem 2.7. Let $\mathfrak{B}_i = \mathfrak{B}$ be a factor different from the scalars. Let $\mathfrak{A} = \overset{*}{\otimes}\mathfrak{B}_i$. Then $I(\mathfrak{A})_{III}$ is dense in $I(\mathfrak{A})$.

We remark that if $\mathfrak{B}$ is a factor of type II or III then the pure states are dense in the state space of $\mathfrak{B}$ [5, Thm.3]. Hence also $I(\mathfrak{A})_I$ is in this case dense $I(\mathfrak{A})$.

<u>3. Symmetric measures on Cartesian products</u>. In the present section we show how the theory in section 2 applies to measure theory. Let X be a compact Hausdorff space, $X_i = X$, $i = 1,2,\ldots$, and $\tilde{X} = \prod_{i=1}^{\infty} X_i$ with the product topology. Let P denote the set of probability measures on X (i.e. positive regular Borel measures of total mass 1), and let $\tilde{P}$ be the set of corresponding product measures on $\tilde{X}$ (see e.g. [8, p.157]). Let G denote the group of finite permutations of the positive integers, and as before identify G with the group of finite permutations of $\tilde{X}$. A probability measure is <u>symmetric</u> if it is G-invariant. Let $\tilde{S}$ denote the set of symmetric probability measures on $\tilde{X}$. We can now state and prove a theorem which together with the results on simplexes with closed extreme boundaries due to Bauer [2], is a restatement of a theorem of Hewitt and Savage [9 , Thms.5.3, 7.2, and 9.4].

<u>Theorem 3.1</u>. In the notation introduced above $\tilde{S}$ is a simplex with closed extreme boundary equal to $\tilde{P}$.

<u>Proof</u>. Let $\mathfrak{A} = \overset{*}{\otimes} C(X_i)$. Then $\mathfrak{A}$ is $*$-isomorphic to $C(\tilde{X})$ [15, Thm.6], the isomorphism being implemented by the homeomorphism $\otimes f_{x_i} \rightarrow (x_i)$ of the pure states of $\mathfrak{A}$ onto $\tilde{X}$, where f_{x_i} denotes the evaluation at the point x_i in X_i [7, p.19]. If Y is a compact Hausdorff space then the probability measures can be identified with the state space of $C(Y)$ [8, pp.247-248]. Thus the theorem is a direct corollary to Theorems 2.4 and 2.5.

4. <u>The anti-commutation relations</u>. We say the elements $b_1, b_2, \dots$ in an algebra with involution and identity I satisfy the anti-commutation relations if $b_i b_j + b_j b_i = 0$, $b_i b_j^* + b_j^* b_i = \delta_{ij} I$ for all i, j. Guichardet [7, Prop.3.3] has shown that the study of the anti-commutation relations is equivalent to that of $\overset{*}{\bigotimes} \mathcal{B}_i$, where $\mathcal{B}_i = M_2$. Hence the results in section 2 may be applied. Since Corollary 2.3 is well known for $n = 2$ we shall rather consentrate on applications of Theorems 2.4 and 2.5, and show how a theorem of Shale and Stinespring 12, Thm.4 can be recovered.

Let $\mathcal{H}$ be a separable real Hilbert space with orthonormal basis $(\omega_1, \omega_2, \dots, \omega_1', \omega_2', \dots)$. Let φ_j denote the canonical imbedding of $\mathcal{B}_j$ into $\overset{*}{\bigotimes} \mathcal{B}_i$. Let

$$e_j = \varphi_j\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \quad f_j = \varphi_j\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad g_j = \varphi_j\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right),$$

for $j = 1, 2, \dots$. By [7, Prop.3.7] we may identify $\mathcal{H}$ with a subspace of $\mathfrak{A} = \overset{*}{\bigotimes} \mathcal{B}_i$, $\mathcal{B}_i = M_2$, by writing

$$\omega_n = e_1 \cdots e_{n-1} \cdot f_n$$
$$\omega_n' = e_1 \cdots e_{n-1} \cdot g_n$$

Then ω_n and ω_n' are self-adjoint unitary operators in $\mathfrak{A}$ satisfying the spin relations

$$\omega_j\omega_k+\omega_k\omega_j = \omega_j'\omega_k'+\omega_k'\omega_j' = 2\delta_{jk}I$$

$$\omega_j\omega_k'+\omega_k'\omega_j = 0,$$

and the operators $\frac{1}{2}(\omega_j+i\omega_j')$ satisfy the anti-commutation relations. Let $\mathcal{K}$ denote the Hilbert space spanned by $\omega_1, \omega_2, \ldots$. If U is a unitary operator on $\mathcal{K}$, say $U\omega_n = \sum_j u_{nj}\omega_j$, then we can extend U to a unitary operator on the complex Hilbert space generated by $\mathcal{H}$ by $U\omega_n' = \sum_j u_{nj}\omega_j'$. U has then a unique extension to an automorphism α_U of $\mathfrak{A}$ satisfying

$$\alpha_U(\sum \prod \lambda_{n_j m_j}\omega_{n_j}\omega_{m_j}') = \sum \lambda_{n_j m_j} U\omega_{n_j} U\omega_{m_j}' ,$$

for polynomials in the ω_j and ω_j'. A state ρ of $\mathfrak{A}$ is said to be <u>universally invariant</u> if $\rho = \rho \circ \alpha_U$ for all unitary operators U on $\mathcal{K}$. We can now restate the quoted theorem of Shale and Stinespring. The proof is omitted because it follows from the way the theorem is stated by somewhat tedious computations together with obvious applications of Theorems 2.4 and 2.5.

<u>Theorem 4.1</u>. Let $\mathfrak{A} = \overset{*}{\otimes}\mathcal{B}_i$, $\mathcal{B}_i = M_2$ be the C^*-algebra of the anti-commutation relations. Let $\mathcal{D}_i$ denote the diagonal matrices in M_2, and let $\mathcal{D} = \overset{*}{\otimes}\mathcal{D}_i$ be considered as a subalgebra of $\mathfrak{A}$. Then the map $\rho \to \rho|\mathcal{D}$ is an affine isomorphism of the universally invariant

states of $\mathfrak{A}$ onto the symmetric states of $\mathfrak{B}$. Hence ρ is an extreme universally invariant state of $\mathfrak{A}$ if and only if $\rho = \overset{*}{\otimes}\rho_\lambda$, where $0 \leq \lambda \leq 1$, and

$$\rho_\lambda(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \lambda a+(1-\lambda)d.$$

In particular, the universally invariant states of $\mathfrak{A}$ are affinely isomorphic to the probability measures on the closed unit interval.

5. The commutation relations. We shall in this section indicate how the results in section 2 can be used to study the commutation relations, and shall do it for the simplest case with infinite degrees of freedom, namely the situation studied by Gårding and Wightman to obtain similar results, see [16]. For simplicity we use the same notation as used by Glimm [6]. Let $\mathcal{K}$ be a separable Hilbert space with orthonormal basis $e_1, e_2, \ldots$. Let $\mathcal{H}_0$ denote the real linear span of $e_1, e_2, \ldots$. Let $\mathcal{H}_0' = i\mathcal{H}_0$. We assume there are two linear maps p and q from $\mathcal{H}_0$ and $\mathcal{H}_0'$ respectively to respective commutative families of (unbounded) self-adjoint operators on a complex Hilbert space (the representation space) such that

2) $\quad e^{ip(x)}e^{iq(x')} = e^{iB(x,x')}e^{iq(x')}e^{ip(x)}$

for arbitrary $x = \sum a_i e_i \in \mathcal{H}_0, x' = i\sum b_j e_j \in \mathcal{H}_0'$, where $B(x,x') = \sum a_i b_i$.

A bounded linear operator T (on the representation space) is said to depend on submanifolds $\mathfrak{M}$ of $\mathcal{H}_0$ and $\mathfrak{M}'$ of $\mathcal{H}_0'$ in case T is in the von Neumann algebra generated by $e^{ip(x)}$ and $e^{iq(x')}$ as x and x' range over $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The von Neumann algebra so obtained is denoted by $\mathfrak{A}(\mathfrak{M},\mathfrak{M}')$. The C^*-algebra $\mathfrak{A}$ generated by all $\mathfrak{A}(\mathfrak{M},\mathfrak{M}')$ with $\mathfrak{M}$ and $\mathfrak{M}'$ finite dimensional is called the <u>representation algebra of field observables</u>, also the <u>Weyl algebra</u>. If $\mathfrak{M}_n$(resp. $\mathfrak{M}_n'$) denotes the linear span of $e_1,\dots,e_n$ in $\mathcal{H}_0$ (resp. $\mathcal{H}_0'$) it is clear from the definition of $\mathcal{H}_0$ that $\mathfrak{A}$ equals the C^*-algebra $\mathfrak{A}_0$ generated by the $\mathfrak{A}(\mathfrak{M}_n,\mathfrak{M}_n')$, $n = 1,2,\dots$. From a theorem of von Neumann [10] $\mathfrak{A}(\mathfrak{M}_1,\mathfrak{M}_1') \cong \mathcal{B}(\mathcal{H}_1)$ with $\mathcal{H}_1$ a separable Hilbert space, hence an application of 2) shows $\mathfrak{A}(\mathfrak{M}_n,\mathfrak{M}_n') = \bigotimes_{i=1}^{n} \mathcal{B}(\mathcal{H}_i) = \mathcal{B}(\bigotimes_{i=1}^{n} \mathcal{H}_i)$, with $\mathcal{H}_i$ separable, the tensor product being the von Neumann algebra tensor product. Thus $\mathfrak{A}_0 \neq \overset{*}{\bigotimes}\mathcal{B}(\mathcal{H}_i)$, but is close enough to it in order to make the techniques in section 2 applicable. It should be remarked that in the more general case Glimm studied, $\mathfrak{A} \neq \mathfrak{A}_0$, but he succeeded in extending results from $\mathfrak{A}_0$ to $\mathfrak{A}$ [6, Thm.8].

Let ω be a normal state of $\mathcal{B}(\mathcal{H}_1)$. Then by continuity ω defines a product state $\bigotimes_{i=1}^{n}\omega$ of $\mathfrak{A}(\mathfrak{M}_n,\mathfrak{M}_n')$. Hence it defines a unique product state, denoted by $\otimes\omega$, of $\mathfrak{A}_o$.

<u>Theorem 5.1</u>. With the notation introduced above $\otimes\omega$ is a factor state of $\mathfrak{A}_o$. Furthermore

1) $\otimes\omega$ is of type I_∞ if and only if ω is a vector state.

2) $\otimes\omega$ is of type II_∞ if and only if $\omega = \frac{1}{k}\sum_{j=1}^{k}\omega_{x_j}$, $1 < k \infty$, with x_j orthogonal unit vectors in $\mathcal{H}_1$.

3) $\otimes\omega$ is of type III if and only if ω is not of the form $\frac{1}{k}\sum_{j=1}^{k}\omega_{x_j}$, $k \geq 1$, x_j orthogonal unit vectors in $\mathcal{H}_1$.

The proof is a trivial modification of the similar one given by Glimm [6, p.608] together with an application of Corollary 2.3.

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