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Introduction. The main result of the present paper is the following

THEOREM 2. Let T be a compact space, let P be a convex cone in the linear space $C(T)$ of all continuous real functions on T. Denote with $M^+(T)$ the set of all positive measures on T, and let μ_0 be such a measure. Then for any strictly positive $g \in C(T)$

$$\sup\{\mu_0(p) : p \in P \ \& \ p \leq g\} = \inf\{\mu(g) : \mu \in M^+(T) \ \& \ \mu \geq \mu_0 \text{ on } P\}$$

Several particular results of this kind is known. Section 1 provides four such examples. We prove Theorem 2 via a geometric version of it, where we assume that the compact set T is a convex subset of a topological linear space. This is explained in more detail in section 2, where we also exhibit an example which shows that Theorem 2 is no longer valid if g is not assumed strictly positive. However, if the cone P contains a strictly negative function, then Theorem 2 is valid for any continuous g. (Theorem 3) In section 3 we apply the preceding results to prove Theorem 4 where we state an equality of inf. and sup. for polar convex cones. The first part of this theorem can be derived from the main theorem of linear programming (see [6]), a fact we became aware of after having finished this paper. We only sketch the proofs. A paper containing complete proofs will appear elsewhere.

1. Examples. In (i) and (ii) below, we assume that K is a convex, compact set in a locally convex Hausdorff space E. E' denotes the topological dual of E.

(i) The first example is a very old one; it states that the gauge function of K equals the support function of the polar set of K.

Otherwise formulated: If $k \in K$, then

$$(1.1) \qquad \sup\{f(k) : f \in E' \ \& \ f \leq 1 \text{ on } K\} = \inf\{\lambda \geq 0 : k \in \lambda K\}.$$

(ii) The next example has been useful in the proof of the Choquet-theorem, see [5]. Let A denote the set of all continuous affine functions on K, and let $g \in C(K)$. Then for any $k \in K$

$$(1.2) \quad \sup\{f(k) : f \in A \ \& \ f \leq g \text{ on } K\} = \inf\{\mu(g) : \mu \in M^+(K) \ \& \ \mu(f) = f(k), \forall f \in A\}.$$

(iii) If L is a linear subspace of $C(T)$, and $\mu_0 \in M^+(T)$, then H. Bauer proved in [1] that

$$(1.3) \quad \sup\{\mu_0(l) : l \in L \ \& \ l \leq 1 \text{ on } T\} = \inf\{\mu(1) : \mu \in M^+(T) \ \& \ \mu = \mu_0 \text{ on } L\}.$$

(iv) The next example is a theorem in potential theory which recently has been proved by B. Fuglede [3]. The setting for this result is as follows: S and T are compact spaces, $k : S \times T \to \mathbb{R}^+$; for simplicity we assume k to be continuous. Let $\mu \in M^+(T)$ and $\lambda \in M^+(S)$ be given. We define the <u>potential</u> of μ and λ to be

$$k(s,\mu) = \int k(s,t) \, d\mu(t)$$
$$k(\lambda,t) = \int k(s,t) \, d\lambda(s).$$

For any $f \in C(T)$ we define

$$\operatorname{cap} f = \sup\{\mu(f) : \mu \in M^+(T) \ \& \ k(s,\mu) \leq 1 \text{ on } S\}$$
$$\operatorname{cont} f = \inf\{\lambda(1) : \lambda \in M^+(S) \ \& \ k(\lambda,t) \geq f(t), \forall t \in T\}.$$

The result of Fuglede states that

$$(1.4) \qquad \operatorname{cap} f = \operatorname{cont} f.$$

We now make the following assumption:

(1.5) There exists a positive measure $\mu_o \in M^+(T)$ such that $k(s,\mu_o) = 1, \forall\, s \in S$.

Using the condition (1.5) we find by integration and by applying the Fubini-theorem that the condition

$$k(s,\mu) \leq 1,\ \forall\, s \in S$$

is equivalent with

$$(1.6) \qquad \mu(k(\lambda,t)) \leq \mu_o(k(\lambda,t)),\quad \forall \lambda \in M^+(S).$$

Put $P = \{-k(\lambda,t) : \lambda \in M^+(S)\}$. Then P is a convex cone in $C(T)$ and the condition (1.6) means that

$$\mu(f) \geq \mu_o(f),\ \forall f \in P.$$

Finally, interchanging the order of integration, we get

$$\lambda(1) = \mu_o(k(\lambda,t)).$$

The equation (1.4) can therefore be written in the following way

$$(1.7) \quad \sup \mu_o(p) : p \in P \ \&\ p \leq -f\} = \inf\{\mu(-f) : \mu \in M^+(T)\ \&\ \mu \geq \mu_o \text{ on } P\}$$

The observation that the equation (1.4) could be expressed in the form given by (1.7) was the starting point of the present paper.

2. The geometric version.

As above we assume that E is a locally convex Hausdorff space and that E' is the topological dual of E. We equip E' with the weak topology. K is a compact, convex subset of E, and Q is a convex cone in E' with zero as vertex. As usual, the set $Q^o = \{x \in E : f(x) \leq 0, \forall f \in Q\}$ is called the polar cone of Q.

THEOREM 1. If Q is closed, then for any $k \in K$

$$(2.1)\quad \sup\{f(k) : f\in Q \ \&\ f \leq 1 \text{ on } K\} = \inf\{\mu(1) : \mu\in M^+(K) \ \&\ \mu(f) \geq f(k), \forall f \in Q\},$$

and the inf is attained.

This theorem looks like a hybrid of the Examples (i) and (ii). It follows from the following lemma that it is actually a generalization of Example (i).

LEMMA 1. For any $k \in K$

$$\{\mu(1) : \mu\in M^+(K) \ \&\ \mu(f) \geq f(k), \forall f \in Q\} = \{\lambda \geq 0 : (k-Q^o) \cap \lambda K \neq \emptyset\}$$

Sketch of the proof of Theorem 1: It is easy to see that sup $\leq$ inf. If the theorem is not true, choose α such that sup $< \alpha <$ inf. Applying Lemma 1 we then get

$$(k-Q^o) \cap \alpha K = \emptyset .$$

Since αK is convex and compact, and $k-Q^o$ is convex and closed, we can use the fundamental separation theorem to assert the existence of an $f \in E'$ and a real number γ such that

$$(2.2)\qquad f(\alpha K) < \gamma < f(k-Q^o).$$

First we assume that $0 \in K$. We then get $\gamma > 0$. Putting $g = \alpha\gamma^{-1} f$, we obtain

$$(2.3)\qquad g(k) \geq \alpha > \sup$$

We can, on the other hand, use (2.2) to infer that $g \in (Q^o)^o = Q$. Together with (2.3) this contradicts the definition of sup.

In order to prove Theorem 2 using Theorem 1, we first assum that the convex cone P is closed and that $g = 1$. We then apply Theorem 1 using the usual imbedding of T in the dual of $C(T)$. If g is not a constant, we introduce the cone

$$P_g = \{fg^{-1} : f \in P\}$$

which enables us to reduce the case of general g to the case $g = 1$. Finally, if P is not closed, we apply the above result to the uniform closure of P, and make use of special properties c the uniform convergence to ascertain that the desired result is not affected by the closure operation.

Theorem 2 includes Examples (i) and (iii), but it does not include Examples (ii) and (iv), because we require g to be strictly positive. This condition can, however, not be relaxed in the general case. In fact the following example shows that Theorem 2 can not be valid for a non-negative function which vanishes in just one point.

Example. Let $T = [0,1]$, let P consists of all polynomials with non-negative coefficients, and let g be defined as follows

$$g(t) = \begin{cases} e^{-\frac{1}{t}}, & 0 < t \leq 1 \\ 0, & t = 0. \end{cases}$$

We have choosen g in this way because we want to conclude that $p = 0$ is the only member of P such that $p \leq g$. Hence we get for any $\mu_0 \in M^+(T)$

$$\sup\{\mu_0(p) : p \in P \ \& \ p \leq g\} = 0.$$

We now choose μ_0 as the measure with unit mass placed in the point $t = 1$. Assume that $\mu \in M^+(T)$ satisfies

$$\mu(t^n) \geq \mu_0(t^n) = 1, \qquad n = 0,1,\ldots .$$

Let v denote the characteristic function of the set $\{1\}$. Since $\lim_{n\to\infty} t^n = v(t)$, it follows from Lebesgue's convergence theorem that $\mu(t^n) \to \mu(v)$. Hence $\mu(v) \geq 1$. Since $g \geq e^{-1}v$, we get

$$\mu(g) \geq e^{-1}\mu(v) \geq e^{-1} .$$

We can therefore conclude that

$$\inf\{\mu(g) : \mu \in M^+(T) \ \& \ \mu \geq \mu_0 \text{ on } P\} \geq e^{-1} .$$

Thus we have got the desired counterexample.

The convex cone P contains in this example the non-negative constants, but not the constant -1. It turns out that the appropriate condition on P is that P shall contain a strictly negative function:

THEOREM 3. If T is compact and $P \subset C(T)$ is a convex cone containing a strictly negative function p_0, then for any $\mu_0 \in M^+(T)$ and $g \in C(T)$

$$\sup\{\mu_0(p) : p \in P \ \& \ p \leq g\} = \inf\{\mu(g) : \mu \in M^+(T) \ \& \ \mu \geq \mu_0 \text{ on } P\},$$

and the inf is attained.

The proof of this theorem is an easy consequence of Theorem 2 if we in addition know that $-p_0 \in P$. However, Theorem 3 can in general be proved independent of Theorem 2 by a technique using the analytic Hahn-Banach theorem. This technique was applied by F.F. Bonsall to prove the Choquet theorem in the metrizable case [2]. The condition that P contains a strictly negative function is used to ascertain that

$$\varphi(g) = \sup\{\mu_0(p) : p \in P \ \& \ p \leq g\}$$

is finite for any $g \in C(T)$. It is then easy to see that $-\varphi$ is a subadditive and positive-homogeneous functional on $C(T)$. A straight forward application of the Hahn-Banach theorem then gives the desired result.

3. Polar convex cones.

We denote with (K) the convex cone generated by the compact convex set K. Hence

$$(K) = \bigcup_{\lambda \geq 0} \lambda K$$

For any $\mu \in M^+(K)$ we denote with $r(\mu)$ the <u>resultant</u> of μ. $r(\mu)$ is uniquely given by the requirement

$$\mu(f) = f(r(\mu)), \ \forall f \in E'.$$

The next lemma should be compared with Lemma 1.

LEMMA 2. Let $\mu_o \in M^+(K)$ be given. Then

$$\{r(\mu) : \mu \in M^+(K) \;\&\; \mu \geq \mu_o \text{ on } Q\} = (r(\mu_o) - Q^o) \cap (K).$$

We also observe that if $h \in E'$, then

$$(q \in Q \;\&\; q \leq h \text{ on } K) \Longleftrightarrow q \in Q \cap (h+(K)^o).$$

Using Lemma 2 and Theorem 2 and 3 we get

LEMMA 3. Assume that $\mu_o \in M^+(K)$. Then:

(i) If $h \in E'$ is strictly positive on K,

(3.1) $\sup\{q(r(\mu_o)) : q \in Q \cap (h+(K)^o)\} = \inf\{h(x) : x \in (r(\mu_o) - Q^o) \cap (K)\}$.

(ii) If Q contains a function which is strictly negative on K, then (3.1) is valid for any $h \in E'$.

Applying Lemma 3 we get

THEOREM 4. Let A, B be convex cones in E such that A is closed and the interior of A, int. A, is non-empty. Let $f \in A^o$ be given. Then:

(i) If $-x \in$ int. A

(3.2) $\sup\{f(y) : y \in (x+A) \cap B\} = \inf\{g(x) : g \in A^o \cap (f - B^o)\}$

(ii) If

$$(\text{int. } A) \cap B \neq \emptyset$$

then (3.2) is valid for any $x \in E$.

<u>Sketch of the proof</u>: Since int. $A \neq \emptyset$, it is known that there exists a weakly compact, convex subset C of E' such that

$$A^o = (C) = \bigcup_{\lambda \geq 0} \lambda C$$

It follows that if $-x \in$ int. A, then

$$g(x) > 0, \quad \forall g \in C.$$

We then consider E as the topological dual of E' and apply Lemma 3.

COROLLARY. Let $f \in A^{o}$ and assume

$$(\text{int. } A) \cap B \neq \emptyset$$

Then the following two statements are equivalent

(i) $$A^{o} \cap (f - B^{o}) = \{f\}.$$

(ii) $$f(x) = \sup\{f(y) : y \in (x+A) \cap B\}, \ \forall x \in E.$$

This corollary should be compared with the characterization of the Choquet boundary given by M. Herve [4].

References.

[1] H. Bauer, Über die Fortsetzung positiver Linearformen, Bayerische Akademie der Wissenschaften, München 1957.

[2] F.F. Bonsall, On the representation of points of a convex set, J.London Math.Soc. 38(1963), 332-334.

[3] B. Fuglede, Le théorème du minimax et la théorie fine du potentiel, Ann.Inst.Fourier, Grenoble 15, 1(1965) 65-88.

[4] M. Hervé, Sur les representation intégrales a l'aide des points extremaux dans un ensemble compact convexe metrisable, C.R. Acad.Sci. (Paris) 253(1961) 366-368.

[5] R.R. Phelps, Lectures on Choquet's theorem,to appear.

[6] K.S. Kretschmer, Programmes in paired spaces, Can.J.Math. vol. XIII(1961) 221-238.