Matematisk Seminar
Universitetet i Oslo

Nr. 3
Mai 1966

Note on cofibrations

by

Arne Strøm

In the first section of this note it is proved that cofibrations are homeomorphisms, and a characterization of closed cofibrations is given. The second section contains the proof of a homotopy lifting-extension theorem generalizing a result on relative CW-complexes.

All functions considered will be continuous.

1.

Definition: A (Hurewicz) fibration is a map $p:E \to B$ with the property that for any map $f:X \to E$ and any homotopy $F:X\times I \to B$ such that $F(x,0) = pf(x)$ for all $x \in X$, there exists a homotopy $\tilde{F}:X\times I \to E$ such that $p\tilde{F} = F$ and $\tilde{F}(x,0) = f(x)$ for all $x \in X$.

A cofibration is a map $j:A \to X$ such that for any map $f:X \to Y$ and any homotopy $\tilde{F}:A\times I \to Y$ such that $\tilde{F}(a,0) = fj(a)$ for all $a \in A$, there exists a homotopy $F:X\times I \to Y$ such that $F(j\times 1_I) = \tilde{F}$ and $F(x,0) = f(x)$ for all $x \in X$.

If A is a subspace of a space X such that the inclusion map $A \subset X$ is a cofibration, the pair (X,A) is called a cofibered pair or is said to possess the absolute homotopy extension property (AHEP). A necessary condition for (X,A) to be a cofibered pair is the existence of a retraction $r:X\times I \to (X\times 0) \cup (A\times I)$. If A is closed, this condition is also sufficient.

The following theorem shows that, essentially, the only cofibrations are cofibered pairs.

Theorem 1: If $j:A \to X$ is a cofibration, then j is a homeomorphism $A \approx j(A)$.

Proof: Let $j: A \to X$ be a cofibration and consider the mapping cylinder Z of j, that is, the quotient space of the topological sum $(X \times 0) \vee (A \times I)$ obtained by identifying $(a,0) \in A \times I$ with $(j(a),0) \in X \times 0$ for each $a \in A$. Denote by q the quotient map $(X \times 0) \vee (A \times I) \to Z$. There is a continuous map $i: Z \to X \times I$ defined by

$$iq(x,0) = (x,0) \qquad (x \in X),$$

$$iq(a,t) = (j(a),t) \qquad (a \in A,\ t \in I).$$

Define maps $f: X \to Z$ and $\bar{F}: A \times I \to Z$ by

$$f(x) = q(x,0), \quad \bar{F}(a,t) = q(a,t).$$

Because j is a cofibration there exists a map $F: X \times I \to Z$ such that $F(j(a),t) = q(a,t)$ and $F(x,0) = q(x,0)$ for all $a \in A$, $t \in I$, and $x \in X$. Then $Fi = 1_Z$, and i is, therefore, a homeomorphism of Z onto $i(Z) = (X \times 0) \cup (j(A) \times I)$. Also, $q|A \times 1$ is a homeomorphism of $A \times 1$ onto $q(A \times 1)$, and consequently $iq|A \times 1$ is a homeomorphism of $A \times 1$ onto $iq(A \times 1) = j(A) \times 1$. Q.E.D.

Next we shall prove a theorem which generalizes 3.1 of [1].

Theorem 2: Let A be a closed subspace of a topological space X. Then (X,A) is a cofibered pair if and only if there exist

(i) a neighbourhood U of A which is deformable in X to A rel A (that is, there exists a homotopy $H: U \times I \to X$ such that $H(x,0) = x$, $H(a,t) = a$, and $H(x,1) \in A$ for all $x \in U$, $a \in A$, $t \in I$), and

(ii) a continuous function $\varphi : X \to I$ such that $A = \varphi^{-1}(0)$ and $\varphi(x) = 1$ for all $x \in X - U$.

Proof: Suppose that (X,A) is a cofibered pair. Then there exists a retraction $r: X \times I \to (X \times 0) \cup (A \times I)$, and U,H and φ may be chosen as follows.

$$U = \{ x \in X \mid pr_1 r(x,1) \in A \} ,$$
$$H = pr_1 r | U \times I$$
$$\varphi(x) = \sup_{t \in I} | t - pr_2 r(x,t) | .$$

Conversely, suppose that U, H and φ are given and satisfy the conditions of the theorem. Since A is closed it suffices to prove the existence of a retraction $r: X \times I \to (X \times 0) \cup (A \times I)$. The required retraction may be constructed as follows.

(i) If $\varphi(x) = 1$, let $r(x,t) = (x,0)$.

(ii) If $\frac{1}{2} \leq \varphi(x) < 1$, let $r(x,t) = (H(x, 2(1-\varphi(x))t), 0)$.

(iii) If $0 < \varphi(x) \leq \frac{1}{2}$ and $0 \leq t \leq 2\varphi(x)$, let

$$r(x,t) = (H(x, \frac{t}{2\varphi(x)}), 0).$$

(iv) If $0 < \varphi(x) \leq \frac{1}{2}$ and $2\varphi(x) \leq t \leq 1$, let

$$r(x,t) = (H(x,1), t - 2\varphi(x)).$$

(v) If $\varphi(x) = 0$, let $r(x,t) = (x,t)$.

(This construction is that of [2].) The proof of continuity is straightforward and will be omitted. Q.E.D.

2. It was remarked in section 1 that if (X,A) is a cofibered pair, then $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$. In fact, we have the following stronger result.

<u>Lemma</u>: If (X,A) is a cofibered pair, then $(X\times 0)\cup(A\times I)$ is a strong deformation retract of $X\times I$.

<u>Proof</u>: Let $i:(X\times 0)\cup(A\times I)\subset X\times I$ be the inclusion map, and let $r:X\times I\to(X\times 0)\cup(A\times I)$ be a retraction. A homotopy $D:ir\simeq 1_{X\times I}$ rel $(X\times 0)\cup(A\times I)$ is given by

$$D(x,t,t') = (pr_1 r(x,(1-t')t),\ (1-t')pr_2 r(x,t) + t't). \quad \text{Q.E.D.}$$

<u>Theorem 3</u>: Suppose that $p:E\to B$ is a fibration, that A is a strong deformation retract of X, and that there exists a map $\varphi:X\to I$ such that $A = \varphi^{-1}(0)$. Then any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ i\cap & & \downarrow p \\ X & \xrightarrow[f']{} & B \end{array}$$

may be filled in with a map $f:X\to E$ such that $pf = f'$ and $fi = f''$. f is unique up to homotopy rel A.

<u>Proof</u>: By hypothesis there exists a retraction $r:X\to A$ and a homotopy $D:ir\simeq 1_X$ rel A. If $f:X\to E$ is such that $fi = f''$, then $f\simeq fir = f''r$ rel A, which proves the last assertion of the theorem. Define $\bar{D}:X\times I\to X$ by

$$\bar{D}(x,t) = \begin{cases} D(x,\frac{t}{\varphi(x)}) & t<\varphi(x) \\ D(x,1) & t\geqq\varphi(x). \end{cases}$$

$\bar{D}$ is easily shown to be continuous. Because p is a fibration there exists a homotopy $\bar{F}:X\times I\to E$ such that $p\bar{F} = f'\bar{D}$ and $\bar{F}(x,0) = f''r(x)$ for each $x\in X$. f is given by $f(x) = \bar{F}(x,\varphi(x))$. Q.E.D.

We are now in a position to prove

<u>Theorem 4</u>: Suppose that $p: E \to B$ is a fibration, that (X,A) is a cofibered pair, and that A is closed. Then any commutative diagram

$$\begin{array}{ccc} (X \times 0) \cup (A \times I) & \xrightarrow{\ f\ } & E \\ \cap & & \downarrow p \\ X \times I & \xrightarrow[F]{} & B \end{array}$$

may be filled in with a homotopy $\bar{F}: X \times I \to E$ such that $p\bar{F} = F$ and $\bar{F} \mid (X \times 0) \cup (A \times I) = f$.

<u>Proof</u>: According to the Lemma $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$, and by Theorem 2 there exists a function $\psi : X \to I$ such that $A = \psi^{-1}(0)$. Define $\varphi : X \times I \to I$ by $\varphi(x,t) = t\,\psi(x)$. Then $(X \times 0) \cup (A \times I) = \varphi^{-1}(0)$, and the theorem follows from Theorem 3. Q.E.D.

The condition that A be closed is not very restrictive. For instance, A will always be closed if X is Hausdorff. Not all cofibrations are closed, however. The most trivial example of a non-closed cofibration is the pair (X,a) where X is the two-point space $\{a,b\}$ with the trivial topology.

References.

1. C.H. Dowker: Homotopy extension theorems. Proc. London Math. Soc. (3) 6 (1956) 100-116.

2. G.S. Young: A condition for the absolute homotopy extension property. Amer. Math. Monthly 71 (1964) 896-897.