Matematisk Seminar
Universitetet i Oslo

Nr. 5
Mai 1966

Integration in partially ordered spaces with the Riesz interpolation property

by

Per Roar Andenæs

1. Introduction.

A partially ordered linear space E *) has the Riesz interpolation property if to every four elements $x_i, y_j \in E$, $i, j = 1, 2$, such that $x_i \leq y_j$, $i, j = 1, 2$, there exists $z \in E$ satisfying $x_i \leq z \leq y_j$, $i, j = 1, 2$. The investigation of spaces of this type goes back to F. Riesz ([10]). Recently they have been discussed by different authors (Fuchs [6], [7], Bauer [2], cf. also Lindenstrauss [8] and Namioka [9]. It should be noted that in a partially ordered linear space the interpolation property mentioned above is equivalent to the more familiar Riesz decomposition property ([8, lemma 6.2]).

We note that a vector lattice always has the Riesz interpolation property. The converse statement, however, is false. We give two examples of partially ordered linear spaces with the Riesz interpolation property that are not vector lattices: (a) The space of polynomials (with real coeffisients) on $[0,1]$, (b) the space of real-valued continuous functions f on $[0,4]$ satisfying $f(2) = f(1) + f(3)$. In these examples the order relation is the usual pointwise one. The verification of the Riesz interpolation property for example (a) can be found in [8, pp. 75-76]. Example (b) is due to Namioka ([9, p. 45]). Another interesting example, which we will study in some detail in part 4 of this paper, is the space of continuous affine functions on a Choquet simplex.

Let H be the (σ-) complete lattice of all extended real-valued functions on some set S. With pointwise addition and scalar multiplication $H_o = \{f \in H : |f(s)| < \infty \text{ for all } s \in S\}$ becomes a linear space. Let v be a positive linear functional on a Riesz space in H, i.e. a directed linear subspace E of H_o with the Riesz interpolation property. The question arises whether it is possible to extend v to a Lebesgue integral within H. In case E is a vector lattice in H it is well known that such an extension exists if and only if the Daniell axiom is satisfied:

(1.1) $x_n \in E$, $n = 1, 2, \dots$, $x_n \uparrow x \in E \Longrightarrow v(x_n) \uparrow v(x)$.

*) All linear spaces considered in this paper are over the reals.

It is the aim of this paper to establish necessary and sufficient conditions for an extension to be possible in the general case. We will, however, attack the problem in a slightly different form. In [1] Alfsen has developed a theory of integration based on order. In this more general setting we are led to consider the following problem: Given a valuation v (defined in (2.1)) on a subset E of a σ-continuous lattice H, E having the Riesz interpolation property, we want to extend v to a full integral within H. This extension theory is contained in part 2 of this paper. In part 3 we add further assumptions on E to obtain essential simplifications of the axioms introduced in part 2. Finally, in part 4, we show that for metrizable Choquet simplexes, and also for simplexes with closed extreme boundary, our theory can be used to give a rather natural and straightforward construction of representing boundary measures, at the same time proving uniqueness.

We would like to express here our thanks to E. Alfsen for valuable help and suggestions during the preparation of this paper.

2. Extension theory. For explanation of terms not defined here the reader is referred to [1]. In the sequel E is a fixed subset of a σ-continuous lattice H. We assume that E has the Riesz interpolation property and also that E is directed, i.e. to $x, y \in E$ there exist $z_1, z_2 \in E$ such that $z_1 \leq x \wedge y$, $z_2 \geq x \vee y$. (All lattice operations are in H.) We use the term valuation to denote a real-valued function v on E satisfying

$$(2.1) \quad v(x) + v(y) = \overline{v}(x \vee y) + \underline{v}(x \wedge y) \quad \text{for all } x, y \in E ,$$

where we have put

$$\overline{v}(x \vee y) = \inf \{ v(z) : x \vee y \leq z \in E \}$$

$$\underline{v}(x \wedge y) = \sup \{ v(z) : x \wedge y \geq z \in E \}$$

As we shall only consider the case when v is increasing, no confusion can arise from our use of the word valuation. If E is a sublattice of H, (2.1) does not generally coincide with the usual lattice valuation identity ([1, p. 421]); the two definitions are, however, identical if v is increasing.

An increasing valuation v is called <u>admissible</u> if it satisfies

$$(2.2)\quad \begin{aligned} \overline{v}(x \vee y) &= \sup \{ \inf_n v(z_n) : z_n \downarrow z \in H, \ z \leq x \vee y, \ z_n \in E, \ n = 1,2\ldots \} \\ \underline{v}(x \wedge y) &= \inf \{ \sup_n v(z_n) : z_n \uparrow z \in H, \ z \geq x \wedge y, \ z_n \in E, \ n = 1,2\ldots \} \end{aligned}$$

for all $x, y \in E$. (2.2) will hereafter be termed condition (B). It will turn out later that if an admissible increasing valuation v is extendable to a full integral within H, then $\hat{v}(x \vee y) = \overline{v}(x \vee y)$, $\hat{v}(x \wedge y) = \underline{v}(x \wedge y)$ for any such full integral $(H, \hat{E}, \hat{v})$. In [1, th. 5, p. 429] it is proved that an increasing valuation v on a sublattice E of H is extendable to a full integral within H if and only if the following condition (A) is satisfied:

$$(2.3)\ (A): \quad \begin{aligned} & x_n, y_n \in E, \ n = 1, 2, \ldots, \ x_n \uparrow x \in H, \ y_n \downarrow y \in H, \ x \geq y \\ & \Longrightarrow \quad \sup_n v(x_n) \geq \inf_n v(y_n). \end{aligned}$$

It is now easily seen that in the lattice case an increasing valuation v is extendable to a full integral within H only if it is admissible. In fact, in this case (B) is equivalent to (1.1) and its dual version (the (extended) Daniell axiom).

In the sequel v denotes a fixed admissible increasing valuation on E. v is supposed to satisfy the following condition (D), which will replace (A) in our extension procedure.

$$(2.4)\ (D): \left. \begin{aligned} & x_n, x_n' \in E, \ n = 1,2,\ldots, \ x_n \uparrow x \in H, \ x_n' \uparrow x' \in H \\ & y_n, y_n' \in E, \ n = 1,2,\ldots, \ y_n \downarrow y \in H, \ y_n' \downarrow y' \in H \end{aligned} \right\} \& \ x \wedge x' \geq y \vee y'$$

$$\Longrightarrow \quad \sup_n \underline{v}(x_n \wedge x_n') \geq \inf_n \overline{v}(y_n \vee y_n').$$

Obviously, (D) always implies (A), and if E is a sublattice of H, (D) and (A) coincide. We put

$$E^o = \{ x \in H : \text{ there exist } x_n \in E, \ n = 1,2,\ldots, \ x_n \uparrow x, \ \sup_n v(x_n) < \infty \},$$

and for each $x \in E^o$ we define

$$(2.5)\qquad v^o(x) = \sup \{ v(z) : x \geq z \in E \}.$$

We make the following observations:

(a) v^o is increasing

(b) $v^o(x) = \sup_n v(x_n)$ for any increasing sequence $\{x_n\}$, $x_n \in E$, $n = 1,2,\dots$ such that $x_n \uparrow x$.

(c) $-\infty < v^o(x) < \infty$ for every $x \in E^o$

(d) $v^o \mid E = v$.

(a) follows from the definition of v^o. If $x \in E^o$ and $x_n \uparrow x$, $x_n \in E$, $n = 1,2,\dots$, it follows from (D) ((A) is sufficient) that $\sup_n v(x_n) \geq v^o(x)$. The reverse inequality is trivial, and so we have proved (b). (c) follows from (b) and the definition of E^o. (d) is trivial.

<u>2.1 Lemma</u>: Let $x, y \in E^o$ and $\varepsilon > 0$ be given, and let $\{x_n\}$, $\{y_n\}$, $x_n, y_n \in E$, $n = 1,2,\dots$, be sequences such that $x_n \uparrow x$, $y_n \uparrow y$. Then there exists a sequence $\{z_n\}$, $z_n \in E$, $n = 1,2,\dots$ such that

(a) $z_n \geq x_n \vee y_n$, $n = 1,2,\dots$

(b) $\{z_n\}$ is increasing.

(c) $\sup_n v(z_n) \leq \sup_n \overline{v}(x_n \vee y_n) + \varepsilon$

(d) $z = \sup_n z_n \in E^o$.

<u>Proof</u>: It follows from the definition of $\overline{v}(x_1 \vee y_1)$ that we can select $z_1 \in E$, $z_1 \geq x_1 \vee y_1$ such that

$$v(z_1) \leq \overline{v}(x_1 \vee y_1) + \frac{\varepsilon}{2}.$$

Suppose now that $z_1, z_2, \dots, z_n$ have been chosen such that $z_1 \leq z_2 \leq \dots \leq z_n$ and $z_i \geq x_i \vee y_i$, $i = 1,2,\dots, n$ and such that

$$v(z_i) \leq \overline{v}(x_i \vee y_i) + \sum_{k=1}^{i} \frac{\varepsilon}{2^k}, \quad i = 1,2,\dots, n.$$

We must find $z_{n+1} \in E$ such that $z_{n+1} \geq z_n$, $z_{n+1} \geq x_{n+1} \vee y_{n+1}$ and

$$v(z_{n+1}) \leq \overline{v}(x_{n+1} \vee y_{n+1}) + \sum_{k=1}^{n+1} \frac{\varepsilon}{2^k} .$$

To this end we choose $z'_{n+1} \in E$, $z'_{n+1} \geq x_{n+1} \vee y_{n+1}$ such that

$$v(z'_{n+1}) \leq \overline{v}(x_{n+1} \vee y_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} .$$

We then choose $z_{n+1} \in E$, $z_{n+1} \geq z_n \vee z'_{n+1}$ such that

$$v(z_{n+1}) \leq \overline{v}(z_n \vee z'_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} .$$

Using (2.1) we can rewrite the last inequality:

$$(2.6) \quad v(z_{n+1}) \leq v(z_n) + v(z'_{n+1}) - \underline{v}(z_n \wedge z'_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} .$$

We have $x_n \vee y_n \leq z_n \wedge z'_{n+1}$, and it follows from the Riesz interpolation property that there exists $w_n \in E$ such that $x_n \vee y_n \leq w_n \leq z_n \wedge z'_{n+1}$. This implies that $\overline{v}(x_n \vee y_n) \leq \underline{v}(z_n \wedge z'_{n+1})$. Using this fact, it follows from (2.6) that

$$\begin{aligned} v(z_{n+1}) &\leq v(z_n) - \overline{v}(x_n \vee y_n) + v(z'_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} \\ &\leq v(z_n) - \overline{v}(x_n \vee y_n) + \overline{v}(x_{n+1} \vee y_{n+1}) + \frac{\varepsilon}{2^{n+1}} . \end{aligned}$$

From the induction hypothesis we conclude that

$$(2.7) \quad v(z_{n+1}) \leq \overline{v}(x_{n+1} \vee y_{n+1}) + \sum_{k=1}^{n+1} \frac{\varepsilon}{2^k} .$$

The constructed sequence $\{z_n\}$ evidently satisfies (a) and (b). (c) follows easily from (2.7). To prove (d) we must show that $\sup_n v(z_n) < \infty$. For every $n = 1,2,\ldots$ we have

$$v(x_n) + v(y_n) = \overline{v}(x_n \vee y_n) + \underline{v}(x_n \wedge y_n)$$

$$\geq \overline{v}(x_n \vee y_n) + \underline{v}(x_1 \wedge y_1).$$

Hence we obtain

$$\sup_n \overline{v}(x_n \vee y_n) \leq v^o(x) + v^o(y) - \underline{v}(x_1 \wedge y_1) < \infty ,$$

and the desired conclusion follows from (c). This completes the proof.

Our next lemma generalizes the first half of lemma 1, p. 429 in [1].

<u>2.2 Lemma</u>: For all $x, y \in E^o$ we have

$$(2.8) \quad v^o(x) + v^o(y) = \inf \{ v^o(z) : x \vee y \leq z \in E^o\} + \sup \{ v^o(z) : x \wedge y \geq z \in E^o\}$$

<u>Proof</u>: Since $x, y \in E^o$ there exist $x_n, y_n \in E$, $n = 1,2,\ldots,$ such that $x_n \uparrow x$, $y_n \uparrow y$. Using the valuation identity (2.1) we obtain

$$v^o(x) + v^o(y) = \sup_n \overline{v}(x_n \vee y_n) + \sup_n \underline{v}(x_n \wedge y_n)$$

The proof will be accomplished if we can prove the following two formulae:

$$(2.9) \quad \inf \{ v^o(z) : x \vee y \leq z \in E^o\} = \sup_n \overline{v}(x_n \vee y_n)$$

$$(2.10) \quad \sup \{ v^o(z) : x \wedge y \geq z \in E^o\} = \sup_n \underline{v}(x_n \wedge y_n).$$

We first prove (2.9). Let $\varepsilon > 0$ be given and choose $z = \sup_n z_n \in E^o$ as in lemma 2.1. Then we have

$$v^o(z) \leq \sup_n \overline{v}(x_n \vee y_n) + \varepsilon ,$$

and, since ε was arbitrary, it follows that

$$\inf \{ v^o(z) : x \vee y \leq z \in E^o\} \leq \sup_n \overline{v}(x_n \vee y_n).$$

The reverse inequality follows easily from (D), and the validity of (2.9) is proved.

To prove (2.10) we observe that

$$(2.11)\quad \sup \{ v^o(z) : x \wedge y \geq z \in E^o \} = \sup \{ v(z) : x \wedge y \geq z \in E \},$$

and it follows immediately that

$$\sup \{ v^o(z) : x \wedge y \geq z \in E^o \} \geq \sup_n \underline{v}(x_n \wedge y_n).$$

We prove the reverse inequality. Let $\varepsilon > 0$ be given. From (2.11) it follows that there exists $z \in E$, $z \leq x \wedge y$, such that

$$\sup \{ v^o(z) : x \wedge y \geq z \in E^o \} \leq v(z) + \varepsilon .$$

Putting $z_k = z_k' = z$, $k = 1,2,\ldots,$ we have that $z_k \vee z_k' \downarrow z$, and application of (D) yields

$$v(z) \leq \sup_n \underline{v}(x_n \wedge y_n).$$

ε was arbitrary, and we conclude that

$$\sup \{ v^o(z) : x \wedge y \geq z \in E^o \} \leq \sup_n \underline{v}(x_n \wedge y_n),$$

q. e. d.

The difficult part in the extension procedure will be to prove the validity of the "Beppo Levi property". The proof in the lattice case, contained in the second half of lemma 1, p. 429 of [1] , rests heavily upon the possibility of performing lattice operations. Nevertheless, it turns out that a rather delicate use of the Riesz interpolation property does the work.

<u>2.3 Lemma</u>: If $x_n \uparrow x \in H$, $x_n \in E^o$, $n = 1,2,\ldots,$ and $\sup_n v^o(x_n) < \infty$, then, given $\varepsilon > 0$, there exists $z \in E^o$, $z \geq x$, such that

$$v^o(z) \leq \sup_n v^o(x_n) + \varepsilon .$$

<u>Proof:</u> For every n there exist $x_{n,k} \in E$, $k = 1,2,\dots,$ such that $x_{n,k} \uparrow x_n$. We put $y_{1,k} = x_{1,k}$, $k = 1,2,\dots,$ and choose, in accordance with lemma 2.1, an increasing sequence $\{y_{2,k}\}$, $y_{2,k} \in E$, $k = 1,2,\dots,$ such that $y_{2,k} \geq y_{1,k} \vee x_{2,k}$, $k = 1,2,\dots,$ and such that

$$\sup_k v(y_{2,k}) \leq \sup_k \overline{v}(y_{1,k} \vee x_{2,k}) + \frac{\varepsilon}{2}$$

From (2.9) and the fact that $x_1 \vee x_2 = x_2$ we have

$$\sup_k \overline{v}(y_{1,k} \vee x_{2,k}) = \inf \{ v^o(z) : x_1 \vee x_2 \leq z \in E^o \} = v^o(x_2),$$

hence it follows that

$$\sup_k v(y_{2,k}) \leq v^o(x_2) + \frac{\varepsilon}{2} .$$

Suppose now that we have found increasing sequences $\{y_{i,k}\}_k$, $y_{i,k} \in E$, $k = 1,2,\dots,$ for $i = 2,3,\dots, n$ such that $y_{i,k} \geq y_{i-1,k} \vee x_{i,k}$, $k = 1,2,\dots,$ $i = 2,3,\dots, n$ and such that

$$\sup_k v(y_{i,k}) \leq v^o(x_i) + \sum_{j=1}^{i-1} \frac{\varepsilon}{2^j} , \quad i = 2,3,\dots, n.$$

We put $y_n = \sup_k y_{n,k}$. Evidently, $y_n \geq x_n$ and $y_n \in E^o$. Using lemma 2.1, we select an increasing sequence $\{y_{n+1,k}\}$, $y_{n+1,k} \in E$, $k = 1,2,\dots,$ such that $y_{n+1,k} \geq y_{n,k} \vee x_{n+1,k}$, $k = 1,2,\dots,$ and such that

$$\sup_k v(y_{n+1,k}) \leq \sup_k \overline{v}(y_{n,k} \vee x_{n+1,k}) + \frac{\varepsilon}{2^n}$$

From (2.10), (2.8) and the fact that $x_n \leq y_n \wedge x_{n+1}$ we obtain

$$\sup_k \overline{v}(y_{n,k} \vee x_{n+1,k}) = \inf \{ v^o(z) : y_n \vee x_{n+1} \leq z \in E^o \}$$

$$= v^o(y_n) + v^o(x_{n+1}) - \sup \{ v^o(z) : y_n \wedge x_{n+1} \geq z \in E^o \}$$

$$\leq v^o(y_n) + v^o(x_{n+1}) - v^o(x_n)$$

$$= \sup_k v(y_{n,k}) - v^o(x_n) + v^o(x_{n+1}) .$$

From the induction hypothesis it then follows that

$$\sup_k v(y_{n+1,k}) \leq v^o(x_{n+1}) + \sum_{j=1}^{n} \frac{\varepsilon}{2^j} .$$

We now select a diagonal sequence $\{z_n\}$ by setting $z_n = y_{n,n}$ for every $n = 1,2,\ldots$. We easily verify that $z_{n+1} = y_{n+1,n+1} \geq y_{n+1,n} \geq y_{n,n} = z_n$, hence $\{z_n\}$ is increasing. We put $z = \sup_n z_n$. For a fixed n we have for all $k \geq n$ that

$$z \geq z_k = y_{k,k} \geq y_{n,k} \geq x_{n,k} ,$$

and it follows that $z \geq \sup_k x_{n,k} = x_n$. This is valid for every n, hence $z \geq \sup_n x_n = x$. The choice of the sequences $\{y_{n,k}\}_k$ ensures that

$$v(z_n) = v(y_{n,n}) \leq \sup_k v(y_{n,k}) \leq v^o(x_n) + \varepsilon .$$

It follows that

$$\sup_n v(z_n) \leq \sup_n v^o(x_n) + \varepsilon < \infty ,$$

and we conclude that $z \in E^o$ and $v^o(z) \leq \sup_n v^o(x_n) + \varepsilon$. This completes the proof.

In the next lemma we use for the first time the fact that v is admissible.

<u>2.4 Lemma</u>: For all $x, y \in E^o$ we have

(2.12) $\inf \{ v^o(z) : x \wedge y \leq z \in E^o \} = \sup \{ v^o(z) : x \wedge y \geq z \in E^o \}$

<u>Proof</u>: v^o is increasing, hence we have

$$\inf \{ v^o(z) : x \wedge y \leq z \in E^o \} \geq \sup \{ v^o(z) : x \wedge y \geq z \in E^o \}$$

Let $\varepsilon > 0$ be given. There exist $x_n, y_n \in E$, $n = 1,2,\ldots,$ such that $x_n \uparrow x$, $y_n \uparrow y$. According to condition (B) we can now find $z_1 \in E^o$, $z_1 \geq x_1 \wedge y_1$, such that

$$v^o(z_1) \leq \underline{v}(x_1 \wedge y_1) + \frac{\varepsilon}{2} .$$

Suppose now that $z_1, z_2, \ldots, z_n \in E^o$ have been chosen such that $z_k \geq x_k \wedge y_k$, $k = 1,2,\ldots, n$ and $z_1 \leq z_2 \leq \ldots \leq z_n$ and such that

$$v^o(z_k) \leq \underline{v}(x_k \wedge y_k) + \sum_{i=1}^{k} \frac{\varepsilon}{2^i}, \quad k = 1,2,\ldots, n.$$

Using (B) again, we choose $z'_{n+1} \in E^o$, $z'_{n+1} \geq x_{n+1} \wedge y_{n+1}$, such that

$$v^o(z'_{n+1}) \leq \underline{v}(x_{n+1} \wedge y_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} .$$

Finally, we pick $z_{n+1} \in E^o$, $z_{n+1} \geq z_n \vee z'_{n+1}$, such that

$$v^o(z_{n+1}) \leq \inf \{ v^o(z) : z_n \vee z'_{n+1} \leq z \in E^o \} + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} .$$

From (2.8) and the fact that $x_n \wedge y_n \leq z_n \wedge z'_{n+1}$ it follows that

$$\begin{aligned} v^o(z_{n+1}) &\leq v^o(z_n) - \underline{v}(x_n \wedge y_n) + v^o(z'_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} \\ &\leq \sum_{i=1}^{n} \frac{\varepsilon}{2^i} + \underline{v}(x_{n+1} \wedge y_{n+1}) + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} + \frac{1}{2} \cdot \frac{\varepsilon}{2^{n+1}} \\ &= \underline{v}(x_{n+1} \wedge y_{n+1}) + \sum_{i=1}^{n+1} \frac{\varepsilon}{2^i} . \end{aligned}$$

It now easily follows that

$$(2.13) \quad \sup_n v^o(z_n) \leq \sup_n \underline{v}(x_n \wedge y_n) + \varepsilon < \infty .$$

We put $z_o = \sup_n z_n$. Evidently we have $z_o \geq x \wedge y$. (H is σ-continuous.) By lemma 2.3 there exists $z' \in E^o$, $z' \geq z_o$, such that

$$v^o(z') \leq \sup_n v^o(z_n) + \varepsilon .$$

From (2.13) we then obtain, using (2.10),

$$\begin{aligned} v^o(z') &\leq \sup_n \underline{v}(x_n \wedge y_n) + 2\varepsilon \\ &= \sup \{ v^o(z) : x \wedge y \geq z \in E^o \} + 2\varepsilon . \end{aligned}$$

ε was arbitrary, and the required inequality follows.

The following lemma completes the necessary ground work in the extension procedure. It generalizes the results of prop. 3.1, p. 427 and lemma 1, p. 429 in [1].

2.5 Lemma: v is extendable to an upper semi-integral (H, E^*, v^*) where the sublattice E^* of H is hereditary from above.

Proof: We define E^* as follows:

$$E^* = \{ x \in H : \text{ there exists } y \in E^o \text{ such that } x \leq y \}$$

It is evident from this definition that if $x, y \in E^*$, then we also have $x \wedge y \in E^*$. An easy application of lemma 2.1 yields that $x \vee y \in E^*$, and so we have proved that E^* is a sublattice of H. E^* is evidently hereditary from above.

We now define $v^* : E^* \to [-\infty, +\infty\rangle$ by

$$v^*(x) = \inf \{ v^o(y) : x \leq y \in E^o \} .$$

We observe that v^* is increasing, and it then immediately follows that $v^* | E = v$. It remains to prove

(a) $v^*(x) + v^*(y) \geq v^*(x \vee y) + v^*(x \wedge y)$ for all $x, y \in E^*$
(i.e. v^* is an upper semi-valuation).

(b) $x_n \in E^*$, $n = 1,2,\ldots$, $x_n \uparrow x \in H$, $-\infty < \sup_n v^*(x_n) < \infty$
$\Longrightarrow$ $x \in E^*$ and $v^*(x) = \sup_n v^*(x_n)$
(i.e. the "upper half" of the Beppo Levi property).

To prove (a), let $x, y \in E^*$ and $\varepsilon > 0$ be given. We assume that $v^*(x)$, $v^*(y)$, $v^*(x \vee y)$, and $v^*(x \wedge y)$ are all finite, otherwise the desired inequality is trivial. There exist x', $y' \in E^o$, $x' \geq x$, $y' \geq y$, such that $v^o(x') \leq v^*(x) + \frac{\varepsilon}{2}$, $v^o(y') \leq v^*(y) + \frac{\varepsilon}{2}$. Application of lemma 2.4 and lemma 2.2 yields

$$v^*(x \vee y) + v^*(x \wedge y)$$

$$\leq \inf \{ v^o(z) : x' \vee y' \leq z \in E^o \} + \inf \{ v^o(z) : x' \wedge y' \leq z \in E^o \}$$

$$= \inf \{ v^o(z) : x' \vee y' \leq z \in E^o \} + \sup \{ v^o(z) : x' \wedge y' \geq z \in E^o \}$$

$$= v^o(x') + v^o(y')$$

$$\leq v^*(x) + v^*(y) + \varepsilon .$$

ε was arbitrary, and (a) is proved.

(b): Let $x_n \in E^*$, $n = 1,2,\ldots$, $x_n \uparrow x \in H$, $-\infty < \sup_n v^*(x_n) < \infty$. Without loss of generality we can assume $v^*(x_n) > -\infty$ for $n = 1,2,\ldots$. We first select $y_1 \in E^o$, $y_1 \geq x_1$, such that

$$v^o(y_1) \leq v^*(x_1) + \frac{\varepsilon}{2} .$$

Then we choose $y_2 \in E^o$, $y_2 \geq y_1 \vee x_2$, such that

$$v^o(y_2) \leq v^*(y_1 \vee x_2) + \frac{\varepsilon}{4} .$$

Using (a) of this lemma we obtain

$$v^o(y_2) \leq v^o(y_1) + v^*(x_2) - v^*(y_1 \wedge x_2) + \frac{\varepsilon}{4} .$$

Now $y_1 \geq x_1$ and $x_2 \geq x_1$, thus $y_1 \wedge x_2 \geq x_1$, and it follows that

$$v^o(y_2) \leq v^*(x_2) + v^o(y_1) - v^*(x_1) + \frac{\varepsilon}{4}$$

$$\leq v^*(x_2) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} .$$

Proceeding by induction, we obtain an increasing sequence $\{y_n\}$, $y_n \in E^o$, $n = 1,2,\ldots$, such that $y_n \geq x_n$ and

$$v^o(y_n) \leq v^*(x_n) + \sum_{k=1}^{n} \frac{\varepsilon}{2^k} .$$

It follows that

$$\sup_n v^o(y_n) \leq \sup_n v^*(x_n) + \varepsilon < \infty,$$

and lemma 2.3 implies that there exists $z \in E^o$ such that $z \geq \sup_n y_n \geq x = \sup_n x_n$ and such that

$$v^o(z) \leq \sup v^o(y_n) + \varepsilon .$$

It follows that $x \in E^*$ and also that

$$v^o(z) \leq \sup_n v^*(x_n) + 2\varepsilon .$$

Since ε was arbitrary, we obtain

$$v^*(x) \leq \sup_n v^*(x_n) .$$

The reverse inequality is trivial, and the proof is complete.

Remark: Evidently the extension theory developed so far has a dual version, and we can thus also extend v to a lower semi-integral (H, E_*, v_*), where the sublattice E_* of H is hereditary from below.

2.6 Theorem: An admissible increasing valuation v on a directed subset E of a σ-continuous lattice H, where E has the Riesz interpolation property, can be extended to a full integral (Lebesgue type integral) within H if and only if condition (D) holds.

Then the common restriction I of the two functions v^*, v_* of (2.14), (2.15) to the set $\tilde{E}$ of those elements for which they are both well defined with the same finite value will be such an extension.

(2.14) $v^*(x) = \inf \{ \sup_n v(y_n) : y_n \in E, \ n = 1,2,\dots, \ y_n \uparrow y \geq x \}$

(2.15) $v_*(x) = \sup \{ \inf_n v(z_n) : z_n \in E, \ n = 1,2,\dots, \ z_n \downarrow z \leq x \}$.

Moreover, E will be dense in $\tilde{E}$ with respect to the pseudo metric d_I.

Proof: (D) is necessary. Assume that v is extendable to a full integral $(H, \hat{E}, \hat{v})$ and let $x_n, x_n', y_n, y_n' \in E$, $n = 1,2,\ldots$ be such that $x_n \uparrow x \in H$, $x_n' \uparrow x' \in H$, $y_n \downarrow y \in H$, $y_n' \downarrow y' \in H$ and $x \wedge x' \geq y \vee y'$. We must prove that

$$(2.16)\quad \sup_n \underline{v}(x_n \wedge x_n') \geq \inf_n \overline{v}(y_n \vee y_n').$$

We can assume that $\sup_n \underline{v}(x_n \wedge x_n')$ and $\inf_n \overline{v}(y_n \vee y_n')$ are both finite, otherwise (2.16) is trivial. $\hat{E}$ is a sublattice of H, hence $x_n \wedge x_n' \in \hat{E}$, $y_n \vee y_n' \in \hat{E}$, $n = 1,2,\ldots$. $(H, \hat{E}, \hat{v})$ is a full integral, and we obtain (from the Beppo Levi property) that

$$\sup_n \hat{v}(x_n \wedge x_n') \geq \inf_n \hat{v}(y_n \vee y_n').$$

Thus, to prove (2.16) it suffices to show that

$$\hat{v}(x_n \wedge x_n') \leq \underline{v}(x_n \wedge x_n'), \quad \hat{v}(y_n \vee y_n') \geq \overline{v}(y_n \vee y_n')$$

for $n = 1,2,\ldots$. Since v is admissible and $E^o \subseteq \hat{E}$, it follows that

$$\begin{aligned}\underline{v}(x_n \wedge x_n') &= \inf \{ v^o(z) : x_n \wedge x_n' \leq z \in E^o \} \\ &= \inf \{ \hat{v}(z) : x_n \wedge x_n' \leq z \in E^o \} \\ &\geq \hat{v}(x_n \wedge x_n').\end{aligned}$$

The dual inequality is proved similarly.

(D) is sufficient. We extend v to an upper semi-integral (H, E^*, v^*) as in lemma 2.5. Similarly we also extend v to a lower semi-integral (H, E_*, v_*). Evidently v^* and v_* will satisfy (2.14) and (2.15) respectively. From (D) it is easily seen that

$$x \in E^* \cap E_* \Longrightarrow v_*(x) \leq v^*(x).$$

It now follows from proposition 3.2, p. 428 in [1] that $(H, \tilde{E}, I)$ is a full integral, where we have put

$$\widetilde{E} = \{ x \in E^* \cap E_* : v^*(x) = v_*(x)$$

and $I = v^* \mid \widetilde{E} = v_* \mid \widetilde{E}$. If $x \in E$, then $v^*(x) = v_*(x) = v(x)$, hence I is an extension of v.

<u>E is dense in $\widetilde{E}$ with respect to d_I</u>. The straightforward proof is omitted (cf. [1, p. 430]).

We saw in the first part of this proof that

$$\hat{v}(x \wedge y) \leq \underline{v}(x \wedge y), \quad \hat{v}(x \vee y) \geq \overline{v}(x \vee y), \quad x, y \in E$$

for any full integral $(H, \hat{E}, \hat{v})$ extending v. The reverse inequalities are both trivial, and we have proved the following

<u>2.7 Corollary</u>: If v is admissible, we have

$$\hat{v}(x \wedge y) = \underline{v}(x \wedge y), \quad \hat{v}(x \vee y) = \overline{v}(x \vee y) \ ; \quad x, y \in E$$

for any full integral $(H, \hat{E}, \hat{v})$ extending v.

<u>Remarks</u>: The most disappointing feature in our extension theory is probably the introduction of the "messy" condition (B). This restriction seems (to us), however, necessary if one wants to carry through the arguments along the same tracks as in the classical Daniell theory and in [1]. (The difficulty is to prove the validity of (2.12)). In a certain sense we can say that in the general case (A) "splits" into <u>two</u> different conditions (B) and (D). The relationship between the conditions (A), (B), (D) and the (extended) Daniell axiom ((1.1) and its dual version), which we here denote by (C), in the various stages of generalization can be illustrated as follows (The almost trivial proofs of the different implications are omitted):

(i) E is a vector lattice in the complete lattice of extended real-valued functions on some set:

$$(A) \Longleftrightarrow (B) \Longleftrightarrow (C) \Longleftrightarrow (D).$$

(ii) E is a sublattice of a σ-continuous lattice H:

$$(D) \Longleftrightarrow (A) \Longrightarrow (C) \ , \quad (B) \Longleftrightarrow (C).$$

(iii) E is a directed subset of a σ-continuous lattice H with the Riesz interpolation property:

$$(D) \Longrightarrow (A) \Longrightarrow (C) \ , \quad (B) \Longrightarrow (C).$$

Finally it should be pointed out that (B) by no means is necessary for an extension to be possible. This is demonstrated by the following example: Let H be the complete lattice of extended real-valued functions on $[0,1]$ and define E to be the linear space consisting of the continuous affine functions on $[0,1]$ (i.e. functions of the form $t \mapsto a\,t + b$, $a, b \in \mathbb{R}$). Obviously E has the Riesz interpolation property. We define v as the restriction of the ordinary Lebesgue integral to E. It is easily verified that v is an increasing valuation, and by the very definition v is extendable to a full integral. Let $x, y \in E$ be defined by $x(t) = t$, $y(t) = 1 - t$, $t \in [0,1]$. Then $\underline{v}(x \wedge y) = 0$, but $\hat{v}(x \wedge y) = \frac{1}{4}$ for the Lebesgue extension. From corollary 2.7 it follows that (B) is not satisfied.

3. <u>The semi-lattice condition</u>. In a given example it may be difficult to prove the validity of the conditions (B) and (D). A very reasonable condition imposed on E will, however, simplify the situation drastically. We shall use the following notation:

$E \wedge E = \{x \wedge y : x, y \in E\}$, $\quad E \vee E = \{x \vee y : x, y \in E\}$.

$E^{\sigma} = \{x \in H : \text{ there exist } x_n \in E, \ n = 1,2,\ldots, \ x_n \uparrow x\}$. E_{σ} is defined dually. A subset K of H is called an <u>upper semi-lattice</u> if $x \vee y \in K$ for all $x, y \in K$. A <u>lower semi-lattice</u> is defined dually. We now assume the following two inclusions to hold:

(3.1) $\quad E \wedge E \subseteq E^{\sigma}, \quad E \vee E \subseteq E_{\sigma}$.

(3.1) could reasonably be called the <u>semi-lattice condition</u> because of the following proposition.

3.1 Proposition:

(a) $E \wedge E \subseteq E^{\sigma} \Longleftrightarrow E^{\sigma}$ is a lower semi-lattice

(b) $E \vee E \subseteq E_{\sigma} \Longleftrightarrow E_{\sigma}$ is an upper semi-lattice.

Proof: Obviously it suffices to prove (a). If E^{σ} is a lower semi-lattice, it follows immediately that $E \wedge E \subseteq E^{\sigma}$, since $E \subseteq E^{\sigma}$. Conversely, assume $E \wedge E \subseteq E^{\sigma}$, and let $x, y \in E^{\sigma}$ be given. There exist $x_n, y_n \in E$, $n = 1,2,\ldots$, such that $x_n \uparrow x$, $y_n \uparrow y$. The assumption $E \wedge E \subseteq E^{\sigma}$ implies that we can find, for every n, $z_{n,k} \in E$, $k = 1,2,\ldots$, such that $z_{n,k} \uparrow x_n \wedge y_n$. We put $z_1 = z_{1,1}$ and choose $z_2 \in E$ such that

$$z_{1,2} \vee z_{2,2} \leq z_2 \leq x_2 \wedge y_2.$$

This choice is, of course, made possible by the Riesz interpolation property. Proceeding by induction we obtain an increasing sequence $\{z_n\}$ from E such that

$$z_{1,n} \vee z_{2,n} \vee \ldots \vee z_{n,n} \leq z_n \leq x_n \wedge y_n \; ; \quad n = 1,2,\ldots .$$

We put $z = \sup_n z_n \in E^{\sigma}$. For a fixed n we now have for every $k \geq n$ that $z \geq z_k \geq z_{n,k}$. It follows that $z \geq \sup_k z_{n,k} = x_n \wedge y_n$. Passing to the limit once more we obtain $z \geq x \wedge y$. On the other hand, $z_n \leq x_n \wedge y_n$ for every n, and we conclude that $z = x \wedge y$. Therefore $x \wedge y \in E^{\sigma}$, and the proof is complete.

Remark: The reader might perhaps suspect the last proof to constitute a simplification of the technique used in lemma 2.3. This is, however, not the case; the Riesz interpolation property is used quite differently in the two proofs in question.

Our next proposition should be compared with the observation (ii) in the remark following corollary 2.7.

3.2 Proposition: Let v be an increasing valuation on E, and assume that $E \wedge E \subseteq E^{\sigma}$, $E \vee E \subseteq E_{\sigma}$. Then the following two equivalences hold:

$$(A) \Longleftrightarrow (D) , \qquad (B) \Longleftrightarrow (C) .$$

Proof: The implications $(D) \Longrightarrow (A)$ and $(B) \Longrightarrow (C)$ are trivial.
$(A) \Longrightarrow (D)$: Let $x_n \uparrow x \in H$, $x_n' \uparrow x' \in H$, $y_n \downarrow y \in H$, $y_n' \downarrow y' \in H, x \wedge x' \geq y \vee y'$, $x_n, x_n', y_n, y_n' \in E$, $n = 1,2,\dots$. According to the previous proposition we can find $z_n \in E$, $z_n \leq x_n \wedge x_n'$, $n = 1,2,\dots$, such that $z_n \uparrow x \wedge x'$. Dually we can find $w_n \in E$, $w_n \geq y_n \vee y_n'$, $n = 1,2,\dots$, such that $w_n \downarrow y \vee y'$. The required inequality now immediately follows from (A).

$(C) \Longrightarrow (B)$: Since $E \wedge E \subseteq E^{\sigma}$, the proof will be accomplished if we can prove the following: If $z_n \in E$, $n = 1,2,\dots$, $z_n \uparrow z \geq z_o \in E$, then $\sup_n v(z_n) \geq v(z_o)$. H is σ-continuous, hence $z_n \wedge z_o \uparrow z_o$. We choose in accordance with proposition 3.1 $z_n' \in E$, $z_n' \leq z_n \wedge z_o$, $n = 1,2,\dots$, such that $z_n' \uparrow z_o$. From (C) it follows that $\sup_n v(z_n) \geq \sup_n v(z_n') = v(z_o)$, and the desired inequality is proved.

3.3 Corollary: Let v be an increasing valuation on E and assume that $E \wedge E \subseteq E^{\sigma}$, $E \vee E \subseteq E_{\sigma}$. Then v is extendable to a full integral within H if and only if condition (A) is satisfied. Moreover, a necessary condition for extendability is that v is admissible.

Proof: The corollary follows directly from proposition 3.2 and theorem 2.6.

Let H denote the (σ-) continuous lattice of extended real-valued functions on some set S. As in the introduction we define $H_o = \{ x \in H : |x(s)| < \infty \text{ for all } s \in S \}$. Let E be a directed linear subspace of H_o, and assume that E has the Riesz interpolation property, i.e. E is a Riesz space in H. To be able to apply our extension theory, we make the following observation (which is trivial in case E is a vector lattice):

3.4 Proposition: A linear functional v on E is a valuation (in the sense of (2.1)).

Proof: Let $x, y \in E$ be given and let $z \in E$ be such that $z \leq x \wedge y$. We know that $x + y = x \vee y + x \wedge y$, hence $x + y - z \geq x \vee y$, and it follows that $v(x + y - z) \geq \overline{v}(x \vee y)$, i.e. $v(x) + v(y) \geq \overline{v}(x \vee y) + v(z)$. This inequality is valid for every $z \in E$ such that $z \leq x \wedge y$, and so we have

$$v(x) + v(y) \geq \overline{v}(x \vee y) + \underline{v}(x \wedge y).$$

The reverse inequality is proved similarly.

Remark: The reader should note that the assumption that E be directed, is essential in the last proof.

Our next proposition should be compared with proposition 3.2 and the observation (i) in the remark following corollary 2.7.

3.5 Proposition: Let v be a positive linear functional on a Riesz space E in H and assume that $E \wedge E \subseteq E^{\sigma}$. Then we have

$$(A) \Longleftrightarrow (B) \Longleftrightarrow (C) \Longleftrightarrow (D) .$$

Proof: In virtue of proposition 3.2 it suffices to prove that (A) follows from the (extended) Daniell axiom, i.e. the implication $(C) \Longrightarrow (A)$. To this end let $x_n \uparrow x \in H$, $y_n \downarrow y \in H$, $x \geq y$, $x_n, y_n \in E$, $n = 1,2,\ldots$. We must prove that $\sup_n v(x_n) \geq \inf_n v(y_n)$. As usual we can assume that $\sup_n v(x_n)$ and $\inf_n v(y_n)$ are both finite. It is now evident that $x_n - y_n \uparrow x - y \geq 0$. ($x$ and y need not be in H_o, nevertheless their difference is well defined in H.) We choose $z_n \in E$, $z_n \leq (x_n - y_n) \wedge 0$, $n = 1,2,\ldots,$ such that $z_n \uparrow 0$. This choice is made possible by the construction given in the proof of proposition 3.1. From the Daniell axiom it now follows that

$$\sup_n v(x_n - y_n) \geq \sup_n v(z_n) = 0 ,$$

and the proof is complete.

Theorem 2.6 and proposition 3.5 immediately yield the following

3.6 Theorem: Let v be a positive linear functional on a Riesz space E in H and assume that $E \wedge E \subseteq E^{\sigma}$. Then v can be extended to a full integral within H if and only if the Daniell axiom (1.1) is satisfied.

Remark: Let v and E be as in the preceding theorem, and let $(H, \tilde{E}, I)$ be the extension constructed in theorem 2.6. $\tilde{E}$ will generally consist of infinite valued functions, thus $\tilde{E}$ is usually not a linear space. It is, however, not hard to prove that $\tilde{E} \cap H_o$ is a vector lattice and that $I \mid \tilde{E} \cap H_o$ is a positive linear functional. If we also assume the Stone axiom to hold ([1, p. 459]), we can prove that each member of $\tilde{E}$ is equivalent to a member of $\tilde{E} \cap H_o$ (mod I). Then $\mathcal{L}_1 = \tilde{E}/[I]$ can be organized into a vector lattice in the natural way.

There is a useful corollary to theorem 3.6 which we shall need when we turn to the study of Choquet simplexes. We now assume that S is a compact Hausdorff space. We denote the class of real-valued continuous functions on S by $\mathcal{C}(S)$. Let K be some subset of H. For a given $s \in S$ the notation $(K \wedge K)(s) \subseteq K^{\sigma}(s)$ means that for any $x, y \in K$ there exist $z_n \in K$, $z_n \leq x \wedge y$, $n = 1,2,\ldots$, such that $z_n(s) \uparrow (x \wedge y)(s)$.

3.7 Corollary: Let S be a compacr Hausdorff space, and let E be a linear subspace of $\mathcal{C}(S)$ with the Riesz interpolation property. If E contains the constant functions and $(E \wedge E(s) \subseteq E^{\sigma}(s)$ for every $s \in S$, then any positive linear functional on E can be extended to a full integral within H.

Proof: Since the function 1 is in E, and since every member of E is continuous, E is directed, hence E is a Riesz space in H.
Let $x, y \in E$ and a positive integer n be given. For every $s \in S$ we choose $z_s \in E$, $z_s \leq x \wedge y$ such that $(x \wedge y)(s) - \frac{1}{2n} \leq z_s(s)$. An easy compactness argument implies that we can find $s_1, \ldots, s_k \in S$ and open neighbourhoods U_i of s_i, $i = 1,2,\ldots, k$, such that $S = \bigcup_{i=1}^{k} U_i$ and such that $(x \wedge y)(t) - \frac{1}{n} \leq z_{s_i}(t)$ for all $t \in U_i$, $i = 1,2,\ldots, k$. Using the Riesz interpolation property we find $z_n' \in E$, $z_n' \leq x \wedge y$ such that $x \wedge y - \frac{1}{n} \leq z_n'$. We use the Riesz interpolation property once more together with induction to obtain an increasing sequence $\{z_n\}$, $z_n \in E$, such that $x \wedge y - \frac{1}{n} \leq z_n \leq x \wedge y$, $n = 1,2,\ldots$. It follows that $x \wedge y \in E^{\sigma}$. Since $1 \in E$, we conclude from Dini's lemma that any positive linear functional on E satisfies the Daniell axiom. Now theorem 3.6 applies.

4. Applications. For explanation of terms not defined here the reader is referred to [3]. Let X be a compact, convex subset of a locally convex Hausdorff space over the reals. The extreme boundary of X is denoted by X_e. $\mathcal{K}$ is the class of real-valued continuous concave functions on X, $\mathcal{A} = \mathcal{K} \cap -\mathcal{K}$ the class of continuous affine functions. $\mathfrak{M}_+^1(X)$ denotes the set of probability measures on X. A measure $\mu \in \mathfrak{M}_+^1(X)$ represents $x \in X$ if $\mu(f) = f(x)$ for all $f \in \mathcal{A}$. A boundary measure is a measure $\mu \in \mathfrak{M}_+^1(X)$ which vanishes off every boundary set ("Bordüre" in [3]). If X is also a simplex, i.e. X is affinely isomorphic to the base of a lattice cone, we shall see how our extension theory can be used to give a very natural construction of representing boundary measures in case (a) X_e is closed or (b) X is metrizable. The connecting link is provided by the following fact proved by Edwards ([4], cf. also [5, th. 2.1]): X is a simplex if and only if $\mathcal{A}$ has the Riesz interpolation property. Note that if $\mathcal{A}$ possesses the Riesz interpolation property, so does $\mathcal{A}|X_e = \{ f|X_e : f \in \mathcal{A} \}$. (This follows from [3, Satz 2.4.4, Vergleichsprinzip].) We recall to the reader the definition of

the u.s.c. (concave) envelope $\overline{f}$ and the l.s.c. (convex) envelope $\underline{f}$ of a bounded real-valued function f:

$$\overline{f} = \inf \{ g : f \leq g \in \mathcal{A} \}, \quad \underline{f} = \sup \{ g : f \geq g \in \mathcal{A} \}.$$

We also note that ([3, Korollar 3.1.4])
$X_e = \bigcap \{ B_f : f \in -\mathcal{K} \}$ where B_f is the boundary set $\{ x : \overline{f}(x) = f(x) \}$.

<u>4.1 Theorem</u>: Let X be a simplex with closed extreme boundary X_e. To every $x \in X$ there exists a unique boundary measure $\mu_x \in \mathcal{M}_+^1(X)$ representing x.

<u>Proof</u>: Let $x \in X$ be given. We define a linear functional v_x on $E = \mathcal{A} \mid X_e$ by setting $v_x(f \mid X_e) = f(x)$ for $f \in \mathcal{A}$. v_x is well defined and positive because of [3, Satz 2.4.4]. $\mathcal{A}$ contains the constant functions, hence E is a Riesz space in the (σ-) continuous lattice H of extended real-valued functions on X_e. If $h_1, h_2 \in \mathcal{A}$, $h_1 \wedge h_2$ is continuous, hence $\underline{h_1 \wedge h_2} \mid X_e = h_1 \wedge h_2 \mid X_e$. We conclude that $(E \wedge E)(y) \subseteq E^{\sigma}(y)$ for every $y \in X_e$. From corollary 3.7 it follows that v_x is extendable to a full integral within H. Let $(H, \tilde{E}, \tilde{v}_x)$ be the extension of theorem 2.6. We now prove that $\mathcal{C}(X) \mid X_e \subseteq \tilde{E}$. Let $\mathcal{J}$ denote the set of all finite joins of elements in $\mathcal{A}$. According to [3, Lemma 3.1.1] $\mathcal{J} - \mathcal{J}$ is dense in $\mathcal{C}(X)$ (sup-norm topology), hence it suffices to prove that $\mathcal{J} \mid X_e \subseteq \tilde{E}$. This inclusion, however, is obvious since $\tilde{E}$ is a sublattice of H. We now define μ_x by putting $\mu_x(f) = \tilde{v}_x(f \mid X_e)$ for every $f \in \mathcal{C}(X)$. Evidently μ_x is a positive linear functional on $\mathcal{C}(X)$, and $\mu_x(1) = \tilde{v}_x(1 \mid X_e) = 1$, hence $\mu_x \in \mathcal{M}_+^1(X)$. Furthermore, $\mu_x(f) = \tilde{v}_x(f \mid X_e) = v_x(f \mid X_e) = f(x)$ for all $f \in \mathcal{A}$, and finally $\mu_x(\overline{f} - f) = 0$ for all $f \in -\mathcal{K}$. Therefore, μ_x is a boundary measure representing x.

To prove uniqueness let μ_1, μ_2 be two boundary measures both representing x. Evidently $\mu_1 | \mathcal{A} = \mu_2 | \mathcal{A}$. It suffices to prove that $\mu_1 | \mathcal{J} = \mu_2 | \mathcal{J}$. Let $f \in \mathcal{J}$ be given. We have $\mu_1(f) = \mu_1(\overline{f})$, $\mu_2(f) = \mu_2(\overline{f})$. We have just proved that $E \wedge E \subseteq E^{\sigma}$, or equivalently, $E \vee E \subseteq E_{\sigma}$. From proposition 3.1 it follows that $\mathcal{G}|X_e \subseteq E_{\sigma}$. Accordingly, let $h_n \in \mathcal{A}$, $n = 1,2,\ldots$, be such that $h_n | X_e \downarrow \bar{f} | X_e = f | X_e$. Since X is a simplex, $\bar{f}$ is affine, hence $h_n \downarrow \bar{f}$. It follows that $\mu_1(\bar{f}) = \lim_n \mu_1(h_n) = \lim_n \mu_2(h_n) = \mu_2(\bar{f})$. This completes the proof.

<u>4.2 Theorem</u>: Let X be a metrizable simplex. To every $x \in X$ there exists a unique boundary measure $\mu_x \in \mathcal{M}_+^1(X)$ representing x.

<u>Proof</u>: Our proof is based upon the following fact: If X is a metrizable, compact, convex subset of a locally convex Hausdorff space, there exist to every l.s.c. affine function f on X $h_n \in \mathcal{A}$, $n = 1,2,\ldots$, such that $h_n \uparrow f$. Dually, if g is an u.s.c. affine function on X, there exist $k_n \in \mathcal{A}$, $n = 1,2,\ldots$, such that $k_n \downarrow g$. The proof of this statement is omitted, we only mention that it is based on an application of the Hahn-Banach theorem and on the fact that a metrizable compact space is 2nd countable. Let $x \in X$ be given. We define v_x on $E = \mathcal{A}|X_e$ just as in the proof of theorem 4.1. To be able to use theorem 3.6 we have to prove that $E \wedge E \subseteq E^{\sigma}$. This, however, is now immediate: If $h_1, h_2 \in \mathcal{A}$, $\underline{h_1 \wedge h_2}$ is l.s.c. and affine since X is a simplex, hence there exist $g_n \in \mathcal{A}$, $n = 1,2,\ldots$, such that $g_n \uparrow \underline{h_1 \wedge h_2}$. It follows that $E \wedge E \subseteq E^{\sigma}$. The rest of the proof is almost identical with the corresponding part of the proof of theorem 4.1.

<u>Remark</u>: It should be noted that we do not make use of the existence of a <u>strictly convex</u> function on X in the last proof, cf. [3, Satz 3.2.4].

References

[1] Alfsen, E.M.: Order Theoretic Foundations of Integration, Math. Annalen, 149, (1963), pp. 419-461.

[2] Bauer, H.: Geordnete Gruppen mit Zerlegungseigenschaft, Sitz.-Ber., Bayerische Akad. d. Wiss., (1958), pp. 25-35.

[3] Bauer, H.: Konvexität in topologischen Vektorräumen, Lecture notes, Hamburg University 1963/64.

[4] Edwards, D.A.: Separation des fonctions réelles definies sur un simplexe de Choquet, Comptes rendus Acad. Sci. Paris 261, (1965), pp. 2798-2800.

[5] Effros, E.G.: Structure in simplexes, Lecture notes, Aarhus University, Oct. 1965.

[6] Fuchs, L.: On partially ordered vector spaces with the Riesz interpolation property, Publ. Math. Debrecen, (1965), pp. 335-343.

[7] Fuchs, L.: Riesz Groups, Ann. Scuola. Norm. Sup. Pisa 19, (1965), pp. 1-34.

[8] Lindenstrauss, J.: Extension of compact operators, Memoirs Am. Math. Soc. 48, (1964).

[9] Namioka, I.: Partially ordered linear topological spaces, Memoirs Am. Math. Soc. 24, (1957).

[10] Riesz, F.: Sur quelques notions fondamentales dans la théorie générale des opérations linéaires, Ann. of Math 41, (1940), pp. 174-206.