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A CERTAIN CLASS OF MAPS BETWEEN OPERATOR ALGEBRAS

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As in most of mathematics our knowledge about operator algebras has to a great extent been obtained through the study of maps between them. The study of two types of maps has been particularly informative: homomorphisms and positive linear functionals. The following objection to restricting ourselves to these maps raises itself, namely, there are too few homomorphisms and the range of a linear functional is too small, being the complex numbers.

Let me first say what a C^*-algebra is; it is an algebra of (bounded) operators acting on a Hilbert space, closed in the uniform topology (the one defined by the norm) and containing the adjoint of each operator in it. We assume for simplicity that the algebra has an identity operator (always denoted by I). An operator A is positive $A \geq 0$, if $(Ax,x) \geq 0$ for all vectors x, and A is self-adjoint if $A = A^*$. Among the positive linear functionals (linear functionals f such that $A \geq 0$ implies $f(A) \geq 0$) two types have distinguished themselves, namely the extreme points (pure states) of the convex set of all positive linear functionals f such that $f(I) = 1$ (f is then called a state), and the "midpoints" of these convex sets - the traces. For example, the spectral theorem is obtained by studying the pure states of an abelian C^*-algebra.

The purpose of the present note is to describe a class of maps including both pure states and homomorphisms. We say a linear map ϕ of one C^*-algebra into another is positive if $A \geq 0$ implies $\phi(A) \geq 0$.

D e f i n i t i o n : Let ϕ be a positive linear map of a C^*-algebra $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$ such that $\phi(I) = I$. We say ϕ is p u r e s t a t e p r e s e r v i n g if $f \circ \phi$ is a pure state of $\mathfrak{A}$ for each pure state f of $\mathfrak{B}$.

R e m a r k . Since the composition of two pure state preserving maps clearly is pure state preserving, and since the identity map of a C^*-algebra is pure state preserving, the C^*-algebras form a category with the pure state preserving maps as the maps.

E x a m p l e 1 . Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and f a pure state of $\mathfrak{A}$. Then the mapping $\tilde{f} : A \to f(A)I$, I being the identity in $\mathfrak{B}$, is pure state preserving. We identify f and $\tilde{f}$ and say f is a pure state of $\mathfrak{A}$ into $\mathfrak{B}$.

This is immediate, for if g is a pure state of $\mathfrak{B}$ then $f(A) = g(\tilde{f}(A))$, so $g \circ \tilde{f} = f$, is a pure state of $\mathfrak{A}$.

E x a m p l e 2 . If ϕ is a $*$-homomorphism or $*$-anti-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ then ϕ is pure state preserving. (ϕ is a $*$-homomorphism means ϕ is a homomorphism and $\phi(A^*) = \phi(A)^*$, and dually for $*$-anti-homomorphisms.)

It is elementary to show that a state f of $\mathfrak{A}$ is pure if and only if whenever g is a positive linear functional of $\mathfrak{A}$ such that $f - g$ is positive, we say $g \leq f$, then g is a scalar multiple of f. Let f be a pure state of $\mathfrak{B}$ and let g be a positive linear functional of $\mathfrak{A}$ such that $g \leq f \circ \phi$. Then $\phi(A) = 0$ implies $\phi(A^*A) = 0$ so $0 \leq g(A^*A) \leq f(\phi(A^*A)) = 0$. In particular $g(A) = 0$, using the Cauchy-Schwarz inequality. It follows easily that $g = g' \circ \phi$, where g' is a positive linear functional of $\mathfrak{B} = \phi(\mathfrak{A})$, $g' \leq f$. Thus $g' = \alpha f$, and $g = \alpha f \circ \phi$, - $f \circ \phi$ is pure.

We will need the following result; if $\mathfrak{A}$ is an irreducible C^*-algebra acting on a Hilbert space H, i.e. $\mathfrak{A}$ has no closed invariant subspaces of H except 0 and H, then the state $A \to (Ax,x)$ is pure on $\mathfrak{A}$ for each unit vector x in H. Now each state f of a C^*-algebra $\mathfrak{A}$ can be written in the form: $f(A) = (\varphi_f(A)x,x)$, where x is a unit vector in a Hilbert space H_f, and φ_f is a $*$-homomorphism of $\mathfrak{A}$ into the operators of H. It is easy to show from what we have seen, that f is pure if and only if $\varphi_f(\mathfrak{A})$ is irreducible. From this we arrive at a class of maps which are close to being pure state preserving.

E x a m p l e 3 . Let $\mathfrak{A}$ be an irreducible C^*-algebra acting on a Hilbert space K . Let H be a Hilbert space and V a linear isometry of H into K . Then the map $A \rightarrow V^*AV$ has the property that the state $A \rightarrow (V^*AVx,x)$ is pure for each unit vector x in H .

The surprising thing is that we get all the pure state preserving maps from a combination of the three examples above.

T h e o r e m 1 . Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a positive linear map of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\phi(I) = I$. Then ϕ is pure state preserving if and only if for each irreducible $*$-representation (i.e. $*$-homomorphism) ψ of $\mathfrak{B}$, $\psi \circ \phi$ is either a pure state of $\mathfrak{A}$ or $\psi \circ \phi = V^* \rho V$, where V is a linear isometry of H - the Hilbert space on which $\psi(\mathfrak{B})$ acts - into a Hilbert space K , and ρ is an irreducible $*$-homomorphism or $*$-anti-homomorphism of $\mathfrak{A}$ into the operators on K .

The sufficiency follows from the previous discussion. If f is a pure state of $\mathfrak{B}$ then $f = \omega_x \varphi_f$, where φ_f is an irreducible representation of $\mathfrak{B}$ and x a unit vector in H_f ($\omega_x(A) = (Ax,x)$) . We may assume $\psi = \varphi_f$ for some pure state f . Then $\varphi_f \circ \phi$ has the property that $\omega_z \circ (\varphi_f \circ \phi)$ is a pure state of $\mathfrak{A}$ for each vector state ω_z due to a unit vector z in H_f . Theorem 1 is thus a corollary of

T h e o r e m 2 . Let $\mathfrak{A}$ be a C^*-algebra and H a Hilbert space. Let ϕ be a positive linear map of $\mathfrak{A}$ into the operators on H such that $\phi(I) = I$. Then $\omega_x \phi$ is a pure state of $\mathfrak{A}$ for each unit vector x in H if and only if either ϕ is a pure state or $\phi = V^* \rho V$, where V is a linear isometry of H into a Hilbert space K , and ρ is an irreducible $*$-homomorphism or $*$-anti-homomorphism of into operators on K .

The first thing we show, is that for two unit vectors x and y in H ,

$\omega_x \phi$ and $\omega_y \phi$ are unitarily equivalent, i.e. there exists a Hilbert space K and an irreducible representation ρ' of $\mathfrak{A}$ on K such that $\omega_x \phi = \omega_{x_1} \rho'$ and $\omega_y \phi = \omega_{y_1} \rho'$. It follows that $\phi = \eta \circ \rho'$ where η is a map of $\rho'(\mathfrak{A})$ into the operators on H such that $\omega_x \eta$ is a vector state ω_w of $\rho'(\mathfrak{A})$ due to a unit vector w in K. If we denote by B(H) all the bounded operators on H, and similarly for K, we use a result by Kadison to show that η has an extension mapping $\bar{\eta}$ of B(K) into B(H) which is ultra-weakly continuous and has the property that $\omega_x \bar{\eta}$ is a vector state for each unit vector x in H. I won't say much about the ultra-weak topology, except that it enables us to work on operators of finite rank, hence to work on matrices.

Instead of saying more about the proof, I will say a few words about applications of Theorem 1. Certain maps of C^*-algebras have attracted attention by several mathematicians, namely C^*-homomorphisms. They are positive linear maps ϕ with the property that $\phi(A^2) = \phi(A)^2$ if A is self-adjoint.

C o r o l l a r y. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and ϕ a C^*-homomorphism of $\mathfrak{A}$ o n t o $\mathfrak{B}$. Then $\psi \circ \phi$ is either a $*$-homomorphism or a $*$-anti-homomorphism for each irreducible $*$-representation ψ of $\mathfrak{B}$.

P r o o f : Using the argument of example 2 it is easy to show that ϕ is pure state preserving. Let ψ be an irreducible $*$-representation of $\mathfrak{B}$ on a Hilbert space H. By Theorem 1 $\psi \circ \phi$ is either a pure state of $\mathfrak{A}$ or is of the form $V^* \rho V$ with V and ρ as in the theorem. Now $\psi \circ \phi$ is clearly a C^*-homomorphism. If $\psi \circ \phi$ is a state it is therefore a homomorphism (a C^*-homomorphism of a C^*-algebra into an abelian C^*-algebra is a homomorphism). We assume $\psi \circ \phi = V^* \rho V$. Since V is a linear isometry $VV^* = P$

is a projection in $B(K)$. Then the map $A \rightarrow PAP$ is a C^*-homomorphism of $\rho(\mathfrak{A})$, since the map $B \rightarrow VBV^*$ is an isomorphism of $B(H)$ into $B(K)$. With A self-adjoint in $\rho(\mathfrak{A})$,

$$(AP - PAP)^*(AP - PAP)$$

$$= (PA^2P - PAPAP) - (PAPAP - PAPPAP)$$

$$= PA^2P - (PAP)^2 = 0 .$$

Thus $AP = PAP$, and taking adjoints, $AP = PA$, for each self-adjoint operator A in $\rho(\mathfrak{A})$. Thus the subspace $P(H)$ is invariant under $\rho(\mathfrak{A})$. $\rho(\mathfrak{A})$ is irreducible, hence $P(H) = H$, $P = I$. The map $A \rightarrow V^*AV$ is an isomorphism of $\rho(\mathfrak{A})$. Thus $\psi \circ \phi$ is either a homomorphism or an anti-homomorphism. QED.

We remark that the assumption made in the above corollary that ϕ be surjective, is much stricter than necessary. However, the proof would, under weaker assumptions, be much more complicated.