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A GENERALIZATION OF THE NOTION OF MODULE

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1. INTRODUCTION

Introducing the notion of an *ideal system* - or *x-system* - in a commutative semi-group Aubert (((1)) , ((2))) has shown that essential parts of commutative ideal theory can be developed on the basis of a set of axioms which are valid for most of the various notions of ideal that appear in the literature. In the present paper a definition of a corresponding generalized notion of module is given, and some results from ideal and module theory are generalized to such modules. For complete proofs we refer to ((3)) . The definition and fundamental properties of an x-system are given in ((1)) .

2. MODULES OVER SEMI-GROUPS

Let M be a set with a semi-group S of operators, and let S be equipped with an (integral) x-system. (See ((1)) or ((2)) .) We shall say that there is defined a *y-system* in M with respect to S if to every subset U of M there corresponds a subset U_y of M such that the following axioms are valid:

(2.1) $$U \subset U_y$$

(2.2) $$U \subset V_y \implies U_y \subset V_y$$

(2.3) $$AU_y \subset U_y$$

(2.4) $$AU_y \subset (AU)_y$$

(2.5) $$A_x U \subset (AU)_y$$

(2.6) $$U_y \neq \emptyset \implies U_y : M \neq \emptyset$$

(Here A denotes any subset of S.) M is then called an (x,y)-module over S or briefly an S-module. The subsets U of M for which we have $U_y = U$ are called y-modules in M. As in the case of x-systems, a y-system is defined by the set of all y-modules. To distinguish between several y-systems (resp. x-systems) we will sometimes speak of y_1-modules, y_2-modules, etc., but in general we shall from now on use the terms ideal and module instead of x-ideal and y-module.

The property of finite character is defined for y-systems exactly as for x-systems: The S-module M is said to be of finite character if both the x-system in S and the y-system in M are of finite character. M is called principal if $(u)_y = Su$ for all $u \in M$. The operations of y-union and y-product, denoted by $\cup_y$ and $\circ_y$, are defined by

$$U \cup_y V = (U \cup V)_y$$

$$A \circ_y V = (AV)_y$$

The axioms (2.4) and (2.5) have equivalent forms corresponding to the various forms of the continuity axiom for x-systems. We list a sample of the most useful ones in the following two theorems:

Theorem 1: The following statements are equivalent under the hypothesis that $U \rightarrow U_y$ is a closure operation:

I : $AU_y \subset (AU)_y$

II : $A \circ_y U_y = A \circ_y U$

III : $A(U \cup_y V) \subset AU \cup_y AV$

IV : $A \circ_y (U \cup_y V) = A \circ_y U \cup_y A \circ_y V$

V : $(U_y : A)_y = U_y : A$

T h e o r e m 2 : The following statements are equivalent under the hypothesis that $A \to A_x$ and $U \to U_y$ are closure operations:

I : $A_x U \subset (AU)_y$

II : $A_x \circ_y U = A \circ_y U$

III : $(A \cup_x B)U \subset AU \cup_y BU$

IV : $(A \cup_x B) \circ_y U = A \circ_y U \cup_y B \circ_y U$

V : $(U_y : V)_x = U_y : V$

3. CONGRUENCE. ADDITIVITY. QUOTIENT MODULES.

The relation

$$v \equiv w \pmod{U_y}$$

is defined by

$$(U_y, v)_y = (U_y, w)_y$$

and called c o n g r u e n c e m o d u l o U_y . It is easily verified that this relation really is a congruence with respect to multiplication by elements of S . The canonical mapping φ of M onto the set $\overline{M}$ of equivalence classes is hence an operator homomorphism. In $\overline{M}$ the set of all subsets $\overline{U}$ for which $\varphi^{-1}(\overline{U})$ is a module in M defines a y-system with respect to S , thus giving rise to a quotient module M/U_y .

The property of additivity can also be defined exactly as for x-systems: A y-system is said to be a d d i t i v e if the following condition holds for all elements and modules:

$$w \in U_y \cup_y V_y \implies (\exists v)(v \in V_y \;\&\; w \equiv v \pmod{U_y})$$

M is then called an additive S-module. Corresponding to theorem 2 in ((2)) we have:

T h e o r e m 3 : If φ denotes the canonical mapping $M \rightarrow M/U_y$ and the y-system in M/U_y is denoted by $\bar{y}$, the following statements are equivalent:

I : M is additive

II : $\varphi(U_y \cup_y V_y) = \varphi(V_y)$

III : $\varphi^{-1}(\varphi(V_y)) = U_y \cup_y V_y$

IV : $\varphi(V_y) = (\varphi(V))_{\bar{y}}$

V : $\varphi(V \cup_y W) = \varphi(V) \cup_{\bar{y}} \varphi(W)$

Each of the statements implies

$$\varphi(A \circ_y V) = A \circ_{\bar{y}} \varphi(V)$$

and is implied by this if M is unitary.

4. PRIMARY DECOMPOSITIONS

By the r a d i c a l of a module U_y in M, denoted by $r(U_y)$, we mean the nilpotent radical of the ideal $U_y : M$. If M is of finite character $r(U_y)$ is an ideal in S. A module U_y is said to be p r i m e (resp. p r i m a r y) if $av \in U_y$ and $v \notin U_y$ implies $a \in U_y : M$ (resp. $a \in r(U_y)$). If U_y is a prime (resp. primary) module then $U_y : M$ is a prime (resp. primary) ideal. Consequently, if M is of finite character, the radical of a primary module is a prime ideal.

A primary decomposition

$$(4.1) \qquad U_y = Q_y^{(1)} \cap Q_y^{(2)} \cap \dots \cap Q_y^{(r)} \qquad r(Q_y^{(i)}) = P_x^{(i)}$$

is called *irredundant* if no $Q_y^{(i)}$ contains the intersection of the others.

Theorem 4: If the decomposition (4.1) is irredundant, then U_y is primary if and only if $P_x^{(1)} = P_x^{(2)} = \dots = P_x^{(r)}$.

In a given primary decomposition we can therefore group together components with the same radical and get a primary decomposition in components with mutually different radicals. Such a decomposition will be called a *shortest* primary decomposition.

With the given definitions one can by some modification of the corresponding parts of ideal theory prove:

Theorem 5: Let U_y be a module admitting an irredundant primary decomposition (4.1). Then a prime ideal P_x is identical to one of the $P_x^{(i)}$ if and only if there exists an element v of $M - U_y$ such that the ideal $U_y : v$ is primary with P_x as radical. The prime ideals $P_x^{(i)}$ are therefore uniquely determined by U_y. P_x is a minimal member of the $P_x^{(i)}$ if and only if P_x is a minimal prime ideal containing $U_y : M$. If (4.1) is a shortest primary decomposition also the components corresponding to those minimal prime ideals are uniquely determined by U_y.

5. NOETHERIAN MODULES

An S-module M of finite character satisfying the ascending chain condition for submodules will be called *noetherian*. An S-module of finite character is noetherian if and only if every module in M has a finite basis. In a noetherian S-module which is additive and principal, one can prove that every irreducible module is primary, and consequently:

T h e o r e m 6 : In a noetherian S-module which is additive and principal every module has a finite primary decomposition.

The next theorem depends also essentially on the condition of additivity:

T h e o r e m 7 : Let M be an additive and principal S-module of finite character having a finite basis. Then if S is noetherian, so is M .

P r o o f : Let U_y be any module in M . Suppose first that M has a basis consisting of one single element v . Since S is noetherian the ideal $U_y : v$ has a finite basis $s_1, \ldots, s_r$, and since M is principal, the elements $s_1v, s_2v, \ldots, s_rv$ form a basis for U_y .

Suppose next that the theorem is valid for all S-modules having a basis consisting of $n - 1$ elements. We put:

$$M = (v_1, \ldots, v_n)_y$$

$$M' = (v_1, \ldots, v_{n-1})_y$$

$$U_y' = U_y \cap M'$$

$$A_x = (U_y \cup_y M') : v_n$$

The ideal A_x has a finite basis $a_1, \ldots, a_p$, and for $i = 1, 2, \ldots, p$ we have

$$a_i v_n \in U_y \cup_y M'$$

Since M is additive, U_y contains elements $u_1, \ldots, u_p$ such that

$$u_1 \equiv a_1 v_n \quad (\mathrm{mod}\ M')$$

$$\cdots\cdots\cdots\cdots$$

$$u_p \equiv a_p v_n \quad (\mathrm{mod}\ M')$$

We shall see that

$$U_y = (U_y', u_1, \ldots, u_p)_y$$

It remains to prove the inclusion $U_y \subset (U_y', u_1, \ldots, u_p)_y$. Let u be any element of U_y . From $u \in (M', v_n)_y$ follows, since M is additive and principal, that there exists an element $s \in S$ satisfying

$$u \equiv sv_n \pmod{M'}$$

and we must have $s \in A_x$. Hence

$$u \in (M', A_x v_n)_y \subset (M', a_1 v_n, \ldots\ldots, a_p v_n)_y = (M', u_1, \ldots, u_p)_y$$

Because of the additivity of M , there exists an element $w \in M'$ satisfying

$$w \equiv u \pmod{(u_1, \ldots, u_p)_y}$$

and since all elements on the right hand side belong to U_y , we have $w \in M' \cap U_y = U_y'$ and consequently

$$u \in (w, u_1, \ldots, u_p)_y \subset (U_y', u_1, \ldots, u_p)_y$$

The theorem is then by induction valid for all S-modules.

As an ideal-theoretic application of the preceding theorem we prove the following theorem, copying the original proof given by I.S. Cohen in the case of ordinary ideals in a noetherian ring.

T h e o r e m 8 : Let S be a commutative semi-group with an x-system of finite character which is additive and principal. If every prime ideal in S has a finite basis, then S is noetherian.

P r o o f : Suppose that the set of ideals in S without finite basis is not empty. This set is inductive and possesses by Zorn's lemma a maximal element A_x. By hypothesis A_x is not prime, and hence is properly contained in two ideals B_x and C_x such that $B_x \circ_x C_x \subset A_x$. By the maximality of A_x, both B_x and C_x have finite bases.

Now we form the quotient semi-groups S/C_x and $B_x/B_x \circ_x C_x$. It is easily verified that the latter can be regarded as a S/C_x-module, which is additive and principal. Every ideal in S containing C_x has a finite basis, thus S/C_x is noetherian, and since B_x has a finite basis, so does $B_x/B_x \circ_x C_x$. Then by the preceding theorem $B_x/B_x \circ_x C_x$ is a noetherian S/C_x-module and consequently $A_x/B_x \circ_x C_x$ has a finite basis in $B_x/B_x \circ_x C_x$. Using theorem 3, and the fact that B_x and C_x and therefore $B_x \circ_x C_x$ has a finite basis in S, this implies that A_x has a finite basis (in S). We have thus reached a contradiction.

It goes without saying that a lot of other results from the ordinary theory of modules can be formulated and proved within the present framework. For a more detailed exposition the reader is referred to ((3)).

References

((1)) K.E. Aubert: Theory of x-ideals. Acta Math. 107 (1962), p. 1-52.

((2)) K.E. Aubert: Additive ideal systems and commutative algebra. Matematisk seminar, Universitetet i Oslo, no. 3, 1963.

((3)) E.R. Hansen: En generalisering av modulbegrepet. Oslo 1963.