Inductive limits of finite dimensional C^*-algebras

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Abstract

Inductive limits of ascending sequences of finite dimensional C^*-algebras are studied. The ideals of such algebras are classified, and a necessary and sufficient condition for isomorphism of two such algebras is obtained. The results of Powers concerning factor states and representations of UHF algebras are generalized to this case. A study of the current algebra of the canonical anticommutation relations are then being made.

Introduction.

In this paper we study C^*-algebras which are the uniform closure of ascending sequences of finite dimensional C^*-algebras. We call these algebras approximately finite dimensional (AF). Similar classes of C^*-algebras have been studied before. In [6] Glimm describes the C^*-algebras which are the uniform closure of strictly ascending sequences of full $n \times n$ matrix algebras, all having the same unit (uniformly hyperfinite algebras). In [4] Dixmier removes the assumption that the matrix algebras have the same unit (Matroid C^*-algebras). In the study of quantum mechanical systems with an infinite number of degrees of freedom the study of unductive limits of nets of factors and their locally normal representations plays an important role, see e.g. [10].

The main algebraic feature which distinguishes the AF algebras from UHF algebras and matroid C^*-algebras is that the latter algebras are always simple, while this is not the case for the former in general. In fact the ideal structure, and even the primitive ideal structure of an AF algebra may be fairly complicated, and it seems that the structure space of an AF algebra may have almost all kinds of topological degeneracies, see e.g. 5.9.

The AF algebras overlap, without exhausting, a great range of the kinds of C^*-algebras which have been systematically studied, for example there exist nontrivial AF algebras which are liminal, postliminal, antiliminal, UHF etc. As the AF algebras are relatively simple to handle without being trivial, they are especially well suited to test conjectures and to provide examples in the theory of C^*-algebras, and I think their principal interest lies herein. As shown in paragraph 5 they may also have some interest in physics.

We give a brief outline of the paper. In paragraph 1 the major tool for analyzing an AF algebra, the diagram, is introduced, (1.8) and a graphical representation which easily reveals the properties of a given AF algebra is devised. In paragraph 2 we give an alternative characterization of AF algebras (2.2), and prove a necessary and sufficient condition for isomorphism of two AF algebras (2.7). In paragraph 3 the ideal structure of an AF algebra is analyzed (3.3), and thus a criterion for simplicity appears (3.5). Then the primitive ideals of an AF algebra are characterized (3.8), and by means of this result and the diagram the topology of the structure space of a given AF algebra may be found. In paragraph 4 criteria for a state to be a factor state is given (4.4), and we find conditions for quasi equivalence of two factor representations (4.5). Then a necessary and sufficient condition for algebraic equivalence of certain representations of an AF algebra is proved (4.12), and a corollary to this result is that the automorphism group of an AF algebra acts transitively on those states of the algebra whose associated Gelfand-Segal representation are faithful (4.15). Another corollary is a simple characterization of the pure states of an AF algebra (4.16). In paragraph 5 the results of the foregoing paragraphs are applied to a specific example, the current algebra or the observable algebra of the algebra of the canonical anticommutation relations. The most striking result obtained is a classification of all the irreducible representations of the current algebra with kernel $\neq \{0\}$. (5.6). These representations are in a natural way divided into two series, one of which is obtained by decomposing the Fock representation and the other by decomposing the anti-Fock representation (5.9).

I wish to thank my supervisor Erling Størmer. Without his many helpful suggestions this work could not have been done. In paragraph 2 I lean heavily on the results of Glimm in [6], and in paragraph 4 on the work by Powers in [12].

1. Definition and elementary properties of approximately finite dimensional C^*-algebras.

1.1. Definition. A C^*-algebra $\mathfrak{A}$ is called approximately finite dimensional (AF) if $\mathfrak{A}$ has a unit e, and there exists an increasing (with respect to inclusion) subsequence $<\mathfrak{A}_n>_{n=1,2,\dots}$ of finite dimensional subalgebras of $\mathfrak{A}$, such that $\mathfrak{A}$ is the norm closure of $\bigcup_n \mathfrak{A}_n$ i.e.

$$\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$$

1.2. If $\mathfrak{A}$ and $\mathfrak{A}_n$ is as in 1.1, then $\mathfrak{A}_n + \mathbb{C}e$ trivially is a finite dimensional C^*-subalgebra of $\mathfrak{A}$, and $\mathfrak{A}_n \subseteq \mathfrak{A}_n + \mathbb{C}e \subseteq \mathfrak{A}_{n+1} + \mathbb{C}e$. We may therefore assume that each $\mathfrak{A}_n$ contains the unit of $\mathfrak{A}$, and this is done in all what follows.

1.3. If $<\mathfrak{A}_n>_n$ is a sequence of finite dimensional C^*-algebras, and $\alpha_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ are morphisms, and each α_n are injective and maps the unit of $\mathfrak{A}_n$ into the unit of $\mathfrak{A}_{n+1}$, then the diagram

$$\mathfrak{A}_1 \xrightarrow{\alpha_1} \mathfrak{A}_2 \xrightarrow{\alpha_2} \mathfrak{A}_3 \longrightarrow ----,$$

has a certain inductive limit $\mathfrak{A}$ by [4]. The algebras $\mathfrak{A}_n$ may be considered as subalgebras of $\mathfrak{A}$. Then $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and since each $\mathfrak{A}_n$ has the same unit e and multiplication is norm continuous e is a unit in $\mathfrak{A}$. Hence $\mathfrak{A}$ is AF, and each diagram of the considered type gives rise to an AF algebra.

1.4. In all what follows, the expression

$$\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n} \quad (\text{resp. } \mathfrak{B} = \overline{\bigcup_n \mathfrak{B}_n} \quad \text{etc. })$$

will mean:

"$\mathfrak{A}$ (resp. $\mathfrak{B}$) is a AF algebra, and $\langle \mathfrak{A}_n \rangle_{n=1,2\cdots}$ (resp. $\langle \mathfrak{B}_n \rangle_{n=1,2\cdots}$) is an increasing sequence of finite dimensional subalgebras of $\mathfrak{A}$ (resp. $\mathfrak{B}$) all containing the identity of $\mathfrak{A}$ (resp. $\mathfrak{B}$), such that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ (resp. $\mathfrak{B} = \overline{\bigcup_n \mathfrak{B}_n}$)".

If $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and e is the unit of $\mathfrak{A}$, we set, for convenience, $\mathfrak{A}_0 = \mathbb{C}e$, so that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots\cdots$, and $\mathfrak{A} = \overline{\bigcup_{n=0}^{\infty} \mathfrak{A}_n}$.

1.5. Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, $\mathfrak{B} = \overline{\bigcup_n \mathfrak{B}_n}$. Then it is trivial to verify that $\mathfrak{A} \oplus \mathfrak{B} = \overline{\bigcup_n (\mathfrak{A}_n \oplus \mathfrak{B}_n)}$, and $\mathfrak{A} \overset{*}{\otimes} \mathfrak{B} = \overline{\bigcup_n (\mathfrak{A}_n \overset{*}{\otimes} \mathfrak{B}_n)}$.

Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and let ρ be a morphism of $\mathfrak{A}$ onto a C^*-algebra $\mathfrak{B}$. Then, since $\|\rho(x)\| \leq \|x\|$ for all $x \in \mathfrak{A}$ we have that $\mathfrak{B} = \overline{\bigcup_n \rho(\mathfrak{A}_n)}$. Since $\mathfrak{A}_n$ is finite dimensional $\mathfrak{B}_n = \rho(\mathfrak{A}_n)$ is a finite dimensional C^*-subalgebra of $\mathfrak{B}$, and since ρ maps the unit of $\mathfrak{A}$, into a unit of $\mathfrak{B}$, $\mathfrak{B}$ is AF.

It follows that the class of AF algebras with their morphisms form a category which is closed under finite sums and tensor products.

1.6. We introduce some notation which will be standard in what follows. Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. Then each $\mathfrak{A}_n$ is a finite dimensional C^*-algebra with unit e. It is then well known that $\mathfrak{A}_n$ is a direct sum of finite dimensional factors:

$$\mathfrak{A}_n = \bigoplus_{k=1}^{n_n} M_{(nk)}$$

The symbol (nk) serves to label the factor $M_{(nk)} \subseteq \mathfrak{A}_n$. The square root of the dimension of $M_{(nk)}$ is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$ such that $M_{(nk)} \cong M_{\begin{bmatrix} n \\ k \end{bmatrix}}$, where $M_{\begin{bmatrix} n \\ k \end{bmatrix}}$ is the full $\begin{bmatrix} n \\ k \end{bmatrix} \times \begin{bmatrix} n \\ k \end{bmatrix}$ complex matrix algebra.

We let $e^{(nk)}$ denote the maximal projection in $M_{(nk)}$. It is then well known that the $e^{(nk)}$; $k=1 \cdots n_n$, are the minimal projections of the center of $\mathfrak{A}_n$, and we have that

$$e = \sum_{k=1}^{n_n} e^{(nk)}$$

We will let $\{e_{ij}^{(nk)}\}_{i,j=1}^{\begin{bmatrix} n \\ k \end{bmatrix}}$ denote a set of matrix units for $M_{(nk)}$. We will say that $\{e_{ij}^{(nk)}\}_{i,j=1}^{\begin{bmatrix} n \\ k \end{bmatrix}}{}_{k=1}^{n_n} \in \mathfrak{A}_n$ is a set of <u>matrix units for</u> $\mathfrak{A}_n$ if the $e_{ij}^{(nk)}$'s span $\mathfrak{A}_n$ linearly, and satisfy

i) $$e_{ij}^{(nk)} e_{sq}^{(np)} = \delta_{kp} \delta_{js} e_{iq}^{(nk)}$$

ii) $$e_{ij}^{(nk)*} = e_{ji}^{(nk)}$$

We always choose the indexes such that $e_{ij}^{(nk)} \in M_{(nk)}$, i.e. such that $\{e_{ij}^{(nk)}\}_{i,j=1}^{\begin{bmatrix} n \\ k \end{bmatrix}}$ are matrix units for $M_{(nk)}$ in the usual sense. If the $e_{ij}^{(nk)}$'s satisfy i) and ii) without necessarily spanning $\mathfrak{A}_n$ they are said to be <u>matrix units in</u> $\mathfrak{A}_n$.

1.7. We now shall study how one finite dimensional C^*-algebra may be embedded into another.

Proposition: Let $\mathfrak{A}_n = \bigoplus_{k=1}^{n_n} M_{(nk)}$, $n=1,2$, be two finite dimensional C^*-algebras with the same unit e, and suppose that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$. Let $\{e_{ij}^{(1k)}\}$ be matrix units for $\mathfrak{A}_1$. Then there exist unique non-negative integers n_{ki}, $k=1,\cdots;n_2$, $i=1\cdots,n_1$ and there exist matrix units $\{e_{ij}^{(2k)}\}$ for $\mathfrak{A}_2$ such that

$$(1) \quad e_{ij}^{(1k)} = \sum_{q=1}^{n_2} \sum_{m=1}^{n_{qk}} e^{(2q)}_{\left(\sum_{p=1}^{k-1} n_{qp}\begin{bmatrix}1\\p\end{bmatrix}+(m-1)n_{qk}+i\right)\left(\sum_{p=1}^{k-1} n_{qp}\begin{bmatrix}1\\p\end{bmatrix}+(m-1)n_{qk}+j\right)}$$

(In unformal,but more illuminating language this proposition says: If we identify $\mathfrak{A}_n$ with $\bigoplus_{k=1}^{n_n} M_{\begin{bmatrix}n\\k\end{bmatrix}}$ and define $pM_q = M_q \otimes \mathbb{C}I_{\mathbb{C}^p}$ then the embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ is of the form

$$\bigoplus_{k=1}^{n_2}\left(\bigoplus_{i=1}^{n_1} n_{ki}\, M_{\begin{bmatrix}1\\i\end{bmatrix}}\right)$$

where we identify $\bigoplus_{i=1}^{n_1} n_{ki}\, M_{\begin{bmatrix}1\\i\end{bmatrix}}$ with a subalgebra of $M_{\begin{bmatrix}2\\k\end{bmatrix}}$.)

Proof: Let α_n be a isomorphism of $\mathfrak{A}_n$ onto $\bigoplus_{k=1}^{n_n} M_{\begin{bmatrix}n\\k\end{bmatrix}}$, and let $\beta = \alpha_2 \circ \alpha_1^{-1}$. Then β is an injective morphism of $\bigoplus_{k=1}^{n_1} M_{\begin{bmatrix}1\\k\end{bmatrix}}$ into $\bigoplus_{k=1}^{n_2} M_{\begin{bmatrix}2\\k\end{bmatrix}}$.

Define $\beta_k = \alpha_2(e^{(2k)})\beta$. Since $\alpha_2(e^{(2k)})$ is a central projection in $\bigoplus_{p=1}^{n_2} M_{\begin{bmatrix}2\\p\end{bmatrix}}$, β_k is a morphism of $\bigoplus_{p=1}^{n_1} M_{\begin{bmatrix}1\\p\end{bmatrix}}$ into $M_{\begin{bmatrix}2\\k\end{bmatrix}}$, and we have $\beta(x) = \bigoplus_{k=1}^{n_2} \beta_k(x)$, $x \in \bigoplus_{k=1}^{n_1} M_{\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]}$.

From [2], Ch.I, § 4, Th.3 it follows that β_k has the form

$\beta_k = \phi_3 \circ \phi_2 \circ \phi_1$ where ϕ_1 is an ampliation, ϕ_2 is an induction

and ϕ_3 is a spatial isomorphism. There exists a Hilbert space κ

such that $\phi_1(x) = x \otimes I_\kappa$; $x \in \alpha_1(\mathfrak{A}_1)$, so ϕ_1 transforms

$\alpha_1(\mathfrak{A}_1)$ onto the algebra $\alpha_1(\mathfrak{A}_1) \otimes \mathbb{C}I_\kappa = (\bigoplus_{p=1}^{n_1} M_{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]}) \otimes \mathbb{C}I_\kappa =$

$= \bigoplus_{p=1}^{n_1} (M_{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]} \otimes \mathbb{C}I_\kappa)$. The commutant of this algebra is

$\bigoplus_{p=1}^{n_1} \mathbb{C}I_{\mathbb{C}^{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]}} \otimes \mathcal{B}(\kappa)$. As ϕ_2 is defined by a projection in this

commutant, $\phi_2 \circ \phi_1$ transforms $\bigoplus_{p=1}^{n} M_{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]}$ into an algebra of the form

$\bigoplus_{p=1}^{n_1} M_{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]} \otimes \mathbb{C}I_{\mathbb{C}^{n_{kp}}}$. Since this algebra is transformed into $M_{\left[\begin{smallmatrix}2\\k\end{smallmatrix}\right]}$

by the spatial isomorphism ϕ_3, all the n_{kp}'s are finite, and in

fact we have $\sum_{p=1}^{n_1} n_{kp}\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\k\end{smallmatrix}\right]$, since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ has the same

unit e. More specifically, $\phi_2 \circ \phi_1$ transforms an element

$x = \bigoplus_{p=1}^{n_1} x_p \in \bigoplus_{p=1}^{n_1} M_{\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]}$ into $\bigoplus_{p=1}^{n_1} (x_p \otimes I_{\mathbb{C}^{n_{kp}}})$. By using the spatial

isomorphism ϕ_3, this last element may be viewed as an element in $M_{\left[\begin{smallmatrix}2\\k\end{smallmatrix}\right]}$ and doing this, we see that

β transforms x into $\bigoplus_{k=1}^{n_2} (\bigoplus_{p=1}^{n_1} x_p \otimes I_{\mathbb{C}^{n_{kp}}})$. Now by choosing set

of matrix units $\{e_{ij}^{(1k)}\}$ for $\mathfrak{A}_1$ and setting $x = \alpha_1(e_{ij}^{(1k)})$ above,

and using the fact that $\alpha_2^{-1} \circ \beta \circ \alpha_1$ is the identity mapping

$\mathfrak{A}_1 \to \mathfrak{A}_2$, one may easily define matrix units $e_{ij}^{(2k)}$ in $\mathfrak{A}_2$

such that (1) is fullfilled.

<u>1.8.</u> The remark after proposition 1.7 makes the following definition natural: With the same notation as in the theorem we say that $M_{(1i)}$ is <u>partially embedded</u> in $M_{(2k)}$ <u>with multiplicity n_{ki}.</u> If $n_{ki} \geq 1$ we say that $M_{(1i)}$ is <u>partially embedded</u> in $M_{(2k)}$. These two relations are written as $M_{(1i)} \overset{n_{ki}}{\searrow} M_{(2k)}$ and $M_{(1i)} \searrow M_{(2k)}$.

From the proof of the proposition it is easily seen that $M_{(1i)} \searrow M_{(2k)}$ iff $e^{(1i)} e^{(2k)} \neq 0$, and that if we define $a = \sup\{m \mid \exists\, m$ mutually orthogonal projections $e_1, \cdots, e_m$ in $\mathfrak{A}_2$ such that $e_i \leq e^{(1i)} e^{(2k)}$, $i=1,\cdots,m\}$, and $b = \sup\{m \mid \exists\, m$ mutually orthogonal projections $e_1, \cdots, e_m$ in $\mathfrak{A}_1$ such that $e_i \leq e^{(1i)}$; $i=1,\cdots,m\}$ then $n_{ki} = a/b$.

Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. Then the <u>diagram</u> $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A}$ is defined as the set of all ordered pairs (nk); $k=1,\cdots,n_n$, $n=0,1,\cdots$, together with a sequence $\langle \overset{p}{\searrow} \rangle_{p=0,\cdots}$ of relations defined by $(n\ k) \overset{p}{\searrow} (m\ q)$ iff $m = n+1$ and $M_{(nk)}$ is partially embedded in $M_{(mq)}$ with multiplicity p.

This definition requires a couple of comments.

It is clear that $\mathfrak{D}(\mathfrak{A})$ depends not only on $\mathfrak{A}$, but on the particular sequence $\langle \mathfrak{A}_n \rangle_n$ which generates $\mathfrak{A}$. This dependence will be implicit in what follows.

A natural question to ask is: If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic AF algebras, what are the relation between $\mathfrak{D}(\mathfrak{A})$ and $\mathfrak{D}(\mathfrak{B})$?

Alternatively, if $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n} = \overline{\bigcup_n \mathfrak{B}_n}$, what are the connection between the associated diagrams ? An answer to this question will be given in theorem 2.7. From that theorem it is in principle easy to deduce an algoritm which gives a method of constructing from a given diagram all diagrams which define AF-algebras which are isomorphic with the original one.

Another question is: Does really the diagram $\mathcal{D}(\mathfrak{A})$ define $\mathfrak{A}$ up to isomorphism ? The answer is the affirmative, for if $\mathfrak{A}$ and $\mathfrak{B}$ are two AF-algebras with the same diagram $\mathcal{D}$, an isomorphism $\alpha: \mathfrak{A} \to \mathfrak{B}$ may be constructed inductively as follows:

- Since $\mathfrak{A}_0 \cong \mathbb{C} \cong \mathfrak{B}_0$ there exists an isomorphism $\alpha_0: \mathfrak{A}_0 \to \mathfrak{B}_0$
- Now suppose we have constructed isomorphisms $\alpha_r: \mathfrak{A}_r \to \mathfrak{B}_r$, $r = 0,1,\cdots,n-1$, such that $\alpha_r | \mathfrak{A}_{r-1} = \alpha_{r-1}$, $r=1,\cdots,n-1$.

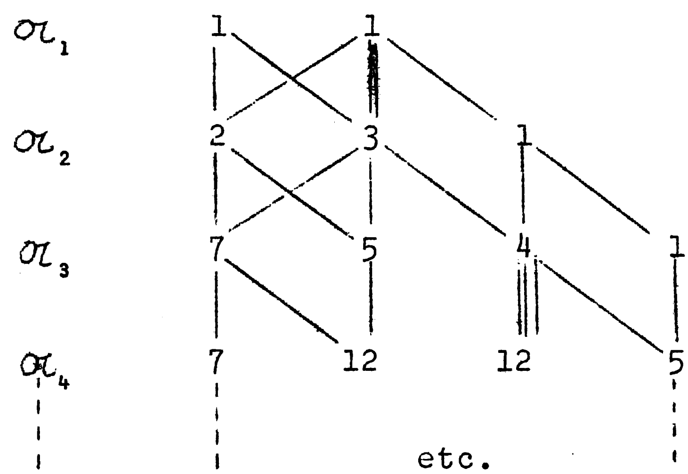
Let $\{e_{ij}^{(n-1,k)}\}$ be a set of matrix units for $\mathfrak{A}_{n-1}$, and let $f_{ij}^{(n-1,k)} = \alpha_{n-1}(e_{ij}^{(n-1,k)})$ be the corresponding matrix units for $\mathfrak{B}_{n-1}$. Let n_{qp} be the non-negative integer such that $(n-1,p) \overset{n_{qp}}{\searrow} (n,q)$. Then, by definition of $\mathcal{D}(\mathfrak{A})$ and prop. 1.7 there exist matrix units $\{e_{ij}^{(n,q)}\}$ for $\mathfrak{A}_n$ such that equation (1) is fulfilled, with $e_{ij}^{(1k)}$ replaced by $e_{ij}^{(n-1,k)}$ and $e_{ij}^{(2k)}$ replaced by $e_{ij}^{(n,k)}$, and $\begin{bmatrix}1\\p\end{bmatrix}$ replaced by $\begin{bmatrix}n-1\\p\end{bmatrix}$. In the same way, there exist matrix units $f_{ij}^{(n,k)}$ for $\mathfrak{B}_n$ such that (1) holds with e replaced by f.

Then one may define $\alpha_n(e_{ij}^{(nk)}) = f_{ij}^{(nk)}$, and extend the the definition of α_n to $\mathfrak{A}_n$ by linearity. Then α_n is an isomorphism

$\mathfrak{A}_n \to \mathfrak{B}_n$ and from (1) it follows that $\alpha_n | \mathfrak{A}_{n-1} = \alpha_{n-1}$.

Now, because of the last relation we may define a *-isomorphism $\alpha: \bigcup_n \mathfrak{A}_n \to \bigcup_n \mathfrak{B}_n$ by $\alpha | \mathfrak{A}_n = \alpha_n$. Since each α_n is an isometry, α is an isometry, and α may therefore be extended to a mapping of $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ onto $\mathfrak{B} = \overline{\bigcup_n \mathfrak{A}_n}$ by continuity. Since all the operations in the definition of a C^*-algebra are norm continuous this extended map is an isomorphism, so $\mathfrak{A} \cong \mathfrak{B}$.

The diagram of an AF-algebra may be given a graphical representation, which we show by an example.



This means that $\mathfrak{A}_1 \cong M_1 \oplus M_1$, $\mathfrak{A}_2 \cong M_2 \oplus M_3 \oplus M_1$, $\mathfrak{A}_3 \cong M_7 \oplus M_5 \oplus M_4 \oplus M_1$, etc. and the number of lines between the numbers indicate the multiplicity of the partial embedding of the factor above into that below. As an example, the second factor in the central decomposition of $\mathfrak{A}_1$ is partially embedded with multiplicity 1 in the first factor of $\mathfrak{A}_2$, with multiplicity 2 in the second factor and with multiplicity 1 in the third factor.

Given a set of $\mathfrak{D}$ of ordered pairs (n,k); $k=1,\cdots,n_n$, $n=0,1,\cdots$, where $n_0 = 1$, and a sequence $\langle \overset{n}{\searrow} \rangle_{p=0,1\cdots}$ of relations on $\mathfrak{D}$, when is $\mathcal{D} = \mathcal{D}(\mathfrak{A})$ for some AF-algebra $\mathfrak{A}$? We list some axioms that $\mathcal{D}$ must satisfy (Define $(n,k)\searrow(m,q) \iff \exists p \geq 1: (n,k)\searrow^p(m,q)$)

i) If $(n,k), (m,q) \in \mathcal{D}$ and $m = n+1$

there exists one and only one non-negative integer p such that $(n,k) \searrow^p (m,q)$.

ii) If $m \neq n+1$ none such integer exists.

iii) If $(n,k) \in \mathcal{D}$ there exists a $q \in \{1,\cdots,n_{n+1}\}$ such that $(n,k)\searrow(n+1,q)$.

(iv) If $(n,k) \in \mathcal{D}$ and $n \geq 1$ there exists a $q \in \{1,\cdots,n_{n-1}\}$ such that $(n-1,q) \searrow (n,k)$.

It is not difficult to see that the diagram of a given AF-algebra satisfies these axioms. We only mention that (iii) expresses the trivial fact that the kernel of the identity morphism $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$ is equal to $\{0\}$, and (iv) expresses the fact that the identity of $\mathfrak{A}_n$ is mapped into the identity of $\mathfrak{A}_{n+1}$ by the identity morphism.

Conversely, if $\mathcal{D}$ satisfies axioms (i)-(iv) one may by induction construct a sequence of finite dimensional C^*-algebras $\langle\mathfrak{A}_n\rangle_n$ and injective morphisms $\alpha_n: \mathfrak{A}_n \to \mathfrak{A}_{n+1}$, such that $\mathfrak{A}_0 = \mathbb{C}$, $\alpha_n|\mathfrak{A}_{n-1} = \alpha_{n-1}$; $n=1,2,\cdots$ and such that for a given set of matrix units $e_{ij}^{(nk)}$ in $\mathfrak{A}_n$ there exists a set of matrix elements $e_{ij}^{(n+1,k)}$ in $\mathfrak{A}_{n+1}$ such that

$$\alpha_n(e_{ij}^{(nk)}) = \sum_{q=1}^{n_{n+1}} \sum_{m=1}^{n_{qk}} \cdot e^{(n+1,q)}_{(\sum_{p=1}^{k-1} n_{qp}\left[{n \atop p}\right]+(m-1)n_{qk}+i)(\sum_{p=1}^{k-1} n_{qp}\left[{n \atop p}\right]+(m-1)n_{qk}+j)}$$

where n_{qk} is such that $(nk) \searrow^{n_{qk}} (n+1,q)$. This is done by choosing the dimensions $\begin{bmatrix} n+1 \\ q \end{bmatrix}$ of the factors $M_{(n+1,q)}$ in an appropriate way; in fact we have $\begin{bmatrix} n+1 \\ q \end{bmatrix} = \sum_{p=1}^{n} n_{qp} \begin{bmatrix} n \\ p \end{bmatrix}$.

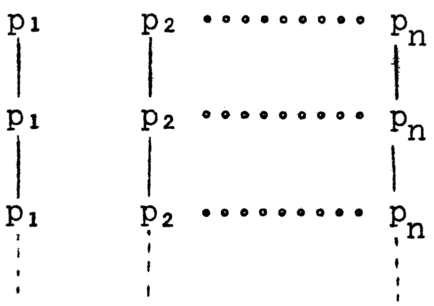
The inductive limit of the diagram

$$\mathfrak{A}_0 \xrightarrow{\alpha_0} \mathfrak{A}_1 \xrightarrow{\alpha_1} \mathfrak{A}_2 \longrightarrow \cdots\cdot$$

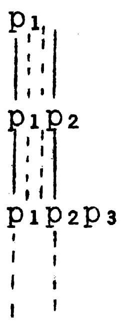
will then have diagram $\mathcal{D}$.

1.9. We mention some examples of AF-algebras $\mathfrak{A}$.

(i) $\mathfrak{A}$ finite dimensional. Then the diagram has the following form:

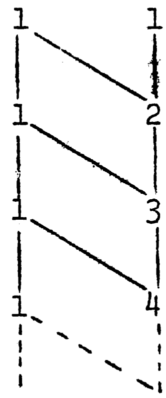


(ii) $\mathfrak{A}$ is an UHF-algebra. Then all $\mathfrak{A}_n$ are factors and the diagram has the form:



The number of lines between $p_1 \cdots p_n$ and $p_1 \cdots p_{n+1}$ is p_{n+1}.

(iii) We next give an example which is closely related to the algebra $M_{1,\infty}$ studied in [4], 5.2. Let κ be an infinite dimensional separable Hilbert space, and let $\mathfrak{A} = \mathcal{LC}(\kappa) + \mathbb{C}I_\kappa$. Then $\mathfrak{A}$ is AF and has a diagram:



This is shown as follows: Let $\langle \xi_n \rangle_{n=1,\dots}$ be an orthonormal basis in κ, and let κ_n be the subspace generated by $\xi_1 \cdots \xi_n$. Let E_n be the orthogonal projection onto κ_n. Define
$\mathfrak{A}_n = \{x \in \mathcal{B}(\kappa) \mid x(1-E_n) = (1-E_n)x \in \mathbb{C}(1-E_n)\} \cong \mathcal{B}(\kappa_n) \oplus \mathbb{C} \cong M_n \oplus M_1$.
Then $\mathfrak{A}_n$ is embedded in $\mathfrak{A}_{n+1}$ as indicated on the diagram, and since each $x \in \mathfrak{A}_n$ is a sum of an operator of finite rank and a multiple of the identity we have that $\mathfrak{A}_n \subseteq \mathcal{LC}(\kappa) + \mathbb{C}I$.

Conversely, by using the fact that the operators of finite rank are norm dense in $\mathcal{LC}(\kappa)$, and that the finite linear combinations of $\xi_1\ \xi_2 \cdots$ are densen in κ, it is easy to show that
$\mathcal{LC}(\kappa) + \mathbb{C}I_\kappa \subseteq \overline{\bigcup_n \mathfrak{A}_n}$.

<u>1.10.</u> An AF algebra is separable, but a separable C^*-algebra with unit does not need to be AF. This follows from the example $\mathfrak{A} = C[0,1]$.

Since $[0,1]$ is connected, $[0,1]$ contains no nontrivial open-closed subsets, hence $C[0,1]$ contains no other projections than 0 and 1. It follows that $C[0,1]$ contains no other finite dimensional *-subalgebras than $\{0\}$ and $\mathbb{C}1$, thus $C[0,1]$ cannot be AF.

2. New definition of AF algebras. Isomorphism of AF algebras.

2.1. Lemma. Let $\mathfrak{A}$ be a C^*-algebra on a Hilbert space κ, let $\varepsilon > 0$ and let n be a positive integer. Then there exists a $\delta(\varepsilon,n) = \delta > 0$ such that if

1) $\{e_{ij}^{(k)}\ ;\ i,j = 1,\cdots n_k,\ k = 1,\cdots,m\}$ is a family of matrix units for a finite dimensional C^*-algebra on κ with unit I_κ, such that $\sum_{k=1}^{n} n_k = n$.

2) There exists $x_{ij}^{(k)} \in \mathfrak{A}$ such that $\|x_{ij}^{(k)} - e_{ij}^{(k)}\| < \delta$,

then there exists a family $\{f_{ij}^{(k)}\}$ of matrix units in $\mathfrak{A}$ such that $\|f_{ij}^{(k)} - e_{ij}^{(k)}\| < \varepsilon$.

Proof: The method of proof of this lemma is the same as that Glimm uses in [6], lemma 1.10; thus the proof will be omitted.

The next theorem is analoguous to theorem 1.13 in [6].

2.2. Theorem. Let $\mathfrak{A}$ be a C^*-algebra with unit e. Then $\mathfrak{A}$ is an AF algebra if and only if the following two conditions are fullfilled.

i) $\mathfrak{A}$ is separable.

ii) If $x_1,\cdots,x_n \in \mathfrak{A}$ and $\varepsilon > 0$, then there exists a finite dimensional C^*-subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ and elements

$y_1, \cdots, y_n \in \mathcal{B}$ such that $\|x_i - y_i\| < \varepsilon$; $i = 1, \cdots, n$.

Furthermore, if $\mathfrak{A}$ is AF, and $\mathfrak{A}_1$ is a finite dimensional C^*-subalgebra of $\mathfrak{A}$, there exists an increasing sequence $\mathfrak{A}_2 \subseteq \mathfrak{A}_3 \subseteq \cdots$ of finite dimensional C^*-subalgebras such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\bigcup_i \mathfrak{A}_i = \mathfrak{A}$.

Proof. The proof is closely related to Glimms proof in [6]. The necessity of conditions i) and ii) is clear.

To show sufficiency, let $\{d_i\}_{i=1,2,\cdots}$ be a dense sequence in the open sphere of radius $\frac{1}{2}$ about the origin in $\mathfrak{A}$. We may, without loss of generality, suppose that the subalgebras we consider contain e. We shall construct an increasing sequence $\langle \mathfrak{A}_n \rangle_n$ of finite dimensional subalgebras of $\mathfrak{A}$ such that for all n there exists $b_k \in \mathfrak{A}_n$, $k=1,\cdots,n$ such that $\|b_k - d_k\| < 2^{-n}$, $k=1,\cdots,n$.

Since $\|d_1\| < \frac{1}{2}$, $\mathfrak{A}_1$ may be chosen arbitrarily.

Suppose as induction hypothesis that $\mathfrak{A}_n$ has been constructed and has the required properties. Let $\{e_{ij}^{(nk)}\}$ be matrix units for $\mathfrak{A}_n$. Define $\varepsilon = 2^{-n-1}(1+4(\sum_{k=1}^{n_n} \begin{bmatrix} n \\ k \end{bmatrix})^2)^{-1}$. By using hypothesis ii) of the theorem and lemma 2.1 it follows that there exists a finite dimensional subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ and a set of matrix units $\{f_{ij}^{(k)}\}$ in $\mathfrak{A}'$ (which does not necessarily generate $\mathfrak{A}'$) such that $\|f_{ij}^{(k)} - e_{ij}^{(nk)}\| < \delta$; $1 \leq i,j \leq \begin{bmatrix} n \\ k \end{bmatrix}$, where δ is the $\delta(\varepsilon, n_k)$ of [6], lemma 1.8, and such that there exists $b_k' \in \mathfrak{A}'$, $k=1,\cdots,n+1$, such that $\|b_k' - d_k\| < \varepsilon$.

By [6], lemma 1.8, there exists a partial isometry $w \in \mathfrak{A}$ such that $wf_{11}^{(k)}$ is a partial isometry between $f_{11}^{(k)}$ and $e_{11}^{(nk)}$, $k=1,\cdots,n_n$, and $\|e_{11}^{(k)} - wf_{11}^{(k)}\| < \varepsilon$; $k=1,\cdots,n_n$. Define $u = \sum_{k=1}^{n_n} \sum_{i=1}^{[n_k]} e_{i1}^{(nk)} wf_{1i}^{(k)}$,. Then $u \in \mathfrak{A}$, and by trivial algebra u is unitary and $uf_{ij}^{(k)} u^* = e_{ij}^{(nk)}$:

Define $\mathfrak{A}_{n+1} = u\mathfrak{A}'u^*$. Then $\mathfrak{A}_{n+1}$ is a finite dimensional subalgebra of $\mathfrak{A}$ isomorphic with $\mathfrak{A}'$, and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$. We must find $b_k \in \mathfrak{A}_{n+1}$ such that $\|b_k - d_k\| < 2^{-n-1}$, $k=1,\cdots,n+1$. Let $b_k = ub_k'u^* \in \mathfrak{A}_{n+1}$. Then

$$\|b_k-d_k\| \leq \|d_k - b_k'\| + \|b_k'-b_k\| < \varepsilon + \|b_k'-ub_k'u^*\|$$

$$= \varepsilon + \|\sum_{kqst} (f_{ss}^{(k)}b_k' f_{tt}^{(q)} - e_{s1}^{(nk)} wf_{1s}^{(k)} b_k' f_{t1}^{(q)} w^* e_{1t}^{(nq)})\|$$

$$\leq \varepsilon + (\sum_{k=1}^{n_n} [n_k])^2 \sup_{kqst} \|f_{ss}^{(k)} b_k'f_{tt}^{(q)} - e_{s1}^{(nk)} wf_{1s}^{(k)} b_k'f_{t1}^{(q)} w^*e_{1t}^{(nq)}\|$$

Now:

$$\|f_{ss}^{(k)} b_k'f_{tt}^{(q)} - e_{s1}^{(nk)} wf_{1s}^{(k)} b_k'f_{t1}^{(q)} w^*e_{1t}^{(nq)}\|$$

$$\leq \|f_{ss}^{(k)}b_k'f_{tt}^{(q)} - f_{ss}^{(k)}b_k'f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| + \|(f_{ss}^{(k)} - e_{s1}^{(nk)}wf_{1s}^{(k)})b_k'f_{t1}^{(q)}w^*e_{1t}^{(nq)}\|$$

$$\leq \|f_{tt}^{(q)} - f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| + \|f_{ss}^{(k)} - e_{s1}^{(nk)}wf_{1s}^{(k)}\| \leq \|f_{1t}^{(q)} - w^*e_{1t}^{(nq)}\|$$

$$+ \|f_{s1}^{(k)} - e_{s1}^{(nk)}w\| \leq \|f_{1t}^{(q)} - e_{1t}^{(nq)}\| + \|e_{t1}^{(nq)} - e_{t1}^{(nq)}w\|$$

$$+ \|f_{s1}^{(k)} - e_{s1}^{(nk)}\| + \|e_{s1}^{(nk)} - e_{s1}^{(nk)}w\| < 4\varepsilon.$$

Hence:

$$\|b_k - d_k\| < \varepsilon + 4\Big(\sum_{k=1}^{n_n} \left[{n \atop k}\right]\Big)^2 \varepsilon = \frac{1}{2^{n+1}}$$

By induction, a sequence $<\mathfrak{A}_n>_n$ with the required properties exists. Then $\{d_i\} \in \overline{\bigcup_n \mathfrak{A}_n}$, so $\overline{\bigcup_n \mathfrak{A}_n} = \mathfrak{A}$.

2.3 Lemma. Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and let $\mathcal{B}$ be a finite dimensional subalgebra of $\mathfrak{A}$. Then for all $\varepsilon > 0$ there exist a unitary operator $u \in \mathfrak{A}$ and a positive integer n such that

(i) $\|u-e\| < \varepsilon$

(ii) $u\mathcal{B}u^* \subseteq \mathfrak{A}_n$

Proof. We may assume that $e \in \mathcal{B}$. Let $\{f_{ij}^{(k)}\}_{k=1}^m$ be matrix units for $\mathcal{B}$, and suppose $1 \leq i,j \leq N$ for all $f_{ij}^{(k)}$. Let $\varepsilon_1 = \frac{\varepsilon}{3mN}$ and let δ be the $\delta(\varepsilon_1, m)$ of lemma 1.8 in [6]. Lemma 2.1 implies that there exist an $\mathfrak{A}_n$ and a family $\{e_{ij}^{(k)}\}_{k=1}^m$ of matrix units in $\mathfrak{A}_n$ such that $\|f_{ij}^{(k)} - e_{ij}^{(k)}\| < \delta$.
From [6], lemma 1.8, it follows that there exists a partial isometry $w \in \mathfrak{A}$ such that $e_{11}^{(k)} w = w f_{11}^{(k)}$ is a partial isometry having $f_{11}^{(k)}$ and $e_{11}^{(k)}$ as initial and final projection, respectively, and such that $\|e_{11}^{(k)} - e_{11}^{(k)} w\| < \varepsilon_1$. Define $u = \sum_k \sum_i e_{i1}^{(k)} w f_{1i}^{(k)}$. Then, since $\sum_{ki} e_{ii}^{(k)} = \sum_{ki} f_{ii}^{(k)} = e$, u is unitary, and we have $e_{ij}^{(k)} = u f_{ij}^{(k)} u^*$, thus $u\mathcal{B}u^* \subseteq \mathfrak{A}_n$. Furthermore:

$$\|e_{ii}^{(k)} - e_{ii}^{(k)}u\|$$

$$= \|e_{ii}^{(k)} - e_{ii}^{(k)}\, e_{i1}^{(k)}\, w\, f_{1i}^{(k)}\|$$

$$\leq \|e_{1i}^{(k)} - e_{11}^{(k)}\, w\, f_{1i}^{(k)}\|$$

$$\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{1i}^{(k)} - e_{11}^{(k)}\, w\, f_{1i}^{(k)}\|$$

$$\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{11}^{(k)} - e_{11}^{(k)}\, w\, f_{11}^{(k)}\|$$

$$\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{11}^{(k)} - e_{11}^{(k)}\| + \|e_{11}^{(k)} - e_{11}^{(k)}\, w\|$$

$$< 2\delta + \varepsilon_1 < 3\varepsilon_1$$

Thus:

$$\|e - u\| = \|\sum_k \sum_i (e_{ii}^{(k)} - e_{ii}^{(k)}\, u)\|$$

$$\leq \sum_k \sum_i \|e_{ii}^{(k)} - e_{ii}^{(k)}u\|$$

$$< m \cdot N \cdot 3\varepsilon_1 = \varepsilon.$$

<u>2.4 Lemma.</u> Let $\mathfrak{A}$ be a C^*-algebra with unit e, let $\mathcal{B}_1$, $\mathcal{B}_2$ be two finite dimensional *-subalgebras of $\mathfrak{A}$ containing e. Let $\alpha: \mathcal{B}_1 \to \mathcal{B}_2$ be a *-isomorphism such that:

$$\|\alpha - I|\mathcal{B}_1\| < 1,$$

where $I: \mathfrak{A} \to \mathfrak{A}$ is the identity map. Then there exists a unitary operator $u \in \mathfrak{A}$ such that:

$$\alpha(x) = u\, x\, u^* \, ; \qquad x \in \mathcal{B}_1.$$

<u>Proof:</u> Let $\{e_{ij}^{(k)}\}$ be a set of matrix units for $\mathcal{B}_1$, and define $f_{ij}^{(k)} = \alpha(e_{ij}^{(k)})$. Then $\{f_{ij}^{(k)}\}$ is a set of matrix units for

$\mathcal{B}_2$. We have

$$\| f_{11}^{(k)} - e_{11}^{(k)} \| = \| (\alpha - I)\, e_{11}^{(k)} \| < \| e_{11}^{(k)} \| = 1$$

for all k. Lemma 1.8 in [6] implies that there exists a partial isometry $w \in \mathfrak{A}$ such that $f_{11}^{(k)}\, w\, e_{11}^{(k)}$ is a partial isometry having $e_{11}^{(k)}$ as initial projection and $f_{11}^{(k)}$ as final projection. Define $u = \sum_k \sum_i f_{i1}^{(k)}\, w\, e_{1i}^{(k)}$. Then u is unitary and

$$u\, e_{ij}^{(k)}\, u^* = f_{ij}^{(k)} = \alpha(e_{ij}^{(k)})$$

so u has the required property.

<u>2.5 Lemma:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and let $\mathcal{B}$ be a finite dimensional *-subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}_1 \subseteq \mathcal{B}$. Then there exists a positive integer n and a unitary operator $u \in \mathfrak{A}$ such that

(i) $u \mathcal{B} u^* \subseteq \mathfrak{A}_n$

(ii) $u x u^* = x \; ; \qquad x \in \mathfrak{A}_1$.

<u>Proof:</u> By lemma 2.3 there exists a unitary $v \in \mathfrak{A}$ and a positive integer n such that $\| v - e \| < 1/3$ and $v \mathcal{B} v^* \subseteq \mathfrak{A}_n$. Define $\mathfrak{A}_1' = v \mathfrak{A}_1 v^* \subseteq \mathfrak{A}_n$ and define an isomorphism $\alpha: \mathfrak{A}_1 \to \mathfrak{A}_1'$ by:

$$\alpha(x) = v x v^* \; ; \quad x \in \mathfrak{A}_1 .$$

Then, for $x \in \mathfrak{A}_1$:

$$\|\alpha(x) - x\| = \|v x v^* - x\| \leq \|v x v^* \quad v^*\|$$
$$+ \|x v^* - x\| \leq 2\|x\| \cdot \|v - e\| < \tfrac{2}{3}\|x\| ,$$

thus

$$\|\alpha - I|\mathfrak{A}_1\| \leq \tfrac{2}{3} < 1$$

By lemma 2.4 there exists a unitary $w \in \mathfrak{A}_n$ such that

$$\alpha(x) = w x w^* ; \qquad x \in \mathfrak{A}_1$$

Let $u = w^* v$. Then

$$u\mathcal{B}u^* = w^* v \mathcal{B} v^* w \subseteq w^* \mathfrak{A}_n w = \mathfrak{A}_n ,$$

since $w \in \mathfrak{A}_n$. For $x \in \mathfrak{A}_1$ we have:

$$u x u^* = w^* v x v^* w = w^* \alpha(x) w = \alpha^{-1}(\alpha(x)) = x ,$$

thus u solves our problem.

<u>2.6 Lemma:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n} = \overline{\bigcup_n \mathcal{B}_n}$. Then there exists an automorphism α of $\mathfrak{A}$ such that:

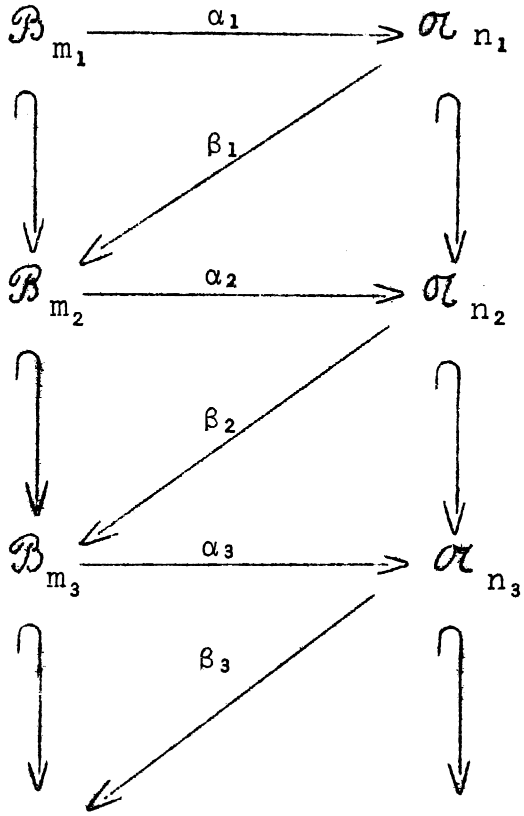
For every positive integer n there exists an positive integer m such that

$$\alpha(\mathcal{B}_n) \subseteq \mathfrak{A}_m$$
$$\mathfrak{A}_n \subseteq \alpha(\mathcal{B}_m) .$$

<u>Proof:</u> By induction we shall find two strictly increasing sequences $m_1 = 1, m_2, m_3, \dots$, and $n_1, n_2, n_3, \dots$, of positive integers, two sequences $u_1, u_2, \dots$, and $v_1, v_2, \dots$, of unitary operators in $\mathfrak{A}$ such that if α_i (resp. β_i) are the isomorphism $\mathfrak{A} \to \mathfrak{A}$ implemented by u_i (resp. v_i), restricted to $\mathcal{B}_{m_i}$ (resp. $\mathfrak{A}_{n_i}$), then $\alpha_i(\mathcal{B}_{m_i}) \subseteq \mathfrak{A}_{n_i}$ $(\beta_i(\mathfrak{A}_{n_i}) \subseteq \mathcal{B}_{m_{i+1}})$ and

the following diagram commutes

(1)



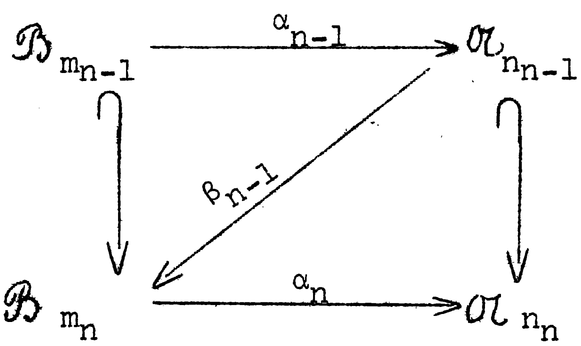
etc.

Here $\hookrightarrow$ denotes the inclusionmap.

We construct $u_1, v_1, u_2, v_2, u_3, \dots$ successively by induction.

By lemma 8 there exists a positive integer n_1 and a unitary operator $u_1 \in \mathfrak{A}$ such that $u_1 \mathcal{B}_1 u_1^* \subseteq \mathfrak{A}_{n_1}$. This is the first step in the induction.

Suppose now that $u_1, v_1, u_2, \dots, u_n$ has been constructed such that the following diagram commutes



We shall construct v_n. Let $\mathfrak{A}' = u_n^* \mathfrak{A}_{n_n} u_n$. Then $\mathfrak{A}'$ is a finite dimensional $*$-subalgebra of $\mathfrak{A}$, and since $u_n \mathcal{B}_{m_n} u_n^* \subseteq \mathfrak{A}_{n_n}$ we have $\mathcal{B}_{m_n} \subseteq \mathfrak{A}'$.

By lemma 2.5 there exists a unitary $v \in \mathfrak{A}$ and a positive integer $m_{n+1} > m_n$ such that $v\mathfrak{A}'v^* \subseteq \mathcal{B}_{m_{n+1}}$, and such that $v x v^* = x$; $x \in \mathcal{B}_{m_n}$.

Let $v_n = vu_n^*$.

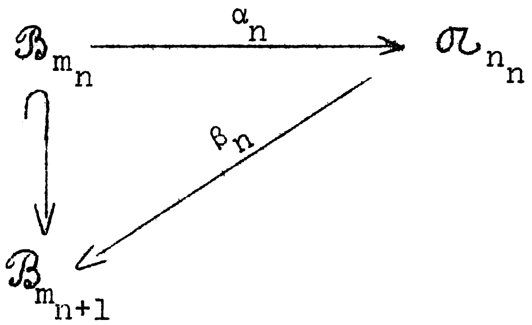
Then

$$v_n \mathfrak{A}_{n_n} v_n^* = vu_n^* \mathfrak{A}_{n_n} u_n v^* = v\mathfrak{A}' v^* \subseteq \mathcal{B}_{m_{n+1}},$$

and if $x \in \mathcal{B}_{m_n}$:

$$\beta_n(\alpha_n(x)) = v_n u_n x u_n^* v_n^* = v u_n^* u_n x u_n^* u_n v^* = v x v^* = x.$$

Hence the following diagram commutes :



u_{n+1} is then constructed in an analogous fashion by "rotating" $v_n^* \mathcal{B}_{m_{n+1}} v_n$ into an algebra $\mathfrak{A}_{n_{n+1}}$ by means of a unitary operator u such that $\mathfrak{A}_{n_n}$ is kept fixed, and define $u_{n+1} = u\, v_n^*$.
By induction we obtain the commutative diagram (1). Because of the commutativity we have $\alpha_{n+1} | \mathcal{B}_{m_n} = \alpha_n$. Hence, we may define a morphism $\alpha: \bigcup_n \mathcal{B}_{m_n} \to \bigcup_n \mathfrak{A}_{n_n}$ by;

$$\alpha | \mathcal{B}_{m_n} = \alpha_n.$$

α is surjective, because if $y \in \mathfrak{A}_{n_k}$ we have $y = \alpha_{k+1}(\beta_k(y)) = \alpha(\beta_k(y))$ and $\beta_k(y) \in \mathcal{B}_{m_{k+1}}$.

Furthermore, since $\alpha|\mathcal{B}_{m_k}$ is injective and hence isometric, α is an isometric isomorphism of $\bigcup_k \mathcal{B}_{m_k}$ onto $\bigcup_k \mathfrak{A}_{n_k}$. Since these two sets are dense in $\mathfrak{A}$, α may be extended to an automorphism of $\mathfrak{A}$.

If n is a positive integer there exists an integer k such that $n \leq m_k$ and $n \leq n_k$. Thus $\alpha(\mathcal{B}_n) \subseteq \alpha(\mathcal{B}_{m_k}) \subseteq \mathfrak{A}_{n_k}$ and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n_k} = \alpha_{k+1}(\beta_k(\mathfrak{A}_{n_k})) \subseteq \alpha_{k+1}(\mathcal{B}_{m_{k+1}}) = \alpha(\mathcal{B}_{m_{k+1}})$, which implies the proposition.

<u>2.7 Theorem:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathcal{B} = \overline{\bigcup_n \mathcal{B}_n}$. Then $\mathfrak{A}$ is isomorphic to $\mathcal{B}$ if and only if $\langle \mathfrak{A}_n \rangle_n$ contains a subsequence $\langle \mathfrak{A}_{n_k} \rangle_k$ and each $\mathfrak{A}_{n_k}$ contains a finite dimensional *-subalgebra $\mathcal{B}'_k$ such that $e \in \mathcal{B}'_k$ and

i) $\langle \mathcal{B}'_n \rangle_n$ is an increasing sequence, and there exists an isomorphism $\alpha: \bigcup_n \mathcal{B}_n \to \bigcup_n \mathcal{B}'_n$ such that $\alpha(\mathcal{B}_n) = \mathcal{B}'_n$ for all n.

ii) For all positive integers n there exists a positive integer k such that

$$\mathfrak{A}_n \subseteq \mathcal{B}'_k$$

<u>Proof:</u> Sufficiency: Suppose that there exists a sequence $\langle \mathcal{B}'_n \rangle_n$ and a *-isomorphism α such that i) and ii) are fulfilled.

Since $\alpha|\mathcal{B}_n$ is an isometry, α is an isometry. By ii) we have $\bigcup_n \mathfrak{A}_n = \bigcup_n \mathcal{B}'_n$. Hence α is an isometric isomorphism between a dense subalgebra of $\mathcal{B}$ and a dense subalgebra of $\mathfrak{A}$ and may be extended by continuity to an isomorphism from $\mathcal{B}$ onto $\mathfrak{A}$.

Necessity: Suppose that $\mathcal{B}$ and $\mathfrak{A}$ are isomorphic, and let $\beta: \mathcal{B} \to \mathfrak{A}$ be a *-isomorphism. Let $\mathcal{B}''_n = \beta(\mathcal{B}_n)$. Since β is an isometry $\bigcup_n \mathcal{B}''_n$ is a dense subset of $\mathfrak{A}$, so $\mathfrak{A} = \overline{\bigcup_n \mathcal{B}''_n}$. Lemma 2.6 then implies that there exist an automorphism γ of $\mathfrak{A}$, and an increasing sequence $\langle n_i \rangle_i$ of positive integers such that $\gamma(\mathcal{B}''_k) \subseteq \mathfrak{A}_{n_k}$; $k = 1,2,\cdots$, and such that for all n there exists a k such that $\mathfrak{A}_n \subseteq \gamma(\mathcal{B}''_k)$. Define $\mathcal{B}'_k = \gamma(\mathcal{B}''_k)$ and $\alpha = \gamma \circ \beta | \bigcup_n \mathcal{B}_n$. Then i) and ii) of the theorem are fulfilled.

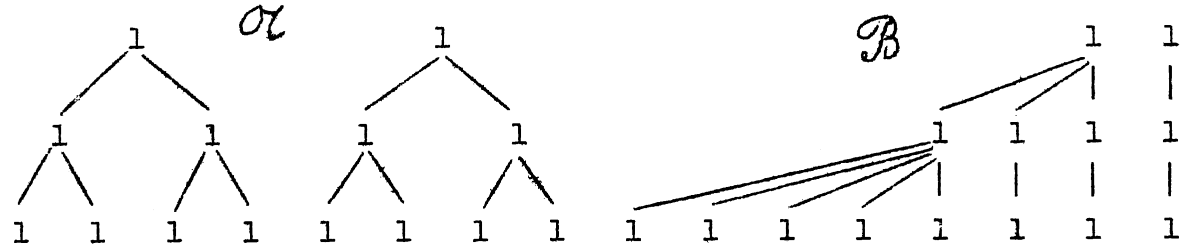
<u>2.8.</u> Glimm has in [6], theorem 1.12, given a necessary and sufficient condition for isomorphism of two uniformly hyperfinite algebras $\mathfrak{A}$ and $\mathcal{B}$. His result is essentially that $\mathfrak{A}$ and $\mathcal{B}$ are isomorphic if and only if the following condition is fulfilled: If $\mathfrak{A}$ contains a type I_n-factor with the same unit as $\mathfrak{A}$, then $\mathcal{B}$ contains a type I_n-factor with the same unit as $\mathcal{B}$ and vice versa. One might suspect that a similar result would be true for an AF-algebra with the condition replaced by: If $\mathfrak{A}$ contains a finite dimensional *-algebra $\mathcal{C}$ with same unit as $\mathfrak{A}$, then

$\mathcal{B}$ contains a *-algebra with the same unit as $\mathcal{B}$ which is isomorphic to $\mathcal{C}$. Such a result is however not true, and the reason is roughly as follows: If $\mathcal{M}_1$ is a factor of type I_n, $\mathcal{M}_2$ is a factor of type I_{nm}, $n,m < \infty$, then $\mathcal{M}_1$ can be embedded in $\mathcal{M}_2$ in essentially only one way. By this is meant that if α_1, α_2 are two injective morphisms $\mathcal{M}_1 \to \mathcal{M}_2$ which maps the unit of $\mathcal{M}_1$ on the unit of $\mathcal{M}_2$, then there exists an automorphism β of $\mathcal{M}_2$ such that $\alpha_1 = \beta \circ \alpha_2$. That this is the case, follows easily from [2], Ch. 1, § 4, theoreme 3. Because of this, if $\mathfrak{A} = \overline{\bigcup_n \mathcal{M}_n}$ is a UHF algebra, where all $\mathcal{M}_n$'s are factors, then the isomorphism class of $\mathfrak{A}$ depends only on the factors themselves and not on the way they are embedded into each other. In fact, the isomorphism α of theorem 2.7 will automatically exist if all the $\mathcal{B}_n$ are factors isomorphic to $\mathcal{B}_n'$, so Glimm's result is a corollary to this theorem.

On the other hand, a finite-dimensional C^*-algebra $\mathfrak{A}_1$ may in most cases be embedded into another finite dimensional C^*-algebra $\mathfrak{A}_2$ in essentially different ways. Thus we may expect that the isomorphism class of an AF-algebra $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ depends not only on the $\mathfrak{A}_n$'s, but also on the way they are embedded into each other. This dependence is reflected in the condition i) of the theorem. Of course condition i) may be replaced by the equivalent condition that all $\mathcal{B}_n$ are isomorphic to $\mathcal{B}_n'$, and that corresponding factors in the central decomposition of $\mathcal{B}_n$ and $\mathcal{B}_n'$ are partially embedded in corresponding factors of $\mathcal{B}_{n+1}$ and $\mathcal{B}'_{n+1}$ with the same partial multiplicities. This will then enable us to construct α by using the method which in 1.8 is used to show that the diagram of an AF-algebra determines the algebra up to isomorphism.

We shall give an explicit example of two AF-algebras $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathfrak{B} = \overline{\bigcup_n \mathfrak{B}_n}$ such that $\mathfrak{A}_n$ is isomorphic to $\mathfrak{B}_n$ for all n, but $\mathfrak{A}$ is not isomorphic to $\mathfrak{B}$. By lemma 2.3, each finite dimensional *-subalgebra of $\mathfrak{A}$ (resp. $\mathfrak{B}$) is isomorphic with a subalgebra of one $\mathfrak{A}_n$ (resp. $\mathfrak{B}_n$) so $\mathfrak{A}$ and $\mathfrak{B}$ contains the same finite dimensional subalgebras. Thus the condition i) of theorem 2.7 is essential.

$\mathfrak{A}$ and $\mathfrak{B}$ have the following diagrams:



For all n $\mathfrak{A}_n \cong \mathfrak{B}_n \cong \bigoplus_{2^n} M_1$, where $\bigoplus_{2^n} M_1$ is the direct sum of 2^n replicas of M_1. From the classification of ideals to be given in § 3 it immediately follows that $\mathfrak{B}$ has ideals of dimension 1 while all the ideals $\neq \{0\}$ in $\mathfrak{A}$ are infinite dimensional. Since the dimension of an ideal is an isomorphism invariant, $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

A little remark at last: At first sight it perhaps does not seem to be essential that the isomorphisms between finite dimensional subalgebras considered in lemmas 2.3 through 2.6 are unitary implemented. And in fact, the only use which is made of this fact is in the proof of 2.6, where it is important that an isomorphism between subalgebras may be extended to an automorphism of the algebra in which they are embedded. The existence of this extention is assured by the unitary implementation.

3. Algebraic structure of an approximately finite dimensional C^*-algebra.

3.1 Lemma: Let $\mathfrak{A}$ be a C^*-algebra and let $\{\mathcal{B}_n\}_{n=1}^{\infty}$ be an increasing sequence of finite dimensional subalgebras of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{B}_n}$. Let $\mathcal{J}$ be a closed twosided ideal in $\mathfrak{A}$. Then

$$\mathcal{J} = \overline{\mathcal{J} \cap (\bigcup_{n=1}^{\infty} \mathcal{B}_n)}.$$

Proof: Set $\mathcal{J}_n = \mathcal{J} \cap \mathcal{B}_n$. Then $\mathcal{J}_n$ is a closed, twosided ideal in $\mathcal{B}_n$ and we must prove that $\overline{\bigcup_{n=1}^{\infty} \mathcal{J}_n} = \mathcal{J}$.

Trivially: $\overline{\bigcup_{n=1}^{\infty} \mathcal{J}_n} \subseteq \mathcal{J}$.

On the other hand, suppose that $x \notin \overline{\bigcup_{n=1}^{\infty} \mathcal{J}_n}$. We must prove that $x \notin \mathcal{J}$.

Let $\rho: \mathfrak{A} \to \mathfrak{A}/\mathcal{J}$ be the quotient mapping. Let $\langle x_n \rangle_n$ be a sequence such that $x_n \in \mathcal{B}_n$ and $x_n \to x$. Since $x \notin \overline{\bigcup_{n=1}^{\infty} \mathcal{J}_n}$ we have that

$$\inf_{y \in \bigcup_n \mathcal{J}_n} \|x-y\| = \varepsilon > 0$$

Since $x_n \to x$ there exists an N such that $n \geq N$ implies :

$$\|x-x_n\| < \varepsilon/2$$

For $n \geq N$ and $y \in \mathcal{J}_n$ we therefore have

$$\|x_n-y\| \geq \|x-y\| - \|x_n-x\| > \varepsilon - \varepsilon/2 = \varepsilon/2.$$

Now, since $\ker \rho|\mathcal{B}_n = \mathcal{J} \cap \mathcal{B}_n = \mathcal{J}_n$ we have

$$\|\rho(x_n)\| = \inf_{y\in \mathcal{Y}_n} \|x_n - y\| \geq \frac{\varepsilon}{2} ,$$

because the norm on the C^*-algebra $\rho(\mathcal{B}_n)$ is the same whether $\rho(\mathcal{B}_n)$ is viewed as a subalgebra of $\rho(\mathfrak{A})$, or as the image of the quotient mapping $\mathcal{B}_n \to \mathcal{B}_n/\mathcal{Y}_n$. Now, since $x_n \to x$ and ρ is continuous $\rho(x_n) \to \rho(x)$. In particular

$$\|\rho(x)\| = \lim_{n\to\infty}\|\rho(x_n)\| \geq \varepsilon/2$$

so $x \notin \mathcal{Y}$.

<u>3.2</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. In the following the term "ideal in $\mathfrak{A}$" will mean "normclosed twosided ideal in $\mathfrak{A}$", while the term "ideal in $\bigcup_n \mathfrak{A}_n$" will mean "twosided ideal in $\bigcup_n \mathfrak{A}_n$". The ideals in $\bigcup_n \mathfrak{A}_n$ are described as follows :

<u>Lemma:</u> Let I be an ideal in $\bigcup_n \mathfrak{A}_n$. Then I has the form:

(1) $$I = \bigcup_{n=1}^{\infty} \bigoplus_{\substack{k \\ (nk)\in\Lambda}} M_{(nk)}$$

where Λ is some subset of $\mathcal{D} = \mathcal{D}(\mathfrak{A})$ satisfying the two conditions:

i) If $(nk) \in \Lambda$ and $(nk) \searrow (n+1\,q)$ then $(n+1\,q) \in \Lambda$

ii) If $(nk) \searrow (n+1\,q)$ implies that $(n+1\,q) \in \Lambda$; $q = 1,---, n_{n+1}$ then $(nk) \in \Lambda$.

Conversely, if $\Lambda \subseteq \mathcal{D}$ satisfies i) and ii) then the subset I of $\bigcup_n \mathfrak{A}_n$ defined by (1) is an ideal in $\bigcup_n \mathfrak{A}_n$ such that $I \cap \mathfrak{A}_n = \bigoplus_{\substack{k \\ (nk) \in \Lambda}} M_{(nk)}$.

<u>Proof:</u> Suppose I is an ideal in $\bigcup_n \mathfrak{A}_n$, and define $I_n = I \cap \mathfrak{A}_n$. Then I_n is an ideal of $\mathfrak{A}_n$, and $I = \bigcup_n I_n$.

It is well known that the ideals of $\mathfrak{A}_n = \bigoplus_{k=1}^{n_n} M_{(nk)}$ is the subsums of this finite direct sum of factors. Hence I_n has the form:

$$I_n = \bigoplus_{\substack{k \\ (nk) \in \Lambda}} M_{(nk)},$$

where Λ is some subset of $\mathcal{D}$, so I has the form (1). We show that Λ satisfies i) and ii).

i) If $(nk) \in \Lambda$ then $M_{(nk)} \subseteq I_n \subseteq I$. In particular $e^{(nk)} \in I$. Now, if $(nk) \searrow (n+1\,q)$ then $e^{(nk)}\, e^{(n+1\,q)} \neq 0$. (see 1.8). Since $e^{(n+1\,q)} \in M_{(n+1\,q)}$ we have that $e^{(nk)}\, e^{(n+1\,q)} \in M_{(n+1\,q)}$. Since I is an ideal: $e^{(nk)}\, e^{(n+1\,q)} \in I$. Hence $M_{(n+1\,q)} \cap I \neq \{0\}$, and since I is an ideal and $M_{(n+1\,q)}$ is a finite dimensional factor: $M_{(n+1\,q)} \subseteq I$, i.e. $(n+1q) \in \Lambda$.

ii) Suppose that $(nk) \searrow (n+1q)$ implies that $(n+1q) \in \Lambda$; $q = 1, ---, n_{n+1}$. This is equivalent to say that if $M_{(nk)}$ is partially embedded in $M_{(n+1\,q)}$, then $M_{(n+1\,q)} \subseteq I$. But since $M_{(nk)}$ is contained in the sum of the factors $M_{(n+1\,q)}$ in which it is partially embedded $M_{(nk)} \subseteq I$; thus $(nk) \in \Lambda$.

Conversely, assume that Λ satisfies i) and ii), and define I by (1). Define:

$$I_n = \bigoplus_{\substack{k \\ (nk)\in\Lambda}} M_{(nk)}$$

From i) it follows that if $M_{(nk)} \subseteq I_n$ and $M_{(nk)}$ is partially embedded in $M_{(n+1\,q)}$, then $M_{(n+1\,q)} \in I_{n+1}$. Hence $M_{(nk)} \subseteq I_{n+1}$ and by this $I_n \subseteq I_{n+1}$. Hence, if $x \in \bigcup_k \mathfrak{A}_k$, $y \in I = \bigcup_k I_k$ there exists an n such that $x \in \mathfrak{A}_n$, $y \in I_n$. Since I_n is an ideal in $\mathfrak{A}_n$ this implies that $xy, yx \in I_n \subseteq I$. Hence I is an ideal in $\bigcup_n \mathfrak{A}_n$.

It remains to show that $I \cap \mathfrak{A}_n = I_n$. Clearly: $I_n \subseteq I \cap \mathfrak{A}_n$. To show equality it is enough to show that if $M_{(nk)} \subseteq I \cap \mathfrak{A}_n$, then $M_{(nk)} \subseteq I_n$. So suppose $M_{(nk)} \subseteq I$. Since $M_{(nk)}$ has a finite basis (as a vector space), and the I_m's are increasing linear subspaces of I, there exists an m such that $M_{(nk)} \subseteq I_m$.

If $m \leq n$ there is nothing more to show, so suppose $m > n$. Suppose ad absurdum that $M_{(nk)} \not\subseteq I_n$. From ii) it follows that there exists a $M_{(n+1\,k_1)}$ such that $M_{(nk)}$ is partially embedded in $M_{(n+1\,k_1)}$ and $(n+1\;k_1) \notin \Lambda$. By ii) again it follows that there exists $M_{(n+2\;k_2)}$ such that $M_{(n+1\;k_1)}$ is partially embedded in $M_{(n+2\;k_2)}$

and $(n+2\ k_2) \in \Lambda$. As partially embedding is a transitive relation $M_{(nk)}$ then partially embedded in $M_{(n+2\ k_2)}$. Proceeding by induction we find a factor $M_{(mq)}$ such that $M_{(nk)}$ is partially embedded in $M_{(mq)}$, but $(mq) \notin \Lambda$. Hence $M_{(mq)} \cap I_m = \{0\}$. Therefore $M_{(nk)}$ is not contained in I_m, which is a contradiction. Thus $I \cap \mathfrak{A}_n = I_n$.

3.3 <u>Theorem.</u> Let $\mathfrak{A} = \bigcup_n \mathfrak{A}_n$, and define

A_1 = set of norm closed ideals in $\mathfrak{A}$

A_2 = set of ideals in $\bigcup_n \mathfrak{A}_n$

A_3 = set of subsets Λ of $\mathfrak{D}(\mathfrak{A})$ satisfying i) and ii) of lemma 3.2.

Then there exists a natural $1 - 1$ correspondence between the elements of A_1, A_2 and A_3. This correspondence may be defined by bijections :

$$\phi_{23}: \quad A_3 \to A_2 \ : \ \Lambda \to \bigcup_{n=1}^{\infty} \bigoplus_{\substack{k \\ (nk)\in\Lambda}} M_{(nk)}$$

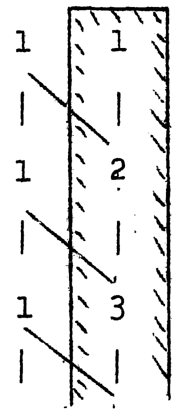
$$\phi_{12}: \quad A_2 \to A_1 \ : \ I \to \bar{I}$$

<u>Proof:</u> From lemma 3.2 it follows that ϕ_{23} is bijective.

Lemma 3.1 implies immediately that ϕ_{12} is surjective. To show that ϕ_{12} is injective suppose that I_1, I_2 are two ideals in $\bigcup_n \mathfrak{A}_n$ such that $I_1 \neq I_2$. From lemma 3.2 it follows that there exists a factor $M_{(nk)}$ which is contained in one but not the other of the ideals I_1 and I_2. Suppose that $M_{(nk)} \subseteq I_1 \sim I_2$.

Then $e^{(nk)} \notin I_2 \cap \mathfrak{A}_m$; $m = n, n+1, ---$ because if $e^{(nk)} \in I_2 \cap \mathfrak{A}_m$ for some m, then $e^{(nk)} \in I_2$ and thus $M_{(nk)} = e^{(nk)} M_{(nk)} \in I_2$. Hence $e^{(nk)}$ is mapped into a projection $\neq 0$ by the canonical mapping $\mathfrak{A}_m \to \mathfrak{A}_m/(I_2 \cap \mathfrak{A}_m)$; $m = n, n+1, ---$. Hence: $\inf_{y \in I_2 \cap \mathfrak{A}_m} \| e^{(nk)} - y \| = 1$, and since $I_2 = \bigcup_m (I_2 \cap \mathfrak{A}_m)$: $\inf_{y \in I_2} \| e^{(nk)} - y \| = 1$. Hence $e^{(nk)} \notin \overline{I}_2$ while $e^{(nk)} \in I_1 \subseteq \overline{I}_1$, thus $\overline{I}_1 \neq \overline{I}_2$, hence ϕ_{12} is injective.

<u>3.4.</u> As an example of the use of theorem 3.3 we look at the algebra $\mathcal{LC}(\kappa) + \mathbb{C}I$ mentioned in 1.9 .



The only ideals of this algebra, except for the trivial ones, is the algebra generated by the factors lying inside the boundary indicated. From the description of this algebra given in 1.9 it follows that this ideal is $\mathcal{LC}(\kappa)$, and we thus get the well known fact that the only nontrivial norm closed ideal in $\mathcal{LC}(\kappa) + \mathbb{C}I$ is $\mathcal{LC}(\kappa)$.

<u>3.5</u> Using theorem 3.3 we shall find a condition for $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ being simple:

<u>Corollary.</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. Then the following conditions are equivalent :

i) $\mathfrak{A}$ is simple

ii) $\mathfrak{A}$ is algebraically simple

iii) If $M_{(nk)}$ is a factor in the central decomposition of $\mathfrak{A}_n$ there exists an $m \geq n$ such that $M_{(nk)}$ is partially embedded in all factors in the central decomposition of $\mathfrak{A}_m$.

iv) For all $e^{(nk)}$ there exists an $m \geq n$ such that

$$e^{(nk)}\, e^{(mq)} \neq 0; \quad q = 1, ---, n_m.$$

(The equivalence of i) and ii) is a well known general result for Banach-algebras with unit, and is stated only for completeness. See [15], Ch. XI, prop. 1.1 and 1.2).

<u>Proof:</u> By 1.8:

iii) <=> iv)

Now,suppose iii) and suppose that I is an ideal of $\mathfrak{A}$ which is not $\{0\}$. By theorem 3.3 I contains some factor $M_{(nk)}$. By iii) and condition i) of lemma 3.2 there exists an $m \geq n$ such that $\mathfrak{A}_m \subseteq I$. Then $e \in I$; hence $I = \mathfrak{A}$; hence iii) => i).

Now, suppose i). We show iv) by using the fact that the ideal generated by some $M_{(nk)}$ is $\mathfrak{A}$. This ideal is:

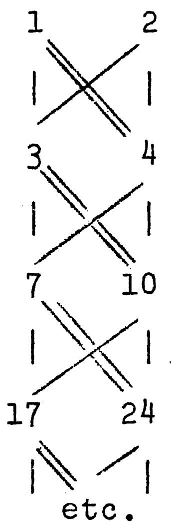
$$I = \overline{\bigcup_{m \geq n} \bigoplus_{\substack{q \\ e^{(mq)} e^{(nk)} \neq 0}} M_{(mq)}} ,$$

as we see in the following manner. Define

$$I_m = \bigoplus_{\substack{q \\ e^{(mq)}e^{(nk)} \neq 0}} M_{(mq)} \quad ; \quad m = n, n+1, \dots$$

Then I_m is an ideal of $\mathfrak{A}_m$, and it is the least ideal which contains $e^{(nk)}$, and thus the least ideal which contains $M_{(nk)}$. If $M_{(nk)}$ is partially embedded in $M_{(mq)}$ and $M_{(mq)}$ is partially embedded in $M_{(m+1p)}$ then $M_{(nk)}$ is partially embedded in $M_{(m+1p)}$; hence $I_m \subseteq I_{m+1}$. It follows that $\bigcup_m I_m$ is an ideal of $\bigcup_m \mathfrak{A}_m$, and it is the least ideal which contains $M_{(nk)}$. Thus $I = \overline{\bigcup_m I_m}$ is the least ideal of $\mathfrak{A}$ which contains $M_{(nk)}$. Since $\mathfrak{A}$ is simple, $\mathfrak{A} = I$. Now, suppose ad absurdum that for all $m \geq n$ exists an q such that $e^{(mq)}e^{(nk)} = 0$. For each m is then $I_m \neq \mathfrak{A}_m$ and thus $\inf_{x \in I_m} \|e-x\| = 1$; hence $\inf_{x \in \bigcup_m I_m} \|e-x\| = 1$; hence $e \notin \overline{\bigcup_m I_m} = I = \mathfrak{A}$, which is a contradiction. Hence i) $\Rightarrow$ iv).

3.6 We show an example of an infinite dimensional AF algebra which are simple but not UHF. Its diagram is



From 3.5 it follows that this algebra $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is simple.

Furthermore

$$\mathfrak{A}_n \cong M_{\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]} \oplus M_{\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]}$$

where $\begin{bmatrix} n \\ 1 \end{bmatrix}$, $\begin{bmatrix} n \\ 2 \end{bmatrix}$ are defined recursively as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 , \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 ,$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = 2\begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}$$

By the Euklidean algorithm we have the following equivalences:

$\begin{bmatrix} n \\ 1 \end{bmatrix}$ and $\begin{bmatrix} n \\ 2 \end{bmatrix}$ are relatively prime

$\Updownarrow$

$\begin{bmatrix} n \\ 2 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} n \\ 1 \end{bmatrix}$ are relatively prime

$\Updownarrow$

$\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = \begin{bmatrix} n-1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}$ are relatively prime

Since 1 and 2 are relatively prime it follows by induction that $\begin{bmatrix} n \\ 1 \end{bmatrix}$ and $\begin{bmatrix} n \\ 2 \end{bmatrix}$ are relatively prime for all n.

Now, suppose that $M \subseteq \mathfrak{A}_n$ is a factor of type I_m with unity e. Then $M e^{(ni)}$ is a type I_m factor in $M_{(ni)}$ with unit $e^{(ni)}$, $i = 1,2$, thus m must divide $\begin{bmatrix} n \\ 1 \end{bmatrix}$ and $\begin{bmatrix} n \\ 2 \end{bmatrix}$; thus $m = 1$, i.e. $M = \mathbb{C}e$. Now if M is a type I_m factor in $\mathfrak{A}$ with unit e, then M is isomorphic with a factor in some $\mathfrak{A}_n$ with the same unit by lemma 2.3. Thus $M = \mathbb{C}e$.

Hence $\mathfrak{A}$ does not contain any factor of type I_m, $m < \infty$, with unit e, except from $\mathbb{C}e$, so $\mathfrak{A}$ is not UHF.

<u>3.7.</u> We now proceed to study the primitive ideals of an AF algebra $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. Since the property of being primitive is not an intrinsic property of the ideal I itself, but in fact is a property which solely depends on $\mathfrak{A}/I$, we first study the structure of $\mathfrak{A}/I$, for $\mathfrak{A}$ and I given.

<u>Proposition.</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and let I be an ideal of $\mathfrak{A}$. Index the factors in the central decomposition of $\mathfrak{A}_n$ in such a way that the subset $\Lambda \subseteq \mathfrak{D}(\mathfrak{A})$ corresponding to I has the form

$$\Lambda = \{(nk);\ m_n+1 \le k \le n_n;\ n = 0,1,---\}$$

Let $\rho: \mathfrak{A} \to \mathfrak{A}/I$ be the quotient mapping. Then $\mathfrak{A}/I = \overline{\bigcup_n \rho(\mathfrak{A}_n)}$ (in the AF-sense) and the central decomposition of $\rho(\mathfrak{A}_n)$ is:

$$\rho(\mathfrak{A}_n) = \bigoplus_{k=1}^{m_n} \rho(M_{(nk)}),$$

where $\rho(M_{(nk)}) \cong M_{(nk)}$ for $(nk) \notin \Lambda$. Furthermore, the diagram of $\mathfrak{A}/I$ consists of the pairs (nk), $k = 1,---,m_n$, $n = 0,1,---$ together with the relations $\searrow^p$ inherited from $\mathfrak{D}(\mathfrak{A})$, i.e. $(nk) \searrow^p (mq)$ in $\mathfrak{D}(\mathfrak{A}/I)$ if and only if $(nk) \searrow^p (mq)$ in $\mathfrak{D}(\mathfrak{A})$.

<u>Proof:</u> $\mathfrak{A} = \overline{\bigcup_n \rho(\mathfrak{A}_n)}$ in the AF-sense by 1.5. By theorem

3.3 $I \cap (\bigcup_n \mathfrak{A}_n) = \bigcup_n \bigoplus_{k=m_n+1}^{n_n} M_{(nk)}$ and then by lemma 3.2 $I \cap \mathfrak{A}_n = \bigoplus_{k=m_n+1}^{n_n} M_{(nk)}$. Hence $\rho|\mathfrak{A}_n$ has kernel $\bigoplus_{k=m_n+1}^{n_n} M_{(nk)}$ and since $\mathfrak{A}_n$ is the direct sum of this kernel and $\bigoplus_{k=1}^{m_n} M_{(nk)}$ the central decomposition of $\rho(\mathfrak{A}_n)$ is $\rho(\mathfrak{A}_n) = \bigoplus_{k=1}^{m_n} \rho(M_{(nk)})$ where $\rho(M_{(nk)}) \cong M_{(nk)} \cong M_{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}$ for $k = 1, ---, m_n$. Indexing the factor $\rho(M_{(nk)})$ by (nk) it is clear that the underlying set of $\mathfrak{D}(\mathfrak{A}/I)$ consist of the pairs (nk); $k = 1, ---, m_n$, $n = 0,1, ---$.

Now, suppose that $(nk), (mq) \notin \Lambda$ and suppose $m \geq n$. Let f be a minimal projection in $M_{(nk)}$ and let $f_1 --- f_p$ be a maximal set of mutually orthogonal minimal projections of $M_{(mq)}$ such that $\sum_{i=1}^{p} f_i \leq f$, i.e. we have $\sum_{i=1}^{p} f_i = e^{(mq)}f$. By proposition 1.7 p is the multiplicity of the partial embedding of $M_{(nk)}$ in $M_{(mq)}$. Now, since $\rho|M_{(nk)}$ and $\rho|M_{(mq)}$ are injective, $\rho(f)$ is a minimal projection in $\rho(M_{(nk)})$, $\rho(f_i)$ are minimal in $\rho(M_{(mq)})$ and $\sum_{i=1}^{p} \rho(f_i) = \rho(e^{(mq)})\rho(f)$. Therefore the multiplicity of the partial embedding of $\rho(M_{(nk)})$ in $\rho(M_{(mq)})$ is also p, and the last assertion of the proposition follows.

<u>3.8. Theorem.</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, let I be an ideal in $\mathfrak{A}$, let Λ be the subset of $\mathfrak{D}(\mathfrak{A})$ associated to I. Then the following conditions are equivalent :

i) I is primitive

ii) There does not exist two ideals I_1, I_2 in $\mathfrak{A}$ such that $I_1 \neq I \neq I_2$ and $I = I_1 \cap I_2$

iii) If $(nk), (mq) \notin \Lambda$ then there exists a $p \geq n,m$ and a $(pr) \notin \Lambda$ such that $M_{(nk)}$ and $M_{(mq)}$ both are partially embedded in $M_{(pr)}$.

(The implication i) => ii) is well known for an arbitrary C^*-algebra, while the implication ii) => i) is proved for separable C^*-algebras by Dixmier in [1]).

Proof: Let $\rho: \mathfrak{A} \to \mathfrak{A}/I$ be the quotient mapping. Then I is primitive in $\mathfrak{A}$ iff $\{0\}$ is primitive in $\mathfrak{A}/I = \rho(\mathfrak{A})$. There is a one-one correspondence between the ideals in $\mathfrak{A}$ containing I and the ideals of $\mathfrak{A}/I$ given by $\mathcal{Y} \to \rho(\mathcal{Y})$; $I \subseteq \mathcal{Y} \subseteq \mathfrak{A}$; $\mathcal{Y}$ ideal in $\mathfrak{A}$. This mapping (and thus its inverse mapping) preserves inclusions, so ii) holds iff $\{0\}$ is not the intersection of two ideals both different from $\{0\}$ in $\mathfrak{A}/I$. By proposition 3.7, iii) holds iff for any two factors of the form $\rho(M_{(nk)})$, $\rho(M_{(mq)})$, (nk), $(mq) \notin \Lambda$ in $\mathfrak{A}/I$ there exist an $p \geq n,m$ and an $(pr) \notin \Lambda$ such that $\rho(M_{(nk)})$ and $\rho(M_{mq})$ both are partially embedded in $\rho(M_{(pr)})$.

From the remarks just stated it follows that we may assume in the rest of the argument that $I = \{0\}$.

That i) => ii) follows from [3], Corollaire 2.8.4 and Lemme 2.11.3. (ii).

ii) => i): This is proved for separable C^*-algebras in [1], Corollaire 1 to Théorème 2. We shall later, in 4.17, give an direct argument for this fact which applies for AF-algebras.

ii) => iii): Assume ii). Then the intersection of any two ideals both $\neq \{0\}$ is $\neq \{0\}$. Now, let $(nk), (mq) \in \mathcal{D}(\mathfrak{A})$. By the argument used in 3.5 the ideal in $\bigcup_n \mathfrak{A}_n$ generated algebraically by $M_{(nk)}$ (resp. $M_{(mq)}$) is $I_1 = \bigcup_{p \geq n} \bigoplus_{\substack{r \\ e^{(nk)} e^{(pr)} \neq 0}} M_{(pr)}$

(resp. $I_2 = \bigcup_{p \geq m} \bigoplus_{\substack{r \\ e^{(mq)} e^{(pr)} \neq 0}} M_{(pr)}$).

By ii) $\bar{I}_1 \cap \bar{I}_2 \neq \{0\}$, and then by 3.1: $\{0\} \neq \bar{I}_1 \cap \bar{I}_2 \cap (\bigcup_n \mathfrak{A}_n) =$

$= (\bar{I}_1 \cap (\bigcup_n \mathfrak{A}_n)) \cap (\bar{I}_2 \cap (\bigcup_n \mathfrak{A}_n)) = I_1 \cap I_2$

where the last equality follows from theorem 3.3. I_1 and I_2 are defined as the union of some subspaces indexed by p, and by the argument in 3.5 these subspaces are increasing with p. Since $I_1 \cap I_2 \neq \{0\}$ there must exist an p such that the intersection of the corresponding subspaces in I_1 and I_2 are not $\{0\}$, i.e. there exists an $(pr) \in \mathcal{D}(\mathfrak{A})$ such that $e^{(mq)} e^{(pr)} \neq 0 \neq e^{(nk)} e^{(pr)}$. Then $M_{(nk)}$ and $M_{(mq)}$ are both partially embedded in $M_{(pr)}$.

iii) => ii). Assume iii) and let I_1, I_2 be two ideals in $\mathfrak{A}$ different from $\{0\}$. Then there exist $(nk), (mq) \in \mathcal{D}(\mathfrak{A})$ such that $M_{(nk)} \subseteq I_1$, $M_{(mq)} \subseteq I_2$. Then the ideal generated by $M_{(nk)}$ (resp. $M_{(mq)}$) is contained in I_1 (resp. I_2) so:

$$\mathcal{J}_1 = \bigcup_{p \geq n} \bigoplus_{\substack{r \\ e^{(nk)}e^{(pr)} \neq 0}} M_{(pr)} \subseteq I_1, \quad \text{and } \mathcal{J}_2 = \bigcup_{p \geq m} \bigoplus_{\substack{r \\ e^{(mq)}e^{(pr)}}} M_{(pr)} \subseteq I_2$$

By iii) there exists a $p \geq n,m$ and a r such that $e^{(mq)}e^{(pr)} \neq 0 \neq e^{(nk)}e^{(pr)}$; thus $M_{(pr)} \subseteq \mathcal{J}_1 \cap \mathcal{J}_2 \subseteq I_1 \cap I_2$; hence ii) holds.

<u>3.9. Corollary.</u> Let $\mathfrak{A} = \bigcup_n \mathfrak{A}_n$. Then the following conditions are equivalent

i) $\mathfrak{A}$ is primitive

ii) There does not exist two ideals in $\mathfrak{A}$ different from $\{0\}$ whose intersection is $\{0\}$.

iii) If $(nk), (mq) \in \mathcal{D}(\mathfrak{A})$ there exist a $p \geq n,m$ and a $(pr) \in \mathcal{D}(\mathfrak{A})$ such that $M_{(nk)}$ and $M_{(mq)}$ both are partially embedded in $M_{(pr)}$.

4. States and representations of approximately finite dimensional C^*-algebras.

4.1. In sections 4.1 - 5 we shall show under which conditions a state ω of $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is a factor state, and we shall find necessary and sufficient conditions for two factor representations of $\mathfrak{A}$ to be quasi-equivalent. As the methods of proof are essentially those used by Powers in [12] to prove the same results for UHF algebras, we will mostly only state the results.

If $\mathcal{B}$ is a C^*-subalgebra of a C^*-algebra $\mathfrak{A}$, then $\mathcal{B}^c$ is the commutant of $\mathcal{B}$ relative to $\mathfrak{A}$.

Lemma. Let $\mathfrak{A}$ be a C^*-algebra with unit e, and let $\mathcal{B} \subseteq \mathfrak{A}$ be a finite dimensional *-subalgebra of $\mathfrak{A}$ such that $e \in \mathcal{B}$. Let Π be a representation of $\mathfrak{A}$. Then

$$\{\Pi(\mathcal{B}^c)\}'' = \{\Pi(\mathcal{B})\}' \cap \{\Pi(\mathfrak{A})\}''$$

Proof: As proof of lemma 2.3 in [12].

4.2. Lemma. Let $\mathfrak{A}$ be an AF algebra, and let $\mathcal{F}$ be the set of finite dimensional *-subalgebras of $\mathfrak{A}$ with unit e. Let Π be a representation of $\mathfrak{A}$ and let $\mathcal{R} = \{\Pi(\mathfrak{A})\}''$. Then the center of $\mathcal{R}$ is:

$$\mathcal{R} \cap \mathcal{R}' = \bigcap_{\mathcal{B} \in \mathcal{F}} \{\Pi(\mathcal{B}^c)\}''$$

Proof: As proof of lemma 2.4 in [12].

<u>4.3. Lemma:</u> Let $\mathfrak{A}$ be a C^*-algebra with unit e, and let I be an ideal in $\mathfrak{A}$. Let $\mathcal{B}$ be a finite dimensional *-subalgebra of $\mathfrak{A}$ containing e. Let $\rho: \mathfrak{A} \to \mathfrak{A}/I = \mathfrak{A}_o$ be the quotient morphism and let $\mathcal{B}_o = (\mathcal{B})$. Then

$$\mathcal{B}_o^{\,c} = \rho(\mathcal{B}^c)$$

<u>Proof:</u> Since ρ is an morphism:

$$\rho(\mathcal{B}^c) \subseteq \mathcal{B}_o^{\,c}$$

Let $\{e_{ij}^{(k)}\}$ be a set of matrix units for $\mathcal{B}$, and let $e_o = \rho(e) = \rho(\sum_k \sum_i e_{i1}^{(k)} e_{1i}^{(k)}) = \sum_k \sum_i \rho(e_{i1}^{(k)})\rho(e_{1i}^{(k)})$ be the unit of $\mathcal{B}_o$. Now, suppose that $x = \rho(y) \in \mathcal{B}_o^{\,c}$. Then

$$x = e_o x = \sum_k \sum_i \rho(e_{i1}^{(k)})\rho(e_{1i}^{(k)})x = \sum_k \sum_i \rho(e_{i1}^{(k)}) x \rho(e_{1i}^{(k)})$$

$$= \sum_k \sum_i \rho(e_{i1}^{(k)} y\, e_{1i}^{(k)}) = \rho(\sum_k \sum_i e_{i1}^{(k)} y\, e_{1i}^{(k)})$$

It is straight forward to verify that $\sum_k \sum_i e_{i1}^{(k)}\, y\, e_{1i}^{(k)} \in \mathcal{B}^c$; thus $x \in \rho(\mathcal{B}^c)$; i.e. $\mathcal{B}_o^{\,c} \subseteq \rho(\mathcal{B}^c)$.

<u>4.4. Theorem:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and suppose that ω is a state of $\mathfrak{A}$ and Π_ω the representation associated to ω by the Gelfard-Segal construction. Then the following conditions are equivalent:

i) ω is a factor state

ii) For all $x \in \mathfrak{A}$ there exists an integer $r > 0$ such that
$|\omega(xy) - \omega(x)\omega(y)| \leq \|\Pi_\omega(y)\|$
for all $y \in \mathfrak{A}_r^{\,c}$.

iii) For all $x \in \mathfrak{A}$ there exists a finite dimensional *-algebra $\mathcal{B} \subseteq \mathfrak{A}$ containing e such that $|\omega(xy) - \omega(x)\omega(y)| \leq \|\Pi_\omega(y)\|$ for all $y \in \mathcal{B}^c$.

Proof: Suppose first that Π_ω is faithful. Then $\|\Pi_\omega(x)\| = \|x\|$ for all $x \in \mathfrak{A}$, and the argument which shows the equivalence of (i), (ii) and (iii) is exactly the same as the argument Powers uses in showing theorem 2.5 in [12] if we replace lemma 2.3 and 2.4 in Powers work by lemma 4.1 and 4.2.

Then, suppose that Π_ω is not faithful. Let $I = \ker \Pi_\omega$, and let $\rho: \mathfrak{A} \to \mathfrak{A}/I$ be the quotient morphism. Then by prop. 3.7 $\mathfrak{A}/I = \overline{\bigcup_n \rho(\mathfrak{A}_n)}$ (in the Af-sense). We may lift ω to a state ω_o of $\mathfrak{A}/I$ and Π_ω to a faithful representation Π_{ω_o} of $\mathfrak{A}/I$ such that $\omega = \omega_o \circ \rho$ and $\Pi_\omega = \Pi_{\omega_o} \circ \rho$. Then Π_{ω_o} is the Gelfand-Segal representation of $\mathfrak{A}/I$ associated to ω_o. Therefore the following conditions are equivalent:

(i)' ω_o is a factor state of $\rho(\mathfrak{A})$

(ii)' For all $x \in \rho(\mathfrak{A})$ there exists an integer $r > 0$ such that
$|\omega_o(xy) - \omega_o(x)\omega_o(y)| \leq \|\Pi_{\omega_o}(y)\|$
for all $y \in \rho(\mathfrak{A}_r)^c$

(iii)' For all $x \in \rho(\mathfrak{A})$ there exists a finite dimensional *-algebra $\mathcal{B} \subseteq \rho(\mathfrak{A})$ such that $\rho(e) \in \mathcal{B}$, and such that
$|\omega_o(xy) - \omega_o(x)\omega_o(y)| \leq \|\Pi_{\omega_o}(y)\|$
for all $y \in \mathcal{B}^c$.

Now, since $\Pi(\mathfrak{A}) = \Pi_0(\rho(\mathfrak{A}))$ we have (i)' <=> (i). Since $\rho(\mathfrak{A}_r)^c = \rho(\mathfrak{A}_r{}^c)$ by lemma 4.3, (ii)' <=> (ii). If $\mathcal{C} \subseteq \mathfrak{A}$ is finite dimensional then $\rho(\mathcal{C})$ is finite dimensional; thus (iii) => (iii)'. Furthermore, by lemma 4.3 again, we have that (ii)' => (iii). We have then established the following implications:

$$\begin{array}{ccc} (i)' & \Longleftrightarrow & (i) \\ \Updownarrow & & \\ (ii)' & \Longleftrightarrow & (ii) \\ \Updownarrow & \searrow & \\ (iii)' & \Longleftarrow & (iii) \end{array}$$

Hence (i), (ii) and (iii) are equivalent.

<u>4.5. Theorem:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and let Π_1 and Π_2 be two factor representations of $\mathfrak{A}$ such that $\ker \Pi_1 = \ker \Pi_2$. Let ω_1 and ω_2 be vector states of Π_1 and Π_2 respectively. Then the following statements are equivalent:

(i) Π_1 and Π_2 are quasi-equivalent

(ii) For all $\varepsilon > 0$ there exists an integer $r > 0$ such that
$|\omega_1(x) - \omega_2(x)| < \varepsilon \|\Pi_1(x)\|$
for all $x \in \mathfrak{A}_r{}^c$

(iii) For all $\varepsilon > 0$ there exists a finite dimentional *-algebra $\mathcal{B} \subseteq \mathfrak{A}$ containing e such that
$|\omega_1(x) - \omega_2(x)| < \varepsilon \|\Pi_1(x)\|$

for all $x \in \mathcal{B}^c$.

(iv) There exists a finite dimensional *-algebra $\mathcal{C} \subseteq \mathfrak{A}$ containing e such that

$|\omega_1(x) - \omega_2(x)| < 2 \|\Pi_1(x)\|$

for all $x \in \mathcal{C}^c$.

<u>Proof:</u> If Π_1 and Π_2 is faithful, $\|\Pi_1(x)\| = \|\Pi_2(x)\| = \|x\|$ for all $x \in \mathfrak{A}$ and the proof goes exactly as the proof of theorem 2.7 in [12].

Suppose then that $\ker \Pi_1 = \ker \Pi_2 = I$, and let $\rho: \mathfrak{A} \to \mathfrak{A}/I$ be the quotient map. Then Π_1 and Π_2 may be lifted to faithful representations of $\rho(\mathfrak{A})$: Let (i)', (ii)', (iii)' and (iv)' be the statements (i) - (iv) expressed for these lifted representations. Then, in the same way as in the proof of theorem 4.4 one may establish the following implications:

$$\begin{array}{ccc}
(i)' & \Longleftrightarrow & (i) \\
\Updownarrow & & \\
(ii)' & \Longleftrightarrow & (ii) \\
\Updownarrow & \searrow & \\
(iii)' & \Longleftrightarrow & (iii) \\
\Updownarrow & \searrow & \\
(iv)' & \Longleftarrow & (iv)
\end{array}$$

This proves the equivalence of (i) - (iv).

<u>4.6.</u> We shall now prove a result concerning algebraic equivalence of representations of AF algebras (4.12) and a result concerning the

orbits of the automorphism group of an AF algebra $\mathfrak{A}$ in the set $P(\mathfrak{A})$ of pure states of $\mathfrak{A}$. (4.15). The results are analogous to some results obtained by Powers in the UHF case in [12], section 3. In the case of AF algebras the methods of Powers have to be modified. This is due to the following facts: Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two isomorphic AF algebras on a Hilbert space κ, and let $\mathcal{B}_1$, $\mathcal{B}_2$ be two isomorphic finite dimensional *-subalgebras of $\mathfrak{A}_1$, $\mathfrak{A}_2$ resp. containing e. Suppose that $\mathfrak{A}_1'' = \mathfrak{A}_2'' = \mathfrak{M}$. Then the following two conditions hold if $\mathfrak{A}_i$ is a UHF algebra and $\mathcal{B}_i$ is a factor, but they do not hold in general:

i) there exists a unitary operator $u \in \mathfrak{M}$ such that

$$u\mathcal{B}_1 u^* = \mathcal{B}_2$$

(see [12], Lemma 3.3)

ii) $\mathcal{B}_1' \cap \mathfrak{A}_1$ and $\mathcal{B}_2' \cap \mathfrak{A}_2$ are isomorphic.

(see [12], Lemma 3.2).

These two facts play an essential role in Powers argument. Since they do not hold in general we must restrict the class of von Neumann algebras $\mathfrak{M}$ to be considered. Furthermore this class must depend on $\mathfrak{A}_1$. Roughly speaking, the simpler $\mathfrak{A}_1$ is the more complicated $\mathfrak{M}$ may be. This is reflected in the following definition.

<u>Definition:</u> Let $\mathfrak{A}$ be an AF algebra and let $\mathfrak{M}$ be a von Neumann algebra. Then $\mathfrak{A}$ is <u>permanently locally unitary equivalently embedded</u> in $\mathfrak{M}$ if there exists a faithful representation Π of $\mathfrak{A}$ such that $\Pi(\mathfrak{A})'' = \mathfrak{M}_1$ and if for any pair Π_1, Π_2 of such representations and any projection $f \in \mathfrak{A}$ we have that

$\Pi_1(f) \sim \Pi_2(f)$ (i.e. the projections $\Pi_1(f)$ and $\Pi_2(f)$ are equivalent relatively to the von Neumann algebra $\mathfrak{M}$). We then write $\mathfrak{A} \,\bar{\subseteq}\, \mathfrak{M}$.

For explanation of the term perm. loc. un. eq. em., see lemma 4.8.

4.7. Proposition: For the following pairs $\mathfrak{A}, \mathfrak{M}$, where $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathfrak{M}$ is a von Neumann algebra we have that $\mathfrak{A} \,\bar{\subseteq}\, \mathfrak{M}$ if there exists a faithful representation Π of $\mathfrak{A}$ such that $\mathfrak{M} = \Pi(\mathfrak{A})''$:

i) $\mathfrak{A}$ is a UHF algebra.
$\mathfrak{M}$ is arbitrary.

ii) $\mathfrak{A}$ is an AF algebra.
$\mathfrak{M}$ is a type I factor.

iii) $\mathfrak{A}$ is an AF algebra.
$\mathfrak{M}$ is a type III factor.

Proof: i) Let $f \in \mathfrak{A}$ be a projection in $\mathfrak{A}$. Then e and f generate a two dimensional subalgebra $\mathcal{C}$ of $\mathfrak{A}$, and by lemma 2.3 there exist a unitary $u \in \mathfrak{A}$ and an n such that $u\,\mathcal{C}\,u^* \subseteq \mathfrak{A}_n$. Define $\mathcal{B} = u^* \mathfrak{A}_n u$. Then $\mathcal{B}$ is a finite dimensional factor in $\mathfrak{A}$ containing f. Let Π_1 and Π_2 be two faithful representations of $\mathfrak{A}$ such that $\Pi_i(\mathfrak{A})'' = \mathfrak{M}$; i = 1,2. Let $\langle e_{ij} \rangle$ be a set of matrix units for $\mathcal{B}$. Then $\langle \Pi_k(e_{ij}) \rangle_{ij}$ is a set of matrix units for $\Pi_k(\mathcal{B})$; k = 1,2,---. Now, by using the technique in the proof of lemma 3.3 in [12] one shows that there exists a partial isometry $W \in \mathfrak{M}$ with initial projection $\Pi_1(e_{11})$ and final projection $\Pi_2(e_{11})$. Define $U = \sum_i \Pi_2(e_{i1}) W \Pi_1(e_{1i})$. Then U is unitary, and $\Pi_2(x) = U\Pi_1(x)U^*$ for all $x \in \mathcal{B}$. By setting $x = f$ one then see that $\Pi_1(f) \sim \Pi_2(f)$.

ii) If $\mathfrak{M}$ is a type I factor, then $\mathfrak{M}$ has the form $\mathcal{B}(\kappa) \otimes \mathbb{C}I$ where κ is some Hilbert space and I is the identity mapping on some other Hilbert space. The map:

$$\mathfrak{M} \to \mathcal{B}(\kappa): \ x \otimes I \to x$$

is then an isomorphism, so we may assume $\mathfrak{M} = \mathcal{B}(\kappa)$, since equivalence of projections is an isomorphisminvariant property. Now let Π_1 and Π_2 be two faithful representations of $\mathfrak{A}$ such that $\Pi_1(\mathfrak{A})'' = \Pi_2(\mathfrak{A})'' = \mathcal{B}(\kappa)$, i.e. Π_1 and Π_2 are irreducible. Then two cases may occur:

1) $\Pi_1(\mathfrak{A})$ contains a compact operator. Then it follows from [3], Corollaire 4.1.10 that Π_1 and Π_2 are unitary equivalent and in particular $\Pi_1(f) \sim \Pi_2(f)$ for all projections $f \in \mathfrak{A}$.

2) $\Pi_1(\mathfrak{A})$ contains no compact operator. Then, by using the same corollaire as in 1), $\Pi_2(\mathfrak{A})$ contains no compact operator. In particular, if f is an projection in $\mathfrak{A}$ and $f \neq 0$, then $\Pi_1(f)$ and $\Pi_2(f)$ are infinite. Now, $\mathfrak{A}$ is separable and Π_1 is isometric and $\Pi_1(\mathfrak{A})$ is strongly dense in $\mathcal{B}(\kappa)$, so by applying Π_1 of a countable dense subset of $\mathfrak{A}$ on a fixed non zero vector of κ one obtain a countable dense subset of κ. Hence κ is separable, so $\Pi_1(f) \sim \Pi_2(f)$.

iii) Suppose that $\mathcal{M}$ is a type III factor on a Hilbert space κ, and that there exists a faithful representation Π of $\mathfrak{A}$ such that $\Pi(\mathfrak{A})'' = \mathcal{M}$. Let ξ be a non-zero vector of κ. Let P be the projection onto $\overline{\mathcal{M}\xi}$. Since $\Pi(\mathfrak{A})$ is norm separable, $\Pi(\mathfrak{A})\xi$ is separable, and since $\Pi(\mathfrak{A})$ is strongly dense in $\mathcal{M}$, $\overline{\mathcal{M}\xi} = \overline{\Pi(\mathfrak{A})\xi}$ is separable. Now $P \in \mathcal{M}' = \Pi(\mathfrak{A})'$ so $\mathcal{M}$ is isomorphic to $\mathcal{M}_P$. $\mathcal{M}_P$ is a type III factor on a separable Hilbert space, so all non-zero projections in $\mathcal{M}_P$ are equivalent. Therefore $\mathfrak{A} \lessapprox \mathcal{M}_P$, and so $\mathfrak{A} \lessapprox \mathcal{M}$.

<u>4.8. Lemma:</u> Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two AF algebras on a Hilbert Space κ, let $\alpha: \mathfrak{A}_1 \to \mathfrak{A}_2$ be an isomorphism, and suppose that $\mathfrak{A}_1'' = \mathfrak{A}_2'' = \mathcal{M}$ and that $\mathfrak{A}_1 \lessapprox \mathcal{M}$. Let $\mathcal{C}_1 \subseteq \mathfrak{A}_1$ be a finite dimensional *-algebra containing I_κ, and let $\mathcal{C}_2 = \alpha(\mathcal{C}_1)$. Then there exists a unitary operator $U \in \mathcal{M}$ such that

$$U x U^* = \alpha(x) \quad ; \qquad \forall x \in \mathcal{C}_1 .$$

<u>Proof:</u> Let $\{e_{ij}^{(k)}\}$ be matrix units for $\mathcal{C}_1$. Since $\mathfrak{A}_1 \lessapprox \mathcal{M}$ there exist partial isometries $U_k \in \mathcal{M}$ such that $U_k U_k^* = \alpha(e_{11}^{(k)})$, and $U_k^* U_k = e_{11}^{(k)}$. Define $U = \sum_k \sum_i \alpha(e_{i1}^{(k)}) U_k e_{1i}^{(k)}$. Then $UU^* = \alpha(I) = I = U^*U$ so U is unitary, and furthermore

$$U e_{ij}^{(k)} U^* = \alpha(e_{ij}^{(k)}).$$

<u>4.9. Lemma:</u> Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, α, $\mathcal{C}_1$, κ and $\mathcal{M}$ be as in lemma 4.8. Suppose that $\mathcal{B} \subseteq \mathfrak{A}_1 \cap \mathfrak{A}_2$ is a finite dimensional *-algebra containing I_κ, that $\mathcal{B} \subseteq \mathcal{C}_1$ and that $\alpha|\mathcal{B}$ is the identity mapping. Let $\varepsilon > 0$ and let $\{\xi_1 \cdots \xi_n\}$ be a finite set of vectors in κ. Then there exists a unitary operator $U \in \mathcal{M}$ and

an isomorphism $\beta: \mathfrak{A}_1 \to \mathfrak{A}_2$ such that

(i) $U \mathcal{C}_1 U^* \subseteq \mathfrak{A}_2$

(ii) $\| Uf_i - f_i \| < \varepsilon$; $i = 1, \cdots, n$

(iii) $U x U^* = x$; $x \in \mathcal{B}$

(iv) $\beta(x) = U x U^*$; $x \in \mathcal{C}_1$

Proof: By lemma 4.8 there exists a unitary operator $V \in \mathcal{M}$ such that:

$$\alpha(x) = V x V^* \; ; \quad x \in \mathcal{C}_1$$

In particular:

$$x = V x V^* \; ; \quad x \in \mathcal{B}$$

so $V \in \mathcal{B}'$. Let $\mathcal{B}^c$ be the relative commutant of $\mathcal{B}$ in $\mathfrak{A}_2$. By lemma 4.1 we have:

$$\mathcal{B}^{c''} = \mathcal{B}' \cap \mathfrak{A}_2'' = \mathcal{B}' \cap \mathcal{M}$$

By [8], Theorem 2, the unitary operators in $\mathcal{B}^c$ lies strongly dense in the unitary operators in $\mathcal{B}^{c''}$. Since $V \in \mathcal{B}' \cap \mathcal{M}$ there exists then a unitary operator $S \in \mathcal{B}^c$ such that

$$\| (S - V^*)(V\xi_i) \| < \varepsilon \; ; \qquad i = 1, \cdots, n.$$

Define $U = SV$. Then, since $S \in \mathfrak{A}_2$

(i) $U \mathcal{C}_1 U^* = SV \mathcal{C}_1 V^* S^* = S\alpha(\mathcal{C}_1) S^* \subseteq S \mathfrak{A}_2 S^* = \mathfrak{A}_2$

Furthermore:

(ii) $\| (U-I)\xi_i \| = \| (SV-I)\xi_i \| = \| (S-V^*)V\xi_i \| < \varepsilon$.

Since $V \in \mathcal{B}'$ and $S \in \mathcal{B}^c$ we have for $x \in \mathcal{B}$

(iii) $\quad U x U^* = SV x V^* S^* = x$

Since $S \in \mathfrak{A}_2$ is unitary, $x \to S x S^*$ is an automorphism of $\mathfrak{A}_2$, so $\beta(x) = S\alpha(x)S^*$; $\quad x \in \mathfrak{A}_1$, defines an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$, such that for $x \in \mathscr{C}_1$:

(iv) $\quad \beta(x) = S\alpha(x)S^* = SV x V^* S^* = U x U^*$.

<u>4.10. Lemma:</u> Let $\mathfrak{A}_1, \mathfrak{A}_2, \kappa, \alpha, \mathfrak{M}, \mathscr{C}_1, \mathscr{C}_2$ and U be as in lemma 4.8. Let $\mathscr{B}_1 \subseteq \mathfrak{A}_1$ be a finite dimensional *-algebra such that $\mathscr{C}_1 \subseteq \mathscr{B}_1$. Let $\varepsilon > 0$, and let $\{\xi_1, \cdots, \xi_n\}$ be a finite set of vectors in κ. Then there exists a finite dimensional *-algebra $\mathscr{B}_2$ on κ , a unitary operator $U_1 \in \mathfrak{M}$ and an isomorphism β: $\mathfrak{A}_1 \to \mathfrak{A}_2$ such that

(i) $\quad \mathscr{C}_2 \subseteq \mathscr{B}_2 \subseteq \mathfrak{A}_2$

(ii) $\quad U_1 \mathscr{B}_1 U_1^* = \mathscr{B}_2$

(iii) $\quad U_1 x U_1^* = U x U^*$; $\quad x \in \mathscr{C}_1$

(iv) $\quad \|U_1 U^* \xi_i - \xi_i\| < \varepsilon$; $\quad i = 1, \cdots, n$

(v) $\quad \beta(x) = U_1 x U_1^*$; $\quad x \in \mathscr{B}_1$

<u>Proof:</u> Define $\mathfrak{A}_1^\circ = U \mathfrak{A}_1 U^*$, and $\mathscr{B}_1^\circ = U \mathscr{B}_1 U^* \supseteq U \mathscr{C}_1 U^* = \mathscr{C}_2$. Define an isomorphism δ: $\mathfrak{A}_1^\circ \to \mathfrak{A}_2$ by : $\delta(x) = \alpha(U^* x U)$; $x \in \mathfrak{A}_1^\circ$. If $x \in \mathscr{C}_2$ then $U^* x U \in \mathscr{C}_1$ thus $\delta(x) = UU^* x UU^* = x$. From lemma 4.9 it follows that there exist a unitary operator $V \in \mathfrak{M}$ and an isomorphism γ: $\mathfrak{A}_1^\circ \to \mathfrak{A}_2$ such that:

$\quad V \mathscr{B}_1^\circ V^* \equiv \mathscr{B}_2 \subseteq \mathfrak{A}_2$

(vi) $\quad \|V\xi_i - \xi_i\| < \varepsilon$; $\quad i = 1, \cdots, n$

(vii) $\quad V x V^* = x$; $\quad x \in \mathscr{C}_2$

(viii) $\quad \gamma(x) = V x V^*$; $\quad x \in \mathscr{B}_1^\circ$

Define $U_1 = VU$. Then U_1 is a unitary operator in $\mathcal{M}$. Define $\beta(x) = \gamma(UxU^*)$; $x \in \mathfrak{A}_1$. Then $\beta: \mathfrak{A}_1 \to \mathfrak{A}_1{}^o \to \mathfrak{A}_2$ is an isomorphism. We verify (i) - (v):

$$\text{(i)}: \mathcal{B}_2 = V\mathcal{B}_1{}^o V^* = VU\mathcal{B}_1 U^* V^* = U_1 \mathcal{B}_1 U_1{}^* \subseteq \mathfrak{A}_2$$

This also shows (ii). By (vii) :

$$\mathcal{C}_2 = V\mathcal{C}_2 V^* = VU\,\mathcal{C}_1 U^* V^* \subseteq V\mathcal{B}_1{}^o V^* = \mathcal{B}_2$$

(iii): If $x \in \mathcal{C}_1$ then $UxU^* \in \mathcal{C}_2$ so by (vii):

$U_1 x U_1{}^* = VUxU^*V^* = UxU^*$

(iv): Since $U_1 U^* = V$ is (iv) an immediate consequence of (vi).

(v): If $x \in \mathcal{B}_1$ then $UxU^* \in \mathcal{B}_1{}^o$, so by (viii):

$\beta(x) = \gamma(UxU^*) = VUxU^*V^* = U_1 x U_1{}^*$.

4.11. Lemma: Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two isomorphic AF-algebras on a separable Hilbert space κ, and assume that $\mathfrak{A}_1{}'' = \mathfrak{A}_2{}'' = \mathcal{M}$ and that $\mathfrak{A}_1 \subseteq \mathcal{M}$. Then there exists a unitary operator $U \in \mathcal{M}$ such that

$U\mathfrak{A}_1 U^* = \mathfrak{A}_2$

<u>Proof:</u> We construct U by using a method which is similar to that used by Powers in [12], lemma 3.6.

Let $\{a_i | i = 1,2,\cdots\}$ and $\{b_i | i = 1,2,\cdots\}$ be sequences which are dense in the unit spheres of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ resp., and let $\{\xi_i | i = 1,2,\cdots\}$ be a sequence dense in the unit sphere of κ.

By induction with respect to r we shall construct increasing sequences $\langle \mathfrak{A}_{1,r} \rangle_r$ and $\langle \mathfrak{A}_{2,r} \rangle_r$ of finite dimensional *-subalgebras of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ resp., and a sequence $\langle\alpha_r\rangle_r$ of isomorphisms from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$, and a sequence $\langle U_r\rangle_r$ of unitary operators in $\mathfrak{M}$ such that

(i) For all $r > 0$ there exists $c_i \in \mathfrak{A}_{1,r}$, $d_i \in \mathfrak{A}_{2,r}$ such that $\|c_i - a_i\| \leq 2^{-r+1}$, and $\|d_i - b_i\| \leq 2^{-r+1}$ for $= 1,\cdots,r$.

(ii) For all $r \geq 0$, $U_r\mathfrak{A}_{1,i}U_r^* = \mathfrak{A}_{2,i}$ for $i = 0,1,\cdots,r$, and if $r \geq 1$, $U_rxU_r^* = U_{r-1}xU_{r-1}^*$ for all $x \in \mathfrak{A}_{1,r-1}$

(iii) For all $r \geq 0$, $\|(U_{r+1} - U_r)\xi_i\| < 2^{-r}$ for $i = 1,\cdots,r$.

(iv) For all $r \geq 0$, $\alpha_r(x) = U_rxU_r^*$ for $x \in \mathfrak{A}_{1,r}$.

For $r = 0$, set $\mathfrak{A}_{1,0} = \mathfrak{A}_{2,0} = \mathbb{C}I$, and $U_0 = I$ and let α_0 be an isomorphism from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. The conditions (i) - (iv) are then trivially satisfied.

Suppose that $\mathfrak{A}_{1,i}$, $\mathfrak{A}_{2,i}$, U_i and α_i are constructed for $i = 1,\cdots,r$, such that (i) - (iv) is satisfied. We shall then construct $\mathfrak{A}_{1,r+1}$, $\mathfrak{A}_{2,r+1}$, U_{r+1} and α_{r+1}. From theorem 2.2 it follows that there exists a finite dimensional *-algebra $\mathfrak{B}$ such that

$\mathfrak{A}_{1,r} \subseteq \mathfrak{B} \subseteq \mathfrak{A}_1$ and elements $c_i \in \mathfrak{B}$ such that $\|a_i - c_i\| \leq 2^{-r}$ for $i = 1,\cdots,r+1$. By lemma 4.10 there exist a finite dimensional *-algebra $\mathfrak{C}$ and an isomorphism $\alpha: \mathfrak{A}_1 \to \mathfrak{A}_2$ and a unitary operator $V \in \mathfrak{M}$ such that

(v) $\mathfrak{A}_{2,r} \subseteq \mathfrak{C} \subseteq \mathfrak{A}_2$

(vi) $V\mathfrak{B}V^* = \mathfrak{C}$

(vii) $VxV^* = U_rxU_r^*$ for $x \in \mathfrak{A}_{1,r}$

(viii) $\|(V U_r^* - I)(U_r \xi_i)\| = \|(V-U_r)\xi_i\| < 2^{-r-1}$ for $i = 1,\cdots,r+1.$

(ix) $\alpha(x) = VxV^*$ for $x \in \mathcal{B}$

By theorem 2.2 there exists a finite dimensional *-algebra $\mathfrak{A}_{2,r+1}$ such that $\mathcal{C} \subseteq \mathfrak{A}_{2,r+1} \subseteq \mathfrak{A}_2$, and such that there exist elements $d_i \in \mathfrak{A}_{2,r+1}$ such that $\|b_i - d_i\| < 2^{-r}$ for $i = 1,\cdots,r+1.$

By a new application of lemma 4.10 there exists a finite dimensional *-algebra $\mathfrak{A}_{1,r+1}$ and an isomorphism $\alpha_{r+1}^{-1}: \mathfrak{A}_2 \to \mathfrak{A}_1$ and a unitary operator $U_{r+1}^* \in \mathfrak{M}$ such that:

(x) $\mathcal{B} \subseteq \mathfrak{A}_{1,r+1} \subseteq \mathfrak{A}_1$

(xi) $U_{r+1}^* \mathfrak{A}_{2,r+1} U_{r+1} = \mathfrak{A}_{1,r+1}$

(xii) $U_{r+1}^* x U_{r+1} = V^* x V$ for $x \in \mathcal{C}$

(xiii) $\|U_{r+1}^* V\xi_i - \xi_i\| < 2^{-r-1}$ for $i = 1,\cdots,r+1$

(xiv) $\alpha_{r+1}^{-1}(x) = U_{r+1}^* x U_{r+1}$ for $x \in \mathfrak{A}_{2,r+1}.$

We now show that (i) - (iv) hold for $r+1$.

(i) holds by the construction of $\mathfrak{A}_{1,r}$ and $\mathfrak{A}_{2,r}$.

(ii): By (xi), $U_{r+1}\mathfrak{A}_{1,r+1} U_{r+1}^* = \mathfrak{A}_{2,r+1}$. If $x \in \mathfrak{A}_{1,r}$ we have:

$$\begin{aligned} U_{r+1} x U_{r+1}^* &= U_{r+1} V^*V x V^*VU_{r+1} \\ &= V x V^* \quad \text{by (vi) and (xii)} \\ &= U_r x U_r^* \quad \text{by (vii).} \end{aligned}$$

By using this and induction hypothesis, (ii) holds.

(iii): $\|(U_{r+1} - U_r)\xi_i\| \leq \|(U_{r+1}-V)\xi_i\| + \|(V-U_r)\xi_i\|$

$= \|U_{r+1}^*(U_{r+1}-V)\xi_i\| + \|(V-U_r)\xi_i\| < 2^{-r-1} + 2^{-r-1} = 2^r,$

by (viii) and (xiii).

(iv): If $x \in \mathfrak{A}_{1,r+1}$, $\alpha_{r+1}(x) = U_{r+1} x U_{r+1}^*$ by (xiv) and (xi).

This ends the induction.

Now, by using (i) - (iii) one may by the method used by Powers to prove lemma 3.6 in [12] show that $\langle U_r \rangle_r$ converges strongly towards a unitary operator $U \in \mathfrak{M}$ which has the property which is required in the lemma. The details of that proof are omitted.

<u>4.12. Theorem:</u> Let $\mathfrak{A}$ be an AF algebra and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be von Neumann algebras such that $\mathfrak{A} \subseteq \mathfrak{M}_i$ for $i = 1,2$. Let Π_1 and Π_2 be faithful representations of $\mathfrak{A}$ such that $\Pi_i(\mathfrak{A})'' = \mathfrak{M}_i$ for $i = 1,2$. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic if and only if there exists an automorphism α of $\mathfrak{A}$ such that Π_1 and $\Pi_2 \circ \alpha$ are quasi-equivalent. This is proved from lemma 4.11 in the same way as Powers proves theorem 3.7 from lemma 3.6 in [12].

<u>4.13. Corollary:</u> Let $\mathfrak{A}$ be an AF algebra and assume that Π_1 and Π_2 are two faithful type III factor representations of $\mathfrak{A}$. Then $\Pi_1(\mathfrak{A})''$ and $\Pi_2(\mathfrak{A})''$ are isomorphic if and only if there exists an automorphism α of $\mathfrak{A}$ such that Π_1 and $\Pi_2 \circ \alpha$ are quasi-equivalent.

<u>Proof:</u> Follows from theorem 4.12 and proposition 4.7 . (iii).

<u>4.14. Corollary:</u> Let $\mathfrak{A}$ be an AF algebra and suppose that Π_1 and Π_2 are two faithful irreducible representations of $\mathfrak{A}$. Then there exists an automorphism α of $\mathfrak{A}$ such that Π_1 and $\Pi_2 \circ \alpha$ are unitary equivalent.

<u>Proof:</u> Since $\mathfrak{A}$ is separable and Π_i are irreducible the representation spaces of Π_i must be separable for $i = 1,2$. If there exists an integer n such that the representation space of Π_1 is isomorphic to $\mathbb{C}^n$, then $\Pi_1(\mathfrak{A}) \simeq M_n$, thus $\mathfrak{A} \simeq M_n$ and so the representation space of Π_2 must also be isomorphic to $\mathbb{C}^n$. If $\mathfrak{A}$ is not finite dimensional the representation spaces of Π_1 and Π_2 must be infinite dimensional, and so isomorphic to $l^2(\mathbb{Z})$. In all cases, $\Pi_1(\mathfrak{A})'' \simeq \Pi_2(\mathfrak{A})''$ (= all bounded operators on the representation space). Hence, by theorem 4.12 and proposition 4.7, ii), there exists an automorphism α of $\mathfrak{A}$ such that Π_1 and $\Pi_2 \circ \alpha$ are quasi-equivalent. Since $\Pi_2(\alpha(\mathfrak{A})) = \Pi_2(\mathfrak{A})$, $\Pi_2 \circ \alpha$ is irreducible, and then by [3], prop. 5.3.3. Π_1 and $\Pi_2 \circ \alpha$ are unitary equivalent.

<u>4.15. Corollary:</u> Let $\mathfrak{A}$ be an AF algebra and let ω_1 and ω_2 be pure states of $\mathfrak{A}$ such that the associated representations Π_1 and Π_2 are faithful. Then there exists an automorphism α of $\mathfrak{A}$ such that

$$\omega_1 = \omega_2 \circ \alpha$$

<u>Proof:</u> By corollary 4.14 there exists an automorphism β of $\mathfrak{A}$ such that Π_1 and $\Pi_2 \circ \beta$ are unitary equivalent. Therefore ω_1 and $\omega_2 \circ \beta$ are vector states of the same irreducible representation, and so by [8], Corollary 8, there exists a unitary operator

$v \in \mathfrak{A}$ such that $\omega_1(x) = \omega_2(v\beta(x)v^*)$ for all $x \in \mathfrak{A}$. Then $\alpha(x) = v\beta(x)v^*$ is the desired automorphism.

<u>4.16. Corollary:</u> Let $\mathfrak{A}$ be an AF algebra and let ω be a state of $\mathfrak{A}$ such that the associated representation Π_ω is faithful. Then ω is pure if and only if there exists an increasing sequence $\langle \mathfrak{A}_n \rangle_n$ of finite dimensional *-subalgebras of $\mathfrak{A}$, all containing e, such that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\omega | \mathfrak{A}_n$ is pure for all n.

<u>Proof:</u> Suppose first that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and that $\omega|\mathfrak{A}_n$ is pure for all n. We show that ω is pure. Suppose $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ where ω_1 and ω_2 are states of $\mathfrak{A}$ and $0 \leq \lambda \leq 1$. Then $\omega|\mathfrak{A}_n = \lambda\omega_1|\mathfrak{A}_n + (1-\lambda)\omega_2|\mathfrak{A}_n$, thus $\omega|\mathfrak{A}_n = \omega_1|\mathfrak{A}_n = \omega_2|\mathfrak{A}_n$, thus $\omega|\bigcup_n \mathfrak{A}_n = \omega_1|\bigcup_n \mathfrak{A}_n = \omega_2|\bigcup_n \mathfrak{A}_n$, and so, by the norm continuity of ω, ω_1 and ω_2: $\omega = \omega_1 = \omega_2$, i.e. ω is pure.

Conversely, suppose that ω is pure and that Π_ω is faithful. Then Π_ω is irreducible, and so $\mathfrak{A}$ is primitive. Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{B}_n}$ where $e \in \mathfrak{B}_n \subseteq \mathfrak{A}$, and $\langle \mathfrak{B}_n \rangle_n$ is an increasing sequence of finite dimensional *-subalgebras. Let $\{e_{ij}^{(nk)}\}_{ijk}$ be matrix units for $\mathfrak{B}_n$. We shall construct a pure state ρ of $\mathfrak{A}$, by defining inductively $\rho|\mathfrak{B}_{n_i}$, $i = 1,2,\cdots$ where $\langle n_i \rangle_i$ is a strictly increasing sequence of integers which are chosen in the course of the induction.

Let $n_1 = 1$ and define

$$\rho(e_{ij}^{(1q)}) = \begin{cases} 1 & \text{if } q = i = j = 1 \\ 0 & \text{in other cases.} \end{cases}$$

Then $\rho|\mathfrak{B}_1$ is pure.

Now, suppose that the matrix units for $\mathcal{B}_{n_k}$ has been chosen in such a way that:

$$\rho(e_{ij}^{(n_k q)}) = \begin{cases} 1 & \text{if } q = i = j = 1 \\ 0 & \text{in other cases.} \end{cases}$$

Since $\mathfrak{A}$ is primitive it follows by repeated application of corollary 3.9 that there exists an $n_{k+1} > n_k$ and a factor $M_{(n_{k+1},p)}$ in the central decomposition of $\mathcal{B}_{n_{k+1}}$ such that all factors in the central decomposition of $\mathcal{B}_{n_k}$ are partially embedded in $M_{(n_{k+1},p)}$. By a suitable choice of indices one may assume $p = 1$, and by a suitable choice of matrix units in $\mathcal{B}_{n_{k+1}}$ one obtains

(1) $$e_{11}^{(n_k 1)} e_{11}^{(n_{k+1} 1)} = e_{11}^{(n_{k+1} 1)}$$

Now, define $\rho | \mathcal{B}_{n_{k+1}}$ by :

$$\rho(e_{ij}^{(n_{k+1} q)}) = \begin{cases} 1 & \text{if } i = j = q \\ 0 & \text{in other cases.} \end{cases}$$

Then $\rho | \mathcal{B}_{n_{k+1}}$ is a pure state, and for $x \in \mathcal{B}_{n_{k+1}}$ we have: $\rho(x) = \rho(e_{11}^{(n_{k+1} 1)} x \, e_{11}^{(n_{k+1} 1)})$. By combining this with (1) we see that $\rho | \mathcal{B}_{n_{k+1}}$ is really an extention of $\rho | \mathcal{B}_{n_k}$. For simplicity we now write $\mathcal{B}_k$ instead of $\mathcal{B}_{n_k}$. Then $\mathfrak{A} = \overline{\bigcup_k \mathcal{B}_k}$. Since $|\rho(x)| \leq \|x\|$ for $x \in \bigcup_k \mathcal{B}_k$ ρ may be extended by continuity to a state of $\mathfrak{A}$. Since $\rho | \mathcal{B}_k$ is pure for all k, ρ is pure by first part of the proof.

We now show that Π_ρ is faithful. By lemma 3.1 it is enough to show that $\ker \Pi_\rho \cap \mathcal{B}_n = \{0\}$, $n = 1,2,\cdots$. We show this by showing that for each minimal projection $e^{(nk)}$ in the center of $\mathcal{B}_n$ there exists an $x \in \mathcal{B}_{n+1}$ such that $\rho(x\, e^{(nk)}\, x^*) \neq 0$. Then, by definition of Gelfand-Segal representation, $e^{(nk)} \notin \ker \Pi_\rho$ and the result is obtained.

Now, by construction of $\mathcal{B}_{n+1}$ we have that $e^{(nk)}\, e^{(n+1,1)}$ is a non-zero projection. This is included in the factor $M_{(n+1,1)}$, so there exists a partial isometry $x \in M_{(n+1,1)}$ such that $xx^* = e_{11}^{(n+1,1)}$, and $x^*x \leq e^{(nk)}\, e^{(n+1,1)}$. Thus :

$\rho(x\, e^{(nk)}\, x^*) = \rho(x\, e^{(n+1,1)}\, e^{(nk)}\, e^{(n+1,1)}\, x^*) = \rho(xx^*) =$

$\rho(e_{11}^{(n+1,1)}) = 1 \neq 0$; thus Π_ρ is faithful.

By corollary 4.15 there exists an automorphism α of $\mathfrak{A}$ such that

$$\rho = \omega \circ \alpha .$$

Let $\mathfrak{A}_n = \alpha(\mathcal{B}_n)$. Since α is an isometry, $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$.
$f_{ij}^{(nk)} = \alpha(e_{ij}^{(nk)})$ are matrix units for $\mathfrak{A}_n$ and

$$\omega(f_{ij}^{(nk)}) = \rho(e_{ij}^{(nk)}) = \begin{cases} 1 & \text{if } i = j = k = 1 \\ 0 & \text{in other cases} \end{cases}$$

thus $\omega|\mathfrak{A}_n$ is pure.

<u>4.17.</u> In the course of the proof of 4.16 we gave in fact a proof for the implication ii) => i) in corollary 3.9 which is independent of Dixmiers proof in [1], i.e. we proved that if the intersection of any two non-zero ideals in an AF Algebra $\mathfrak{A}$ is non-zero, then $\mathfrak{A}$ is primitive. This is because the equivalence

ii) <=> iii) in 3.9 was established independently of Dixmiers result, and the only property of $\mathfrak{A}$ which was used in the construction of the pure state ρ in the proof of 4.16 was iii). Since Π_ρ is faithful, iii) implies that $\mathfrak{A}$ is primitive.

4.18. By using techniques closely related to those in 4.16 one may give a direct proof for the fact that if $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is a simple, infinite dimensional AF algebra then the closure of the set of pure states of $\mathfrak{A}$ in the w^*-topology is the set of all states of $\mathfrak{A}$. This is proved in general for simple, antiliminal C^*-algebras by Glimm in [7], se also [3] Lemme 11.2.4. The argument is roughly as follows: Since $\mathfrak{A}$ is simple, $\mathfrak{A}$ is primitive. By using the characterization of these two concepts given in 3.5 and 3.9 resp., and an induction argument one may prove that for any n there exists a $m > n$ such that all the factors in the central decomposition of $\mathfrak{A}_n$ are partially embedded in one single factor $M_{(mk)}$ in the central decomposition on $\mathfrak{A}_m$ in such a way that $M_{(nq)}$ is embedded in $M_{(mk)}$ with partial multiplicity $\geq \begin{bmatrix} n \\ q \end{bmatrix}$. Then it is not difficult to show by methods similar to those in [6] Theorem 2.8, that if ω is a state of $\mathfrak{A}_n$ there exists a pure state ρ of $M_{(mk)}$ such that $\omega(x) = \rho(e^{(mk)}x)$ for $x \in \mathfrak{A}_n$. ρ may be extended to a pure state of $\mathfrak{A}_m$ by $\rho(x) = \rho(e^{(mk)}x)$ for $x \in \mathfrak{A}_m$, and still : $\rho|\mathfrak{A}_n = \omega$. Then ρ may be extended to a pure state of $\mathfrak{A}$ by [3], Lemme 2.10.1. In short, each state of $\mathfrak{A}_n$ has a pure extention to $\mathfrak{A}$. Since $\bigcup_n \mathfrak{A}_n$ is dense in $\mathfrak{A}$ it then follows that the set of pure states of $\mathfrak{A}$ is w^*-dense in the set of states of $\mathfrak{A}$.

5. An example. The current algebra.

5.1. In this section we shall apply the machinery developed in sections 1 to 4 to one specific AF algebra. This will be the algebra of all gauge invariant elements of the algebra of the canonical anticommutation relations. This algebra is named the fermion current algebra in [9].

We recall some basic facts from [11]. Let $\mathcal{K}$ be a separable infinite dimensional complex Hilbert Space. Then $\mathfrak{A}(\mathcal{K})$, the CAR algebra of $\mathcal{K}$, is the C^*-algebra generated by elements a(f), where $f \to a(f)$ is a linear map of $\mathcal{K}$ into $\mathfrak{A}(\mathcal{K})$ satisfying the canonical anticommutation relations

$$a(f)a(g) + a(g)a(f) = 0$$

$$a(f)^*a(g) + a(g)a(f)^* = (g,f)I$$

(We adapt the convention that the inner product on $\mathcal{K}$ is linear in the first factor). If U is a unitary operator on $\mathcal{K}$, then by [11] there exists a unique automorphism ϕ of $\mathfrak{A}(\mathcal{K})$ such that $\phi(a(f)) = a(Uf)$, and this defines a homomorphism from the unitary group on $\mathcal{K}$ into the automorphism group of $\mathfrak{A}(\mathcal{K})$. The unitary group on $\mathcal{K}$ has a subgroup isomorphic to the circle group, namely the unitaries of the form $f \to e^{i\theta}f$, $0 \leq \theta < 2\Pi$. The corresponding automorphisms of $\mathfrak{A}(\mathcal{K})$, which we shall denote by χ_θ, are called the gauge group of automorphisms. The elements $x \in \mathfrak{A}(\mathcal{K})$ such that $\chi_\theta(x) = x$ for all $\theta \in [0,2\Pi\rangle$ form a C^*-algebra which we shall denote by $\mathfrak{A}^\circ(\mathcal{K})$, and call the current algebra.

If $x = a(f_1)^* \cdots a(f_n)^* a(g_1) \cdots a(g_m)$ one has that $\chi_\theta(x) = e^{i\theta(m-n)}x$, so $x \in \mathfrak{A}^\circ(\mathcal{K})$ if and only if $m = n$. We shall see in 5.4 that the linear span of the x's of this form with $m = n$ lies dense in $\mathfrak{A}^{o}(\mathcal{K})$.

We shall now use the fact that $\mathfrak{A}(\mathcal{K})$ is a UHF algebra to deduce that $\mathfrak{A}^{o}(\mathcal{K})$ is an AF algebra. We use the description of $\mathfrak{A}(\mathcal{K})$ given in [11]. Let $\{f_n\}_{n=1,2}$ be an orthonormal basis in $\mathcal{K}$. Define:

$$V_o = I$$

$$V_n = \prod_{i=1}^{n-1} (1-2a(f_i)^* a(f_i)) \; ; \qquad n \geq 1$$

$$e_{11}^{(n)} = a(f_n)a(f_n)^* \qquad e_{12}^{(n)} = a(f_n)V_n$$

$$e_{21}^{(n)} = a(f_n)^* V_n \qquad e_{22}^{(n)} = a(f_n)^* a(f_n)$$

Then it follows from the anticommutation relations that the $\{e_{ij}^{(n)}\}$ form a set of 2×2 matrix units, which commute for different n's. The set of all $e_{i_1 j_1}^{(1)} \; e_{i_2 j_2}^{(2)} \; --- \; e_{i_n j_n}^{(n)}$, where $(i_1, j_1, i_2, \cdots, j_n)$ runs through all 2n-tuples consisting of the elements 1 and 2, therefore constitutes, by suitable indexing, a set of $2^n \times 2^n$ matrix units. These matrix units generate the algebra of all polynomials in the field operators $a(f)$ and $a(f)^*$, where f runs through the linear span of $f_1, \cdots f_n$. We denote this algebra by $\mathfrak{A}_n$. Then $\mathfrak{A}(\mathcal{K}) = \overline{\bigcup_n \mathfrak{A}_n}$, so $\mathfrak{A}(\mathcal{K})$ is a UHF algebra. Let $\mathfrak{A}_n^{o}$ be the gauge invariant elements in $\mathfrak{A}_n$. In the next lemmas we shall study the structure of $\mathfrak{A}_n^{o}$, and the embedding of $\mathfrak{A}_n^{o}$ into $\mathfrak{A}_{n+1}^{o}$. Then we shall show that

$\mathfrak{A}^{\circ}(\mathcal{K}) = \overline{\bigcup_n \mathfrak{A}_n^{\circ}}$, and thus establish that $\mathfrak{A}^{\circ}(\mathcal{K})$ is AF.

<u>5.2. Lemma:</u> $\mathfrak{A}_n^{\circ}$ has the central decomposition :

$$\mathfrak{A}_n^{\circ} = \bigoplus_{k=0}^{n} M_{(nk)}$$

where $M_{(nk)}$ are factors of type $I_{\binom{n}{k}}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. $M_{(nk)}$ is partially embedded in $M_{(n+1\ q)}$ with partial multiplicity $\begin{cases} 1 & \text{if } q = k \text{ or } q = k+1 \\ 0 & \text{in other cases.} \end{cases}$

<u>Proof:</u> It is easily verified that the matrix units $e_{kj}^{(n)}$ mentioned in 5.1 transform under the gauge-group by the formula $\chi_\theta(e_{kj}^{(n)}) = e^{i(j-k)\theta}\, e_{kj}^{(n)}$. Hence

$$\chi_\theta(e_{i_1j_1}^{(1)} \cdots e_{i_nj_n}^{(n)}) = e^{i(\sum_{k=1}^{n} j_k - \sum_{k=1}^{n} i_k)\theta}\, e_{i_1j_1}^{(1)} \cdots e_{i_nj_n}^{(n)}$$

Since the elements $\left\{e_{i_1j_1}^{(1)} \cdots e_{i_nj_n}^{(n)}\right\}_{i_k,j_k = 1,2}$ form a basis for the vector space $\mathfrak{A}_n$. It follows that $\mathfrak{A}_n^{\circ}$ is the algebra spanned by those elements $e_{i_1j_1}^{(1)}\, e_{i_2j_2}^{(2)} \cdots e_{i_nj_n}^{(3)}$ for which $\sum_{k=1}^{n} i_k = \sum_{k=1}^{n} j_k$.

Now, define $\Lambda_{n,k}$ as the set of functions ϕ from $\{1,2,\cdots,n\}$ into $\{1,2\}$ such that ϕ assumes the value 1 exactly k times; $k = 0,1,\cdots,n$. If $\phi,\psi \in \Lambda_{nk}$, define $f_{\phi\psi}^{(nk)} = e_{\phi(1)\psi(1)}^{(1)}\, e_{\phi(2)\psi(2)}^{(2)}$ ----- $e_{\phi(n)\psi(n)}^{(n)}$. Then $f_{\phi\psi}^{(nk)} \in \mathfrak{A}_n^{\circ}$.

Define $M_{(nk)}$ as the linear span of $\{f_{\phi\psi}^{(nk)} : \phi,\psi \in \Lambda_{n,k}\}$. It is then clear that $\mathfrak{A}_n^o = \bigoplus_{k=0}^{n} M_{(nk)}$ as a direct sum of vector spaces. If ϕ,ψ are functions from $\{1,\cdots,n\}$ into $\{1,2\}$, we define

$$\delta_{\phi\psi} = \begin{cases} 1 & \text{if } \phi = \psi \\ 0 & \text{if } \phi \neq \psi . \end{cases}$$

By straight forward computations one verifies $f_{\phi\psi}^{(nk)} f_{\chi\omega}^{(nq)} = \delta_{kq}\, \delta_{\psi\chi}\, f_{\phi\omega}^{(nk)}$, $f_{\phi\psi}^{(nk)*} = f_{\psi\phi}^{(nk)}$, and $\sum_{k=0}^{n} \sum_{\phi \in \Lambda_{n,k}} f_{\phi\phi}^{(nk)} = e$. Thus the $f_{\phi\psi}^{(nk)}$'s form a set of matrix units for $\mathfrak{A}_n^o$. The $M_{(nk)}$'s are factors, and the square root of their dimensions are equal to the number of elements in $\Lambda_{n,k}$, which is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$. Thus the first part of the lemma is established. To prove the second part, assume $\phi,\psi \in \Lambda_{n,k}$. Since $e_{11}^{(n+1)} + e_{22}^{(n+1)} = e$ we have

$$f_{\phi\psi}^{(nk)} = f_{\phi\psi}^{(nk)} e_{11}^{(n+1)} + f_{\phi\psi}^{(nk)} e_{22}^{(n+1)} = f_{\phi_1\psi_1}^{(n+1,k+1)} + f_{\phi_2\psi_2}^{(n+1,k)} ,$$

where $\phi_r,\ \psi_r \in \Lambda_{n+1,k+1}$ are defined by:

$$\phi_r(q) = \begin{cases} \phi(q) & \text{for } q = 1,\cdots,n \\ r & \text{for } q = n+1 \end{cases}$$

$$\psi_r(q) = \begin{cases} \phi(q) & \text{for } q = 1,\cdots,n \\ r & \text{for } q = n+1 \end{cases}$$

where $r = 1,2$.

Thus a matrix unit in $M_{(nk)}$ is a sum of one matrix unit in $M_{(n+1,k+1)}$ and one in $M_{(n+1,k)}$, hence the lemma follows.

<u>5.3. Lemma:</u> Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ be an AF algebra, G compact group, α a strongly continuous representation of G as a group of automorphisms of $\mathfrak{A}$. Suppose that $\alpha_g(\mathfrak{A}_n) \subseteq \mathfrak{A}_n$ for all $g \in G$ and $n \geq 1$. Let $\mathfrak{A}_n^o$ be the G-invariant elements in $\mathfrak{A}_n$ and $\mathfrak{A}^o$ the G-invariant elements in $\mathfrak{A}$. Then

$$\mathfrak{A}^o = \overline{\bigcup_n \mathfrak{A}_n^o}$$

<u>Proof:</u> Since $g \to \alpha_g(x)$ is continuous, the Bochner integral $P(x) = \int_G \alpha_g(x)dg$ exist for all $x \in \mathfrak{A}$, where dg is normalized Haar measure on G. (See [15], V, 5). Since α_g is isometric for all g $\|P(x)\| \leq \int_G \|\alpha_g(x)\|dg = \|x\|$, so

(1) $$\|P\| \leq 1$$

Furthermore, if $g \in G$, $x \in \mathfrak{A}$ then $\alpha_g P(x) = \alpha_g \int_G \alpha_k(x)dk =$ $= \int_G \alpha_{gk}(x)dk = \int_G \alpha_k(x)dk = P(x)$.

so

(2) $$P(\mathfrak{A}) \subseteq \mathfrak{A}^o$$

If $x \in \mathfrak{A}^o$, then:

(3) $$P(x) = \int_G x\,dk = x .$$

If $x \in \mathfrak{A}_n$, then $\alpha_g(x) \in \mathfrak{A}_n$ for all $g \in G$, and so by combining with (2) and (3) :

(4) $$P(\mathfrak{A}_n) = \mathfrak{A}_n^o .$$

Now, let $x \in \mathfrak{A}^o$. Then there exists a sequence $\langle x_n \rangle_n$, with $x_n \in \mathfrak{A}_n$, such that $x = \lim_{n\to\infty} x_n$. Then, by (3) and (1), $x = P(x) = \lim_{n\to\infty} P(x_n)$. Then, by (4): $x \in \overline{\bigcup_n \mathfrak{A}_n^o}$ and so: $\mathfrak{A}^o \subseteq \overline{\bigcup_n \mathfrak{A}_n^o}$. Since trivially $\bigcup_n \mathfrak{A}_n^o \subseteq \mathfrak{A}^o$ and $\mathfrak{A}^o$ is a C^*-algebra, the lemma is obtained.

<u>5.4. Corollary:</u> $\mathfrak{A}^o(\mathcal{K}) = \overline{\bigcup_n \mathfrak{A}_n^o}$.

<u>Proof:</u> The circle group is compact and in the proof of lemma 5.2 we verified that $\chi_\theta(\mathfrak{A}_n) \subseteq \mathfrak{A}_n$ for all n, so by lemma 5.3 we have only to prove that if $x \in \mathfrak{A}(\mathcal{K})$ then $\theta \to \chi_\theta(x)$ is continuous. In the proof of lemma 5.2 we saw that

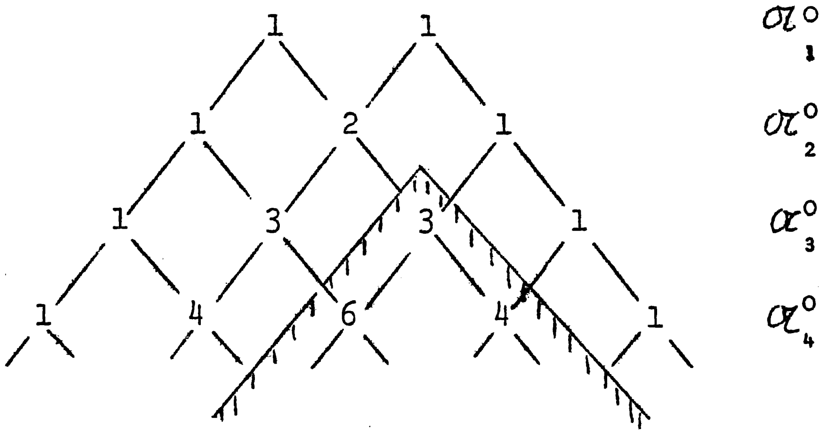
$$\chi_\theta(e^{(1)}_{i_1 j_1} \cdots e^{(n)}_{i_n j_n}) = e^{i(\sum_{k=1}^{n} j_k - \sum_{k=1}^{n} i_k)\theta} e^{(1)}_{i_1 j_1} \cdots e^{(n)}_{i_n j_n})$$

and since each element $y \in \mathfrak{A}_n$ is a finite linear combination of such matrix elements, $\theta \to \chi_\theta(y)$ is in fact uniformly continuous for $y \in \mathfrak{A}_n$. Let $x \in \mathfrak{A}(\mathcal{K})$ and let $\varepsilon > 0$. Then there exists an $\mathfrak{A}_n$ and a $y \in \mathfrak{A}_n$ such that $\|x-y\| < \frac{\varepsilon}{3}$ and then there exists a $\delta > 0$ such that $|\theta_1 - \theta_2| < \delta$ implies $\|\chi_{\theta_1}(y) - \chi_{\theta_2}(y)\| < \frac{\varepsilon}{3}$. Then, since all χ_θ are isometries, $|\theta_1 - \theta_2| < \delta$ implies:

$$\|\chi_{\theta_1}(x)-\chi_{\theta_2}(x)| \leq \|\chi_{\theta_1}(x-y)\| + \|\chi_{\theta_1}(y)-\chi_{\theta_2}(y)\| + \|\chi_{\theta_2}(x-y)\| < 2\|x-y\| + \frac{\varepsilon}{3} < \varepsilon.$$

thus $\theta \to \chi_\theta(x)$ is continuous, i.e. $\theta \to \chi_\theta$ is strongly continuous.

<u>5.5.</u> From 5.2 and 5.4 it follows that the current algebra is and AF algebra with diagram looking like Pascals triangle:



This may now be used to reveal the algebraic structure of $\mathfrak{A}^o(\mathcal{K})$. Theorem 3.3 implies that the ideals of $\mathfrak{A}^o(\mathcal{K})$ except $\{0\}$ are represented by "pyramids" on the diagram, starting from one point in $\mathcal{D}(\mathfrak{A}^o(\mathcal{K}))$. I.e. the most general ideal in $\mathfrak{A}^o(\mathcal{K})$ except $\{0\}$ is:

$$ {}_nI_m = \overline{\bigcup_{k=m+n}^{\infty} \bigoplus_{j=n}^{k-m} M_{(kj)}} \quad ; \quad n,m = 0,1,2, \cdots . $$

These ideals are all distinct. On the figure we have indicated the ideal ${}_2I_1$. The ideal ${}_nI_m$ may be characterized in a couple of other ways.

i) ${}_nI_m$ is the ideal in $\mathfrak{A}^o(\mathcal{K})$ generated by $M_{(n+m,n)}$. This is immediate from the definition.

ii) We may also describe the ideal ${}_nI_m$ directly in terms of the annihilators $a(f)$ and the creators $a(g)^*$ of the field algebra $\mathfrak{A}(\mathcal{K})$. Let p be a polynomial in the field operators such that each addend in p contains equally many creators and annihilators. Then p is gauge invariant. Using the anticommutation relations one may order each addend in p such that all creators are standing to the left of all annihilators. We then say that p is in normal form. If the creators and annihilators are in reverse order in each addend we say that p is in anormal form. Now, if p is a gauge invariant polynomial in the field operators, we may by integrating over the gauge group as in the proof of lemma 5.3 assume that each addend of p contains equally many creators and annihilators. Consider the set of gauge invariant polynomials p such that each addend of p in the normal form contains at least m creators, and each addend of p in the anormal form contains at least n creators. From the anticommutation relations it follows that this set is an ideal in the algebra of gauge invariant polynomials. The matrix units $f_{\phi\psi}^{(qr)}$ for $M_{(qr)}$ constructed in 5.2 are polynomials in the field operators and it is not difficult to verify that $f_{\phi\psi}^{(qr)}$ in normal form has an addend of minimal "degree" in the field operators which contains q-r creators, while $f_{\phi\psi}^{(qr)}$ in anormal form has a term of minimal "degree" r in the creators. It follows that the closure of the ideal of the algebra of field operators described above is ${}_nI_m$.

<u>5.6. Proposition:</u> The primitive ideals of $\mathfrak{A}^o(\mathcal{K})$ are the following

(i) ${}_nI_o$; $n = 1,2,\ldots$

(ii) ${}_oI_n$; $n = 1,2,\ldots$

(iii) $\{0\}$

One has that

$$\mathfrak{A}^{\circ}(\mathcal{K})/{}_1I_o \cong \mathfrak{A}^{\circ}(\mathcal{K})/{}_oI_1 \cong \mathbf{C} ,$$

and

$${}_nI_o \Big/ {}_{n+1}I_o \cong {}_oI_n \Big/ {}_oI_{n+1} \cong \mathcal{L}\mathcal{C}(\mathcal{K}),$$

for $n = 1,2,\cdots.$

Within unitary equivalence there exists for each n only one irreducible representation ${}_o\Pi_n$ with kernel ${}_oI_n$ and only one irreducible representation ${}_n\Pi_o$ with kernel ${}_nI_o$.

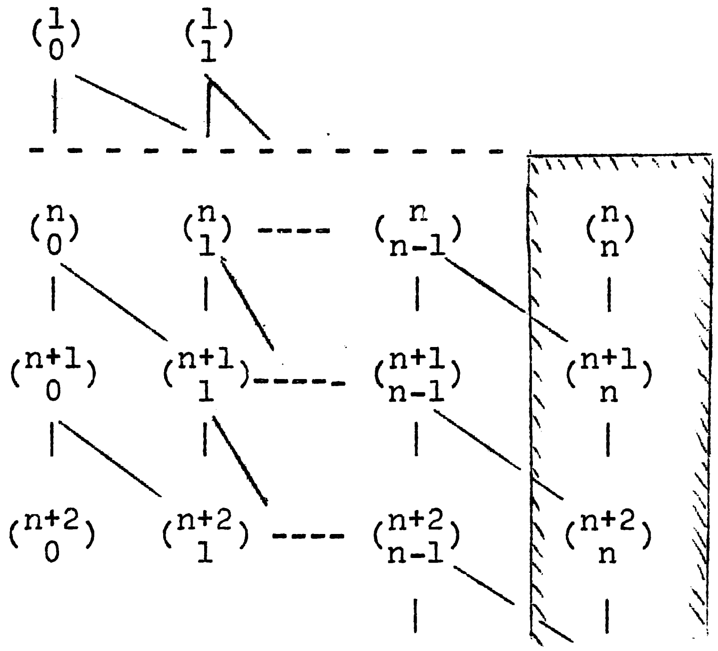
<u>Proof:</u> From the figure in 5.5 and theorem 3.8 it follows that the list (i), (ii), (iii) exhausts the set of primitive ideals in $\mathfrak{A}^{\circ}(\mathcal{K})$.

Using proposition 3.7 we see that both $\mathfrak{A}^{\circ}(\mathcal{K})/{}_1I_o$ and $\mathfrak{A}^{\circ}(\mathcal{K})/{}_oI_1$ have the diagram

$$\begin{array}{c} 1 \\ | \\ 1 \\ | \\ 1 \\ | \end{array}$$

so they are both isomorphic to $M_1 = \mathbb{C}$.

By proposition 3.7 again $\mathfrak{A}^{\circ}(\mathcal{K})/{}_{n+1}I_o$ has the diagram



The ideal ${}_{n}I_{o}/{}_{n+1}I_{o}$ of $\mathfrak{A}^{o}(\mathcal{K})/{}_{n+1}I_{o}$ is then represented by that part of the diagram which is lying inside the shaded boundary. By using exactly the same kind of argument as in 1.10 example (iii), we then show that ${}_{n}I_{o}/{}_{n+1}I_{o} \cong \mathcal{L}\mathcal{C}(\mathcal{K})$. By a similar argument, ${}_{o}I_{n}/{}_{o}I_{n+1} \cong \mathcal{L}\mathcal{C}(\mathcal{K})$.

Now, if Π is an irreducible representation of $\mathfrak{A}^{o}(\mathcal{K})$ with kernel ${}_{n}I_{o}$, then Π may be lifted to a faithful irreducible representation of $\mathfrak{A}^{o}(\mathcal{K})/{}_{n}I_{o} = \mathcal{B}$. As shown above, $\mathcal{B}$ contains an ideal isomorphic to the compact operators on some Hilbert space (which is $\mathbb{C}$ if $n = 1$ and $\mathcal{K}$ if $n > 1$). Since the only irreducible representation of the compact operators is the identity representation (except for unitary equivalence) (see [3], Corollaire 4.1.5), and there is a one-one correspondence between faithful irreducible representations of $\mathcal{B}$ and faithful irreducible representations of the ideal given simply by restriction of representations ([3], Lemme 2.11.3, and the fact that the ideal is minimal), it follows from [3], Corollaire 4.1.10, that each irreducible representation of $\mathfrak{A}^{o}(\mathcal{K})$ with kernel ${}_{n}I_{o}$ is unitary equivalent to Π. An analogous argument for the ideals ${}_{o}I_{n}$ establishes the proposition.

<u>5.7.</u> We shall now prove that the representations ${}_{o}\Pi_{n}$ and ${}_{n}\Pi_{o}$ are subrepresentations of the Foch representation ${}_{o}\Pi$ and the anti-Foch representation Π_{o} resp. (See [11], 1.3 for definitions). For the sake of completeness we state a lemma, the constituents of which are well known.

<u>Lemma:</u> Let $\mathfrak{A}$ be a C^{*}-algebra, G a compact abelian group, α a strongly continuous representation of G as a group of automorphisms of $\mathfrak{A}$, $\mathfrak{A}^{o}$ the algebra of G-invariant elements in $\mathfrak{A}$, ω a pure G-invariant state of $\mathfrak{A}$, Π the irreducible representation of $\mathfrak{A}$ associated with ω, κ the Hilbert space of Π, ξ a cyclic vector in κ such that $\omega(x) = (\Pi(x)\xi,\xi)$ for all $x \in \mathfrak{A}$, $\hat{G}$ the character group of G, dg normalized Haar measure on G, Π_{o} the restriction of Π to $\mathfrak{A}^{o}$.

Then there exists a unique strongly continuous representation U of G on κ such that

$$(1) \qquad U_{g}\Pi(x)U_{g}^{*} = \Pi(\alpha_{g}(x))$$

for all $g \in G$ and $x \in \mathfrak{A}$, and such that $U_{g}\xi = \xi$ for all $g \in G$. If $\chi \in \hat{G}$ define:

$$(2) \qquad E_{\chi} = \int_{G} \overline{\chi(g)}\, U_{g}\, dg.$$

(the integral being taken in strong topology). Then E_χ is an orthogonal projection such that

$$(3) \qquad U_g = \sum_{\chi \in \hat{G}} \chi(g) E_\chi ,$$

for all $g \in G$. Moreover the projections E_χ, $\chi \in \hat{G}$ from a set of mutually orthogonal minimal projections in $\Pi_0(\mathfrak{A}^0)'$ such that $\sum_{\chi \in \hat{G}} E_\chi = I$. Hence

$$(4) \qquad \Pi_0 = \bigoplus_{\substack{\chi \in \hat{G} \\ E_\chi \neq 0}} E_\chi \Pi_0$$

is a decomposition of Π_0 into irreducible subrepresentations.

<u>Proof:</u> The existence of the representation U with the given properties is a wellknown result of Segal, see [13]. Since G is compact and abelian, U has the decomposition (3), see [3], Théorème 15.1.3, and (2) then follows from the orthogonality relations for characters, see [3], Théorème 14.3.7. From [5], Lemma 3.1 and Lemma 3.2 it follows that the weak closure of $\Pi(\mathfrak{A}^0)$ is equal to the commutant of U_G, and hence the commutant of $\Pi_0(\mathfrak{A}^0)$ is equal to the von-Neumann algebra generated by U_G. By (3) the projections E_χ are minimal in this algebra, and thus the last assertion of the lemma follows.

<u>5.8.</u> We now study the decomposition of the Fock and anti-Fock representation, when these representations are restricted to $\mathfrak{A}^0(\mathcal{K})$. We remind the reader of some facts from [11]. The Fock representation ${}_0\Pi$ and anti-Fock representation Π_0 are both operating on the Hilbert space $\kappa = \bigoplus_{n=0}^{\infty} A^n \mathcal{K}$, where $A^0\mathcal{K} \cong \mathbb{C}$,

and $A^n\mathcal{K}$ consists of those vectors in $\mathcal{K}\otimes\mathcal{K}\otimes---\otimes\mathcal{K}$ (n-times) which lie in the closure of those vectors in the algebraic tensorproduct of $\mathcal{K}$ with itself n times, which are antisymmetric under permutation of the factors in $\mathcal{K}$, ${}_o\Pi$ (Π_o) then have the property that ${}_o\Pi(a(f))(A^o\mathcal{K}) = 0$, $\Pi_o(a^*(f))(A^o\mathcal{K}) = 0$ and

(1)
$$\begin{aligned} {}_o\Pi(a(f)) &: A^{n+1}\mathcal{K} \to A^n\mathcal{K} \\ {}_o\Pi(a(f)^*) &: A^n\mathcal{K} \to A^{n+1}\mathcal{K} \\ \Pi_o(a(f)) &: A^n\mathcal{K} \to A^{n+1}\mathcal{K} \\ \Pi_o(a(f)^*) &: A^{n+1}\mathcal{K} \to A^n\mathcal{K} \end{aligned}$$

for $n = 0,1,\cdots$, and $f\in\mathcal{K}$.

${}_o\Pi$ and Π_o are irreducible, and if Ω is a unit vector in $A^o\mathcal{K}$, then the associated vector state is gauge invariant and pure in both representations. This state is called respectively the Fock state ${}_o\omega$ and the anti-Fock state ω_o in the two representations. We shall soon see that ${}_o\omega$ and ω_o, restricted to $\mathfrak{A}^o(\mathcal{K})$ are the multiplicative linear functionals corresponding to the two ideals ${}_oI_1$ and ${}_1I_o$ resp., both having codimension 1 in $\mathfrak{A}^o(\mathcal{K})$.

The homogeneous polynomials of degree n in the creators $a(f)^*$, applied to Ω in the Fock representation generate a dense subset of $A^n\mathcal{K}$. Since X_θ acts on these polynomials by multiplication by $e^{-in\theta}$, it follows that if U_θ is the unitary operator on κ associated X_θ by lemma 5.7, then:

$$U_\theta \xi = e^{-n\theta}\xi \qquad \text{for} \quad \xi \in A^n\mathcal{K}.$$

It follows from lemma 5.7 that the subspace $A^n\mathcal{K}$ is invariant and irreducible under ${}_o\Pi$ restricted to $\mathfrak{A}^o(\mathcal{K})$. Using (1), one easily deduces that the kernel of the corresponding subrepresentation is generated by those gauge invariant polynomials for which each addend in their normal form contains at least n+1 annihilators, thus this kernel is ${}_oI_{n+1}$ and thus by proposition 5.6 the subrepresentation is unitarily equivalent to ${}_o\Pi_{n+1}$. Using analogous arguments for the anti-Fock representation we obtain

<u>Proposition:</u> Let ${}_o\Pi$ (resp. Π_o) be the Fock representation (resp. the anti-Fock representation) restricted to $\mathfrak{A}^o(\mathcal{K})$, acting on the direct sum $\kappa = \bigoplus_{n=0}^{\infty} A^n\mathcal{K}$ of n-particle subspaces. Then each subspace $A^n\mathcal{K}$ is invariant and irreducible in both representations, and the corresponding decomposition into irreducible subrepresentations is

$${}_o\Pi = \bigoplus_{n=0}^{\infty} {}_o\Pi_{n+1}$$

$$\Pi_o = \bigoplus_{n=0}^{\infty} {}_{n+1}\Pi_o.$$

<u>5.9.</u> In [1], Théorème 3, Dixmier gives an example of a primitive separable C^*-algebra $\mathfrak{A}$ such that its structure space Prim($\mathfrak{A}$) contains no nonempty open sets which is separated. (U is separated if for each point $p \in U$ we have that for all points q not lying in the closure of {p} that p and q have

a pair of disjoint neighbourhoods). $\mathfrak{A}^{\circ}(\mathcal{K})$ provides another such example. Indeed, from the diagram of $\mathfrak{A}^{\circ}(\mathcal{K})$ we see that the open nonempty sets of $\mathrm{Prim}(\mathfrak{A}^{\circ}(\mathcal{K}))$ are of the form $\{{}_nI_o | n \geq n_o\} \cup \{{}_oI_m | m \geq m_o\} \cup \{0\}$. Thus all neighbourhoods of ${}_nI_o$ contain ${}_{n+1}I_o$ although ${}_{n+1}I_o$ does not lie in the closure of ${}_nI_o$, which is $\{{}_mI_o | 1 \leq m \leq n\}$.

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