Isometries on Irreducible Triangular Operator Algebras

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In [1] Arveson associates a norm closed irreducible triangular operator algebra to each ergodic, invertible and measure preserving transformation of the unit interval with Lebesgue measure. He then proves that two such transformations are conjugate if and only if the associated operator algebras are unitarily equivalent. The purpose of this note is to describe the situation in which the associated operator algebras are merely assumed to be isometric. It turns out in this case that either one of the transformations or its inverse is conjugate to the other.

Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$ with respect to normalized Lebesgue measure m. Let $\mathcal{M}$ be the maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all multiplications by bounded measurable functions. Since the projections in $\mathcal{M}$ correspond to the characteristic functions of measurable subsets of $[0,1]$, we may lift the measure m to a measure on the Boolean algebra of projections in $\mathcal{M}$. We may then define a $*$-automorphism α of $\mathcal{M}$ to be <u>measure preserving</u> if $m(\alpha(P)) = m(P)$ for each projection P in $\mathcal{M}$ and to be <u>ergodic</u> if $\alpha(P) = P$ only for the projections 0 and I. Two ergodic measure preserving $*$-automorphisms α and β are <u>conjugate</u> if there is a

*-automorphism τ such that $\tau \circ \alpha = \beta \circ \tau$. There are other essentially equivalent settings for ergodic theory, but for the purposes of operator theory this one is particularly convenient.

Since $\mathcal{M}$ can be identified with $L^\infty[0,1]$, a measure preserving ergodic *-automorphism α of $\mathcal{M}$ gives rise to a multiplicative and L^2-norm isometric linear mapping of $L^\infty[0,1]$ onto itself. This mapping has a unique extension to a unitary operator U_α in $\mathcal{B}(\mathcal{H})$ and U_α has the property that $\alpha(A) = U_\alpha A U_\alpha^*$ for all $A \in \mathcal{M}$. Any other unitary operator V with the property $\alpha(A) = VAV^*$ for all $A \in \mathcal{M}$ is of the form $V = U_\alpha M$, where M is a unitary in $\mathcal{M}$. See [1] for more details.

In [4] Kadison and Singer proved that the algebra of operators $\mathcal{S}(\alpha)$ generated by $\mathcal{M}$ and U_α is an irreducible triangular algebra. (By irreducible we mean that the only closed subspaces of $\mathcal{H}$ left invariant by the algebra are $\mathcal{H}$ and (0). An operator algebra $\mathcal{T}$ is triangular if $\mathcal{T} \cap \mathcal{T}^*$ is a maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$.) In [1] Arveson proved further that the norm closed algebra $\mathcal{T}(\alpha)$ generated by $\mathcal{M}$ and U_α is also triangular. He then obtained the following theorem :

Let α and β be ergodic measure preserving *-automorphisms of $\mathcal{M}$. Then α and β are conjugate if and only if there is a unitary operator W such that $\mathcal{T}(\alpha) = W\mathcal{T}(\beta)W^*$

We should also remark that the relation $\alpha(A) = U_\alpha A U_\alpha^*$ implies that the algebra $\mathcal{S}(\alpha)$ consists of all elements of the form $\sum_{n=0}^{k} A_n U^n$, where $A_n \in \mathcal{M}$.

Lemma. Suppose V is a unitary operator in $\mathcal{J}(\alpha)$ such that $\mathcal{M}$ and V generate $\mathcal{J}(\alpha)$ as a norm closed algebra. Suppose further that $VAV^* \in \mathcal{M}$ for each $A \in \mathcal{M}$. Then there exists a unitary operator $M \in \mathcal{M}$ such that $V = U_\alpha M$.

Proof. Let σ be the automorphism of $\mathcal{M}$ into itself defined by $\sigma(A) = VAV^*$. (That σ is surjective follows from the fact that $V^*AV \in \mathcal{M}$ whenever $A \in \mathcal{M}$. And this in turn is shown by observing that both V^*AV and its adjoint commute with every element of $\mathcal{M}$, which is maximal abelian.) To prove the lemma it will suffice to prove that σ is freely acting in the sense that for each non-zero projection P in $\mathcal{M}$ there exists a projection Q in $\mathcal{M}$, $0 \neq Q \leq P$, such that $\sigma(Q)$ is orthogonal to Q. This is suffcient because the corollary to lemma 1.7 in [1] applies in exactly these circumstances to give our lemma.

Suppose σ is not freely acting. Then there exists a projection $P \neq 0$ in $\mathcal{M}$ such that for each non-zero sub-projection Q of P, the projections Q and VQV^* are not orthogonal. Since any sub-projection of P retains this property we may also assume $P \neq I$. Now P and VPV^* are a pair of commuting projections (both lie in $\mathcal{M}$), so we may write $VPV^* = PVPV^* + (I-P)VPV^*$, where both summands are projections. Let $R = (I-P)VPV^*$ and let $S = V^*RV$. Then S is a sub-projection of P with the property that S is orthogonal to VSV^*. Therefore we must have $S = 0$ and hence $R = 0$. So we obtain $VPV^* = PVPV^*$ and, after multiplying on the right by V, we finally get $VP = PVP$. Thus P is left invariant by V. Since P is left invariant by each operator of $\mathcal{M}$ and V

and $\mathfrak{M}$ generate $\mathcal{T}(\alpha)$, P is left invariant by each element of $\mathcal{T}(\alpha)$ and, in particular, by U_α. Therefore $\alpha(P) = U_\alpha P U_\alpha^* \leq P$. But α is measure preserving so we must have $\alpha(P) = P$. This contradicts the ergodicity of α, and thus σ must be freely acting.

Theorem. Let ϕ: $\mathcal{T}(\alpha) \to \mathcal{T}(\beta)$ be a linear isometry of $\mathcal{T}(\alpha)$ onto $\mathcal{T}(\beta)$ such that $\phi(I) = I$. Then either α is conjugate to β or α is conjugate to β^{-1}.

Proof. Let $C^*(\mathcal{T}(\alpha))$ denote the C^*-algebra generated by $\mathcal{T}(\alpha)$. Then $C^*(\mathcal{T}(\alpha))$ is the norm closure of the linear subspace $\mathcal{T}(\alpha) + \mathcal{T}(\alpha)^*$. (This is seen immediately by observing that the algebra generated by $\mathfrak{M}$, U_α, and U_α^* is contained in $\mathcal{T}(\alpha) + \mathcal{T}(\alpha)^*$ and is dense in the C^*-algebra.) We may therefore apply proposition 1.2.8 of [2] to conclude that ϕ has a unique extension to a positive linear map ψ from $C^*(\mathcal{T}(\alpha))$ into $C^*(\mathcal{T}(\beta))$. In the same fashion ϕ^{-1} has a unique extension to a positive linear map θ of $C^*(\mathcal{T}(\beta))$ into $C^*(\mathcal{T}(\alpha))$. Since $\theta \circ \psi$ is a bounded self-adjoint linear map of $C^*(\mathcal{T}(\alpha))$ into itself which agrees with the identity mapping on $\mathcal{T}(\alpha) + \mathcal{T}(\alpha)^*$, $\theta \circ \psi$ is the identity mapping. Likewise $\psi \circ \theta$ is the identity on $C^*(\mathcal{T}(\beta))$. Therefore $\theta = \psi^{-1}$ and ψ is an order isomorphism of $C^*(\mathcal{T}(\alpha))$ onto the irreducible C^*-algebra $C^*(\mathcal{T}(\beta))$. By a theorem of Størmer, [5, Theorem 6.4], it follows that ψ is either a $*$-isomorphism or a $*$-anti-isomorphism.

In either case $\phi(\mathfrak{M}) = \mathfrak{M}$. Indeed, since ϕ preserves adjoints, $\phi(\mathfrak{M})$ is a self-adjoint sub-algebra of $\mathcal{T}(\beta)$. But then the triangularity of $\mathcal{T}(\beta)$ implies that $\phi(\mathfrak{M}) \subseteq \mathfrak{M}$. By the same argument $\phi^{-1}(\mathfrak{M}) \subseteq \mathfrak{M}$ also, and hence $\phi(\mathfrak{M}) = \mathfrak{M}$.

We also have, regardless of whether ψ is a $*$-isomorphism or a $*$-anti-isomorphism, that $\phi(U_\alpha)$ is a unitary element in $\mathcal{T}(\beta)$ with the property that $\mathfrak{M}$ and $\phi(U_\alpha)$ generate $\mathcal{T}(\beta)$ as a norm closed algebra. We may apply the lemma as soon as we observe that $\phi(U_\alpha)A\,\phi(U_\alpha)^*$ lies in $\mathfrak{M}$ for each A in $\mathfrak{M}$. But $U_\alpha\phi^{-1}(A)U_\alpha^* = \alpha(\phi^{-1}(A))$ is in $\mathfrak{M}$ for each A in $\mathfrak{M}$, and hence $\phi(U_\alpha)A\phi(U_\alpha)^* = \psi(U_\alpha)\psi(\phi^{-1}(A))\psi(U_\alpha)^* = \psi(U_\alpha\phi^{-1}(A)U_\alpha^*)$ $= \phi(U_\alpha\phi^{-1}(A)U_\alpha^*)$ is in $\mathfrak{M}$. So, applying the lemma, we obtain $\phi(U_\alpha) = U_\beta M$ for some unitary M in $\mathfrak{M}$.

We now treat separately the isomorphism and anti-isomorphism cases.

<u>Case 1.</u> ψ is an isomorphism. Then $\phi|\mathfrak{M}$ implements the conjugacy of α and β. Indeed, for any $A \in \mathfrak{M}$,

$$\begin{aligned} \phi \circ \alpha(A) &= \phi(U_\alpha A U_\alpha^*) = \psi(U_\alpha A U_\alpha^*) \\ &= \psi(U_\alpha)\psi(A)\psi(U_\alpha)^* = \phi(U_\alpha)\phi(A)\phi(U_\alpha)^* \\ &= U_\beta M\phi(A)M^*U_\beta^* = U_\beta\phi(A)U_\beta^* = \beta \circ \phi(A) \end{aligned}$$

Thus $\phi \circ \alpha = \beta \circ \phi$ on $\mathfrak{M}$ and α is conjugate to β.

<u>Case 2.</u> ψ is an anti-isomorphism. Then $\phi|\mathfrak{M}$ implements the conjugacy of α and β^{-1}. Again, for any $A \in \mathfrak{M}$,

$$\begin{aligned} \phi \circ \alpha(A) &= \phi(U_\alpha A U_\alpha^*) = \psi(U_\alpha A U_\alpha^*) = \psi(U_\alpha)^*\psi(A)\psi(U_\alpha) \\ &= \phi(U_\alpha)^*\phi(A)\phi(U_\alpha) = M^*U_\beta^*\phi(A)U_\beta M \,. \end{aligned}$$

Since $U_\beta^* \phi(A) U_\beta$ belongs to $\mathcal{M}$ it commutes with M, hence $\phi \circ \alpha(A) = U_\beta^* \phi(A) U_\beta = \beta^{-1} \circ \phi(A)$. Thus $\phi \circ \alpha = \beta^{-1} \circ \phi$ on $\mathcal{M}$ and α is conjugate to β^{-1}.

Remark 1. $\mathcal{T}(\alpha)$ is always isometric and anti-isomorphic to $\mathcal{T}(\alpha^{-1})$. Indeed, if for each f in $L^\infty[0,1]$ we let L_f denote the operator "multiplication by f" then $\mathcal{M} = \{L_f | f \in L^\infty[0,1]\}$. Transfer α to an L^2-norm isometry of $L^\infty[0,1]$ onto itself by letting $\alpha(f)$ be the unique element of $L^\infty[0,1]$ such that $L_{\alpha(f)} = \alpha(L_f)$. Then $\alpha(\bar{f}) = \overline{\alpha(f)}$ and $U_\alpha(f) = \alpha(f)$ for each $f \in L^\infty[0,1]$. Consequently we have $U_\alpha(\bar{f}) = \overline{U_\alpha(f)}$ and, by iteration, $U_\alpha^n(\bar{f}) = \overline{U_\alpha^n(f)}$ for each positive integer n.

Now suppose $T = \sum_{n=0}^{k} L_{g_n} U_\alpha^n$ with $g_n \in L^\infty[0,1]$. We claim that $R = \sum_{n=0}^{k} U_\alpha^{-n} L_{g_n}$ has the same norm as T. It will suffice to prove that $R^* = \sum_{n=0}^{k} L_{g_n}^* U_\alpha^n$ has the same norm as T. But for any vector f in $L^2[0,1]$ a routine calculation shows that $\|T(f)\| = \|R^*(\bar{f})\|$, and hence $\|T\| = \|R^*\|$.

Define a linear mapping ϕ from the algebra $\mathcal{S}(\alpha)$ onto the algebra $\mathcal{S}(\alpha^{-1})$ by the formula

$$\phi\left(\sum_{n=0}^{k} L_{g_n} U_\alpha^n\right) = \sum_{n=0}^{k} \alpha^{-n}(L_{g_n}) U_{\alpha^{-1}}^n$$

Since $(U_\alpha)^{-1} = U_{\alpha^{-1}}$ and $U_{\alpha^{-1}} L_g U_{\alpha^{-1}}^* = \alpha^{-1}(L_g)$ we see that $\phi\left(\sum_{n=0}^{k} L_{g_n} U_\alpha^n\right) = \sum_{n=0}^{k} U_\alpha^{-n} L_{g_n}$. Therefore ϕ is an isometry and consequently has an extension to a linear isometry of $\mathcal{T}(\alpha)$ onto $\mathcal{T}(\alpha^{-1})$. To prove that the extension is an anti-isomorphism we need merely show that ϕ is an anti-isomorphism and this follows easily once we show that $\phi(AU_\alpha^n BU_\alpha^m) = \phi(BU_\alpha^m)\phi(AU_\alpha^n)$ for any $A, B \in \mathfrak{M}$ and m,n positive integers. But

$$\phi(AU_\alpha^n BU_\alpha^m) = \phi(A\alpha^n(B)U_\alpha^{n+m}) = U_\alpha^{-n-m}A\alpha^n(B) = U_\alpha^{-m}U_\alpha^{-n}U_\alpha^n BU_\alpha^{-n}A$$

$$= U_\alpha^{-m}BU_\alpha^{-n}A = \phi(BU_\alpha^m)\phi(AU_\alpha^n)$$

<u>Remark 2.</u> Anzai [3] has constructed an example of an ergodic measure preserving *-automorphism α with the property that α is not conjugate to α^{-1}. As a consequence of this and the remark above the possibility that α might be conjugate to β^{-1} but not conjugate to β cannot be eliminated.

References

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