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Note on colliprations

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In the first section of this note it is proved that cofibrations are homeomorphisms, and a characterization of closed cofibrations is given. The second section contains the proof of a homotopy lifting-extension theorem generalizing a result on relative CW-complexes.

All functions considered will be continuous.

<u>1.</u>

<u>Definition</u>: A (Hurewicz) <u>fibration</u> is a map $p:E \rightarrow B$ with the property that for any map $f:X \rightarrow E$ and any homotopy $F:X \times I \rightarrow B$ such that F(x,0) = pf(x) for all $x \in X$, there exists a homotopy $\overline{F}:X \cdot I \rightarrow E$ such that $p\overline{F} = F$ and $\overline{F}(x,0) = f(x)$ for all $x \in X$.

A <u>cofibration</u> is a map $j:A \to X$ such that for any map $f:X \to Y$ and any homotopy $F:A \ltimes I \to Y$ such that F(a,0) = fj(a)for all $a \in A$, there exists a homotopy $F:X \land I \to Y$ such that $F(j \times 1_T) = \overline{F}$ and F(x,0) = f(x) for all $x \in X$.

If A is a subspace of a space X such that the inclusion map $A \subset X$ is a cofibration, the pair (X,A) is called a <u>cofibered pair</u> or is said to possess the <u>absolute homotopy</u> <u>extension property (AHEP)</u>. A necessary condition for (X,A) to be a cofibered pair is the existence of a retraction $r:X\times I \rightarrow (X\times O) \cup (A\times I)$ If A is closed, this condition is also sufficient.

The following theorem shows that, essentially, the only cofibrations are cofibered pairs.

<u>Theorem 1</u>: If $j:A \rightarrow X$ is a cofibration, then j is a homeomorphism $A \approx j(A)$.

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<u>Proof</u>: Let $j:A \rightarrow X$ be a cofibration and consider the mapping cylinder Z of j, that is, the quotient space of the topological sum $(Z \times O) \vee (A \times I)$ obtained by identifying $(a, 0) \in A \times I$ with $(j(a), 0) \in X \times O$ for each $a \in A$. Denote by q the quotient map $(X \times O) \vee (A \times I) \rightarrow Z$. There is a continuous map $i:Z \rightarrow X \times I$ defined by

> $iq(x,0) = (x,0) \qquad (x \in X),$ $iq(a,t) = (j(a),t) \qquad (a \in A, t \in I).$ Define maps $f:X \longrightarrow Z$ and $\overline{F}:A \times I \longrightarrow Z$ by $f(x) = q(x,0), \quad \overline{F}(a,t) = q(a,t).$

Because j is a cofibration there exists a map $F:X \times I \longrightarrow Z$ such that F(j(a),t) = q(a,t) and F(x,0) = q(x,0) for all $a \in A$, $t \in I$, and $x \in X$. Then $Fi = 1_Z$, and i is, therefore, a homeomorphism of Z onto $i(Z) = (X \times 0) \cup (j(A) \times I)$. Also, $q \mid A \times 1$ is a homeomorphism of $A \times 1$ onto $q(A \times 1)$, and consequently $iq \mid A \times 1$ is a homeomorphism of $A \times 1$ onto $iq(A \times 1) = j(A) \times 1$, Q.E.D.

Next we shall prove a theorem which generalizes 3.1 of [1].

<u>Theorem 2</u>: Let A be a closed subspace of a topological space X. Then (X,A) is a cofibered pair if and only if there exist

(i) a neighbourhood U of A which is deformable in X to A rel A (that is, there exists a homotopy $H: U \times I \longrightarrow X$ such that H(x,0) = x, H(a,t) = a, and $H(x,1) \in A$ for all $x \in U$, $a \in A$, $t \in I$), and <u>Proof</u>: Suppose that (X, A) is a cofibered pair. Then there exists a retraction $r: X \times \mathbb{Z} \to (X \times C) \cup (A \times I)$, and U,H and φ may be chosen as follows.

$$U = \{ x \in X \mid pr_1 r(x, 1) \in \Lambda \},$$

$$H = pr_1 r | U \times I$$

$$\varphi(x) = \sup_{t \in I} | t - pr_2 r(x, t) |.$$

Conversely, suppose that U, H and φ are given and satisfy the conditions of the theorem. Since A is closed it suffices to prove the existence of a retraction $r:X \times I \longrightarrow (X \times 0) \cup (A \times I)$ The required retraction may be constructed as follows.

(i) If
$$\varphi(x) = 1$$
, let $r(x,t) = (x,0)$.
(ii) If $\frac{1}{2} \leq \varphi(x) < 1$, let $r(x,t) = (H(x,2(1-\varphi(x))t),0)$.
(iii) If $0 < \varphi(x) \leq \frac{1}{2}$ and $0 \leq t \leq 2\varphi(x)$, let
 $r(x,t) = (H(x, \frac{t}{2\varphi(x)}),0)$.
(iv) If $0 < \varphi(x) \leq \frac{1}{2}$ and $2\varphi(x) \leq t \leq 1$, let
 $r(x,t) = (H(x,1), t-2\varphi(x))$.
(v) If $\varphi(x) = 0$, let $r(x,t) = (x,t)$.

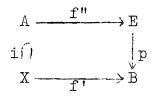
(This construction is that of [2].) The proof of continuity is straightforward and will be omitted. Q.E.D.

2. It was remarked in section <u>1</u> that if (X,A) is a cofibered pair, then $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$. In fact, we have the following stronger result. Lemma: If (X,A) is a cofibered pair, then $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$.

<u>Proof</u>: Let $i:(X \times 0) \cup (A \times I) \subset X \times I$ be the inclusion map, and let $r:X \times I \longrightarrow (X \times 0) \cup (A \times I)$ be a retraction. A homotopy $D:ir \simeq 1_{X \times I}$ rel $(X \times 0) \cup (A \times I)$ is given by

$$D(x,t,t') = (pr_1r(x,(1-t')t), (1-t')pr_2r(x,t) + t't). Q.E.D.$$

<u>Theorem 3</u>: Suppose that $p:E \rightarrow B$ is a fibration, that A is a strong deformation retract of X, and that there exists a map $\varphi: X \rightarrow I$ such that $A = \varphi^{-1}(0)$. Then any commutative diagram



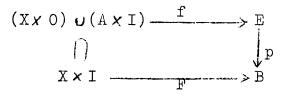
may be filled in with a map $f:X \rightarrow E$ such that pf = f' and fi = f''. f is unique up to homotopy rel A.

<u>Proof</u>: By hypothesis there exists a retraction $r:X \rightarrow A$ and a homotopy $D:ir \cong 1_X$ rel A. If $f:X \rightarrow E$ is such that fi = f", then $f \cong$ fir = f"r rel A, which proves the last assertion of the theorem. Define $\overline{D}:X \times I \rightarrow X$ by

$$\overline{D}(x,t) = \begin{cases} D(x,\frac{t}{\varphi(x)}) & t < \varphi(x) \\ D(x,1) & t \ge \varphi(x). \end{cases}$$

 \overline{D} is easily shown to be continuous. Because p is a fibration there exists a homotopy $\overline{F}:X \times I \longrightarrow E$ such that $p\overline{F} = f'\overline{D}$ and $\overline{F}(x,0) = f''r(x)$ for each $x \notin X$. f is given by $f(x) = \overline{F}(x, \varphi(x))$. Q.E.D. We are now in a position to prove

<u>Theorem 4</u>: Suppose that $p:E \rightarrow B$ is a fibration, that (X,A) is a cofibered pair, and that A is closed. Then any commutative diagram



may be filled in with a homotopy $\overline{F}: X \times I \longrightarrow E$ such that $p\overline{F} = F$ and $\overline{F} \mid (X \times O) \cup (A \times I) = f$.

<u>Proof</u>: According to the Lemma $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$, and by Theorem 2 there exists a function $\psi: X \longrightarrow I$ such that $A = \psi^{-1}(0)$. Define $\varphi: X \times I \longrightarrow I$ by $\varphi(x,t) = t \psi(x)$. Then $(X \times 0) \cup (A \times I) = \varphi^{-1}(0)$, and the theorem follows from Theorem 3. Q.E.D.

The condition that A be closed is not very restrictive. For instance, A will always be closed if X is Hausdorff. Not all cofibrations are closed, however. The most trivial example of a non-closed cofibration is the pair (X,a) where X is the two-point space $\{a,b\}$ with the trivial topology.

References.

- 1. C.H. Dowker: Homotopy extension theorems. Proc. London Math. Soc. (3) 6 (1956) 100-116.
- 2.. G.S. Young: A condition for the absolute homotopy extension property. Amer. Math. Monthly 71 (1964) 896-897.