Invariant measures and von Neumann algebras.

by

Erling Størmer

University of Oslo, Oslo, Norway.

1. Introduction. It is well known, see e.g. [2, Ch.I, § 9] that there is a close relationship between the existence problem for invariant measures in ergodic theory and von Neumann algebra theory. In the present paper we shall elaborate on this relationship in order to study the partial ordering on measurable sets defined by Hopf [5]. He showed that "finiteness" (or boundedness) of the partial ordering was equivalent to the existence of a finite invariant measure equivalent to the given one. Later Kawada [8] and Halmos [3] showed the equivalence of "semi-finiteness" (or σ-boundedness) and the existence of a σ-finite measure. Our main result, which is stated in the language of von Neumann algebras, consists of two characterizations for measurable sets to be bounded in the sense of Hopf. The result has as a straightforward consequence the semi-finite results of Kawada and Halmos, and gives also more information on the close relationship between the measure space in question and von Neumann algebras (viz the canonically defined von Neumann algebra $\mathcal{B}$ below). The reader is referred to the books of Dixmier [2] and Jacobs [6] for the theory of von Neumann algebras and ergodic theory.

2. Hopf's equivalence relation. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Suppose G is a discrete group operating on the left on X by $\zeta \to s\zeta$, $\zeta \in X$, and assume μ is quasi-invariant, i.e. $\mu(s(E)) = 0$ if and only if $\mu(E) = 0$ for $E \in \mathcal{S}$. We

say two sets E and F in $\mathcal{S}$ are equivalent in the sense of Hopf, written $E \sim F$, if there is for each $s \in G$ a set $E_s \in \mathcal{S}$ such that $E_s = \emptyset$ except for a countable number of $s \in G$ and such that the families $\{E_s\}_{s \in G}$ and $\{s(E_s)\}_{s \in G}$ consist of pairwise disjoint sets with unions E and F respectively. If E and F are in $\mathcal{S}$ we write $E \prec F$ if there is $F_o \in \mathcal{S}$ such that $E \sim F_o \subset F$. We say a set $F \in \mathcal{S}$ is Hopf finite (also called bounded) if $E \subset F$ and $E \sim F$ implies $\mu(F-E) = 0$. $(X, \mathcal{S}, \mu)$ is said to be Hopf finite if X itself is Hopf finite. It is well known and first proved by Hopf for G cyclic, that there is a finite G-invariant measure on X equivalent to μ if and only if $(X, \mathcal{S}, \mu)$ is Hopf finite [1,5,8,9]. At this point it should be remarked that Yeadon's short proof of the existence of a trace in a finite von Neumann algebra [13] can be modified almost ad verbatim, using lemmas 2.3 and 2.4 below, to yield a new proof of the existence of a finite invariant measure in the Hopf finite case. Hence we shall feel free to quote the result in the finite case in the general situation we shall consider. We say $(X, \mathcal{S}, \mu)$ is Hopf semi-finite (also called σ-bounded) if every set $E \in \mathcal{S}$ of positive measure contains a Hopf finite subset in $\mathcal{S}$ of positive measure. It was shown by Kawada [8], and independently by Halmos [3] for cyclic groups, that $(X, \mathcal{S}, \mu)$ is Hopf semi-finite if and only if there is a σ-finite G-invariant measure on X equivalent to μ. .

We shall now translate the above discussion into the language of von Neumann algebras. Let $\mathcal{R} = L^{\infty}(X, \mathcal{S}, \mu)$ be the space of all essentially bounded μ-measurable complex functions on X. Then $\mathcal{R}$ is an abelian von Neumann algebra acting by left multiplication on $\mathcal{H} = L^2(X, \mathcal{S}, \mu)$. For each $s \in G$ let $r_s(\zeta)$ be the Radon Nikodym derivative of $\mu \circ s$ with respect to μ, i.e. $d\mu(s\zeta) = r_s(\zeta)d\mu(\zeta)$. For $f \in \mathcal{H}$ let $(U_s f)(\zeta) = r_{s^{-1}}(\zeta)^{\frac{1}{2}} f(s^{-1}\zeta)$. Then, see [2, Ch. I. § 9, no.3], $s \to U_s$ is a unitary representation of G on $\mathcal{H}$ such that if χ_E is the characteristic function of $E \in \mathcal{S}$, $U_s^* \chi_E U_s = \chi_{s(E)}$. We now generalize the above definitions as follows. Let $\mathcal{R}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a discrete group and $s \to U_s$ a unitary representation on $\mathcal{H}$ such that $U_s^* \mathcal{R} U_s = \mathcal{R}$ for $s \in G$. If E and F are projections in $\mathcal{R}$ we say E and F are equivalent in the sense of Hopf, written $E \sim F$ if there is for each $t \in G$ a projection $E_t \in \mathcal{R}$ such that $\sum_{t \in G} E_t = E$, $\sum_{t \in G} U_t^* E_t U_t = F$. In particular the families $\{E_t\}$ and $\{U_t^* E_t U_t\}$ consist of pairwise orthogonal projections. If $E \sim F_0 \leq F$ we write $E \prec F$. F is said to be <u>Hopf finite</u> if $E \leq F$ and $E \sim F$ implies $E = F$. $\mathcal{R}$ is said to be <u>Hopf semi-finite</u> if every non-zero projection in $\mathcal{R}$ majorizes a non-zero Hopf finite projection.

<u>Remark 2.1.</u> An equivalent definition of Hopf equivalence is as follows. We say E and F are equivalent if there are projections $\{E_\alpha\}$ and $t_\alpha \in G$ such that $E = \Sigma E_\alpha$ and $\Sigma U_{t_\alpha}^* E_\alpha U_{t_\alpha} = F$. But if $E_t = \sum_{t_\alpha = t} E_\alpha$ then $E_t \in \mathcal{R}$ and $E = \sum_{t \in G} E_t$, $F = \sum_{t \in G} U_t^* E_t U_t$, so the two definitions are equivalent. From this equivalent definition it is immediate that $\sim$ is indeed an equivalence relation, see e.g. [11, Lem.2.2]. Notice that since G might be uncountable it is here an advantage that $\mathcal{R}$ is a von Neumann algebra, so we can conclude that $E_t \in \mathcal{R}$. For measurable sets this is not clear.

Let notation be as above and let $\mathfrak{A} = \{A \in \mathcal{R}: U_s^* A U_s = A$ for $s \in G\}$. Then $\mathfrak{A}$ is an abelian von Neumann subalgebra of $\mathcal{R}$.

Remark 2.2. If $\{E_\alpha\}$ and $\{F_\alpha\}$ are each orthogonal families of projections in $\mathcal{R}$ and $E_\alpha \sim F_\alpha$ for all α then $\Sigma E_\alpha \sim \Sigma F_\alpha$. Furthermore, if $E \sim F$ in $\mathcal{R}$ and H is a projection in $\mathfrak{A}$ then $EH \sim FH$. This is immediate from definition and Remark 2.1.

The following result is quite useful, for proofs see [8, Lemma 16] or [11, Lemma 2.7].

Lemma 2.3. (The comparison theorem). Let E and F be two projections in $\mathcal{R}$. Then there exists a projection $H \in \mathfrak{A}$ such that $HE \prec HF$ and $(I-H)F \prec (I-H)E$.

Lemma 2.4. Let E be a Hopf finite projection in $\mathcal{R}$. Let P and Q be projections in $\mathcal{R}$ majorized by E such that $P \sim Q$. Then we have

i) $E-P \sim E-Q$.

ii) $P-PQ \sim Q-PQ$.

Proof. By Lemma 2.3 there is a projection $H \in \mathfrak{A}$ such that $H(E-P) \prec H(E-Q)$ and $(I-H)(E-Q) \prec (I-H)(E-P)$. Then $H(E-P) \sim F \leq H(E-Q)$. By Remark 2.2 we have

$$HE = H(E-P) + HP \sim F + HQ \leq H(E-Q) + HQ = HE.$$

Since E is finite so is HE, hence $F = H(E-Q)$, and $H(E-P) \sim H(E-Q)$. Similarly, $(I-H)(E-Q) \sim (I-H)(E-P)$, and i) follows. Since clearly $PQ \sim PQ$ we have by i) and Remark 2.2 that $E-P+PQ \sim E-Q+PQ$, i.e. $E-(P-PQ) \sim E-(Q-PQ)$. By i) $P-PQ \sim Q-PQ$, and ii) is proved.

3. Abelian von Neumann algebras. We recall from [2, Ch.I, § 9] the construction of the cross product of $\mathcal{R}$ and G, letting as before $\mathcal{R}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$, G a discrete group, and $s \to U_s$ a unitary representation of G on $\mathcal{H}$ such that $U_s^* \mathcal{R} U_s = \mathcal{R}$ for $s \in G$. For $s \in G$ let $\mathcal{H}_s$ be a Hilbert space of the same dimension as $\mathcal{H}$, let J_s be an isometry of $\mathcal{H}$ onto $\mathcal{H}_s$ and let $\mathcal{H}_e = \mathcal{H}$, $J_e = I$, where e is the identity in G. Let $\tilde{\mathcal{H}} = \sum_{s \in G} \oplus \mathcal{H}_s$. We write an operator $R \in \mathcal{B}(\tilde{\mathcal{H}})$ - the bounded operators on $\tilde{\mathcal{H}}$ - as a matrix $(R_{s,t})_{s,t \in G}$, where $R_{s,t} = J_s^* R J_t \in \mathcal{B}(\mathcal{H})$. For each $T \in \mathcal{R}$ let $\Phi(T)$ denote the element in $\mathcal{B}(\tilde{\mathcal{H}})$ with matrix $(R_{s,t})$ where $R_{s,t} = 0$, $s \neq t$, and $R_{s,s} = T$ for all $s \in G$. Then Φ is a *-isomorphism of $\mathcal{R}$ onto a von Neumann algebra $\tilde{\mathcal{R}}$ acting on $\tilde{\mathcal{H}}$. For $y \in G$ let $\tilde{U}_y$ be the operator in $\mathcal{B}(\tilde{\mathcal{H}})$ with matrix $(R_{s,t})$, where $R_{s,t} = 0$ if $st^{-1} \neq y$, $R_{yt,t} = U_y$ for $t \in G$. Then, see [2, Ch.I, § 9], $y \to \tilde{U}_y$ is a unitary representation of G on $\tilde{\mathcal{H}}$ such that $\tilde{U}_y^* \Phi(T) \tilde{U}_y = \Phi(U_y^* T U_y)$, $y \in G$, $T \in \mathcal{R}$. Let $\mathcal{B}$ denote the von Neumann algebra generated by $\tilde{\mathcal{R}}$ and the $\tilde{U}_y$, $y \in G$. Then each operator in $\mathcal{B}$ is represented by a matrix $(R_{s,t})$, where $R_{s,t} = T_{st^{-1}} U_{st^{-1}}$, $T_{st^{-1}} \in \mathcal{R}$. $\mathcal{B}$ is called the <u>cross product</u> of $\mathcal{R}$ and G, and Φ the <u>canonical isomorphism</u> of $\mathcal{R}$ into $\mathcal{B}$.

<u>Lemma 3.1</u>. Let $\mathfrak{S}$ denote the set of operators $V \in \mathcal{B}$ of the form $V = (E_{st^{-1}} U_{st^{-1}})$, where both $\{E_t\}_{t \in G}$ and $\{U_t^* E_t U_t\}_{t \in G}$ are orthogonal families of projections in $\mathcal{R}$. Then we have:

i) $\mathfrak{S}$ is self-adjoint and closed under multiplication.

ii) If $V \in \mathfrak{S}$ and $T \in \mathcal{R}$ then $V^* \Phi(T) V \in \tilde{\mathcal{R}}$.

iii) If E and F are projections in $\mathcal{R}$ then $E \sim F$ if and only if there is $V \in \mathfrak{S}$ such that $VV^* = \Phi(E)$, $V^*V = \Phi(F)$.

<u>Proof</u>. Let $V = (E_{st^{-1}} U_{st^{-1}}) \in \mathfrak{S}$. Then $V^* = (F_{st^{-1}} U_{st^{-1}})$, where $F_s = U^*_{s^{-1}} E_{s^{-1}} U_{s^{-1}}$, so both $\{F_s\}$ and $\{U^*_s F_s U_s\}$ are orthogonal families of projections. Hence $V^* \in \mathfrak{S}$, and $\mathfrak{S}$ is self-adjoint.

Let $V = (E_{st^{-1}} U_{st^{-1}})$ and $U = (F_{st^{-1}} U_{st^{-1}})$ belong to $\mathfrak{S}$. Then $VU = (G_{st^{-1}} U_{st^{-1}})$ where

$$G_s = \sum_r E_{sr^{-1}} U_{sr^{-1}} F_r U_{rs^{-1}} .$$

Then G_s is the sum of orthogonal projections, so it is itself a projection. A straightforward but somewhat tedious computation now shows that $\{G_s\}$ and $\{U^*_s G_s U_s\}$ are orthogonal families of projections, hence $VU \in \mathfrak{S}$, and i) follows.

Let $V = (E_{st^{-1}} U_{st^{-1}}) \in \mathfrak{S}$ and $T \in \mathcal{R}$. Then another straightforward computation shows

$$1) \qquad V^* \Phi(T)V = \Phi(\sum_r U^*_r E_r T U_r) \in \tilde{\mathcal{R}}.$$

Thus ii) follows.

Let E and F be projections in $\mathcal{R}$ such that $E \sim F$. Then we have projections $E_t \in \mathcal{R}$ such that $E = \Sigma E_t$ and $F = \Sigma U^*_t E_t U_t$, hence both families $\{E_t\}$ and $\{U^*_t E_t U_t\}$ are orthogonal families of projections. Let $V = (E_{st^{-1}} U_{st^{-1}})$. Then it is easy to see that $V \in \mathcal{B}$, hence by 1) $V^*V = \Phi(F)$, and similarly $VV^* = \Phi(E)$. Conversely if $V = (E_{st^{-1}} U_{st^{-1}}) \in \mathfrak{S}$ and $VV^* = \Phi(E)$, $V^*V = \Phi(F)$ then by 1) $\Sigma U^*_t E_t U_t = F$ and similarly $\Sigma E_t = E$. Hence $E \sim F$, and the proof is complete.

If E is a projection in $\mathcal{R}$ we denote by $\mathcal{R}E$ the von Neuman algebra consisting of operators TE, $T \in \mathcal{R}$, acting on $E\mathcal{R}$. Similarly we have $\tilde{\mathcal{R}}\Phi(E) = \Phi(\mathcal{R}E)$.

Lemma 3.2. Let E be a projection in $\mathcal{R}$. Let $\mathfrak{G}_E$ denote the set of $V \in \mathfrak{G}$ such that $V^*V = VV^* = \Phi(E)$. Then we have:

i) $\mathfrak{G}_E$ is a group of $*$-automorphisms of $\tilde{\mathcal{R}}\,\Phi(E)$.

ii) If P and Q are projections in $\mathcal{R}E$ such that there is $V \in \mathfrak{G}_E$ such that $V^*\,\Phi(P)V = \Phi(Q)$ then $P \sim Q$.

iii) If P and Q are orthogonal projections in $\mathcal{R}E$ such that $P \sim Q$ then there is $V \in \mathfrak{G}_E$ such that $V^*\,\Phi(P)V = \Phi(Q)$.

Suppose further that E is Hopf finite. Then we have:

iv) If P and Q are projections in $\mathcal{R}E$ and $P \sim Q$ then there is $V \in \mathfrak{G}_E$ such that $V^*\,\Phi(P)V = \Phi(Q)$.

v) $\tilde{\mathcal{R}}\,\Phi(E)$ is Hopf finite with respect to the group $\mathfrak{G}_E$ of $*$-automorphisms.

Proof. i) is immediate from Lemma 3.1.

If P and Q and V are as in ii) then since $\Phi(P) \in \mathfrak{G}$ we have from Lemma 3.1 that $U = \Phi(P)V \in \mathfrak{G}$. Since $U^*U = \Phi(Q)$ and $UU^* = \Phi(P)$, $P \sim Q$ by Lemma 3.1. Thus ii) is proved.

Let P and Q be projections in $\mathcal{R}E$ such that $P \sim Q$. If $PQ \neq 0$ assume that E is Hopf finite. Then by Lemma 2.4 $P - PQ \sim Q - PQ$. Hence there are projections $F_t \in \mathcal{R}$ such that $\Sigma\, F_t = P - PQ$ and $\Sigma\, U_t^*\, F_t\, U_t = Q - PQ$. Put

$$E_e = E - (P-PQ) - (Q-PQ)$$

$$E_t = F_t + U^*_{t^{-1}}\, F_{t^{-1}}\, U_{t^{-1}} \qquad \text{for } t \neq e.$$

Let $V = (E_{st^{-1}}\, U_{st^{-1}})$. Then it is easy to see that $V \in \mathfrak{G}_E$ and that $V^*\,\Phi(P)V = \Phi(Q)$. Thus iii) and iv) are proved.

Let P and Q be projections in $\mathcal{R}E$ such that $\Phi(P)$ and $\Phi(Q)$ are equivalent in the sense of Hopf with respect to the group $\mathfrak{G}_E$ of *-automorphisms of $\tilde{\mathcal{R}}\Phi(E)$. Then there is an orthogonal family $\{P_\alpha\}$ of projections in $\mathcal{R}$ such that $\Sigma\,\Phi(P_\alpha) = \Phi(P)$, and there are $V_\alpha \in \mathfrak{G}_E$ such that $\Sigma\, V_\alpha^* \Phi(P_\alpha)V_\alpha = \Phi(Q)$. In particular $\{V_\alpha^* \Phi(P_\alpha)V_\alpha\}$ is an orthogonal family of projections in $\tilde{\mathcal{R}}\Phi(E)$. Let Q_α be the projection in $\tilde{\mathcal{R}}$ such that $\Phi(Q_\alpha) = V_\alpha^* \Phi(P_\alpha)V_\alpha$. Then the Q_α's are all orthogonal and $\Sigma\, Q_\alpha = Q$. By ii) $P_\alpha \sim Q_\alpha$, hence by Remark 2.2, $P \sim Q$. In particular, if $Q = E$ then we have $P \sim E$, so $P = E$ since E is Hopf finite. Thus $\Phi(P) = \Phi(E)$, and v) follows. The proof is complete.

<u>Theorem 3.3</u>. Let $\mathcal{R}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a discrete group and $t \to U_t$ a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for $t \in G$. Let $\mathcal{B}$ be the cross product of $\mathcal{R}$ and G and Φ the canonical isomorphism of $\mathcal{R}$ into $\mathcal{B}$. Let E be a projection in $\mathcal{R}$, and let ω be a faithful normal semi-finite trace on $\mathcal{R}^+$ such that $\omega(E) < \infty$. Then the following conditions are equivalent.

i) E is Hopf finite.

ii) $\Phi(E)$ is a finite projection in $\mathcal{B}$.

iii) Given $\epsilon > 0$ there is $\delta = \delta(\epsilon, E) > 0$ such that if P is a projection in $\mathcal{R}E$ and $\omega(P) < \delta$, then $\omega(Q) < \epsilon$ for all projections $Q \in \mathcal{R}E$ for which $Q \sim P$.

<u>Proof</u>. We show i) => ii) => iii) => i).

i) => ii). Let E be Hopf finite, and suppose $E \neq 0$. By Lemma 3.2 $\tilde{\mathcal{R}}\Phi(E)$ is Hopf finite with respect to the group $\mathfrak{G}_E$. Let ω_E be the faithful normal finite trace on $\mathcal{R}E$ defined by

$\omega_E(A) = \omega(AE)$ for $A \in \mathcal{R}E$. Then $\tilde{\omega}_E$ defined by $\tilde{\omega}_E(\Phi(A)) = \omega_E(A)$ is the same on $\tilde{\mathcal{R}}\Phi(E)$. The theorems on finite invariant measures on Hopf finite algebras now apply to give a faithful normal finite G-invariant trace φ on $\tilde{\mathcal{R}}\Phi(E)$ such that $\varphi(\Phi(E)) = 1$. Let $\mathcal{B}_E$ denote the von Neumann algebra $\Phi(E)\mathcal{B}\Phi(E)$ acting on $\Phi(E)\mathcal{H}$. For $S = (T_{st^{-1}} U_{st^{-1}}) \in \mathcal{B}_E$ define $\Psi(S) = \varphi(\Phi(T_e))$. Then Ψ is a faithful normal finite trace on $\mathcal{B}_E$, see proof of [2, Ch.I, § 9, Prop.1]. Therefore $\mathcal{B}_E$ is a finite von Neumann algebra, in particular $\Phi(E)$ is a finite projection in $\mathcal{B}$.

ii) => iii). Suppose $\Phi(E)$ is a finite projection in $\mathcal{B}$. Then in particular E is Hopf finite in $\mathcal{R}$, cf. Lemma 3.1. Thus by Lemma 3.2 (iv) if P and Q are projections in $\mathcal{R}E$ and $P \sim Q$ then there is $V \in \mathfrak{G}_E$ such that $V^* \Phi(P)V = \Phi(Q)$. Furthermore from the proof of i) => ii) we have a faithful normal finite trace Ψ on $\mathcal{B}_E$. Then Ψ is $\mathfrak{G}_E$-invariant, so the conclusion in iii) follows from [12].

iii) => i). Assume given $\varepsilon > 0$ then there is $\delta > 0$ such that if P is a projection in $\mathcal{R}E$ and $\omega(P) < \delta$ then $\omega(Q) < \varepsilon$ for all projections $Q \in \mathcal{R}E$ such that $Q \sim P$. Since Φ is a $*$-isomorphism of $\mathcal{R}E$ onto $\tilde{\mathcal{R}}\Phi(E)$ we shall for simplicity of notation identify $\mathcal{R}E$ and $\tilde{\mathcal{R}}\Phi(E)$, and consider $\mathfrak{G}_E$ as a group of $*$-automorphisms of $\mathcal{R}E$. Let ω_E be as above. By Lemma 3.2 (ii) and [12] there is a faithful normal finite $\mathfrak{G}_E$-invariant trace ρ on $\mathcal{R}E$. We show that if P and Q are projections in $\mathcal{R}E$ with $P \sim Q$, then $\rho(P) = \rho(Q)$. By Lemma 3.2 (iii) this holds if P is orthogonal to Q. Let P and Q be arbitrary in $\mathcal{R}E$ and $P \sim Q$, say $P = \Sigma P_t$ and $Q = \Sigma U_t^* P_t U_t$ with P_t projections in $\mathcal{R}E$. Since ρ is normal it suffices to show $\rho(U_t^* P_t U_t) = \rho(P_t)$, or in general, if P is a projection in

$\mathcal{R}E$ such that $U_t^* P U_t \in \mathcal{R}E$ then $\rho(P) = \rho(U_t^* P U_t)$ for a given $t \in G$. Let E_t and F_t be projections in $\mathcal{R}$ with sum I such that $U_t^* T U_t = T$ for all $T \in \mathcal{R}E_t$, and the automorphism $T \to U_t^* T U_t$ is freely acting on $\mathcal{R}F_t$, see e.g. [7]. By freely acting we mean that given a projection $F \neq 0$ then there is a subprojection $H \neq 0$ of F such that H and $U_t^* H U_t$ are orthogonal. Clearly $\rho(PE_t) = \rho(U_t^* P E_t U_t)$, so it suffices to consider PF_t, i.e. we may assume $U_t^* \cdot U_t$ is freely acting. Let by Zorn's Lemma $\{P_\alpha\}$ be a maximal orthogonal family of subprojections of P such that $U_t^* P_\alpha U_t$ is orthogonal to P_α. Then $\Sigma P_\alpha = P$, for if not then there is a non-zero projection $R \leq P - \Sigma P_\alpha$ such that $U_t^* R U_t$ is orthogonal to R, contradicting the maximality of $\{P_\alpha\}$. Now $P_\alpha \sim U_t^* P_\alpha U_t$, and they are orthogonal and contained in $\mathcal{R}E$. Thus $\rho(P_\alpha) = \rho(U_t^* P_\alpha U_t)$, and by the normality of ρ, $\rho(P) = \rho(\Sigma P_\alpha) = \Sigma \rho(P_\alpha) = \Sigma \rho(U_t^* P_\alpha U_t) = \rho(U_t^* P U_t)$. Hence we have shown that if P and Q are projections in $\mathcal{R}E$ such that $P \sim Q$, then $\rho(P) = \rho(Q)$. Now suppose F is a projection in $\mathcal{R}$ such that $F \leq E$ and $F \sim E$. Then $F \in \mathcal{R}E$ so $\rho(F) = \rho(E)$, hence $\rho(E-F) = 0$. Since ρ is faithful on $\mathcal{R}E$, $E = F$. Thus E is Hopf finite. The proof is complete.

<u>Corollary 3.4.</u> Let $\mathcal{R}$ and G be as in Theorem 3.3. Then $\mathcal{R}$ is Hopf semi-finite if and only if there exists a faithful normal semi-finite G-invariant trace τ on $\mathcal{R}^+$. Furthermore, if E is a non-zero Hopf finite projection in $\mathcal{R}$ then we can choose τ such that $\tau(E) = 1$.

<u>Proof</u>. If such a trace τ exists it is well known and easy to see that $\mathcal{R}$ is Hopf semi-finite. Conversely, assume $\mathcal{R}$ is Hopf

semi-finite, and let E be a non-zero Hopf finite projection in $\mathcal{R}$. Let D_E be the smallest projection in $\mathcal{O}l$, the fixed point algebra in $\mathcal{R}$, such that $D_E \geq E$. If we can find a faithful normal semi-finite G-invariant trace φ on $\mathcal{R}^+ D_E$ such that $\varphi(E) = 1$, we can by Zorn's Lemma find a family $\{E_\alpha\}$ containing E with $\Sigma D_{E_\alpha} = I$ and a faithful normal semi-finite G-invariant trace τ_α on $\mathcal{R} D_{E_\alpha}$. Then $\tau = \Sigma \tau_\alpha$ is the desired trace. We may therefore assume $D_E = I$. By Theorem 3.3 $\Phi(E)$ is finite in $\mathcal{B}$. Suppose F is any non-zero projection in $\mathcal{R}$. Then it is easy to see, see e.g. [11, Lem. 2.3], that there is a non-zero subprojection F_o of F in $\mathcal{R}$ such that $F_o \prec E$. By Lemma 3.1 $\Phi(F_o) \precsim \Phi(E)$ (in the usual sense for projections in a von Neumann algebra). In particular $\Phi(F_o)$ is finite in $\mathcal{B}$, and the identity in $\mathcal{B}$ is the sup of finite projections, so $\mathcal{B}$ is semi-finite. Let Ψ be a faithful normal semi-finite trace on $\mathcal{B}$ such that $\Psi(\Phi(E)) = 1$. Let $\tau(T) = \Psi(\Phi(T))$ for $T \in \mathcal{R}^+$. Then τ is a faithful normal trace on $\mathcal{R}^+$ such that $\tau(E) = 1$. From the argument with F and F_o above we see that τ is semi-finite, and if $t \in G$ and $T \in \mathcal{R}$ we have

$$\tau(U_t^* \, T \, U_t) = \Psi(\Phi(U_t^* \, T \, U_t)) = \Psi(\tilde{U}_t^* \, \Phi(T) \, \tilde{U}_t) = \Psi(\Phi(T)) = \tau(T).$$

Hence τ is G-invariant. The proof is complete.

<u>Remark 3.5</u>. Theorem 3.3 and its corollary can be generalized as follows. Let $\mathcal{R}$ be an abelian von Neumann algebra and G a discrete group. Suppose $t \to \alpha_t$ is a representation of G as $*$-automorphisms of $\mathcal{R}$. Generalize the definition of equivalence in the sense of Hopf for two projections E and F in $\mathcal{R}$ to, $E \sim F$ if $E = \sum_{t \in G} E_t$ for E_t projections in $\mathcal{R}$, and

$F = \sum_{t\in G} \alpha_t(E_t)$. Then $\sim$ is an equivalence relation and both Theorem 3.3 and Corollary 3.4 generalize. Indeed, $\mathcal{R}$ is *-isomorphic to a maximal abelian von Neumann algebra $\mathcal{C}$, see [2, Ch. I, § 7, no.3]. Since every group of *-automorphisms of a maximal abelian von Neumann algebra is implemented by a group of unitary operators, see [4], Theorem 3.3 and Corollary 3.4 hold for $\mathcal{C}$, and thus the generalized versions hold for $\mathcal{R}$.

4. <u>Invariant measures</u>. Theorem 3.3 and Corollary 3.4 are immediately applicable to the case of σ-finite measures. We shall do it for Corollary 3.4 and leave the application of Theorem 3.3 to the reader.

<u>Corollary 4.1</u>. (Kawada, Halmos). Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Suppose G is a discrete group operating on the left on X such that μ is quasi-invariant. Then $(X, \mathcal{S}, \mu)$ is Hopf semi-finite if and only if there is a G-invariant σ-finite measure ν on $(X, \mathcal{S})$ which is equivalent to μ.

<u>Proof</u>. Since μ is σ-finite there is a finite measure on $(X, \mathcal{S})$ equivalent to μ and quasi-invariant. We may therefore assume μ is finite. Let $\mathcal{R} = L^\infty(X, \mathcal{S}, \mu)$. Then $\mathcal{R}$ is an abelian von Neumann algebra acting on $L^2(X, \mathcal{S}, \mu)$, and there is a faithful normal finite trace ω on $\mathcal{R}$ such that $\omega(\chi_E) = \mu(E)$ for $E \in \mathcal{S}$. In particular $\mathcal{R}$ is countably decomposable, hence if $\{E_\alpha\}$ is an orthogonal family of projections in $\mathcal{R}$ then $E_\alpha = 0$ for all α except a countable number of α's. We can therefore use any one of the different definitions of equivalence in the sense of Hopf. Therefore by the discussion in § 2, Corollary 3.4 is directly applicable. Thus if $(X, \mathcal{S}, \mu)$ is Hopf semi-finite then there is a faithful normal semi-finite G-invariant trace τ on $\mathcal{R}^+$.

Let $E \in \mathcal{S}$ and define $\nu(E) = \tau(\chi_E)$. Then ν is a countably additive, G-invariant measure on $(X, \mathcal{S})$. ν is equivalent to μ because if $E \in \mathcal{S}$, then $\nu(E) = 0$ if and only if $\tau(\chi_E) = 0$ if and only if $\chi_E = 0$ if and only if $\mu(E) = 0$. Finally, ν is σ-finite because $\mathcal{R}$ is countably decomposable, so there is an orthogonal sequence of projections E_n in $\mathcal{R}$ with $\tau(E_n) < \infty$ such that $\Sigma E_n = I$. If $E_n = \chi_{X_n}$ with $X_n \in \mathcal{S}$, then $\cup X_n = X$ (except perhaps for a null set) and $\nu(X_n) < \infty$. The proof is complete.

Remark 4.2. In order to obtain a G-invariant measure it is unnecessary to assume $(X, \mathcal{S}, \mu)$ is σ-finite. Indeed, it suffices to assume the measure space is localizable, i.e. is a direct sum of finite measure spaces. Under this assumption $L^\infty(X, \mathcal{S}, \mu)$ is a maximal abelian von Neumann algebra acting by left multiplication on $L^2(X, \mathcal{S}, \mu)$, see [10, 2.93]. Then the transformations of X defined by elements in G define *-automorphisms of $L^\infty(X, \mathcal{S}, \mu)$, so an application of Remark 3.5 completes the argument. This remark is also applicable to the application of Theorem 3.3 to measure spaces.

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