CORRECTION TO : A NEW PROOF OF THE CLASSICAL HERBRAND AND SKOLEM THEOREMS
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There is one serious omission in my definition of CLASSICAL TREE on Page 7.

In addition to conditions i - iii one must add:

iv) there is a well-order of the parameters such that for any parameter a introduced at a node ν by $\rightarrow \forall$ all parameters occurring at nodes below ν are strictly less than a in the well-order.

This condition expresses in a strong way that we build the classical trees from the bottom and upwards.

This addition does not affect the analyzing lemma and the strong analyzing lemma (pages 11-14). We just fix a well-order of the parameters in advance. Then in the constructions at each step where we need a new parameter, we pick the smallest parameter greater than all parameters used in the construction so far.

The only place where this strengthening of 'classical trees' is needed is in the definition of the Skolem morphism $\mathfrak{S}^x_\pi$ on pages 19-20. Let A be the set of new parameters introduced by occurrences of $\forall x$ in T . The point is that in going from T to $\dot{\mathfrak{S}}^x_\pi(T)$ we want to get rid of all occurrences of elements of A and replace it by terms built up by using one new functionsymbol f . If we do the procedure described in the definition we get rid of all occurrences of $\forall x$ by using f and symbols possibly from A . Say $a \in A$ is assigned $f(t_1,\ldots,t_N)$. Now by the new condition iv on classical trees only parameters strictly less than a occurs in $f(t_1,\ldots,t_N)$. Hence by induction on the well-order of the parameters in A we get rid of all uses of symbols from A by systematically substituting their Skolem terms. - We end up with a tree $\mathfrak{S}^x_\pi(T)$ where there are no occurrences of A .

The extra condition is needed to make the above construction go through. Just consider the tree below. There are no way to construct the result of applying $\mathcal{S}$ on it :

$$\dfrac{\dfrac{\dfrac{\dfrac{\to Aab}{\to \forall y\ Aay}}{\neg \forall y\ Aay \to}}{\forall x \neg \forall y\ Axy \to} \qquad \dfrac{\dfrac{\dfrac{\to Aba}{\to \forall y\ Aby}}{\neg \forall y\ Aby \to}}{\forall x \neg \forall y\ Axy \to}}{\forall x \neg \forall y\ Axy \to}$$

There are two other errors:

1. On page 9 in case v. of the definition of ANALYZING BRANCH instead of 'as a successor to $\forall xFx$' write 'as a successor to a formula in the same strand of formulae as $\forall xFx$'.

2. On page 25 in the downmost lemma instead of '-- function-symbols in T must ---' write '-- functionsymbols in $\mathcal{S}_{\pi}^{x}(T)$ must ---'.

A new proof of the classical herbrand and Skolem theorem

by

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1. INTRODUCTION

In this paper I will give a new proof of the classical Skolem and Herbrand theorems. As formulated here the proofs belong to general proof theory and can be seen as a development of the works of Beth [1], Hintikka [7] and Smullyan [9] on the completeness of the cut-free rules of classical first order logic. - There are a number of proofs of the Herbrand theorem. One could mention Herbrands original proof [5] as corrected by Dreben and Denton [2], Hilbert and Bernays's proof of the ϵ-theorem [6], and Gentzen's proof of the mid-sequent theorem [4]. What do I hope to gain by giving this new proof ? First of all I want to give Herbrand theorems for other logics. So this paper will be followed by other papers on higher order logic, infinitary logic, and intuitionistic logic. Secondly I divide the Herbrand theorem up into two theorems - the Skolem theorem and the Herbrand theorem. The Skolem theorem gives the Skolem normal form of formulae. See Skolem [8]. The theorem I call the Herbrand theorem gives the usual Herbrand theorem for formulae in Skolem normal form. Surprisingly enough it turns out that the difficulties come in proving the Skolem theorem and not the Herbrand theorem. For example in first order intuitionistic logic the Skolem theorem is false while the Herbrand theorem (for formulae in Skolem normal form) is true. Thirdly instead of letting the theorems talk about connections between formulae and transformed formulae, I treat them as giving connection between prooftrees and transformed prooftrees. In this way both the constructive character of the theorems and the uniformity between proofs of formulae and proofs of transformed formulae come out in a nice way.

2. THE FORMAL SYSTEM

We introduce our logic:

LANGUAGE. The language is a usual one with function-symbols. We do not include equality yet, but will treat it later. Instead of just binary conjunctions we have arbitrary finite ones. Among the constants we have a particular one, e , which will play some importance later. So the language consists of

connectives $\bigwedge$ (finite conjunction) $\neg$

quantifier $\forall$

parameters $a_1, a_2, \ldots, b, c, \ldots$

variables $x_1, x_2, \ldots, y, z, \ldots$

functionsymbols and constants $e, f_1, f_2, \ldots, g, h, \ldots$

predicate symbols $P_1, P_2, \ldots, Q, R, \ldots$

In the usual way we build up

terms $t_1, t_2, \ldots, u, v, \ldots$

atomic formulae $A_1, A_2, \ldots, B, C, \ldots$

formulae $F_1, F_2, \ldots, G, H, \ldots$

finite (and empty) sequences of formulae $\Gamma_1, \Gamma_2, \ldots, \Delta, \Lambda, \ldots$

sequents $\Gamma \to \Delta$ where Γ and Δ are finite or empty sequences of formulae; Γ is called the antecedent and Δ the succedent; for Γ or Δ empty we may write $\to \Delta$, $\Gamma \to$, $\to$.

We do not have $\bigvee$, $\supset$, $\exists$ among the symbols. That is just to save a few cases in the definitions and the proofs below. It will be clear that we could have included $\bigvee$, $\supset$, $\exists$ with no extra problems.

In the formulae we do not have free variables. Instead we have parameters. To each predicate symbol and function-symbol we have assigned a number which gives the number of argument places. We write $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ,F etc. for the obvious concatenations of sequences of formulae.

THE CALCULUS LK. On the language we build the sequential calculus LK in the usual way:

Axioms $\quad \Gamma_1,A,\Gamma_2 \rightarrow \Delta_1,A,\Delta_2$ for A atomic

STRUCTURAL RULES

Permutation $\quad \dfrac{\Gamma \rightarrow \Delta}{\Gamma^* \rightarrow \Delta^*}$, where Γ^* is obtained from Γ by a permutation of formulae, and similarly Δ^* from Δ.

Thinning $\quad \dfrac{\Gamma \rightarrow \Delta}{\Gamma,\Gamma_1 \rightarrow \Delta,\Delta_1}$

Contraction $\quad \dfrac{\Gamma,F,F \rightarrow \Delta}{\Gamma,F \rightarrow \Delta} \qquad \dfrac{\Gamma \rightarrow G,G,\Delta}{\Gamma \rightarrow G,\Delta}$

Trivial rule $\quad \dfrac{\Gamma \rightarrow \Delta \text{ ----- } \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$, where we have a finite number of premisses.

LOGICAL RULES

$\bigwedge \rightarrow \quad \dfrac{\Gamma,F_j \rightarrow \Delta}{\Gamma,\bigwedge_{i\in I} F_i \rightarrow \Delta}$ where F_j is one of the conjuncts in $\bigwedge_{i\in I} F_i$.

$\rightarrow \bigwedge \quad \dfrac{\text{---}\ \Gamma \rightarrow F_j,\Delta\ \text{---}}{\Gamma \rightarrow \bigwedge_{i\in I} F_i,\Delta}$ where we have as premisses $\Gamma \rightarrow F_j,\Delta$ for all $j\in I$.

$\neg \rightarrow \quad \dfrac{\Gamma \rightarrow F,\Delta}{\Gamma,\neg F \rightarrow \Delta}$

$\rightarrow \neg \quad \dfrac{\Gamma,F \rightarrow \Delta}{\Gamma \rightarrow \neg F,\Delta}$

$\forall \rightarrow$ $\qquad \dfrac{\Gamma, Ft \rightarrow \Delta}{\Gamma, \forall x Fx \rightarrow \Delta}$ $\qquad$ t is a term.

$\rightarrow \forall$ $\qquad \dfrac{\Gamma \rightarrow Fa.\Delta}{\Gamma \rightarrow \forall x Fx, \Delta}$ $\qquad$ a is a parameter not in $\Gamma \rightarrow \forall x Fx, \Delta$.

This completes the description of our formal system LK. We write $\vdash_{LK} \Gamma \rightarrow \Delta$ for there is a derivation of $\Gamma \rightarrow \Delta$ in LK.

It is well known that

i) $\vdash_{LK} \Gamma, F \rightarrow F, \Delta$

ii) If $\vdash_{LK} \Gamma_1, F \rightarrow \Delta_1$ and $\vdash_{LK} \Gamma_2 \rightarrow F, \Delta_2$ then $\vdash_{LK} \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2$.

3. COMPLETENESS THEOREM

TREES. Below we will work with trees. All our trees are thought of as having a unique downmost node and are then spreading upwards. At the nodes we have associated sequents. The sequents at one node and the sequents at the successornodes will be connected as premisses and conclusion of rules of LK. So the trees will be as the tree of sequents in a derivation except that we do not assume the trees to be well-founded. To talk about such trees we need a few notions. First consider the rules of LK. Observe that to each formula in one of the premisses we can in a natural way associate a unique formula in the conclusion. We say that the formula in the conclusion immediately precedes (or is immediately succeeded by) the formula in the premiss. The details should be clear by the examples below. This correspondence extends to parts of formulae. So if formula F immediately precedes formula G then to each part of formula G we associate in a natural way a part of F as immediately preceding it (or being immediately succeeded by it). Except for a possible change of terms the parts in F and G are equal. We have two notions of immediately precedes (or of immediately succeeds) - one for formula and another for formula parts. Whenever it is not clear from the context which of the two notions we use, we will mention it explicitly. We now define "immediately precedes" and "immediately succeeds" in trees and take "precedes" and "succeeds" to be their transitive and reflexive closure. Two formula parts are in the same strand if they have a common predecessor. They will then be equal except for a possible change of terms. To get better control over the terms introduced for variables in a tree we define <u>the analysis of a formula F</u> as the list of terms introduced for variables in formulae which precedes F.

EXAMPLES

We write $A \wedge B$ for the conjunction of A and B, etc.
The numbers above the arrows indicate the nodes.

$$\underline{A, B, \forall y R(fa,y), R(fa,ffa) \xrightarrow{10} C} \qquad \underline{A, B, \forall y R(fa,y), R(fa,fe) \xrightarrow{13} D}$$

$$\underline{A, B, \forall y R(fa,y), \forall y R(fa,y) \xrightarrow{9} C} \qquad \underline{A, B, \forall y R(fa,y), \forall y R(fa,y) \xrightarrow{12} D}$$

$$\underline{A, B, \forall y R(fa,y) \xrightarrow{8} C} \qquad \underline{A, B, \forall y R(fa,y) \xrightarrow{11} D}$$

$$\underline{A, B, \forall y R(fa,y) \xrightarrow{1} C \wedge D}$$

$$\underline{A, B \xrightarrow{6} \neg \forall y R(fa,y), C \wedge D}$$

$$\underline{A, A \wedge B \xrightarrow{5} \neg \forall y R(fa,y), C \wedge D}$$

$$\underline{A \wedge B, A \xrightarrow{4} \neg \forall y R(fa,y), C \wedge D}$$

$$\underline{A \wedge B, A \wedge B \xrightarrow{3} \neg \forall y R(fa,y), C \wedge D}$$

$$\underline{A \wedge B, A \wedge B \xrightarrow{2} \forall \neg \forall y R(fx,y), C \wedge D}$$

$$\underline{A \wedge B \xrightarrow{1} \forall x \neg \forall y R(fx,y), C \wedge D}$$

As formula A in 10 succeeds A in 9,8,7,6,5,4
and $A \wedge B$ in 3,2,1.

As formula part A in 10 succeeds A in 9,8,7,6,5,4,3,2,1.

All A's are in the same strand.

As formula $R(fa,fe)$ in 13 succeeds $\forall y R(fa,y)$ in 12,11,7
and $\neg \forall y R(fa,y)$ in 6,5,4,3
and $\forall x - \forall y R(fx,y)$ in 2,1.

As formula part R(fa.fe) in 13 succeeds R(fa.y) in 12,11,7,6,5,4,3
and R(fx,y) in 2,1.

R(fa,fe) in 13 has as analysis a for x and fe for y.

R(fa,ffa) in 10 has as analysis a for x and ffa for y.

POSITIVE AND NEGATIVE. We define positive and negative occurrences in a sequent $\Gamma \to \Delta$ inductively by

i) Any formula in Δ occurs positively in $\Gamma \to \Delta$.

ii) Any formula in Γ occurs negatively in $\Gamma \to \Delta$.

iii) If $\bigwedge F_i$ occurs positively (negatively) in $\Gamma \to \Delta$, then each F_i occurs positively (negatively) in $\Gamma \to \Delta$.

iv) If $\neg F$ occurs positively (negatively) in $\Gamma \to \Delta$, then F occurs negatively (positively) in $\Gamma \to \Delta$.

v) If $\forall x Fx$ occurs positively (negatively) in $\Gamma \to \Delta$, then Fx occurs positively (negatively) in $\Gamma \to \Delta$.

GENERAL AND RESTRICTED. A quantifier $\forall x$ in $\Gamma \to \Delta$ is general (restricted) if it occurs as $\forall x Fx$ with $\forall x Fx$ positive (negative) in $\Gamma \to \Delta$.

CLASSICAL TREE. A classical tree over a sequent $\Gamma \to \Delta$ is a tree of sequents with $\Gamma \to \Delta$ at the downmost node and such that

i) a sequent at any node and the sequent at its successor-nodes are related as one of the rules of LK;

ii) the term introduced at a node by $\forall \to$ is built up from constants, parameters, and functionsymbols in $\Gamma \to \Delta$, the constant e, and from parameters introduced by $\to \forall$ somewhere in the tree; and

iii) parameters introduced in $\to \forall$ are distinct if we analyze quantifiers not in the same strand or with distinct analyses.

The classical trees will be our working material. They are meant to generalize the notion of a prooftree, or rather attempted prooftree. In our mind iii is quite natural. It says that for parameters introduced by $\rightarrow \forall$, for distinct quantifiers we introduce distinct parameters. - It may be clear from our use of 'introduce' that we consider a prooftree as starting with the downmost node and then spreading upwards.

We now develop in a rather sketchy fashion the theory of Beth [1], Hintikka [7], Smullyan [9], and show where our theory starts.

BRANCH. A branch in a tree is a path going from the downmost node and as far as possible upwards.

SECURED. A node in a classical tree is secured if the sequent at it is an axiom. A branch is secured if it contains a secured node. A classical tree is secured if all its branches are secured.

LEMMA. Given a classical tree. Then if the tree is secured, then there is a finite classical tree over the same sequent which is secured. (The finite tree can of course be taken as a subtree of the given tree.)

PROVABILITY THEOREM. If we have a secured classical tree over $\Gamma \rightarrow \Delta$, then $\vdash_{LK} \Gamma \rightarrow \Delta$.

Proof:
By the lemma we can assume the tree to be finite. The theorem follows by induction over the nodes of the tree.

Q.E.D.

ANALYZING BRANCH. A branch β in a classical tree T is an analyzing branch when:

i) if $\bigwedge F_i$ occurs (as a formula) in an antecedent in β, then each F_i occurs as a successor to $\bigwedge F_i$ in an antecedent in β;

ii) if $\bigwedge F_i$ occurs in a succedent in β, then at least one of the F_i occurs as a successor to $\bigwedge F_i$ in a succedent in β;

iii) if $\neg F$ occurs in an antecedent in β, then F occurs as a successor to $\neg F$ in a succedent in β;

iv) if $\neg F$ occurs in a succedent in β, then F occurs as a successor to F in an antecedent in β;

v) if $\forall xFx$ occurs in an antecedent in β, then for every term t built up from constants, parameters, and functionsymbols in T, Ft occurs as a successor to $\forall xFx$ in an antecedent in β; and

vi) if $\forall xFx$ occurs in a succedent in β, then there is a term t such that Ft occurs as a successor to $\forall xFx$ in a succedent in β.

ANALYZING TREE. A tree is analyzing if every branch is.

Now in the classical theory we derive:

ANALYZING LEMMA. To any sequent we can find an analyzing classical tree over it.

Proof:

See below.

and then

FALSIFIABILITY LEMMA. If we have a not-secured analyzing branch β in a classical tree over $\Gamma \to \Delta$, then we can find a falsifying model for $\Gamma \to \Delta$ (i.e. a model in which all the formulae in Γ are true and all those in Δ false).

Proof:

Assume β as above.

We construct the model as follows:

The universe consists of all terms which can be built up from constants, parameters, and functionsymbols in formulae in β. An atomic formula is true if and only if it occurs in an antecedent in β.

By induction over the length of formulae we prove that every formula occurring in an antecedent in β is true in the model, every formula occurring in a succedent in β is false in the model.

This gives the lemma.

Q.E.D.

SOUNDNESS LEMMA. For any sequent $\Gamma \to \Delta$, if $\vdash_{LK} \Gamma \to \Delta$, then there are no falsifying models of $\Gamma \to \Delta$.

Proof:

By inductions over the derivations in LK.

Q.E.D.

COMPLETENESS THEOREM. For any sequent $\Gamma \to \Delta$, $\vdash_{LK} \Gamma \to \Delta$ if and only if there are no falsifying models of $\Gamma \to \Delta$.

Proof :

The soundness lemma gives the 'only if' part.

Assume there are no falsifying models of $\Gamma \to \Delta$.

By analyzing lemma we have an analyzing classical tree over $\Gamma \to \Delta$

By falsifiability lemma and the assumption all
branches in the analyzing tree must be secured.
By provability lemma $\vdash_{LK} \Gamma \rightarrow \Delta$.

Q.E.D.

CONSISTENCY THEOREM. For any sequent $\Gamma \rightarrow \Delta$ we have exactly one of i and ii below

i) a secured classical tree over $\Gamma \rightarrow \Delta$.

ii) a classical tree over $\Gamma \rightarrow \Delta$ with not-secured analyzing branch.

Proof:
We have either i or ii by the analyzing lemma.
We cannot have both since then both $\vdash_{LK} \Gamma \rightarrow \Delta$ and not $\vdash_{LK} \Gamma \rightarrow \Delta$.

Q.E.D.

Our theory starts with the observation that the usual proofs of the analyzing lemma prove more than stated. We first give a proof of it:

ANALYZING LEMMA. For any sequent we have an analyzing classical tree over it.

Proof:
We start with the one sequent tree consisting of the given sequent and then tack on new nodes. At each finite stage we have a finite tree. The limit tree will be analyzing.
We have the following possibilities of tacking on new nodes:

i)

$$
\begin{array}{c}
\Gamma_1,\Gamma_2,\Gamma_1,\ldots,F_N \longrightarrow \Delta \\
\hline
\vdots \\
\Gamma_1,\Gamma_2, \bigwedge F_i, F_1 \longrightarrow \Delta \\
\hline
\Gamma_1,\Gamma_2, \bigwedge F_i, \bigwedge F_i \longrightarrow \Delta \\
\hline
\Gamma_1,\Gamma_2, \bigwedge F_i \longrightarrow \Delta \\
\hline
\Gamma_1, \bigwedge F_i, \Gamma_2 \longrightarrow \Delta
\end{array}
$$

Here $F_1,\ldots,F_N$ are all the conjuncts of $\bigwedge F_i$. We assume $\Gamma_1, \bigwedge F_i, \Gamma_2 \rightarrow \Delta$ to be at a top-most node and then tack on the nodes to the left.

ii)

$$
\begin{array}{c}
\Gamma \rightarrow F_1,\Delta_1,\Delta_2 \;\; --- \;\; \Gamma \rightarrow F_N,\Delta_1,\Delta_2 \\
\hline
\Gamma \longrightarrow \bigwedge F_i,\Delta_1,\Delta_2 \\
\hline
\Gamma \longrightarrow \Delta_1, \bigwedge F_i,\Delta_2
\end{array}
$$

Here $F_1,\ldots,F_N$ are the conjuncts of $\bigwedge F_i$.

iii)

$$
\begin{array}{c}
\Gamma_1,\Gamma_2 \longrightarrow F,\Delta \\
\hline
\Gamma_1,\Gamma_2 \neg F \longrightarrow \Delta \\
\hline
\Gamma_1, \neg F,\Gamma_2 \longrightarrow \Delta
\end{array}
$$

iv)

$$
\begin{array}{c}
\Gamma,F \longrightarrow \Delta_1,\Delta_2 \\
\hline
\Gamma \longrightarrow \neg F,\Delta_1,\Delta_2 \\
\hline
\Gamma \longrightarrow \Delta_1, \neg F,\Delta_2
\end{array}
$$

v)

$$
\begin{array}{c}
\Gamma_1,\Gamma_2,\forall xFx,Ft \longrightarrow \Delta \\
\hline
\Gamma_1,\Gamma_2,\forall xFx,\forall xFx \longrightarrow \Delta \\
\hline
\Gamma_1,\Gamma_2 \; \forall xFx \longrightarrow \Delta \\
\hline
\Gamma_1, \forall xFx,\Gamma_2 \longrightarrow \Delta
\end{array}
$$

t is a term built up from parameters, constants, functionsymbols in the tree constructed so far and also the constant e. We assume that there are noe formula Ft in the same strand as the Fx in $\forall xFx$ in a node below $\Gamma_1, \forall xFx,\Gamma_2 \rightarrow \Delta$.

vi) $$\dfrac{\dfrac{\Gamma \longrightarrow Fa, \Delta_1, \Delta_2}{\Gamma \longrightarrow \forall xFx, \Delta_1, \Delta_2}}{\Gamma \longrightarrow \Delta_1, \forall xFx, \Delta_2}$$

a is a parameter not in $\Gamma \to \forall xFx, \Delta_1, \Delta_2$. If a is introduced for $\to \forall$ somewhere else, it is only as analysis for a formula in the same strand and with the same analysis as $\forall xFx$.

This completes the possibilities for tacking on new nodes. We come now to the construction of the analyzing classical tree. The construction goes by stages. It is easily seen that we tack on only finitely many new nodes at each stage.

STAGE 1. Put down the given sequent as a one sequent tree.

STAGE 3N-1. Apply possibilities i-iv above to extend the tree as many times as possible.

STAGE 3N. Apply possibility v above with t a term of length $\leq N$ as many times as possible.

STAGE 3N+1. Apply possibility vi above as many times as possible.

This completes the construction. It should be clear that we get an analyzing classical tree.

Q.E.D.

We give two easy lemmata about the construction.

LEMMA. Consider a topmost node in the tree we get after having applied one of the possibilities in one of the stages. Then for each strand and each analysis, there is at most one formula at the node in the strand with the analysis.

LEMMA. If we have two occurrences in the resulting analyzing tree of a non-atomic formula in the same strand and with the same analysis, then in the construction of the tree the two formulae will be analyzed at the same stage.

Both lemmata are obvious by inspection of the construction. Now consider the definition of a classical tree. We strengthen it to define:

STRONG CLASSICAL TREE. A classical tree is a strong classical tree if:
parameters introduced by $\rightarrow \forall$ are equal if and only if we analyze quantifiers in the same strand and with the same analysis.

The lemmata above enables us to strengthen the construction of the analyzing tree to get:

STRONG ANALYZING LEMMA. To any sequent we can find a strong, analyzing, classical tree over it.

Proof:
We change the construction by:

STAGE* 3N+1. Apply possibility vi with the extra proviso that formulae in the same strand and with the same analyses are analyzed by the same parameter.

The other stages as before.
The lemmata tell us that we can perform this construction.
We clearly get an analyzing classical tree which is strong.

Q.E.D.

4. SKOLEM THEOREM

We come now to the Skolem theorem. First an example.

EXAMPLE. We give a secured classical tree for

$$\longrightarrow \neg \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)].$$

To save space we perform more than one rule in a step in a few obvious places below.

The most important point about the example is to get some grasp on the function of the quantifiers. We indicate the nodes by numbers above the arrows.

$$Bb \xrightarrow{17} Bb$$

$$Aa \xrightarrow{15} Aa \qquad\qquad \xrightarrow{16} Bb, \neg Bb$$

$$Aa \xrightarrow{14} Bb, Aa \wedge \neg Bb$$

$$\neg Bb \qquad , \neg (Aa \wedge \neg Bb), Aa \xrightarrow{13}$$

$$Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc), Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb), Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{12}$$

$$Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb), Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{11} \neg [Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc)]$$

$$Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb), Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{10} \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)]$$

$$\neg \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)], Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{9} \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{8} \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \xrightarrow{7} \forall y \neg [Ay \wedge \neg Ba \wedge \neg (Aa \wedge \neg By)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \neg \forall y \neg [Ay \wedge \neg Ba \wedge \neg (Aa \wedge \neg By)] \xrightarrow{6} \neg [Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \xrightarrow{5} \neg [Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \xrightarrow{4} \forall y \neg [Ay \wedge \neg Be \wedge \neg (Ae \wedge \neg By)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \neg \forall y \neg [Ay \wedge \neg Be \wedge \neg (Ae \wedge \neg By)] \xrightarrow{3}$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \xrightarrow{2}$$

$$\xrightarrow{1} \neg \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)]$$

We will discuss our example informally. In the bottomsequent 1 we have a restricted quantifier $\forall x$ and a general quantifier $\forall y$. The classical tree is strong, but certainly not analyzing. We do not analyze $\forall x$ with respect to c. The $\forall y$'s are used to introduce new parameters. We are allowed to have a new parameter whenever we have a new analysis of $\forall x$. We introduce new terms for $\forall x$ in 2,5,8. Corresponding to these we introduce new parameters in 4,7,10. That is e.g. we introduce c for y in 10 to get $\neg [Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc)]$. This formula succeeds $\neg \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)]$ in 9 which we have got by introducing b for x. We can summarize.

In 2 e for x and then in 4 a for y.
In 5 a for x and then in 7 b for y.
In 8 b for x and then in 10 c for y.

We only introduce new parameters for general quantifiers in a formula when we have a new analysis of the formula. This will be made precise with the Skolem-functions we define below. There are two ways of regarding Skolem-functions, either as indices for parameters or as new terms. We do the latter. We will develop the Skolem theorem in a more general setting than usual - not only as about transformations of sequents (or formulae) but as about transformations of classical trees. To do this we need a little notation.

MORPHISM. A classical morphism is a transformation of classical trees into classical trees preserving the tree structure. A provability morphism is a classical morphism which transforms secured trees into secured trees. An analyzing morphism transforms analyzing trees into analyzing trees. A falsifiability morphism transforms analyzing not-secured trees into analyzing not-secured trees. A classical isomorphism is a classical morphism which is both a provability and a falsifiability morphism.

The classical morphisms we use below will be simple. It will be clear that we preserve more than the tree-structure. Most of the rules are also preserved. The transformations will also be quite constructive. We do not think it worthwhile here to express these things with a sharper definition of classical morphism.

Our main example of a non-trivial classical isomorphism will be the Skolem morphism, $\mathcal{S}$, defined below. For a one sequent tree it coincides with the usual Skolem transformation. Skolem defined originally his transformation as a transformation to get rid of general quantifiers [8]. It is now more common to treat it as a transformation to get rid of restricted quantifiers. This is more natural when one treats the Skolem theorem semantically. We will follow Skolem, but the other way can be read into our treatment by stressing the falsifiability aspect of the theorem.

We assume that we have a way of denoting positions in sequents.

SKOLEM TRANSFORM. Given a sequent $\Gamma \to \Delta$. The Skolem transform of $\Gamma \to \Delta$ with respect to the variable x and the position π, denoted by $S_\pi^x(\Gamma \to \Delta)$ is defined as:

i) if x does not appear as general variable $\forall x$ in position π in $\Gamma \to \Delta$, then $S_\pi^x(\Gamma \to \Delta) = \Gamma \to \Delta$.

ii) Say x appears as general variable $\forall x$ in position π in $\Gamma \to \Delta$. Let $y_1, \ldots, y_N$ $(N \geq 0)$ be the variables such that the restricted variables $\forall y_1, \ldots, \forall y_N$ bind $\forall x$ in $\Gamma \to \Delta$. Then get a new functionsymbol f with N argumentplaces, called the Skolem-function of x in position π in $\Gamma \to \Delta$. We get $S^x_\pi(\Gamma \to \Delta)$ by putting $f(y_1, \ldots, y_N)$ for x in the range of $\forall x$ and then delete $\forall x$.

We get $S(\Gamma \to \Delta)$, the Skolem transform of $\Gamma \to \Delta$, by repeatedly applying S^x_π for various x and π until we get a sequent without general variables. It is easily seen to be well defined except for the names of the Skolem-functions.

<u>SKOLEM MORPHISM</u>. Given a classical tree T over $\Gamma \to \Delta$ we define $\mathfrak{S}^x_\pi(T)$, the Skolem morphism of T with respect to x and π, by

i) If x does not appear as general variable $\forall x$ in position π in $\Gamma \to \Delta$, then $\mathfrak{S}^x_\pi(T) = T$.

ii) Say x appears as general variable $\forall x$ in position π in $\Gamma \to \Delta$. Let f be the Skolem-function and say it has N arguments. To each occurrence of $\forall x$ in T in the same strand as π in $\Gamma \to \Delta$ we assign a term $f(t_1, \ldots, t_N)$ called Skolem term. Now fix such an $\forall x$. Say that in the sequent it occurs, it is bound by the restricted variables $\forall y_{M+1}, \ldots, \forall y_N$ and that the $\forall x$ in position π is bound by the restricted variables $\forall y_1, \ldots, \forall y_M$, $\forall y_{M+1}, \ldots, \forall y_N$. Now consider the analysis of the formula with the fixed $\forall x$. Say we have analyzed y_1 as $s_1, \ldots, y_M$ as s_M. Then to the fixed $\forall x$ we assign the Skolem term $f(s_1, \ldots, s_M, y_{M+1}, \ldots y_N)$. We now get $\mathfrak{S}^x_\pi(T)$ from T by putting the Skolem terms for x in the range of any occurrence of $\forall x$ (in the same strand as π); then delete each such $\forall x$; and lastly for each parameter a introduced by analysis of such an $\forall x$ we put the Skolem term of the $\forall x$ for a whenever a occurs.

EXAMPLE. Consider the sequent $\Gamma \to \Delta = \to \forall x \neg \forall y \neg \forall z \neg \forall u R(x,y,z,u)$

Say x appears as $\forall x$ in position π in $\Gamma \to \Delta$

and z appears as $\forall z$ in position p in $\Gamma \to \Delta$.

Let x have Skolem-function f and y have g with respectively 0 and 1 argument.

Then

$$S^x_\pi(\Gamma \to \Delta) = \longrightarrow \neg \forall y \neg \forall z \neg \forall u\ R(f,y,z,u)$$

$$S^z_p(\Gamma \to \Delta) = \longrightarrow \forall x \neg \forall y \neg \neg \forall u\ R(x,y,gy,u)$$

$$S^x_p(\Gamma \to \Delta) = S^z_\pi(\Gamma \to \Delta) = S^y_\pi(\Gamma \to \Delta) = \Gamma \to \Delta$$

$$S\ (\Gamma \to \Delta) = S^x_{\pi'}\ S^z_p(\Gamma \to \Delta) = S^z_{p'}\ S^x_\pi(\Gamma \to \Delta)$$

$$= \longrightarrow \neg \forall y \neg \neg \forall u\ R(f,y,gy,u).$$

π' is the position of $\forall x$ in $S^z_p(\Gamma \to \Delta)$

p' is the position of $\forall z$ in $S^x_\pi(\Gamma \to \Delta)$.

Let T be the classical tree over $\Gamma \to \Delta$ below. To save space we have omitted some of the nodes. We indicate the nodes by numbers above the arrows.

$$
\begin{array}{r@{\;}l}
R(a,e,b,a),\ R(a,b,c,c),\ \forall u\, R(a,a,d,u) & \xrightarrow{13} \\ \hline
R(a,e,b,a),\ R(a,b,c,c) & \xrightarrow{12} \neg \forall u\, R(a,a,d,u) \\ \hline
\forall u\, R(a,e,b,u),\ \forall u\, R(a,b,c,u) & \xrightarrow{11} \neg \forall u\, R(a,a,d,u) \\ \hline
\forall u\, R(a,e,b,u),\ \forall u\, R(a,b,c,u) & \xrightarrow{10} \forall z \neg \forall u\, R(a,a,z,u) \\ \hline
\neg \forall z \neg \forall u\, R(a,a,z,u) & \xrightarrow{9} \neg \forall u\, R(a,e,b,u),\ \neg \forall u\, R(a,b,c,u) \\ \hline
\neg \forall z \neg \forall u\, R(a,a,z,u) & \xrightarrow{8} \neg \forall u\, R(a,e,b,u),\ \forall z \neg \forall u\, R(a,b,z,u) \\ \hline
\neg \forall z \neg \forall u\, R(a,b,z,u),\ \neg \forall z \neg \forall u\, R(a,a,z,u) & \xrightarrow{7} \neg \forall u\, R(a,e,b,u) \\ \hline
\forall y \neg \forall z \neg \forall u\, R(a,y,z,u) & \xrightarrow{6} \neg \forall u\, R(a,e,b,u) \\ \hline
\forall y \neg \forall z \neg \forall u\, R(a,y,z,u) & \xrightarrow{5} \forall z \neg \forall u\, R(a,e,z,u) \\ \hline
\neg \forall z \neg \forall u\, R(a,e,z,u),\ \forall y \neg \forall z \neg \forall u\, R(a,y,z,u) & \xrightarrow{4} \\ \hline
\forall y \neg \forall z \neg \forall u\, R(a,y,z,u) & \xrightarrow{3} \\ \hline
 & \xrightarrow{2} \neg \forall y \neg \forall z \neg \forall u\, R(a,y,z,u) \\ \hline
 & \xrightarrow{1} \forall x \neg \forall y \neg \forall z \neg \forall u\, R(x,y,z,u)
\end{array}
$$

We then get $\mathfrak{S}_p^z(T)$

$$\underline{R(a,e,ge,a),\ R(a,ge,gge,gge),\ \forall u\, R(a,a,ga,u) \xrightarrow{13}}$$

$$\underline{R(a,e,ge,a)\ ,\ R(a,ge,gge,gge) \xrightarrow{12} \neg \forall u\, R(a,a,ga,u)}$$

$$\underline{\forall u\, R(a,e,ge,u),\ \forall u\, R(a,ge,gge,u) \xrightarrow{11} \neg \forall u\, R(a,a,ga,u)}$$

$$\underline{\forall u\, R(a,e,ge,u),\ \forall u\, R(a,ge,gge,u) \xrightarrow{10} \neg \forall u\, R(a,a,ga,u)}$$

$$\underline{\neg \neg \forall u\, R(a,a,ga,u) \xrightarrow{9} \neg \forall u\, R(a,e,ge,u),\ \neg \forall u\, R(a,ge,gge,u)}$$

$$\underline{\neg \neg \forall u\, R(a,a,ga,u) \xrightarrow{8} \neg \forall u\, R(a,e,ge,u),\ \neg \forall u\, R(a,ge,gge,u)}$$

$$\underline{\neg \neg \forall u\, R(a,ge,gge,u),\ \neg \neg \forall u\, R(a,a,ga,u) \xrightarrow{7} \neg \forall u\, R(a,e,ge,u)}$$

$$\underline{\forall y \neg \neg \forall u\, R(a,y,gy,u) \xrightarrow{6} \neg \forall u\, R(a,e,ge,u)}$$

$$\underline{\forall y \neg \neg \forall u\, R(a,y,gy,u) \xrightarrow{5} \neg \forall u\, R(a,e,ge,u)}$$

$$\underline{\neg \neg \forall u\, R(a,e,ge,u),\ \forall y \neg \neg \forall u\, R(a,y,gy,u) \xrightarrow{4}}$$

$$\underline{\forall y \neg \neg \forall u\, R(a,y,gy,u) \xrightarrow{3}}$$

$$\underline{\xrightarrow{2} \neg \forall y \neg \neg \forall u\, R(a,y,gy,u)}$$

$$\xrightarrow{1} \forall x \neg \forall y \neg \neg \forall u\, R(x,y,gy,u)$$

Let us see how this fits the definition. Consider the occurrence of $\forall z$ in the antecedent of node 8. It has assigned Skolem term ga. It has as predecessor $\forall z$ in node 7 with Skolem term ga, $\forall z$ in nodes 6,5,4,3,2,1 with Skolem term gy. Now follow the $\forall z$ upwards from node 8. In nodes 9 and 10 it has Skolem term ga. In node 11 it has vanished. Instead we have the new parameter d. In going from T to $\mathfrak{S}_p^z(T)$ we put ga for d. Similarly with the other parameters:

the parameter	a	a
the parameter	b	ge
the parameter	c	gge.

We delete all $\forall z$ and for each occurrence of z we put the Skolem term.

We get $\mathfrak{S}(T)$ by deleting $\forall x$ and putting f for a and x in $\mathfrak{S}_p^z(T)$.

LEMMA. Let T be a classical tree over $\Gamma \to \Delta$. Consider the formulae containing no $\forall x$ in the same strand as an $\forall x$ in position π in $\Gamma \to \Delta$. We get the corresponding formulae in $\mathfrak{S}_\pi^x(T)$ by a transformation of terms. If T is a strong classical tree, the transformation is 1-1.

Proof:

We only need to consider the new parameters introduced by an $\forall x$ (in the same strand as a general $\forall x$ in position π in $\Gamma \to \Delta$). Since T is classical, the parameters introduced by the $\forall x$ are distinct from the parameters introduced by analysis of other variables. Parameters introduced by analysis of two occurrences are distinct if the occurrences have different analyses. If T is strong we get if and only if the occurrences have different analyses.

Since it is clear that the only difference between the formulae in T described and those in $\varsigma^x_\pi(T)$ is the terms, we have the lemma.

Q.E.D.

LEMMA. ς^x_π is a classical morphism for all x and π.

Proof:

Let T be a classical tree.

$\varsigma^x_\pi(T)$ have the same treestructure. We must prove that it is a classical tree.

The only problem comes in the applications of $\rightarrow\forall$.

So say we have sequents below some place in T:

$$\frac{\Gamma \rightarrow Fa, \Delta}{\Gamma \rightarrow \forall y Fy, \Delta} \qquad \text{a is a new parameter}$$

with y a variable not being affected by ς^x_π.

Since T is classical, a is only introduced by variables not being affected by ς^x_π. Hence a is not being changed in going to $\varsigma^x_\pi(T)$. Γ, F, Δ must be changed in the same way in premiss and conclusion. So we have still an application of $\rightarrow\forall$ in $\varsigma^x_\pi(T)$.

Now consider the sequents

$$\frac{\Gamma \rightarrow Gb, \Delta}{\Gamma \rightarrow \forall x Gx, \Delta} \qquad \text{b is a new paramter}$$

with x a variable being affected by ς^x_π.

Say t is the Skolem term of $\forall x$ in the conclusion above, and $\Gamma \rightarrow \forall x Gx, \Delta$ is transformed to $\Gamma^* \rightarrow G^*t, \Delta^*$. Then $\Gamma \rightarrow Gb, \Delta$ is also transformed to $\Gamma^* \rightarrow G^*t, \Delta^*$ and we get an application of the trivial rule.

So the applications of $\rightarrow\forall$ are OK. $\varsigma^x_\pi(T)$ is a classical tree.

Q.E.D.

LEMMA ς^x_π is a provability morphism for all x and π.

Proof:

From the lemmata above we get that the secured nodes in T are transformed into secured nodes in $\varsigma^x_\pi(T)$.

Q.E.D.

LEMMA. Let T be an analyzing classical tree containing at least one restricted variable. Then every term built up from parameters, constants, and functionsymbols in T must actually occur in $\varsigma^x_\pi(T)$. (For every x and π).

Proof:

Assume not.

Let T be an analyzing classical tree containing at least one restricted variable. Then every term built up from paramters, constants, and functionsymbols in T must occur as an analysis of the restricted variable in T.

Let t be a term built up from parameters, constants, and functionsymbols in $\varsigma^x_\pi(T)$, and not occurring itself in $\varsigma^x_\pi(T)$, and of minimal length.

Then t must be of the form $f(t_1,\ldots,t_N)$ when f is the Skolem function of $\forall x$ and $N > 0$. By the minimality of t, $t_1,\ldots,t_N$ must occur in $\varsigma^x_\pi(T)$. There must be terms $s_1,\ldots,s_N$ in T which are transformed over into $t_1,\ldots,t_N$ in $\varsigma^x_\pi(T)$. Since T is analyzing there must be a formula $\forall xFx$ in succedent with analysis $s_1,\ldots,s_N$ of the restricted variables binding $\forall x$ in the bottom sequent. We analyze $\forall xFx$ with Fa in T where a is a new parameter. In going from T to $\varsigma^x_\pi(T)$ a is transformed into $f(t_1,\ldots,t_N)$ and hence $t = f(t_1,\ldots,t_N)$ does occur in $\varsigma^x_\pi(T)$. Contradiction.

Q.E.D.

LEMMA. ς^x_π is an analyzing morphism for all x and π.

Proof:

Let T be an analyzing classical tree and assume we have a branch β which is not analyzing in $\varsigma^x_\pi(T)$. There must be a term t built up from parameters, constants, and functionsymbols in $\varsigma^x_\pi(T)$ and a formula $\forall yF^*y$ in an antecedent in β in $\varsigma^x_\pi(T)$ but with not F^*t in an antecedent in β. In T say the formula corresponds to $\forall yFy$.

By the lemma above t must actually occur in $\varsigma^x_\pi(T)$. Hence there is a term s in T which are transformed over into t in $\varsigma^x_\pi(T)$.

Since T is analyzing Fs occurs in an antecedent in β in T. In the transformation to $\varsigma^x_\pi(T)$ Fs gets over into F^*t. Hence F^*t ooccurs in an antecedent in β in T.

Contradiction. And we get $\varsigma^x_\pi(T)$ analyzing.

Q.E.D.

LEMMA. Let T be a classical tree which is strong, not-secured and analyzing. Then for every x and π $\varsigma^x_\pi(T)$ is not-secured.

Proof:

Assume β is a branch which is not secured in T but secured in $\varsigma^x_\pi(T)$.

Let ν be a secured node in β in $\varsigma^x_\pi(T)$.

Say at ν we have the sequent $\Gamma^*, At \to At, \Delta^*$ in $\varsigma^x_\pi(T)$.

Then in T either both At's correspond to atomic formula or the one in the succedent is $\forall xAx$ while the one in the antecedent is atomic.

Take the first case.

Since T is strong, atomic formulae in T are transformed into formulae in $\mathcal{S}^x_\pi(T)$ by a 1-1 transformation of terms. Hence ν is secured in T in this case.

Now the second case.

We will prove that this cannot occur. The sequent at ν in T must be of the form $\Gamma, Ab \rightarrow \forall xAx, \Delta$. By the assumption $\forall xAx$ must be analyzed with a formula succeding it and using the parameter b since T is strong. But this contradicts the parameter-condition in $\rightarrow\forall$.

We get β secured in T which is a contradiction.

Hence $\mathcal{S}^x_\pi(T)$ is not secured.

Q.E.D.

LEMMA. $\mathcal{S}^x_\pi$ is a falsifiability morphism for every x and π.

Proof:

Let T be a not-secured analyzing tree over $\Gamma \rightarrow \Delta$.

Then $\mathcal{S}^x_\pi(T)$ is analyzing.

By the strong analyzing lemma and the lemma above we can find an analyzing classical tree T^* over $\Gamma \rightarrow \Delta$ with $\mathcal{S}^x_\pi(T^*)$ not-secured if T^* is not-secured.

By consistency theorem T^* must be not-secured.

Hence $\mathcal{S}^x_\pi(T^*)$ not-secured.

It is also analyzing since T^* is.

We have an analyzing not-secured tree over $S^x_\pi(\Gamma \rightarrow \Delta)$.

By consistency theorem $\mathcal{S}^x_\pi(T)$ must be not-secured.

Q.E.D.

Putting the lemmata together.

THEOREM $\mathcal{S}^{x}_{\pi}$ is a classical isomorphism for every x and π.

We can also define a morphism $\mathcal{S}$ by repeated applications of $\mathcal{S}^{x}_{\pi}$ for various x and π. It is well defined except for the names of the Skolem functions. Using $\mathcal{S}$ we get rid of all general quantifiers in a classical tree.

THEOREM. $\mathcal{S}$ is a classical isomorphism.

We get the ordinary Skolem theorem by considering what happens with the bottom sequent. By the completeness theorem and the analyzing lemma we get

THEOREM. For any Γ, Δ, x, π

$$\vdash_{LK} \Gamma \to \Delta \iff \vdash_{LK} S^{x}_{\pi}(\Gamma \to \Delta) \iff \vdash_{LK} S(\Gamma \to \Delta).$$

5. HERBRAND THEOREM

Now having got rid of the general quantifiers we try to get rid of the restricted quantifiers. To do this we construct the Herbrand morphisms. It is certainly impossible to get a classical isomorphism to get rid of all restricted quantifiers. We would then get a decisionprocedure for provability in LK. We will construct a sequence of falsifiability morphisms which "in the limit" is a provability morphism.

HERBRAND DOMAIN. A set $\mathcal{D}$ of terms is an Herbrand domain if for any finite set of parameters, constants and functionsymbols the set of terms built up from them and in $\mathcal{D}$ is finite.

HERBRAND TRANSFORM. Let $\mathcal{D}$ be an Herbrand domain, x a variable, π a position. We then define the Herbrand transform of a sequent $\Gamma \rightarrow \Delta$ with respect to $\mathcal{D}$, x, and π, denoted by $H^{x}_{\pi,\mathcal{D}}(\Gamma \rightarrow \Delta)$, as

i) If $\Gamma \rightarrow \Delta$ contains a general variable or if $\forall x$ does not occur as a restricted variable in position π then $H^{x}_{\pi,\mathcal{D}}(\Gamma \rightarrow \Delta) = \Gamma \rightarrow \Delta$.

ii) Now assume $\Gamma \rightarrow \Delta$ contains no general variables and $\forall x$ occurs as a restricted variable in position π in $\Gamma \rightarrow \Delta$.

Let $\mathcal{D}_0$ be the finite subset of $\mathcal{D}$ of terms built up from parameters, constants, and functionsymbols in $\Gamma \rightarrow \Delta$. We get $H^{x}_{\pi,\mathcal{D}}(\Gamma \rightarrow \Delta)$ from $\Gamma \rightarrow \Delta$ by replacing $\forall x$ in position π by $\underset{x \in \mathcal{D}_0}{\bigwedge}$.

HERBRAND MORPHISM. Let $\mathcal{D}$ be an Herbrand domain, x a variable and π a position. For T a classical tree over $\Gamma \rightarrow \Delta$, we define the Herbrand morphism on T, $\mathcal{H}^x_{\pi,\mathcal{D}}(T)$, by:

i) If either $\Gamma \to \Delta$ contains a general variable or $\forall x$ does not occur as a restricted variable in position π in $\Gamma \to \Delta$, then $\mathcal{H}^x_{\pi,\mathcal{D}}(T) = T$.

ii) Now assume $\Gamma \to \Delta$ (and hence also T) contains no general variables and that $\forall x$ occurs as a restricted variable in position π in $\Gamma \to \Delta$. Let $\mathcal{D}_0$ be the finite subset of $\mathcal{D}$ built up from constants, variables and functionsymbols in $\Gamma \to \Delta$. We then get $\mathcal{H}^x_{\pi,\mathcal{D}}$ by

a) Replace every $\forall x$ in T in the same strand as the $\forall x$ in position π in $\Gamma \to \Delta$ by $\bigwedge\limits_{x \in \mathcal{D}_0}$.

b) If a formula F in T succeeds a formula Gt which again immediately succeeds $\forall x Gx$ with $\forall x$ in the same strand as the $\forall x$ in position π in $\Gamma \to \Delta$ and $t \notin \mathcal{D}_0$, then delete F.

c) For every other formula not affected by a or b we do not change it.

$\mathcal{H}_{\mathcal{D}}$ is the morphism and $H_{\mathcal{D}}$ the transform we get by repeated applications of $\mathcal{H}^x_{\pi,\mathcal{D}}$ and $H^x_{\pi,\mathcal{D}}$ for various x and π. Let $\mathcal{D}_n$ be the set of terms of length $\leq n$. $\mathcal{D}_n$ is an Herbrand domain. We write $H^x_{\pi,n}$, H_n, $\mathcal{H}^x_{\pi,n}$, $\mathcal{H}_n$ for $H^x_{\pi,\mathcal{D}_n}$, $H_{\mathcal{D}_n}$, $\mathcal{H}^x_{\pi,\mathcal{D}_n}$, $\mathcal{H}_{\mathcal{D}_n}$.

EXAMPLE. This is a continuation of an old example we had on a Skolem morphism. We had a classical tree T over $\to \forall x \neg \forall y \neg \forall z \neg \forall u\, R(x,y,z,u)$. Let T^* be tree over $\to \neg \forall y \neg \neg \forall u\, R(f,y,gy,u)$. $T^* = \mathcal{S}(T)$. Say $\forall y$ have position σ in $\to \neg \forall y \neg \neg \forall u\, R(f,y,gy,u)$.

We then have $\mathcal{H}^{y}_{\sigma,\{e,f\}}(T^*) = \mathcal{H}^{y}_{\sigma,1}(T^*) =$

$$\begin{aligned}
& R(f,e,ge,f),\ \forall u\ R(f,f,gf,u) \xrightarrow{13} \\
& R(f,e,ge,f) \xrightarrow{12} \neg \forall u\ R(f,f,gf,u) \\
& \forall u\ R(f,e,ge,u) \xrightarrow{11} \neg \forall u\ R(f,f,gf,u) \\
& \neg\neg \forall u\ R(f,f,gf,u) \xrightarrow{10} \neg \forall u\ R(f,e,ge,u) \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{9} \neg \forall u\ R(f,e,ge,u) \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{8} \neg \forall u\ R(f,e,ge,u) \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{7} \neg \forall u\ R(f,e,ge,u) \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{6} \neg \forall u\ R(f,e,ge,u) \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{5} \neg \forall u\ R(f,e,ge,u) \\
& \neg\neg \forall u\ R(f,e,ge,u),\ \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{4} \\
& \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \xrightarrow{3} \\
& \xrightarrow{2} \neg \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u) \\
& \xrightarrow{1} \neg \bigwedge_{y\in\{e,f\}} \neg\neg \forall u\ R(f,y,gy,u)
\end{aligned}$$

LEMMA. $\mathcal{H}^{x}_{\pi,\mathfrak{D}}$, $\mathcal{H}_{\mathfrak{D}}$, $\mathcal{H}^{x}_{\pi,n}$ and $\mathcal{H}_{n}$ are classical morphisms.

Proof:

Obvious from definition.

Q.E.D.

LEMMA. $\mathcal{H}^{x}_{\pi,\mathfrak{D}}$, $\mathcal{H}_{\mathfrak{D}}$, $\mathcal{H}^{x}_{\pi,n}$ and $\mathcal{H}_{n}$ are analyzing morphisms.

Proof:

Obvious.

Q.E.D.

The following obvious lemma is the crucial step towards the Herbrand theorem.

LEMMA. Let $\mathcal{D}_1 \subseteq \mathcal{D}_2$, T, y, π be given. Consider the formulae not containing a part in the same strand as the position π in the bottom sequent. Call those π-formulae. Then for any node ν -

the π-formulae in antecedent (succedent) at ν in $\mathcal{H}^{y}_{\pi,\mathcal{D}_1}(T)$

$\subseteq$ the π-formulae in antecedent (succedent) at ν in $\mathcal{H}^{y}_{\pi,\mathcal{D}_2}(T)$

$\subseteq$ the π-formulae in antecedent (succedent) at ν in T.

There is a finite set of terms $\mathcal{D}_3$ such that

the π-formulae in antecedent (succedent) at ν in $\mathcal{H}^{y}_{\pi,\mathcal{D}_3}(T)$

$=$ the π-formulae in antecedent (succedent) at ν in T.

From the lemma

THEOREM. For any $x, y, \pi, \mathcal{D}$, $\mathcal{D}_1 \subseteq \mathcal{D}_2$

1. $\mathcal{H}^{x}_{\pi,\mathcal{D}}$ is a falsifiability morphism.
2. $\mathcal{H}^{x}_{\pi,\mathcal{D}_1}(T)$ secured $\Rightarrow$ ${}^{x}_{\pi,\mathcal{D}_2}(T)$ secured
3. If T is secured we can find finite $\mathcal{D}_3$ with $\mathcal{H}^{x}_{\pi,\mathcal{D}_3}(T)$ secured.

[Similarly for $\mathcal{H}_{\mathcal{D}}$, $\mathcal{H}_n$, $\mathcal{H}^{x}_{\pi,n}$].

For the proof of 3 in the theorem we only need to note that if T is secured we can find a finite secured subtree of T over the same sequent.

Using the completeness theorem and disregarding everything except the bottom sequent we get the usual Herbrand theorem.

THEOREM. For any sequent $\Gamma \to \Delta$

$$\vdash_{LK} \Gamma \to \Delta \iff \exists n \vdash_{LK} H_n S(\Gamma \to \Delta).$$

6. EQUALITY

We come now to LK with equality, LK =. We add $=$ to the language and as extra axioms

$$r = s,\ Arr \rightarrow Ars \quad \text{and} \quad r = s,\ Ass \rightarrow Ars$$

with Ars atomic. We have written Arr, Ass, Ars to indicate that we substitute r for s or s for r in some but not necessarily all occurrences. We write $\vdash_{LK=}$ for derivability in LK=.

Now we can develop the theories above for LK= with very few changes from LK. We define secured in LK= as an extension of secured in LK to include the new axioms. The morphisms are defined as before. To show that ς is a provability morphism in LK= we only need to note that the axioms are still axioms after a transformation of terms.

THEOREM. ς (and ς^{x}_{π}) is a provability morphism in LK= .

Since we do not change the rules, ς is an analyzing classical morphism in LK= . As in LK the following is the crucial step to prove that ς is a falsifiability morphism in LK= .

LEMMA. Let T be a strong analyzing tree such to every general quantifier formula $\forall xFx$ in T, there is an Fa with a parameter in T and $\forall xFx$ precedes Fa. Then if T is not secured in LK= , then also $\varsigma^{y}_{\pi}(T)$ for any y, π.

Proof:

Assume we have a branch β , not-secured (LK=) in T but secured (LK=) in $\varsigma^{y}_{\pi}(T)$.

Let ν be a secured (LK=) node in β in $\varsigma^{y}_{\pi}(T)$.

Say at ν we have a sequent in $\varsigma^y_\pi(T)$:

$$r = s,\ Arr \rightarrow Ars.$$

If the sequent in T only had atomic formulae, we would have ν secured (LK=) in T, since atomic formulae in T go over into atomic formulae in $\varsigma^y_\pi(T)$ by a 1-1 transformation of terms.
So suppose the sequent in ν in T not only had atomic formulae.
The only possibilities are

$$r' = s',\ Ar'r' \rightarrow \forall y\ Ar'y$$

or

$$r' = s',\ Ar'r' \rightarrow \forall y\ Ays'.$$

But then either r' or s' must be the parameter which we are supposed to use in the analysis of $\forall y$ further up in T. This clearly contradicts the parameter condition in $\rightarrow \forall$.
So we have a contradiction.
We conclude $\varsigma^y_\pi(T)$ not-secured.

Q.E.D.

And we get

THEOREM. ς is a falsifiability morphism in LK=.
(The same with ς^y_π.)

THEOREM. ς and ς^y_π are classical isomorphisms in LK=

The whole chapter on Herbrand theorem in LK can be carried over without change.

THEOREM. In LK= we have

1. $\mathcal{H}^x_{\pi,\mathcal{D}}$, $\mathcal{H}_\mathcal{D}$, $\mathcal{H}^x_{\pi,n}$, $\mathcal{H}_n$ are falsifiability morphisms.
2. If $\mathcal{H}_\mathcal{D}(T)$ is secured and $\mathcal{D} \subseteq \epsilon$ then $\mathcal{H}_\epsilon(T)$ also. (Similarly with $\mathcal{H}^x_{\pi,\mathcal{D}}$, $\mathcal{H}^x_{\pi,n}$, $\mathcal{H}_n$).
3. If T is secured, we can find finite $\mathcal{D}$ with $\mathcal{H}_\mathcal{D}(T)$ secured.

7. CONCLUSION

In their paper on Herbrand-style consistency proofs [3] Dreben and Denton formulated the Herbrand therorem as:

(a) There is a uniform way to find (primitive recursively) a tautologous validity expansion for any logical theorem A from any logical proof of A.

(b) There is a uniform way to find (primitive recursively) a logical proof for a formula A from any tautologous validity expansion of A.

I think that the notion of morphism explains this uniformity in a nice way. By a slight change of the theory above the constructive aspect becomes more transparent. Namely change ii in the definition of classical tree to become "--- somewhere in the branch through the node;" instead of "--- somewhere in the tree". The morphisms defined will then be such that the transformation of a node depends constructively on the node and the nodes below it. - Our reason for not doing that is that the theory of the Herbrand morphism becomes slightly more clumsy. $\mathcal{H}$ will then not necessarily be a falsifiability morphism. Instead we could prove that for any sequent $\Gamma \to \Delta$ if $\mathcal{T}^*$ is a secured tree over $H_{\mathcal{D}}(\Gamma \to \Delta)$, then we can find a secured tree $\mathcal{T}$ over $\Gamma \to \Delta$ with $\mathcal{H}_{\mathcal{D}}(\mathcal{T}) = \mathcal{T}^*$.

The Herbrand theorem is usually formulated by the Herbrand expansions. We get those by taking the prenex normalform of the Skolem transform and then taking the Herbrand transforms of that. The reason for not working with Herbrand expansions is of course that it destroys the uniformity expressed by the morphisms.

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