On projection maps of von Neumann algebras

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An important class of maps in the theory of von Neumann algebras is the positive linear maps of a von Neumann algebra R onto a von Neumann sub-algebra M which are the identity on M. Such maps are called projection maps (or projections of norm one, or expectations). Very often such maps will not exist, se e.g. [10]. In the present note we shall show that if R is of type I and M contains the center C of R then the existence of "sufficiently many" projection maps of R onto M is equivalent to M being of type I with center totally atomic over C.

Following [5] we say a set Λ of projection maps of R onto M is complete if for each non zero positive operator A in R then there is $\phi \in \Lambda$ such that $\phi(A) \neq 0$. A positive linear functional ρ on R is said to be singular if there is no ultra-weakly continuous positive linear functional ψ on R with $\psi \leq \rho$. If ϕ is a positive linear map from R to another von Neumann algebra M then ϕ is said to be singular if its transpose map ϕ^* carries normal states of M to singular positive linear functionals on R. As pointed out by Tomiyama [9] singular maps play an important role in the study of projection maps.

Lemma. Let R be a von Neumann algebra of type I. Let C denote the center of R and suppose ϕ is a positive singular C-module homomorphism of R into C. Then $\phi(E) = 0$ for every abelian projection E in R.

Proof. Let E be an abelian projection in C. Suppose $\phi(E) \neq 0$. Considering FR instead of R for a central projection F in R we may assume E has central carrier I, $\phi(E)$ is invertible in C, and that R is homogeneous. Say $R = C \otimes B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the bounded linear operators on the Hilbert space $\mathcal{H}$. Since all abelian projections with the same central carrier are equivalent [2, p. 251] there is a unitary operator $U \in R$ such that $UEU^{-1} = I \otimes [x]$, where $[x]$ denotes the one dimensional projection on the subspace of $\mathcal{H}$ spanned by the unit vector x [2, p. 243]. Let ϕ_U denote the positive singular C-module homomorphism $A \to \phi(U^{-1}AU)$. Then $\phi(E) = \phi_U(I \otimes [x])$. Replacing ϕ by ϕ_U and E by $I \otimes [x]$, we may thus assume $E = I \otimes [x]$. Let $\psi(A) = \phi(E)^{-1}\, \phi(EAE)$ for $A \in R$. Then ψ is a positive linear map of R into C such that for $A \in C$, $\psi(A) = \phi(E)^{-1}\, \phi(AE) = \phi(E)^{-1}\, A\phi(E) = A$, so indeed ψ is a projection of R onto C. Let ω be a normal state of C, and let $A \in C$, $B \in B(\mathcal{H})$. Then we have

$$
\begin{aligned}
\omega\circ\psi(A \otimes B) &= \omega(A\phi(E)^{-1}\, \phi(E(I \otimes B)E)) \\
&= \omega(A\phi(E)^{-1}\, \phi(E\omega_x(B))) \\
&= \omega(A)\,\omega_x(B) = \omega\otimes\omega_x\,(A \otimes B)\ .
\end{aligned}
$$

Thus $\omega\circ\psi$ is the normal state $\omega\otimes\omega_x$. By assumption $\omega\circ\phi$ is singular, thus the map $A \to \omega(\phi(EAE))$ is a singular positive linear functional, as follows from [8]. But if $A \geq 0$ then

$$
\omega\circ\psi(A) = \omega(\phi(E)^{-1}\, \phi(EAE)) \leq \|\phi(E)^{-1}\|\,\omega(\phi(EAE))\ ,
$$

hence $\omega\circ\psi$ is singular. This is a contradiction, so $\phi(E) = 0$. The proof is complete.

Let Z be an abelian von Neumann algebra and C a von Neumann subalgebra of Z (containing the identity of Z). A projection E in Z is said to be <u>minimal in</u> Z <u>relative to</u> C if $EZ = EC$. Z is said to be <u>totally atomic over</u> C if every non zero projection in Z majorizes a non zero projection in Z minimal in Z relative to C. These two concepts and their generalizations where introduced independently in [3] and [4].

<u>Theorem</u>. Let R be a von Neumann algebra of type I. Let M be a von Neumann subalgebra of R containing the center C of R. Then the following five conditions are equivalent.

1) (resp. 2) There exists a complete set of normal projection maps of R onto M (resp. $M' \cap R$).

3) (resp. 4) There exists a complete set of projection maps of R onto M (resp. $M' \cap R$).

5) M is of type I and its center is totally atomic over C.

<u>Proof</u>. By [7, Lemma 3.4] M is of type I if and only if $M' \cap R$ is of type I. Thus, if we have shown $1) \Longleftrightarrow 3) \Longleftrightarrow 5)$ then an application of these equivalences to $M' \cap R$ yields the equivalences $2) \Longleftrightarrow 4) \Longleftrightarrow 5)$. We shall show $1) \Longrightarrow 3) \Longrightarrow 5) \Longrightarrow 1)$. Clearly $1) \Longrightarrow 3)$.

$3) \Longrightarrow 5)$. Assume there is a complete set Λ of projection maps of R onto M. Let Z denote the center of M. We first assume M is abelian, hence $M = Z$. If ω is a normal state of Z with support E when restricted to C and A a positive operator in EZ the functional $B \rightarrow \omega(AB)$ on EC has a Radon-Nikodym derivative $\Phi(A) \in EC$ with respect to ω, so $\omega(AB) = \omega(\Phi(A)B)$. One easily sees that Φ is a normal projection map of EZ onto EC, see e.g. [1, p. 635]. Adding up the different Φ's obtained from a separating family of ω's with orthogonal supports when restricted to C we see there is a complete family Γ of normal projection maps of Z onto C. Let G denote the group of inner automorphisms $A \rightarrow UAU^{-1}$ of R defined by the unitary operators in Z. Then all the maps in Λ are G-invariant, since if $A \in R$, $U \cdot U^{-1} \in G$, and $\emptyset \in \Lambda$, then

$\emptyset(UAU^{-1}) = U\emptyset(A)U^{-1} = \emptyset(A)$.

Let E be an abelian projection in R. By assumption there is $\emptyset \in \Lambda$ such that $\emptyset(E) \neq 0$. By [9, Cor. 1,1] $\emptyset$ is uniquely decomposed into the sum of a positive singular Z-module homomorphism $\emptyset_S$ and a positive normal Z-module homomorphism $\emptyset_n$ of R to Z . Then, if $\psi \in \Gamma$, $\psi \circ \emptyset_S$ and $\psi \circ \emptyset_n$ are respectively positive singular and normal C-module homomorphisms of R to C . Choose ψ such that $\psi(\emptyset(E)) \neq 0$. By the Lemma $\psi \circ \emptyset_S(E) = 0$, hence $\psi \circ \emptyset_n(E) = \psi \circ \emptyset(E) \neq 0$. Let ω be a normal positive linear functional of C such that $\omega \circ \psi \circ \emptyset_n$ is a normal state of R with $\omega \circ \psi \circ \emptyset_n(E) \neq 0$. Since $\emptyset_n$ is a Z-module homomorphism of R to Z , $\omega \circ \psi \circ \emptyset_n$ is G-invariant. Now if A is a non zero positive operator in R then A majorizes a positive multiple of an abelian projection, hence we have shown the existence of a normal G-invariant state ρ of R for which $\rho(A) \neq 0$. Thus R is G-finite in the sense of [6]. By [7, Thm. 3.5] Z is totally atomic over C .

We next consider the general case. If M is not of type I there is a central projection E in M such that EME has no type I portion. Considering ERE , EME , and the projections $A \rightarrow \emptyset(EAE)$ we have a complete set of projection maps. By [9, Thms. 3 and 4] every projection map from ERE to EME is singular. Now every von Neumann algebra possesses a complete set of normal projections onto its center B . Indeed, it suffices to show that there is a complete set of normal projections of B' onto B . But by [7, Lem. 4.11] and the remarks following it there is a faithful normal projection of B' onto a maximal abelian subalgebra D . Compose this projection with a complete set of faithful normal projections from D onto B as constructed above to obtain the desired set. We thus obtain a complete set of singular projection maps from ERE to the center of EME and thus to EC . But these projections annihilate all abelian projections in ERE by the Lemma. Thus every projection map in Λ annihilates every abelian projection majorized by E , hence Λ is not complete, contrary to assumption. Thus M is of type I .

As shown above there is a complete set Γ of normal projections of M onto Z. Then the set $\{\psi \circ \emptyset : \psi \in \Gamma, \emptyset \in \Lambda\}$ is a complete set of projection maps of R onto Z. By the first part of the proof, Z is totally atomic over C. We have thus shown that 3) $\Longrightarrow$ 5).

5) $\Longrightarrow$ 1). Assume M is of type I and its center Z is totally atomic over C. Then $Z \supset C$. Let $B = Z' \cap R$, and let G denote the group of inner automorphisms $A \longrightarrow UAU^{-1}$ of R defined by unitaries $U \in Z$. Then B is the fixed point algebra of G, and $B' \cap R = Z$ is finite of type I, and its center (= Z) is totally atomic over C. By [7, Thm. 3.5] R is G-finite, so there is a faithful normal G-invariant projection Φ from R onto B [6]. Thus in order to construct a complete set of normal projections from R to M it suffices to do this for R replaced by $B = Z' \cap R$. Therefore we may assume $R = Z' \cap R$, hence $Z = C$. If we can construct a complete set of normal projection maps from $E_\alpha R$ to $E_\alpha M$ for an orthogonal family of central projections in R with sum I, then we can add up the different projection maps to obtain a complete set of normal projections from R to M, see e.g. [5]. Therefore we may assume R homogeneous, and by cutting down by central projections in M (so by projections in C) we may also assume M is homogeneous. Say $M = C \otimes B(K)$ and $R = C \otimes B(\mathcal{H})$. Since $M \subset R$ we may assume $\mathcal{H} = K \otimes K'$ and

$$M = C \otimes B(K) \otimes \mathbb{C}_{K'} \subset C \otimes B(K) \otimes B(K') = R .$$

If ω is a normal state of $B(K')$ and ι is the identity map of $C \otimes B(K)$ onto itself, then $\iota \otimes \omega$ is a normal projection map from R to M. Indeed, if ρ is a state of M and ρ' its restriction to $C \otimes B(K)$, let $A_i \in C \otimes B(K)$, $B_i \in B(K')$, $i = 1,\dots,n$. Then

$$\rho(\iota \otimes \omega(\sum A_i \otimes B_i)) = \sum \rho(A_i \otimes \omega(B_i)I) =$$
$$\sum \rho'(A_i)\omega(B_i) = \rho' \otimes \omega(\sum A_i \otimes B_i) .$$

Thus $\rho \circ (\iota \otimes \omega)$ is a state for each state ρ of M, hence $\iota \otimes \omega$ is positive. Clearly $\iota \otimes \omega$ is a projection map, and it is normal, for if ρ is normal then ρ' is normal, and therefore $\rho \circ (\iota \otimes \omega) = \rho' \otimes \omega$ is normal. Since $\iota \otimes \omega$ is a normal projection for each normal state ω of $B(K')$, we have obtained a set of normal projection maps from R onto M, which is easily seen to be complete. This completes the proof of the theorem.

Remark 1. If $R = B(\mathcal{H})$ with $\mathcal{H}$ a separable Hilbert space the theorem was shown by de Korvin [5] by different methods. He conjectured that it was also true for non separable $\mathcal{H}$ when $R = B(\mathcal{H})$.

Remark 2. With the assumptions as in the theorem and with G the group of inner automorphisms of R defined by the unitaries in $M' \cap R$ [7, Thm. 3.5] states the equivalence of the following three conditions

i) R is G-finite,

ii) $M' \cap R$ is finite and there exists a faithful normal projection of R onto M,

iii) $M' \cap R$ is finite of type I, and its center is totally atomic over C.

Thus, with a proper definition of G-semi-finite our theorem should be viewed as a G-semi-finite extension of [7, Thm. 3.5].

References

1. W. Arveson, Analyticity in operator algebras, Amer. J. Math., 89 (1967), 578-642.

2. J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, Paris, Gauthier-Villars, 1957.

3. A. Guichardet, Une caractérisation des algèbres de von Neumann discrètes, Bull Soc. math. France, 89 (1961), 77-101.

4. R. V. Kadison, Normalcy in operator algebras, Duke Math. J., 29 (1962), 459-464.

5. A. de Korvin, On complete sets of expectations, The Quarterly J. Math., 22 (1971), 135-143.

6. I. Kovács and J. Szücs, Egodic type theorems in von Neumann algebras, Acta Sc. Math., 27 (1966), 233-246.

7. E. Størmer, States and invariant maps of operator algebras, J. Functional Anal., 5 (1970), 44-65.

8. M. Takesaki, On the conjugate space of an operator algebra, Tôhoku Math. J., 10 (1958), 194-203.

9. J. Tomiyama, On the projection of norm one in W^*-algebras, III, Tôhoku Math. J., 11 (1959), 125-129.

10. J. Tomiyama, The extension property of von Neumann algebras and a class of C^*-algebras associated to them, To appear.

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