AN INTRODUCTION TO THE FINE STRUCTURE OF THE CONSTRUCTIBLE HIERARCHY

(Results of Ronald JENSEN)

by

Keith J. DEVLIN

§ 0. Introduction

We shall work in Zermelo-Fraenkel set theory (including the axiom of choice) throughout, and denote this theory by ZFC. We shall adopt the usual, well-known, notations and conventions of contemporary set theory (e.g. an ordinal is defined to be the set of all smaller ordinals, cardinals are initial ordinals, etc.)
The paper is entirely self-contained, but some familiarity with the usual definition of the constructible universe, L, in terms of definability, and the proof that L is a model of $ZFC + GCH + V = L$, will be helpful.
The exposition is based, with permission, very strongly on a set of notes[(1] written by Ronald Jensen and entitled "The Fine Structure of the Constructible Hierarchy". Except where otherwise stated, the results are entirely those of Professor Jensen.
It is convenient at this point for us to express our appreciation of several illuminating discussions with Professor Jensen on his work in general.

Previously, Jensen worked, as did most other people, with the usual "constructible hierarchy". Thus, one defines, inductively, sets L_α , $\alpha \in OR$, by setting $L_0 = \emptyset$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ if $\lim(\lambda)$, and $L_{\alpha+1}$ = the set of all $x \subset L_\alpha$ such that for some $\in$-formula φ and some $a_1, \ldots, a_n \in L_\alpha$, $x = \{z \in L_\alpha \mid L_\alpha \models \varphi[z, a_1, \ldots, a_n]\}$. One then defines the <u>constructible universe</u> as the class $L = \bigcup_{\alpha \in OR} L_\alpha$. Now, the important facts concerning this definition which one uses when studying L, are, firstly, that the construction is (in a strong way, to be made precise later) Σ_1-definable, and thus has certain absoluteness properties, and, secondly, that $L_{\alpha+1}$ contains all and only those subsets of L_α which are L_α-definable (and which, therefore, must be in L if L is to be a model of ZFC). But if, indeed, these are the only conditions which we require (and loosely speaking they are), then it is clear that our above definition is unnecessarily restrictive. For instance, there are many simply definable functions or sets under which L must be closed, but which increase rank - and these functions will lead out of the sets L_α. For instance, unless $\lim(\alpha)$, L_α will not be closed under the formation of ordered pairs. Since this function plays a central role in even the most elementary parts of set theory, we see that this defect becomes quite important (though not unavoidable) when we try to study the fine structure of L rather than L itself. So, following Jensen, we define a new hierarchy of "constructible sets", which is sufficiently like the L-hierarchy to preserve the two properties mentioned above, but which has the extra property that each level in the hierarchy is closed under ordered pairs, etc. More precisely, we first define a certain class of set functions (called "rudimentary functions"), and then define a hierarchy $\langle J_\alpha \mid \alpha \in OR\rangle$ (the <u>Jensen hierarchy</u>) such that each J_α is closed under the rudimentary functions, $L = \bigcup_{\alpha \in OR} J_\alpha$, and the two properties above hold for this hierarchy. In most cases, J_α will be a "constructibly inessential" extension of L_α , and in fact, if $\langle V_\alpha | \alpha \in OR \rangle$ denotes the familiar rank-hierarchy, the precise relationship between the J- and the L-hierarchies is easily seen to be $J_0 = L_0 = \emptyset$ and $L_{\omega+\alpha} = V_{\omega+\alpha} \cap J_{1+\alpha}$ for all α. Hence we have $J_\alpha = L_\alpha$ iff $\omega\alpha = \alpha$.

In § 1 we give some basic definitions. In § 2 we define the class of rudimentary functions and develop the elementary theory of this class. The reader may, if he wishes, safely skip all the proofs in this section without affecting the reading of the later parts. § 3 is devoted to a very brief discussion of the concept of an admissible set. In § 4 the Jensen hierarchy is defined and its elementary properties discussed. In § 5 we investigate the fine structure of the Jensen hierarchy. A corresponding theory may also be developed for the L-hierarchy, the only difference being that some akward complications arise because of the above mentioned defects in this definition.

§ 1. Preliminaries

We shall be concerned with first-order structures of the form $\underset{\sim}{M} = \langle M, \in, A \rangle$, where M is a non-empty set and $A \subset M$. In general, we shall write $\langle M, A \rangle$ for $\langle M, \in, A \rangle$. The (first-order) language for such structures consists of the following:

(i) variables v_j , $j \in \omega$ (generally denoted by v,w,x,y,z, etc.) (Vbl.)

(ii) predicates $=$, $\in$, A .

(iii) bounded quantifiers $(\forall v_i \in v_j)$, $(\exists v_i \in v_j)$, $i,j \in \omega, i \neq j$.

(iv) unbounded quantifiers $(\forall v_i)$, $(\exists v_i)$, $i \in \omega$.

(v) connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$.

Finite strings of variables (or of elements of M) are denoted by $\vec{v}$, $\vec{x}$, etc. We write $\vec{a} \in X$ for $a_1 \in X \wedge \ldots \wedge a_n \in X$, where we have $\vec{a} = a_1, \ldots a_n$. Similarly for $\exists\vec{v}$, $\forall\vec{v}$, etc.

The notions of primitive formula (PFml), formula (Fml), free variable, statement, and satisfaction are assumed known.
A formula of this language is Σ_0 (or Π_0) if it contains no unbounded quantifiers. Let $n \geq 1$, and let Q_n denote $\forall$ if n is even and $\exists$ if n is odd. A formula is $\Sigma_n(\Pi_n)$ if it is of the form $\exists\vec{x}_1 \forall\vec{x}_2 \exists\vec{x}_3 \ldots Q_n \vec{x}_n \varphi (\forall\vec{x}_1 \exists\vec{x}_2 \forall\vec{x}_3 \ldots Q_{n+1}\vec{x}_n\varphi)$ where φ is Σ_0.
A formula in which the predicate A does not occur is called an <u>$\in$-formula</u>.

$\models_{\underset{\sim}{M}}$ denotes the satisfaction relation for $\underset{\sim}{M}$. Thus, $\models_{\underset{\sim}{M}}$ is the set of all $\langle\varphi,\langle\vec{z}\rangle\rangle$ such that φ is a formula of the above language and $\vec{z} \in M$ and φ holds in $\underset{\sim}{M}$ at the point $\langle\vec{z}\rangle$. We generally write $\models_{\underset{\sim}{M}}\varphi[\vec{z}]$ for $\langle\varphi,\langle\vec{z}\rangle\rangle \in \models_{\underset{\sim}{M}}$. $\models_{\underset{\sim}{M}}^{\Sigma_n}$ denotes the set of all $\langle\varphi,\langle\vec{z}\rangle\rangle \in \models_{\underset{\sim}{M}}$ such that φ is Σ_n.

Let $N \subset M$. A set $R \subset M$ is $\Sigma_n^{\underset{\sim}{M}}(N)$. $(\Pi_n^{\underset{\sim}{M}}(N))$ iff there is a Σ_n (Π_n) formula $\varphi(u,\vec{v})$ and elements $\vec{a} \in N$ such that for all $\vec{x} \in M$, $R(x) \leftrightarrow \models_{\underset{\sim}{M}} \varphi[x,\vec{a}]$. The set of all such R is also denoted by $\Sigma_n^{\underset{\sim}{M}}(N)$ $(\Pi_n^{\underset{\sim}{M}}(N))$.

Set $\Sigma_\omega^{\underset{\sim}{M}}(N) = \bigcup_{n\in\omega} \Sigma_n^{\underset{\sim}{M}}(N)$, $\Delta_n^{\underset{\sim}{M}}(N) = \Sigma_n^{\underset{\sim}{M}}(N) \cap \Pi_n^{\underset{\sim}{M}}(N)$.

Write $\Sigma_n^{\underset{\sim}{M}}$ for $\Sigma_n^{\underset{\sim}{M}}(\emptyset)$ and $\Sigma_n(\underset{\sim}{M})$ for $\Sigma_n^{\underset{\sim}{M}}(M)$. Similarly for Π, Δ.

If φ is a formula, $\varphi^{\underset{\sim}{M}}$ denotes the relation $\{x \in M \mid \models_{\underset{\sim}{M}} \varphi[x]\}$. Similarly, and more generally, define $\varphi^{\underset{\sim}{M}}[\vec{a}]$ for $\vec{a} \in M$ as $\{x \in M \mid \models_{\underset{\sim}{M}} \varphi\ [x, \vec{a}]\}$.

Let F be a class of structures of the form $\underset{\sim}{M} = \langle M,A \rangle$. A relation R is <u>uniformly</u> $\Sigma_n(\underset{\sim}{M})$ <u>for</u> $\underset{\sim}{M} \in F$ iff there is a Σ_n formula $\varphi(u,\vec{v})$ and elements $\vec{a} \in \cap\{M \mid \underset{\sim}{M} \in F\}$ such that whenever $\underset{\sim}{M} \in F$, $R \cap M = \varphi^{\underset{\sim}{M}}[\vec{a}]$.

§ 3. Rudimentary Functions

A function $f : V^n \to V$ is <u>rudimentary</u> (<u>rud</u>) iff it is generated by the following schemata:

(i) $f(x_1,\dots,x_n) = x_i$, $1 \leq i \leq n$.

(ii) $f(x_1,\dots,x_n) = x_i - x_j$, $1 \leq i, j \leq n$.

(iii) $f(x_1,\dots,x_n) = \{x_i,x_j\}$, $1 \leq i, j \leq n$.

(iv) $f(x_1,\dots,x_n) = h(g_1(x_1,\dots,x_n),\dots,g_k(x_1,\dots,x_n))$, where $g_1,\dots,g_k,h$ are rud.

(v) $f(x_1,\dots,x_n) = \cup_{y \in x_1} h(y,x_2,\dots,x_n)$, where h is rud.

For example, the following functions are clearly rud:

$f(\vec{x}) = \cup x_i$

$f(\vec{x}) = x_i \cup x_j$ ($= \cup\{x_i,x_j\}$)

$f(\vec{x}) = \{\vec{x}\}$

$f(\vec{x}) = \langle \vec{x} \rangle = \{\{x_1\},\{x_1,\langle x_2,\dots,x_n\rangle\}\}$.

And if $f(y,\vec{x})$ is rud, so is $g(y,\vec{x}) = \langle f(z,\vec{x}) \mid z \in y \rangle$ ($= \cup_{z \in y}\{\langle f(z,\vec{x}),\ z\rangle\}$).

We say that $R \subset V^n$ is <u>rudimentary</u> (rud) iff there is a rud function $r : V^n \to V$ such that $R = \{\langle\vec{x}\rangle | r(\vec{x}) \neq \emptyset\}$.
For example, $\notin$ is rud, since $y \notin x \leftrightarrow \{y\} - x \neq \emptyset$.

We list some basic properties of rudimentary functions and relations.

(1) If f, R are rud, so is $g(\vec{x}) = \begin{cases} f(\vec{x}), & \text{if } R(\vec{x}) \\ \emptyset, & \text{if } \neg R(\vec{x}). \end{cases}$

<u>Proof:</u> By definition, there is a rud r such that $R(\vec{x}) \leftrightarrow r(\vec{x}) \neq \emptyset$. Then $g(\vec{x}) = \bigcup_{y \in r(\vec{x})} f(\vec{x})$.

(2) Let χ_R be the characteristic function of R. R is rud if χ_R is rud.

<u>Proof</u>: By (1).

(3) R is rud iff $\neg R$ is rud.

<u>Proof</u>: By (2), since $\chi_{\neg R}(\vec{x}) = 1 - \chi_R(\vec{x})$.

(4) Let $f_i \cdot V^n \to V$ be rud, $i = 1,\dots,n$. Let $R_i \subset V^n$ be rud and mutually disjoint, $i = 1,\dots,m$, and such that $\bigcup_{i=1}^{m} R_i = V^n$. Define $f : V^n \to V$ by $f(\vec{x}) = f_i(\vec{x})$ iff $R_i(\vec{x})$. Then f is rud.

<u>Proof</u>: Set $\bar{f}_i(\vec{x}) = \begin{cases} f_i(\vec{x}), & \text{if } R_i(\vec{x}) \\ \emptyset, & \text{if } \neg R_i(\vec{x}) \end{cases}$, $i = 1,\dots,m$.

By (1), $\bar{f}_i$ is rud. But $f(\vec{x}) = \bigcup_{i=1}^{m} \bar{f}_i(\vec{x})$.

(5) If $R(y,\vec{x})$ is rud, so is $f(y,\vec{x}) = y \cap \{z | R(z,\vec{x})\}$

<u>Proof</u>: Set $h(y,\vec{x}) = \begin{cases} \{y\}, & \text{if } R(y,\vec{x}) \\ \emptyset, & \text{otherwise.} \end{cases}$

Then h is rud. But $f(y,\vec{x}) = \bigcup_{z \in y} h(z,\vec{x})$.

(6) Suppose $R(y,\vec{x})$ is rud and $(\forall\vec{x})(\exists! y)R(y,\vec{x})$. Set

$$f(y,\vec{x}) = \begin{cases} \text{the unique } z \in y \text{ such that } R(z,\vec{x}), \\ \qquad \text{if such a } z \text{ exists.} \\ \emptyset \text{, otherwise.} \end{cases}$$

Then f is rud.

<u>Proof</u>: $f(y,\vec{x}) = \bigcup[y \cap \{z \mid R(z,\vec{x})\}]$

(7) If $R(y,\vec{x})$ is rud, so is $(\exists z \in y)R(z,\vec{x})$.

<u>Proof</u>: Take r rud so that $R(y,\vec{x}) \leftrightarrow r(y,\vec{x}) \neq \emptyset$. Then

$$(\exists z \in y)R(z,\vec{x}) \leftrightarrow \bigcup_{z \in y} r(z,\vec{x}) \neq \emptyset.$$

(8) If $R_i(\vec{x})$ are rud, $i = 1,\ldots,m$, then so are

$$\bigcup_{i=1}^{m} R_i, \quad \bigcap_{i=1}^{m} R_i, \quad \text{(Trivial).}$$

(9) Let $(-)_0, (-)_1$, denote the inverse functions to $\langle -,- \rangle$. Then $(-)_0$, $(-)_1$ are rud. More generally, let $(-)^n_0, \ldots, (-)^n_{n-1}$ denote the inverse functions to $\langle x_1,\ldots,x_n \rangle$. Then $(-)^n_0, \ldots, (-)^n_{n-1}$ are rud.

<u>Proof</u>: $$(x)_0 = \begin{cases} \text{the unique } z \in \bigcup x \text{ such that} \\ (\exists u_1,u_2 \in \bigcup x)(x = \langle u_1,u_2 \rangle \wedge u_1 = z) \\ \emptyset \text{, if no such } z \text{ exists.} \end{cases}$$

etc.

(10) The function $$f(x,y) = x(y) = \begin{cases} \text{the unique } z \in \bigcup\bigcup x \text{ such} \\ \text{that } \langle z,y \rangle \in x \\ \emptyset, \text{ if no such } z \text{ exists.} \end{cases}$$

is rud (By definition.)

(11) dom and ran are rud.

<u>Proof</u>: $\mathrm{dom}(x) = \{z \in \bigcup\bigcup x \mid (\exists w \in \bigcup\bigcup x)(\langle w,z \rangle \in x)\}$

$\mathrm{ran}(x) = \{z \in \bigcup\bigcup x \mid (\exists w \in \bigcup\bigcup x)(\langle z,w \rangle \in x)\}$.

(12) $f(x,y) = x \times y = \bigcup_{u \in x} \bigcup_{v \in y} \{\langle u,v \rangle\}$ is rud.

(13) $f(x,y) = x \restriction y = x \cap (\mathrm{ran}(x) x y)$ is rud.

(14) $f(x,y) = x \text{ " } y = \mathrm{ran}(x | y)$ is rud.

(15) $f(x) = x^{-1}$ is rud.

Proof: Set $h(z) = \langle (z)_1, (z)_0 \rangle$. Then h is rud. But clearly,

$$f(x) = x^{-1} = h\text{"}(x \cap (\mathrm{ran}(x) \times \mathrm{dom}(x))).$$

Recalling our preliminary discussion (§ 0), we observe that though rud functions increase rank, they only do so by a finite amount. More precisely, by induction on the rud definition* of a given rud function f, we see that there is a $p \in \omega$ such that for all $x_1, \ldots, x_n$, $\mathrm{rank}(f(x_1, \ldots, x_n)) < \max\{\mathrm{rank}(x_1), \ldots, \mathrm{rank}(x_n)\} + p$.

* Note: In future, we shall often refer to "the rud definition of f", or simply "the definition of f". We mean an arbitrary such definition, the actual choice being irrelevant, and hence assumed made once and for all.

We now prove that the rud functions do in fact encompass all of the "simply definable" functions we spoke about in § 0. First, let us call a function $f : V^n \to V$ simple iff whenever $\varphi(z, \vec{y})$ is a Σ_0 $\in$-formula, there is a Σ_0 $\in$-formula ψ such that $\models_V \varphi(f(\vec{x}), \vec{y}) \leftrightarrow \psi(\vec{x}, \vec{y})$. A useful characterisation of this concept is given by the following:

Proposition

A function $f : V^n \to V$ is simple iff

(i) the predicate $x \in f(\vec{y})$ is Σ_0^V; and

(ii) whenever $A(x)$ is Σ_0^V, so is $(\forall x \in f(\vec{y}))A(x)$.

<u>Proof</u>: (→) By definition.

(←) Let f satisfy (i) and (ii), and let $\varphi(x,\vec{y})$ be a Σ_0 $\in$-formula. An easy induction on the length of φ shows that $\varphi(f(\vec{x}),y)$ is equivalent to a Σ_0 $\in$-formula; so f is simple.

Using this proposition, and easy induction on the definition of f yields:

<u>Lemma</u> 1

If f is rud then f is simple.

Now, since there are Σ_0^V functions which increase rank by an infinite amount, it is clear that the converse to the above lemma is false. However, we do have:

<u>Lemma</u> 2

$R \subset V^n$ is Σ_0^V iff R is rud.

<u>Proof</u>: (→) Let R be Σ_0^V. By (3), (7), and (8) above, an easy induction on the Σ_0^V definition of R shows that R is rud.

(←) Let R be rud. Then χ_R is rud. So by lemma 1, χ_R is simple. Using our above proposition, an easy induction on the rud definition of χ_R shows that χ_R, and hence R, is Σ_0^V.

We require some generalisations of these concepts.

Let $A \subset V$. We say that a function f is <u>rud in</u> A iff f is generated by the schemata for rud functions and the function χ_A.

Let $p \in V$. We say that a function f is rud in <u>parameter</u> p iff f is generated by the schemata for rud functions and the constant function $h(\vec{x}) = p$.

Lemma 3

If f is rud in $A \subset V$, there are rud functions $g_1, \dots, g_n$ such that f is expressible (in a uniform way with respect to the rud definition of f) as a composition of $g_1, \dots, g_n$ and the function $h(x) = A \cap x$.

Proof: By induction on the (rud) definition of f.

A set X is said to be rud closed if for all rud functions $f, f''X^n \subset X$.

A structure $\underset{\sim}{M} = \langle M, A\rangle$ is said to be rud closed if for all functions f which are rud in A, $f''M^n \subset M$.

We say a structure $\underset{\sim}{M} = \langle M, A\rangle$ is amenable if $u \in M \rightarrow A \cap u \in M$.

Lemma 4

A structure $\underset{\sim}{M} = \langle M, A\rangle$ is rud closed iff the set M is rud closed and $\underset{\sim}{M}$ is amenable.

Proof: By lemma 3.

Lemma 5

Let $A \subset V$. If f is rud in A, then $f \upharpoonright M^n$ is uniformly $\Sigma_0(\langle M, A \cap M\rangle)$ for all transitive, rud closed $\underset{\sim}{M} = \langle M, A \cap M\rangle$.

Proof: By lemmas 2 and 3.

The next result will be of considerable use to us later on.

Lemma 6

Every rud function is a composition of some of the following rud functions:

$F_0(x,y) = \{x,y\}$

$F_1(x,y) = x - y$

$F_2(x,y) = x \times y$

$F_3(x,y) = \{\langle u,z,v\rangle \mid z \in x \wedge \langle u,v\rangle \in y\}$

$F_4(x,y) = \{\langle u,v,z\rangle \mid z \in x \wedge \langle u,v\rangle \in y\}$

$F_5(x,y) = \bigcup x$

$F_6(x,y) = \mathrm{dom}(x)$

$F_7(x,y) \doteq \in \cap\, x^2$

$F_8(x,y) = \{x''\{z\} \mid z \in y\}$.

Proof: Let $\mathcal{C}$ denote the class of all functions obtainable from $F_0,\dots,F_8$ by composition. We must show that f rud $\rightarrow f \in \mathcal{C}$.

For each $\in$-formula $\varphi(x_1,\dots,x_n)$, set

$$t_\varphi(u) = \{\langle x_1,\dots,x_n\rangle \mid x_1,\dots,x_n \in u \wedge \models_{\langle u,\in\rangle} \varphi[x_1,\dots,x_n]\}.$$

By induction on φ, we show that for all φ, $t_\varphi \in \mathcal{C}$. (The required result will then be proved using this fact.)

(a) $\varphi(\vec{x}) \equiv x_i \in x_j$, $1 \leq i < j \leq n$.

Write $F_x(y)$ for $F_3(x,y)$. Then $t_\varphi(u) = u^{i-1} \times F_u^{j-i}(F_4(\in \cap u^2, u^{n-j}))$ so $t_\varphi \in \mathcal{C}$.

(b) Let $\varphi_1(\vec{x}),\dots,\varphi_p(\vec{x})$ be such that $t_{\varphi_1},\dots,t_{\varphi_p} \in \mathcal{C}$.

Let $\varphi(\vec{x})$ be any propositional combination of $\varphi_1,\dots,\varphi_p$.

Since $x - y$, $x \cup y$ $(= \bigcup\{x,y\})$, $x \cap y$ $(= x - (x-y)) \in \mathcal{C}$, we clearly have $t_\varphi \in \mathcal{C}$.

(c) Let $\bar{\varphi}(y,\vec{x})$ be such that $t_{\bar{\varphi}} \in \mathcal{C}$. Let $\varphi(\vec{x}) \equiv \exists y \bar{\varphi}(y,\vec{x})$ or $\forall y \bar{\varphi}(y,\vec{x})$.

Clearly, $t_{\exists y \bar{\varphi}}(u) = \mathrm{dom}(t_{\bar{\varphi}}(u))$ and $t_{\forall y \bar{\varphi}}(u) = u^n - \mathrm{dom}(u^n - t_{\bar{\varphi}}(u))$. So in either case, $t_\varphi \in \mathcal{C}$.

(d) $\varphi(\vec{x}) \equiv x_i = x_j$.

Let $\theta(y,\vec{x}) \equiv y \in x_i \leftrightarrow y \in x_j$. By (a), (b), $t_\theta \in \mathcal{C}$. But look, $\models_{\langle u,\in\rangle} \varphi[\vec{x}]$ iff $(\forall y \in \cup u)[\models_{\langle u \cup (\cup u),\in\rangle} \theta[y,\vec{x}]]$.

Hence $t_\varphi(u) = u^n \cap t_{\forall y \theta}(u \cup (\cup u))$, so $t_\varphi \in \mathcal{C}$, by (c).

(e) $\varphi(\vec{x}) \equiv x_i \in x_i$, $1 \leq j < i \leq n$.

Let $\psi(y,z,\vec{x}) \equiv y \in z \wedge y = x_i \wedge z = x_j$. By (a), (b), (d), $t_\psi \in \mathcal{C}$. But $\varphi(\vec{x}) \leftrightarrow \exists y \exists z \psi(y,z,\vec{x})$, so by (c), $t_\varphi \in \mathcal{C}$.

Hence, for any $\in$-formula φ , $t_\varphi \in \mathcal{C}$.

If $f : V^n \to V$, define $f^* : V \to V$ by $f^*(u) = f''u''$. Using our above result, we prove by induction on the rud definition of f , that f rud $\to f^* \in \mathcal{C}$. This easily implies the required result.

(a) $f(\vec{x}) = x_i$.

$f^*(u) = f''u^n = u = u - (u-u) \in \mathcal{C}$.

(b) $f(\vec{x}) = x_1 - x_j$.

$f^*(u) = f''u^n = \{x-y \mid x,y \in u\}$. Let $\varphi(z,x,y) \equiv z \in x - y$.
Let $F(u) = t_\varphi(u \cup (\cup u)) \cap (\cup u \times u^2) = \{\langle z,x,y\rangle \mid x,y \in u \wedge z \in x-y\}$.
Then $f^*(u) = F_8(F(u),u^2) \in \mathcal{C}$, since $t_\varphi \in \mathcal{C}$.

(c) $f(\vec{x}) = \{x_i,x_j\}$.

$f^*(u) = f''u^n = \{\{x,y\} \mid x,y \in u\} = \cup u^2 \in \mathcal{C}$.

(d) $f(\vec{x}) = h(y_1(\vec{x}),\ldots,g_k(\vec{x}))$.

Let $G(u) = \bigcup_{i=1}^{k} g_i^*(u)$, $H(u) = h^*(\bigcup_{i=1}^{k} g_i^*(u)) = h^*(G(u))$, and

$K(u) = u^n \cup G(u) \cup H(u)$. By hypothesis, $G, H, K \in \mathscr{C}$. By lemma 1, let $\varphi(y,\vec{x})$ be an $\in$-formula equivalent to the formula

$$\exists z_1 \ldots \exists z_k(z_1 = g_1(\vec{x}) \wedge \ldots \wedge z_k = g_k(\vec{x}) \wedge y = h(z_1,\ldots,z_k)).$$

Clearly, $f^*(u) = F_8(([t_\varphi(K(u))] \cap [H(u) \times u^n]) , u^n) \in \mathscr{C}$.

(e) $f(y,\vec{x}) = \bigcup_{z \in y} g(z,\vec{x})$.

Let $G(u) = \{\langle z,y,\vec{x}\rangle \mid (\exists v \in y)[z \in g(v,\vec{x})] \wedge \vec{x} \in u\}$. As above $G \in \mathscr{C}$. But $f^*(u) = F_8(G(u), u^{n+1}) \in \mathscr{C}$.

Hence f rud $\rightarrow f^* \in \mathscr{C}$, for all f.

Finally, let f be rud. We show that $f \in \mathscr{C}$.

Set $\overline{f}(\langle\vec{z}\rangle) = f(\vec{z})$, $\overline{f}(y) = \emptyset$ in all other cases. Thus $\overline{f}$ is rud. So by the above, $\overline{f}^* \in \mathscr{C}$. Let $P(\vec{x}) = \{\langle\vec{x}\rangle\}$. Thus $P \in \mathscr{C}$. But look, $f(\vec{x}) = \bigcup\bigcup\{\{f(\vec{x})\}\} = \bigcup\bigcup\overline{f}``\{\langle\vec{x}\rangle\}\} = \bigcup\bigcup F_8(\overline{f}^*(P(\vec{x})),P(\vec{x})) \in \mathscr{C}$.

As an immediate corollary of lemmas 3 and 6 we have:

<u>Lemma</u> 7

Let $A \subset V$ and define F_9 by $F_9(x,y) = A \cap x$. Every function rud in A may be expressed as a composition of some of the (rud in A) functions $F_0,\ldots,F_9$.

We shall make immediate use of lemma 7 in investigating the logical complexity of the predicates $\models_{\underset{\sim}{M}}^{\Sigma n}$ for suitable $\underset{\sim}{M}$. We assume, once and for all, that we have a fixed arithmetisation of our language.

<u>Lemma 8</u>

$\models^{\Sigma_0}_{\underset{\sim}{M}}$ is uniformly $\Sigma_1^{\underset{\sim}{M}}$ for transitive, rud closed $\underset{\sim}{M} = \langle M, A\rangle$.

<u>Proof</u>: Let $\mathcal{L}$ be the language consisting of:

(i) variables w_i , $i \in \omega$.

(ii) function symbols (binary) $f_0, \ldots, f_9$.

We shall assume we have a fixed arithmetisation of $\mathcal{L}$. We also assume that the reader understands what is meant by a "term" of $\mathcal{L}$. Henceforth, let $\underset{\sim}{M} = \langle M, A\rangle$ be arbitrary, transitive, and rud closed.

We first define precisely how $\mathcal{L}$ is to be interpreted in $\underset{\sim}{M}$.

Let Q be the set of functions ρ mapping a finite subset of $\{w_i \mid i \in \omega\}$ into M. We may clearly assume Q is rud. Let C be the (rud) function which to each term τ of $\mathcal{L}$ assigns the set of all component terms of τ, including variables. Let $\mathrm{VbL}_{\mathcal{L}}$ be the rud predicate defining the set $\{w_i \mid i \in \omega\}$.
Let P be the predicate

$$P(u,g,v) \leftrightarrow [\mathrm{dom}(g) = u] \wedge (\forall x \in u)[[x \in \mathrm{VbL} \rightarrow x \in \mathrm{dom}(v) \wedge g(x) = v(x)] \wedge \bigwedge_{i=0}^{9} (\forall t_0, t_1 \in u)[x = f_i(t_0,t_1) \rightarrow g(x) = F_i(g(t_0), g(t_1))]].$$

Thus P is rud in A.

We may now define the interpretation of a term τ of $\mathcal{L}$ at a "point" $\rho \in Q$ by:

$$y = \tau^{\underset{\sim}{M}}[\rho] \leftrightarrow \text{"}\tau \text{ is an } \mathcal{L}\text{-term"} \wedge \rho \in C \wedge \exists g[P(C(\tau), g, \rho) \wedge g(\tau) = y]$$

Hence the function $f(\tau,\rho) = \begin{cases} \tau^{\underset{\sim}{M}}[\rho], & \text{if } \tau \text{ is an } \mathcal{L}\text{-term and } \rho \in Q \\ \emptyset, & \text{otherwise} \end{cases}$

is (uniformly) $\Sigma_1^{\underset{\sim}{M}}$ (for transitive, rud closed $\underset{\sim}{M}$).

Since $\underset{\sim}{M}$ is rud closed we can use the above result to define $\models^{\Sigma_0}_{\underset{\sim}{M}}$ as an $\underset{\sim}{M}$-predicate.

Let $\varphi \in \mathrm{Fml}^{\Sigma_0}$. By lemma 2, $\varphi^{\underset{\sim}{M}}$ is rud in A. Hence the function Γ defined by

$$\Gamma(\vec{x}) = \begin{cases} 1, & \text{if } \varphi^{\underset{\sim}{M}}[\vec{x}] \\ 0, & \text{otherwise} \end{cases}$$

is rud in A. So, by lemma 7, we may assume $\Gamma = \tau^{\underset{\sim}{M}}$, where τ is a term of $\mathscr{L}$, under the above interpretation (i.e. with F_i interpreting f_i for each i). In fact, we may clearly pick a recursive function σ mapping Fml^{Σ_0} into the terms of $\mathscr{L}$ so that whenever $\varphi \in \mathrm{Fml}^{\Sigma_0}$, $\varphi^{\underset{\sim}{M}}[\vec{x}] \leftrightarrow [\sigma(\varphi)]^{\underset{\sim}{M}}[x] = 1$. But by our above result, this implies that $\models^{\Sigma_0}_{\underset{\sim}{M}}$ is (uniformly) $\Sigma_1^{\underset{\sim}{M}}$ (for transitive, rud closed $\underset{\sim}{M}$).

As an immediate consequence of this result, we have

<u>Lemma 9</u>

Let $n \geq 1$. Then $\models^{\Sigma_n}_{\underset{\sim}{M}}$ is uniformly $\Sigma_n^{\underset{\sim}{M}}$ for transitive, rud closed $\underset{\sim}{M} = \langle M,A\rangle$.

We conclude this section with a few miscellaneous results of use later. The first two are technical, and will often be used without mention.

<u>Lemma 10</u>

Let $\underset{\sim}{M} = \langle M,A\rangle$ be rud closed. If $R \subset M$ is $\Sigma_n(\underset{\sim}{M})$, there is a $\Sigma_0(\underset{\sim}{M})$ relation P such that $R(x) \leftrightarrow \exists x_1 \forall x_2 \exists x_3 \ldots Q_n x_n P(x,x_1,\ldots,x_n)$.

<u>Proof</u>: Suppose $R(x) \leftrightarrow \models_{\underset{\sim}{M}} \exists \vec{v}_1\ \forall \vec{v}_2\ \exists \vec{v}_3 \ldots Q_n\ \vec{v}_n\ \ \varphi(v,\vec{v}_1,\ldots,\vec{v}_n)[x]$, where φ is a Σ_0-formula. Using the rud functions $\langle -,\ldots,-\rangle$, $(-)^m_0,\ldots,(-)^m_{m-1}$, we can easily obtain, via

lemma 1, a Σ_0-formula ψ such that
$R(x) \leftrightarrow \models_{\underset{\sim}{M}} \exists v_1 \forall v_2 \ldots Q_n v_n \psi(v,v_1,\ldots,v_n)[x]$.
Then $R(x) \leftrightarrow \exists x_1 \forall x_2 \ldots Q_n x_n[\models_{\underset{\sim}{M}} \psi[x,x_1,\ldots,x_n]]$,
as required.

Lemma 11

Let $\underset{\sim}{M} = \langle M,A\rangle$ be rud closed. If $R \subset M$ is $\Sigma_n(\underset{\sim}{M})$, there is a single element $p \in M$ such that R is $\Sigma_n^{\underset{\sim}{M}}(\{p\})$.

Proof: If R is $\Sigma_n^{\underset{\sim}{M}}(\{p_1,\ldots,p_n\})$, then R is also $\Sigma_n^{\underset{\sim}{M}}(\{\langle p_1,\ldots p_n\rangle\})$.

Let $\underset{\sim}{M} = \langle M,A\rangle$ $n \geq 0$. Write $X \prec_{\Sigma_n} \underset{\sim}{M}$ iff $X \subset M$ and for every Σ_n formula φ and every $\vec{x} \in X$,

$$\models_{\langle X,A\cap X\rangle} \varphi[\vec{x}] \text{ iff } \models_{\underset{\sim}{M}} \varphi[\vec{x}].$$

Clearly, if X,M are transitive and $X \subset M$, we always have $X \prec_{\Sigma_0} \underset{\sim}{M}$. And for $n > 0$, we have $X \prec_{\Sigma_n} \underset{\sim}{M}$ iff $X \subset M$ and for every $P \in \Sigma_n^{\underset{\sim}{M}}(X)$, $P \neq \emptyset \rightarrow P \cap X \neq \emptyset$.

Recall that if $\langle X,\in\rangle$ satisfies the axiom of extensionality, there is a unique isomorphism $\pi : \langle X,\in\rangle \cong \langle W,\in\rangle$, where W is a unique transitive set. Furthermore, if $Z \subset X$ is transitive, then $\pi\restriction Z = \text{id}\restriction Z$. In fact, π is defined by $\in$-induction thus: $\pi(x) = \{\pi(y) | y \in x \cap X\}$ for each $x \in X$. The next result is of considerable importance.

Lemma 12

Let $\underset{\sim}{M}$ be transitive and rud closed. Let $X \prec_{\Sigma_1} \underset{\sim}{M}$. Then $\langle X,A\cap X\rangle$ satisfies the axiom of extensionality and is rud closed. Let $\pi : \langle X,A\cap X\rangle \cong \langle W,B\rangle$, where W is transitive. Let $f : M \rightarrow M$

be rud in A. Then for all $\vec{z} \in X$, $\pi(f(\vec{z})) = f(\pi(\vec{z}))$.

Proof: Since $\underset{\sim}{M}$ is transitive, $\underset{\sim}{M}$ satisfies the axiom of extensionality. Hence as $X \prec_{\Sigma_1} \underset{\sim}{M}$, so does $\langle X, A\cap X\rangle$. Similarly, by lemma 5, $\langle X, A\cap X\rangle$ is rud closed. Hence, in particular, $\vec{z} \in X \to f(\vec{z}) \in X$ for $f : M \to M$ rud in A. By induction on the (rud in A) definition of f, $\pi(f(\vec{z})) = f(\pi(\vec{z}))$ for each $\vec{z} \in X$.

§ 3. Admissible Sets.

Let $\underset{\sim}{M} = \langle M, A\rangle$ be non-empty and transitive. We say $\underset{\sim}{M}$ is admissible iff $\underset{\sim}{M}$ is rud closed and satisfies the Σ_0-Replacement Axiom: for all Σ_0 formulas φ and all $\vec{a} \in M$,
$\models_{\underset{\sim}{M}} [\forall x\, \exists y\, \varphi \to \forall u\, \exists v (\forall x \in u)(\exists y \in v)\varphi]\, [\vec{a}]$.

In case $A = \emptyset$ in the above, we call M an admissible set.

More generally, $\underset{\sim}{M}$ is Σ_n-admissible iff $\underset{\sim}{M}$ is rud closed and satisfies the (analogous) Σ_n-Replacement Axiom. Likewise a Σ_n-admissible set. We prove below that $\underset{\sim}{M}$ is admissible iff $\underset{\sim}{M}$ is Σ_1-admissible. All our results extend trivially from admissibility to Σ_n-admissibility, with "Σ_n" everywhere replacing Σ_1, etc.

Roughly speaking, an admissible set (or structure) behaves like the universe as far as Σ_1 concepts are concerned. We give a few elementary results which set the tone for the rest of this exposition.

Convention: For the whole of this paper, we shall adopt the following abuse of notation. Suppose $\underset{\sim}{M}$ is a structure, $\varphi(\vec{v})$

is a formula, and $\vec{x} \in M$. We shall write $\models_{\underset{\sim}{M}} \varphi(\vec{x})$ rather than $\models_{\underset{\sim}{M}} \varphi(\vec{v})\ [\vec{x}]$. Clearly, this is purely a notational convenience.

Firstly, we give the promised "stronger" form of the admissibility definition.

<u>Lemma 13</u> (Σ_1-Replacement)

Let $\underset{\sim}{M}$ be admissible, and let φ be a Σ_1-formula, $\vec{a} \in M$. Then

$$\models_{\underset{\sim}{M}} \forall x\, \exists y\, \varphi(x,y,\vec{a}) \rightarrow \forall u\, \exists v (\forall x \in u)(\exists y \in v)\varphi(x,y,\vec{a}).$$

<u>Proof</u>: Let ψ be a Σ_0-formula such that
$\models_{\underset{\sim}{M}} \varphi(x,y,\vec{a}) \leftrightarrow \exists z \psi(x,y,\vec{a})$.

$$\begin{aligned}
\text{Then } \models_{\underset{\sim}{M}} \forall x\, \exists y \varphi(x,y,\vec{a}) &\rightarrow \forall x\, \exists y\, \exists z \psi(x,y,z,\vec{a}) \\
&\rightarrow \forall x\, \exists w \psi(x,(w)_0,(w)_1,\vec{a}) \\
&\rightarrow \forall u\, \exists v (\forall x \in u)(\exists w \in v)\psi(x,(w)_0,(w)_1,\vec{a}), \\
&\qquad\qquad \text{by } \Sigma_0\text{-Replacement} \\
&\rightarrow \forall u\, \exists v (\forall x \in u)(\exists y \in v)\varphi(x,y,\vec{a}).
\end{aligned}$$

<u>Convention</u>: The essentially superfluous role played by $\vec{a}$ in the above theorem leads us to extend our previous convention slightly by allowing formulas to contain members of $\underset{\sim}{M}$ as parameters. Again, this is clearly an avoidable convenience.

<u>Lemma 14</u>

Let $\underset{\sim}{M}$ be admissible. If $R(\vec{x},y)$ is $\Sigma_1(\underset{\sim}{M})$, so is $(\forall y \in z)R(\vec{x},y)$.

<u>Proof</u>: Let φ be a Σ_0-formula with parameters from ("w.p.f.")M such that $R(\vec{x},y) \leftrightarrow \models_{\underset{\sim}{M}} \exists w\, \varphi(\vec{x},y)$.

$$\begin{aligned}
\text{Then } (\forall y \in z)R(\vec{x},y) &\leftrightarrow \models_{\underset{\sim}{M}} (\forall y \in z)\exists w \varphi(\vec{x},y) \\
&\leftrightarrow \models_{\underset{\sim}{M}} \forall y \exists\, w[(y \in z \wedge \varphi(\vec{x},y)) \vee (y \notin z)].
\end{aligned}$$

So by Σ_0- Replacement,

$$(\forall y \in z)R(\vec{x},y) \leftrightarrow \models_{\underset{\sim}{M}} \exists v(\forall y \in z)(\exists w \in v)\varphi(\vec{x},y),$$

which is $\Sigma_1(\underset{\sim}{M})$.

<u>Lemma 15</u> (Δ_1-comprehension)

Let $\underset{\sim}{M}$ be admissible, $P \in \Delta_1(\underset{\sim}{M})$. Then $u \in M \rightarrow P \cap U \in M$.

<u>Proof:</u> Let φ, ψ be Σ_0-formulas w.p.f. M such that

$$P(z) \leftrightarrow \models_{\underset{\sim}{M}} \forall x \varphi (x,z) \leftrightarrow \models_{\underset{\sim}{M}} \exists y \varphi(y,z).$$

Then,

$$\models_{\underset{\sim}{M}} \forall w_1 \exists w_2[[w_1 \in u \wedge (\exists y(\psi \wedge w_2 = y) \vee \exists x(\neg \varphi \wedge w_2 = x))] \vee [w_1 \notin u \wedge w_2 = \emptyset]]$$

So by Σ_1-Replacement there is $v \in M$ such that

$$\models_{\underset{\sim}{M}} (\forall w_1 \in u)(\exists w_2 \in v)[\exists y(\psi \wedge w_2 = y) \vee \exists x(\neg \varphi \wedge w_2 = x)].$$

So,

$$P \cap u = \{z| \models_{\underset{\sim}{M}} \forall x \varphi (x,z)\} \cap u = \{z| \models_{\underset{\sim}{M}} (\exists y \in v)\psi(y,z)\} \cap u.$$

But $\underset{\sim}{M}$ is rud closed (so satisfies what might be called the Σ_0-comprehension axiom), and therefore we conclude that $P \cap u = \{z \in u| \models_{\underset{\sim}{M}}(\exists y \in v)\psi(y,z)\} \in M$.

The next result has nothing specifically to do with admissibility, but is of considerable value. Let $f : \subset M \rightarrow M$ mean that $f : X \rightarrow M$ for some $X \subset M$.

<u>Lemma 16</u>

Let $\underset{\sim}{M}$ be arbitrary, $f : \subset M \rightarrow M$ be $\Sigma_1(\underset{\sim}{M})$. If dom(f) is $\Pi_1(M)$, then in fact f and dom(f) are $\Delta_1(\underset{\sim}{M})$.

<u>Proof</u>: (a) $\underset{\Sigma_1(\underset{\sim}{M})}{\underline{f(x) = y}} \leftrightarrow \underset{\Pi_1(\underset{\sim}{M})}{\underline{x \in \mathrm{dom}(f) \wedge \forall z(z \neq y \rightarrow f(x) \neq z)}}$

(b) $\underset{\Pi_1(\underset{\sim}{M})}{\underline{x \in \mathrm{dom}(f)}} \leftrightarrow \underset{\Sigma_1(\underset{\sim}{M})}{\underline{\exists y(f(x) = y)}}$

It was necessary to state the above result explicitly because we shall frequently have to deal with functions which, though definable, are not total functions. A particular case of the above theorem would of course occur when $\mathrm{dom}(f) \in M$, (when $\mathrm{dom}(f)$ is $\Sigma_0(\underset{\sim}{M})$).

As usual, we shall use the notation $g(x) \simeq y(x)$ for partial functions, with its usual meaning (i.e. $f(x)$ is defined iff $g(x)$ is defined, in which case $f(x) = g(x)$.).

<u>Lemma 17</u>

Let $\underset{\sim}{M}$ be admissible, $f: \subset M \rightarrow M$ be $\Sigma_1(\underset{\sim}{M})$. If $u \in M$ and $u \subset \mathrm{dom}(f)$, then $f''u \in M$.

<u>Proof</u>: Since $\underset{\sim}{M}$ is rud closed and $f''u = \mathrm{ran}(f \restriction u)$, it suffices to prove that $f \restriction u \in M$.

Now, as $u \in M$, $f \restriction u$ is $\Delta_1(\underset{\sim}{M})$ by lemma 16.

Let $\varphi(x,y)$ be a Σ_1-formula w.p.f. M such that $f(x) = y \leftrightarrow \models_{\underset{\sim}{M}} \varphi(x,y)$.

Then $\models_{\underset{\sim}{M}} \forall x \exists y[(x \in u \wedge \varphi(x,y)) \vee (x \notin u)]$, so by Σ_1-Replacement there is $v \in M$ such that $\models_{\underset{\sim}{M}} (\forall x \in u)(\exists y \in v)\varphi(x,y)$. Hence $f \restriction u \subset v \times u$. So, by Δ_1-Comprehension, $f \restriction u = (f \restriction u) \cap (v \times u) \in M$.

<u>Theorem 18</u> (Recursion Theorem)

Let $\underset{\sim}{M}$ be admissible. Let $h : M^{n+1} \rightarrow M$ be a $\Sigma_1(\underset{\sim}{M})$ function such that for all $\vec{x} \in M$, $\{\langle z,y \rangle | z \in h(y,\vec{x})\}$ is well-founded.

Let $G = M^{n+2} \to M$ be $\Sigma_1(\underset{\sim}{M})$. Then there is a unique $\Sigma_1(\underset{\sim}{M})$ function F such that

(i) $\langle y,\vec{x}\rangle \in \mathrm{dom}(F) \leftrightarrow \{\langle z,\vec{x}\rangle \mid z \in h(y,\vec{x})\} \subset \mathrm{dom}(F)$

(ii) $F(y,\vec{x}) \simeq G(y,\vec{x}\ ,\ \langle F(z,\vec{x}) \mid z \in h(y,\vec{x})\rangle)$.

<u>Proof</u>: Let Φ be the predicate

$$\Phi(f,\vec{x}) \leftrightarrow \text{"f is a function"} \wedge (\forall y \in \mathrm{dom}(f))(\forall z \in h(y,\vec{x}))(z \in \mathrm{dom}(f)) \wedge (\forall y \in \mathrm{dom}(f))(f \upharpoonright y) = G(y,\vec{x},\ f\upharpoonright h(y,\vec{x})).$$

By lemma 16, h, G are $\Delta_1(\underset{\sim}{M})$, so Φ is $\Delta_1(\underset{\sim}{M})$.

Let φ be a Σ_1-formula w.p.f. M such that $\Phi(f,\vec{x}) \leftrightarrow \models_{\underset{\sim}{M}} \varphi(f,\vec{x})$.

Define a $\Sigma_1(\underset{\sim}{M})$ predicate F by (using notation which will later be justified)

$$F(y,\vec{x}) = u \leftrightarrow \exists f[\Phi(f,\vec{x}) \wedge f(y) = u]$$

We verify (i) for this F. Suppose first that $\langle y,\vec{x}\rangle \in \mathrm{dom}(F)$. Then, by definition, $\exists f[\Phi(f,\vec{x}) \wedge y \in \mathrm{dom}(f)]$. By definition of Φ, for such an f we must have $(\forall z \in h(y,\vec{x}))(z \in \mathrm{dom}(f))$. Hence $z \in h(y,\vec{x}) \to \langle z,\vec{x}\rangle \in \mathrm{dom}(F)$. Now suppose that $z \in h(y,\vec{x}) \to \langle z,\vec{x}\rangle \in \mathrm{dom}(F)$. Note that as M is transitive, $h(y,\vec{x}) \subset M$.
By our supposition,

$$\models_{\underset{\sim}{M}} \forall z\, \exists f[(z \in h(y,\vec{x}) \wedge z \in \mathrm{dom}(f) \wedge \varphi(f,\vec{x})) \vee (z \notin h(y,\vec{x}) \wedge f = \emptyset)].$$

so by Σ_1-Replacement,

$$\models_{\underset{\sim}{M}} \exists v(\forall z \in h(y,\vec{x}))(\exists f \in v)[z \in \mathrm{dom}(f) \wedge \varphi(f,\vec{x})].$$

pick such a v. As Φ is $\Delta_1(\underset{\sim}{M})$, by Δ_1-Comprehension we see that $w = v \cap \{f \mid \Phi(f,\vec{x})\} \in M$. Hence $\bigcup w \in M$. It is easily seen that $\Phi(\cup w,\vec{x})$. Noting that $h(y,\vec{x}) \subset \mathrm{dom}(\cup w)$, note that $\cup w\restriction h(y,\vec{x}) \in M$. Set $f = \cup w\restriction h(y,\vec{x}) \cup \{\langle G(y,\vec{x}, \cup w\restriction h(y,\vec{x})),y\rangle\}$. Clearly, $\Phi(f,\vec{x})$, so $\langle y,\vec{x}\rangle \in \mathrm{dom}(F)$. Hence (i) holds for this F.

We now show that F is a function and is unique. By (i), dom(F) is already uniquely determined, so for both of these it suffices to prove the following:

$$\Phi(f,\vec{x}) \wedge \Phi(f',\vec{x}) \wedge y \in \mathrm{dom}(f) \cap \mathrm{dom}(f') \rightarrow f(y) = f'(y)$$

To this end, suppose not. Then $P = \{y \mid y \in \mathrm{dom}(f) \cap \mathrm{dom}(f') \wedge f(y) \neq f'(y)\} \neq \emptyset$. Let y_0 be an h-minimal element of P. Since $y_0 \in P$, $f(y_0) \neq f'(y_0)$. But $\Phi(f,\vec{x})$, $\Phi(f',\vec{x})$, so clearly $f(y_0) = f'(y_0)$ by the h-minimality of $y_0 \in P$. This contradiction suffices (and thus justifies our notation somewhat).

Finally, it is trivial to note that (ii) must hold, virtually by definition.

In view of the many set theoretic concepts defined by a recursion of the above type, it is clear that admissible sets play an important role in set theory.

Say $\underset{\sim}{M}$ is <u>strongly admissible</u> iff $\underset{\sim}{M}$ is non-empty, transitive, rud closed, and satisfies the <u>Strong Σ_0-Replacement Axiom</u> : for all Σ_0 formulas φ w.p.f. M, $\models_{\underset{\sim}{M}} \forall u \exists v (\forall x \in u)[\exists y \varphi(x,y) \rightarrow (\exists y \in u)\varphi(x,y)]$. (Clearly, such an $\underset{\sim}{M}$ will also satisfy the "Strong Σ_1-Replacement Axiom".)

Strongly admissible structures $\underset{\sim}{M}$ are (for reasons to be indicated later) also called <u>non-projectible admissible structures</u>. The difference between admissibility and strong admissibility is closely connected with the difference between Σ_n predicates and Δ_n predicates, which is in turn closely connected with the difference between a function being partial and total. We shall have more to say on this matter later.

§ 4. The Jensen Hierarchy

Let X be a set. The rudimentary closure of X is the smallest set $Y \supset X$ such that Y is rud closed.

Lemma 19

If U is transitive, so is its rud closure.

Proof: Let W be the rud closure of U. Since rud functions are closed under composition, we clearly have $W = \{f(\vec{x}) \mid \vec{x} \in U \wedge f \text{ is rud}\}$. An easy induction on the rud definition of any rud f shows that $\vec{x} \in U \rightarrow TC(f(\vec{x})) \subset W$. Hence W is transitive.

For U transitive, let $\text{rud}(U)$ = the rud closure of $U \cup \{U\}$. Of crucial importance is:

Lemma 20

Let U be transitive. Then $\mathcal{P}(U) \cap \text{rud}(U) = \Sigma_\omega(U)$.

Proof: Clearly, $\mathcal{P}(U) \cap \Sigma_0(U \cup \{U\}) = \Sigma_\omega(U)$, so it suffices to show that $\mathcal{P}(U) \cap \Sigma_0(U \cup \{U\}) = \mathcal{P}(U) \cap \text{rud}(U)$.

Let $X \in \mathcal{P}(U) \cap \Sigma_0(U \cup \{U\}$. Then, exactly as in the proof of lemma 2, $X \in \text{rud}(U)$ (by induction on the Σ_0 definition of X). Now let $X \in \mathcal{P}(U) \cap \text{rud}(U)$. Then X is a $\Sigma_0(\text{rud}(U))$ subset of U. By lemma 1, we may in fact assume that X is $\Sigma_0^{\text{rud}(U)}(U\cup\{U\})$. But $X \subset U\cup\{U\} \subset \text{rud}(U)$ and $U\cup\{U\}$, $\text{rud}(U)$ are transitive, so X is actually $\Sigma_0^{U\cup\{U\}}(U\cup\{U\}) = \Sigma_0(U\cup\{U\})$.

Also very relevant is:

Lemma 21

There is a rud function $\underset{\sim}{S}$ such that whenever U is transitive, $\underset{\sim}{S}(U)$ is transitive, $U\cup\{U\} \subset \underset{\sim}{S}(U)$ and $\bigcup_{n\in\omega} \underset{\sim}{S}^n(U) = \text{rud}(U)$.

Proof: Set $\underset{\sim}{S}(U) = (U\cup\{U\}) \cup (\bigcup_{i=0}^{8} F_i{}''(U\cup\{U\})^2)$. The result follows by lemma 6.

Lemma 22

There is a rud function $\underset{\sim}{Wo}$ such that whenever r is a well-ordering of u, $\underset{\sim}{Wo}(r,u)$ is an end-extension of r which well-orders $\underset{\sim}{S}(u)$.

Proof: Define i^u, j_1^u, j_2^u by:-

$i^u(x)$ = the least $i \leq 8$ such that $(\exists x_1,x_2 \in u)[F_i(x_1,x_2) = x]$

$j_1^u(x)$ = the r-least $x_1 \in u$ such that $(\exists x_2 \in u)[F_{i^u(x)}(x_1,x_2)=x]$

$j_2^u(x)$ = the r-least $x_2 \in u$ such that $F_{i^u(x)}(j_1^u(x),x_2) = x$.

Clearly, i^u, j_1^u, j_2^u are rud functions of u, x.

$$\begin{aligned}\text{Define } \underset{\sim}{Wo}(r,u) = {} & \{\langle x,y\rangle \mid x,y \in u \wedge xry\} \\ & \cup \{\langle x,y\rangle \mid x \in u \wedge y \notin u\} \\ & \cup \{\langle x,y\rangle \mid x \notin u \wedge y \notin u \wedge [i^u(x) < i^u(y) \vee \\ & \quad i^u(x) = i^u(y) \wedge [j_1^u(x)\, r\, j_1^u(y) \vee (j_1^u(x) = \\ & \quad j_1^u(y) \wedge j_2^u(x)\, r\, j_2^u(y))]]\}.\end{aligned}$$

The Jensen hierarchy, $\langle J_\alpha \mid \alpha \in OR\rangle$, is defined as follows:

$$J_0 = \emptyset$$
$$J_{\alpha+1} = \text{rud}(J_\alpha)$$
$$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha \text{, if } \lim(\lambda).$$

Lemma 23

(i) Each J_α is transitive.

(ii) $\alpha \leq \beta \rightarrow J_\alpha \subset J_\beta$

(iii) $\mathrm{rank}(J_\alpha) = \mathrm{OR} \cap J_\alpha = \omega\alpha$.

Proof: (i) By lemma 19.

(ii) Immediate.

(iii) By induction: $\mathrm{rank}(J_{\alpha+1}) = \mathrm{rank}(\mathrm{rud}(J_\alpha)) = \mathrm{rank}(J_\alpha) + \omega$ (by an earlier remark, this last step is easily verified.)

To facilitate our handling of the hierarchy, we "stratify" the J_α's by defining an auxiliary hierarchy $\langle S_\alpha | \alpha \in \mathrm{OR} \rangle$ as follows:

$$S_0 = \emptyset$$

$$S_{\alpha+1} = \underset{\sim}{S}(S_\alpha)$$

$$S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha \text{ , if } \lim(\lambda).$$

Clearly, the J_α's are just the limit points of this sequence. In fact:

Lemma 24

(i) Each S_α is transitive

(ii) $\alpha \leq \beta \rightarrow S_\alpha \subset S_\beta$

(iii) $J_\alpha = \bigcup_{\nu<\omega\alpha} S_\nu = S_{\omega\alpha}$.

Proof: (i) By lemma 21.

(ii) Immediate.

(III) By induction: $J_{\alpha+1} = \mathrm{rud}(J_\alpha) = \bigcup_{n\in\omega} \underset{\sim}{S}^n(J_\alpha) = \bigcup_{n\in\omega} \underset{\sim}{S}^n(S_{\omega\alpha}) = \bigcup_{n\in\omega} S_{\omega\alpha+n} = S_{\omega\alpha+\omega} = S_{\omega(\alpha+1)}$.

Lemma 25

$\langle S_\nu | \nu < \omega\alpha \rangle$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.

<u>Proof:</u> Set $\Phi(f) \equiv$ "f is a function" $\wedge\ \mathrm{dom}(f) \in \mathrm{OR} \wedge f(0) = \emptyset \wedge$

$$(\forall\nu \in \mathrm{dom}(f))[(\mathrm{succ}(\nu) \to f(\nu) = \underset{\sim}{S}(f(\nu-1))) \wedge [\lim(\nu) \to f(\nu) = \bigcup_{\alpha\in\nu} f(\alpha)]].$$

Clearly, Φ is uniformly $\Sigma_0^{J_\alpha}$. And by definition, $y = S_\nu \leftrightarrow \exists f(\Phi(f) \wedge y = f(\nu))$. Thus it suffices to show that for any α, $\nu < \omega\alpha$, the existential quantifier here can be restricted to J_α. In other words, we must show that whenever $\tau < \omega\alpha$, then $\langle S_\nu | \nu < \tau\rangle \in J_\alpha$. This is proved by induction on α. For $\alpha = 0$ it is trivial. For limit α the induction step is immediate. So assume $\alpha = \beta + 1$ and that $\tau < \omega\beta \to \langle S_\nu | \nu < \tau\rangle \in J_\beta$. Then, by our above remarks, it is clear that $\langle S_\nu | \nu < \omega\beta\rangle$ is $\Sigma_1^{J_\beta}$. So by lemma 20, $\langle S_\nu | \nu < \omega\beta\rangle \in J_\alpha$. Thus for all $n < \omega$, $\langle S_\nu | \nu < \omega\beta + n\rangle = \langle S_\nu | \nu < \omega\beta\rangle \cup \{\langle \underset{\sim}{S}^m(J_\beta), \omega\beta + m\rangle | m < n\} \in J_\alpha$ as J_α is rud closed.).

<u>Lemma 26</u>

$\langle J_\nu | \nu < \alpha\rangle$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.

<u>Proof</u>: By an easy induction, $\langle \omega\nu | \nu < \alpha\rangle$ is uniformly $\Sigma_1^{J_\alpha}$ for all α. Since $J_\nu = S_{\omega\nu}$, the result follows by lemma 25.

<u>Lemma 27</u>

There are well-orderings $<_\nu$ of the S_ν such that:

(i) $\nu_1 < \nu_2 \to <_{\nu_1} \subset <_{\nu_2}$;

(ii) $<_{\nu+1}$ is an end-extension of $<_\nu$;

(iii) $\langle <_\nu | \nu < \omega\alpha\rangle$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.

<u>Proof</u>: We use lemma 22. Set $<_0 = \emptyset$, and by induction:

$$<_{\nu+1} = \underset{\sim}{Wo}\ (<_\nu, S_\nu)$$

$$<_\lambda = \bigcup_{\nu\in\lambda} <_\nu\ ,\ \text{if}\ \lim(\lambda).$$

(i) and (ii) are immediate and (iii) is proved like lemma 25.

Lemma 28

There we well-orderings $<_{J_\alpha}$ of the J_α such that:

(i) $\alpha_1 < \alpha_2 \rightarrow <_{J_{\alpha_1}} \subset <_{J_{\alpha_2}}$;

(ii) $<_{J_{\alpha+1}}$ is an end-extension of $<_{J_\alpha}$;

(iii) $\langle <_{J_\beta} | \beta < \alpha \rangle$ is uniformly $\Sigma_1^{J_\alpha}$;

(iv) $<_{J_\alpha}$ is uniformly $\Sigma_1^{J_\alpha}$;

(v) the function $pr_\alpha(x) = \{z | z <_{J_\alpha} x\}$ is uniformly $\Sigma_1^{J_\alpha}$.
("pr" stands for "predecessors" of course.)

Proof: Set $<_{J_\alpha} = <_{\omega\alpha}$ (i)-(iii) are immediate by lemma 27. For (iv), note simply that $x <_{J_\alpha} y \leftrightarrow \exists \nu (x <_\nu y)$. Finally, for (v), note that $y = pr_\alpha(x) \leftrightarrow \exists \nu [x \in S_\nu \wedge y = \{z | z <_\nu x\}]$ (and that $<_\nu \in J_\alpha$) , and use lemma 27.

Lemmas 12 and 26 enable us to prove the following extremely powerful result(due in its original form to Gödel, the present version being Jensen's.):

Theorem 29 (Condensation Lemma)
Let $X \prec_{\Sigma_1} J_\alpha$. Then for some $\beta \leq \alpha$, $X \cong J_\beta$.

Proof: Let $X \prec_{\Sigma_1} J_\alpha$. Then by lemma 12, let $\pi : X \cong W$, where W is transitive. We prove by induction on α that $W = J_\beta$ for $\beta = \pi''(X \cap \alpha)$.

Assume, therefore, that whenever $\nu < \alpha$ and $X^\nu \prec_{\Sigma_1} J_\nu$, the unique isomorphism π^ν of X^ν onto a transitive set W^ν yields $W^\nu = J_{\pi^\nu{}''(X^\nu \cap \nu)}$.
Note that, as $\langle J_\nu | \nu < \alpha \rangle$ is $\Sigma_1^{J_\alpha}$, $\nu \in X \cap \alpha \leftrightarrow J_\nu \in X$.

Claim 1: For all $\nu \in X \cap \alpha$, $\pi(J_\nu) = J_{\pi(\nu)}$.

To see this, note first that for $\nu \in X\cap\alpha$, $X \cap J_\nu \prec_{\Sigma_1} J_\nu$. [For, let $A \in \Sigma_1^{J_\nu}(X \cap J_\nu)$. Since $J_\nu \in X$, $A \in \Sigma_1^{J_\alpha}(X)$. So, if $A \neq \emptyset$, then as $X \prec_{\Sigma_1} J_\alpha$, $A \cap X \neq \emptyset$. But $A \subset J_\nu$, so $A \cap (X \cap J_\nu) \neq \emptyset$.] Hence by induction hypothesis, $\pi' : X \cap J_\nu \cong J_{\pi'"(X\cap J_\nu \cap \nu)}$ for some unique π'. But look, J_α is an $\in$-end extension of J_ν, so π maps $X \cap J_\nu$ isomorphically onto a transitive set also. In other words, $\pi' = \pi \upharpoonright X \cap J_\nu$, and $\pi" X \cap J_\nu = J_{\pi"(X\cap\nu)}$. So, $\pi(J_\nu) = \pi"(X \cap J_\nu) = J_{\pi"(X\cap\nu)} = J_{\pi(\nu)}$, by the definition of π, as claimed.

For $\nu < \alpha$, define $\mathrm{rud}_X(J_\nu)$ = the rud closure of $X \cap (J_\nu \cup \{J_\nu\})$.

Claim 2: $X = \bigcup_{\nu\in X\cap\alpha} \mathrm{rud}_X(J_\nu)$.

To establish this claim, note that as $X \prec_{\Sigma_1} J_\alpha$, X is rud closed, so $\supset$ is obvious. For the converse, let $x \in X$. Then $x \in J_\alpha = \bigcup_{\nu<\alpha} \mathrm{rud}(J_\nu)$, so for some rud function f, $\models_{J_\alpha} (\exists \nu)(\exists p \in J_\nu)(x = f(p, J_\nu))$. But $X \prec_{\Sigma_1} J_\alpha$, so $(\exists \nu \in X \cap \alpha)(\exists p \in J_\nu \cap X)(x = f(p, J_\nu))$. In other words. $x \in \bigcup_{\nu\in X\cap\alpha} \mathrm{rud}_X(J_\nu)$. Hence claim 2.

Claim 3: For $\nu \in X \cap \alpha$, $\pi" \mathrm{rud}_X(J_\nu) = \mathrm{rud}(J_{\pi(\nu)})$.

To see this, let $\nu \in X \cap\alpha$. Suppose first that $x \in \mathrm{rud}_X(J_\nu)$. Then for some rud function f and some $p \in J_\nu \cap X$, $x = f(p, J_\nu)$. By lemma 12 and claim 1, $\pi(x) = f(\pi(p), J_{\pi(\nu)})$. But $p \in J_\nu \cap X$ so $\pi(p) \in J_{\pi(\nu)}$. Hence $\pi(x) \in \mathrm{rud}(J_{\pi(\nu)})$. This proves $\subset$. Conversely, suppose $y \in \mathrm{rud}(J_{\pi(\nu)})$. Then $y \in \mathrm{rud}(\pi(J_\nu))$, by claim 1, so for some rud function f and some $p \in \pi(J_\nu)$, $y = f(p, \pi(J_\nu))$. Now, $\pi(J_\nu) = \pi"(J_\nu \cap X)$, so for some $q \in J_\nu \cap X$, $p = \pi(q)$ and we have $y = f(\pi(q), \pi(J_\nu)) = \pi(f(q, J_\nu)) \in \pi" \mathrm{rud}_X(J_\nu)$. Hence $\supset$, and claim 3 is proved.

By claims 2 and 3, we have $W = \pi''X = \pi''(\bigcup_{\nu \in X \cap \alpha} \mathrm{rud}_X(J_\nu)) = \bigcup_{\nu \in X \cap \alpha} \pi''\mathrm{rud}_X(J_\nu) = \bigcup_{\nu \in X \cap \alpha} \mathrm{rud}(J_{\pi(\nu)}) = \bigcup_{\eta < \beta} \mathrm{rud}(J_\eta) = J_\beta$, where $\beta = \pi''(X \cap \alpha)$.

<u>Note</u>. It is easily seen that we may regard the following as part of the statement of theorem 29: If $Y \subset X$ is transitive, then $\pi \upharpoonright Y = \mathrm{id} \upharpoonright Y$. And for $\nu \in X \cap \alpha$, $\pi(\nu) \leq \nu$, and for all $x \in X$, $\pi(x) \leq_{J_\alpha} x$.

By an argument well known to all set theorists, it is easily shown that $J = \bigcup_{\alpha \in OR} J_\alpha$ is a model of ZFC. (In fact, setting $<_J = \bigcup_{\alpha \in OR} <_{J_\alpha}$, $<_J$ is a J-definable well-ordering of the entire class J, so J satisfies the axiom of choice in a strong way.). Using the condensation lemma, an equally well-known argument shows that $J \models GCH$. However, in the next section we will prove (and have already indicated this fact in our preamble) that $J = L$, so all that the above says is that we can use the Jensen hierarchy in place of the L-hierarchy in order to establish the classical results on the constructible universe.

<u>§ 5. On The Fine Structure of the Jensen Hierarchy</u>.

As mentioned in the introduction, a theory similar to the one following can be developed for the usual L-hierarchy, if desired.

Central in our discussion will be the concept of a "uniformising function" for a relation, which is a sort of "choice function" for a given relation. Specifically, a function r is said to <u>uniformise</u> a relation R iff $\mathrm{dom}(r) = \mathrm{dom}(R)$ and for all $\vec{x}$, $\exists y R(y,\vec{x}) \leftrightarrow R(r(\vec{x}),\vec{x})$.

Let $\underset{\sim}{M} = \langle M,A\rangle$, $n \geq 1$. We say $\underset{\sim}{M}$ is <u>Σ_n-uniformisable</u> iff every $\Sigma_n(\underset{\sim}{M})$ relation on M is uniformised by a $\Sigma_n(\underset{\sim}{M})$ function. A few moments reflection will reveal that Σ_n-uniformisability is a very strong condition to demand of an arbitrary structure $\underset{\sim}{M}$, since in the more obvious cases, the definition of a uniformisary function for a given relation would appear to increase the logical complexity by one or more quantifier switches. However, it will turn out that for all α, all $n \geq 1$, J_α <u>is</u> Σ_n-uniformisable. For $n = 1$, this will be easy to prove, but for $n > 1$, the corresponding argument will only work when J_α is Σ_{n-1}-admissible, so a more indirect approach will be necessary. We shall outline the approach required after we dispose of some of the more easy results. First, Σ_1-uniformisability. The $\Sigma_1(J_\alpha)$ well-ordinary of each J_α gives us this with little effort. In fact, we have a much stronger result, of importance in applications of Σ_1-uniformisability.

Let F be a class of structures $\underset{\sim}{M} = \langle M,A\rangle$, $n \geq 1$. Say F is <u>uniformly Σ_n-uniformisable</u> if, whenever φ is a Σ_n-formula w.p.f. $\cap\{M | \underset{\sim}{M} \in F\}$ such that $\varphi^{\underset{\sim}{M}}$ is a relation on M for each $\underset{\sim}{M} \in F$, there is a Σ_n-formula ψ (w.p.f. $\cap\{M|\underset{\sim}{M} \in F\}$) such that for each $\underset{\sim}{M} \in F$, $\psi^{\underset{\sim}{M}}$ is a function uniformising $\varphi^{\underset{\sim}{M}}$.

<u>Theorem 30</u>

$\langle J_\alpha, A\rangle$ is Σ_1-uniformisable. In fact, the class of all $\langle J_\alpha, A\rangle$ is uniformly Σ_1-uniformisable.

<u>Proof</u>: Let φ be a Σ_1-formula w.p.f. J_α such that $[\varphi(y,\vec{x})]^{\langle J_\alpha, A\rangle}$ is a Σ_1 relation on J_α. By contraction of quantifiers, we can, in a uniform way, find a Σ_0 formu-

la ψ (w.p.f. J_α) such that $\models_{\langle J_\alpha, A\rangle} \varphi(y,\vec{x}) \leftrightarrow \exists z.\psi(z,y,\vec{x})$. Define g by: $g(\vec{x}) \simeq$ the $<_J$-least w such that $\models_{\langle J_\alpha, A\rangle} \psi((w)_0,(w)_1,\vec{x})$. Then g is (uniformly) $\Sigma_1(\langle J_\alpha, A\rangle)$, since if

$$w = g(\vec{x}) \leftrightarrow \models_{\langle J_\alpha, A\rangle} \psi((x)_0,(w)_1,\vec{x})$$
$$\wedge \exists t[t = pr_\alpha(w) \wedge (\forall w' \in t) \neg \psi((w')_0,(w')_1,\vec{x})]$$

Set $r(\vec{x}) \simeq (g(\vec{x}))_1$. Then r is (uniformly) $\Sigma_1(\langle J_\alpha, A\rangle)$. and clearly uniformises $[\varphi(y,\vec{x})]^{\langle J_\alpha, A\rangle}$.

Remark. We call the above construction the canonical Σ_1-uniformisation procedure. Observe that if $R(y,\vec{x})$ us a $\Sigma_1(\langle J_\alpha, A\rangle)$ predicate, then the canonical Σ_1-uniformisation of R is a function whose $\Sigma_1(\langle J_\alpha, A\rangle)$ definition involves only those parameters which occur in the definition of R.

Let us take a little time off to examine the above construction more closely. Suppose $R(y,\vec{x})$ is a given Σ_1 relation, say $R(y,\vec{x}) \leftrightarrow \exists z P(z,y,\vec{x})$, where P is Σ_0. To obtain the Σ_1 uniformisation of R, we first obtain a Σ_1 uniformisation of the Σ_0 relation $\{\langle w,\vec{x}\rangle | P((w)_0,(w)_1,\vec{x})\}$, and then simply pick out the requisite component of the result as our required function. And since $<_J$ is a Σ_1 well order of J_α the result is also Σ_1. However, returning now to the notation of theorem 30, we see that, if we try to extend this procedure to the case $n > 1$, we cannot conclude that the function g is Σ_n, the problem being the last conjunct in the explicit definition of g. Let $\Psi(w,\vec{x})$ denote the predicate $[\neg \psi((w)_0,(w)_1,\vec{x})]^{\langle J_\alpha, A\rangle}$. For $n = 1$, there was no problem, since $\Psi(w,\vec{x})$ is Σ_0, so is $(\forall w \in t)\Psi(w,\vec{x})$. However, for $n > 1$, $\Psi(w,\vec{x})$ is Σ_{n-1}, and we can only conclude that $(\forall w \in t)\Psi(w,\vec{x})$ is Σ_{n-1} if $\langle J_\alpha A\rangle$ is Σ_{n-1}-admissible. Otherwise it is merely Π_n of course, and so the resulting uniformisation of the original Σ_n relation turns out to be Σ_{n+1}. So, in order to establish the general Σ_n-uniformisation lemma, it is not altogether unreasonable to try and "reduce" all Σ_n relations on an arbitrary J_α to Σ_n relations on some Σ_{n-1}-admissible J_β, for which we have a Σ_n-uniformisation procedure. In practice, it will turn out that this hint is slightly off target, but in its general tone it is worth bearing in mind.

Closely connected with Σ_n-uniformisability is the notion of a "Σ_n skolem function".

Let $\underset{\sim}{M} = \langle M,A\rangle$ be transitive and rud closed. By a <u>Σ_n skolem function for $\underset{\sim}{M}$</u> we mean a $\Sigma_n(\underset{\sim}{M})$ function h with $\mathrm{dom}(h)\subset \omega\times M$, such that for some $p\in M$, h is $\Sigma_n^{\underset{\sim}{M}}(\{p\})$, and whenever $P\in \Sigma_n^{\underset{\sim}{M}}(\{x,p\})$ for some $x\in M$, then $\exists y P(y) \rightarrow (\exists i\in\omega)P(h(i,x))$. (With h,p as above, we say that p is a <u>good</u> parameter for h.). Note that Σ_n skolem functions need not be (and in general are not) total! As far as existence of Σ_n skolem functions is concerned, we can get away with slightly less than might first appear.
In fact:

<u>Lemma 31</u>

Let $\underset{\sim}{M} = \langle M,A\rangle$ be transitive and rud closed. Let h be a $\Sigma_n^{\underset{\sim}{M}}(\{p\})$ function with $\mathrm{dom}(h)\subset\omega\times M$. Suppose that whenever $P\in\Sigma_n^{\underset{\sim}{M}}(\{x\})$ for some $x\in M$, then $\exists y P(y)\rightarrow(\exists i\in\omega)P(h(i,x))$. Then $\underset{\sim}{M}$ has a Σ_n skolem function.

<u>Proof</u>: Set $\tilde{h}(i,x)\simeq h(i,\langle x,p\rangle)$. It is easily seen that $\tilde{h}$ is a Σ_n skolem function for $\underset{\sim}{M}$.

Note that in the above, if h is actually $\Sigma_n^{\underset{\sim}{M}}$, then $\tilde{h} = h$. This is used in establishing the following result:

Lemma 32

If $\langle J_\alpha, A\rangle$ is amenable, then it has a Σ_1 skolem function. In fact, there is a Σ_1 skolem function $h_{\alpha,A}$ for $\langle J_\alpha, A\rangle$ which is uniformly $\Sigma_1^{\langle J_\alpha, A\rangle}$ for all amenable $\langle J_\alpha, A\rangle$.

Proof: Let $\langle \varphi_i \mid i < \omega\rangle$ be a recursive enumeration of Fml^{Σ_1}.

Let $\langle J_\alpha, A\rangle$ be amenable. By lemma 9, $\models^{\Sigma_1}_{\langle J_\alpha, A\rangle}$ is (uniformly) $\Sigma_1^{\langle J_\alpha, A\rangle}$. Let $h = h_{\alpha,A}$ be the canonical Σ_1-uniformisation of the $\Sigma_1^{\langle J_\alpha, A\rangle}$ relation $\{\langle y,i,x\rangle \mid \models^{\Sigma_1}_{\langle J_\alpha, A\rangle} \varphi_i[y,x]\}$. (By lemma 30 and the ensuing remark, h is thus uniformly $\Sigma_1^{\langle J_\alpha, A\rangle}$ for amenable $\langle J_\alpha, A\rangle$.). By the remark following lemma 31, it is clear that h is a Σ_1 skolem function for $\langle J_\alpha, A\rangle$.

We refer to $h_{\alpha,A}$ as the canonical Σ_1 skolem function for (amenable) $\langle J_\alpha, A\rangle$.

By a similar argument, we have:

Lemma 33

If $\langle J_\alpha, A\rangle$ is amenable and Σ_n-uniformisable, it has a Σ_n skolem function.

The following lemmas indicate our reason for using the word "skolem" here.

Lemma 34

Let $\underset{\sim}{M}$ be transitive and rud closed, and let h be a Σ_n skolem function for $\underset{\sim}{M}$. Then whenever $x \in M$, $x \in h''(\omega \times \{x\}) \prec_{\Sigma_n} \underset{\sim}{M}$.

Proof: Set $X = h''(\omega\times\{x\})$. Clearly, $x \in X$. Let $P \in \Sigma_n^{\underset{\sim}{M}}(X)$, $P \neq \emptyset$. We must show that $P \cap X \neq \emptyset$. Let p be a good parameter for h, and pick $y_1,\dots,y_m \in X$ with $P \in \Sigma_n^{\underset{\sim}{M}}(\{y_1,\dots,y_m\})$. By definition of X, there are $j_1,\dots,j_m \in \omega$ such that $y_1 = h(j_1,x),\dots,y_m = h(j_m,x)$. Since h is $\Sigma_n^{\underset{\sim}{M}}(\{p\})$, it follows that $P \in \Sigma_n^{\underset{\sim}{M}}(\{p,x\})$. Hence, $P \neq \emptyset \to \exists y P(y) \to (\exists i \in \omega)P(h(i,x)) \to (\exists y \in X)P(y)$.

Lemma 35

Let $\underset{\sim}{M}$ be transitive and rud closed, and let h be a Σ_n skolem function for $\underset{\sim}{M}$. If $X \subset M$ is closed under ordered pairs, then $X \subset h''(\omega\times X) \prec_{\Sigma_n} \underset{\sim}{M}$.

Proof: Set $Y = h''(\omega\times X)$. By lemma 34, $X \subset Y$. Let $P \in \Sigma_n^{\underset{\sim}{M}}(Y)$, $P \neq \emptyset$. We must show that $P \cap Y \neq \emptyset$. Let p be a good parameter for h, and pick $y_1,\dots,y_m \in Y$ with $P \in \Sigma_n^{\underset{\sim}{M}}(\{y_1,\dots,y_m\})$. Pick $j_1,\dots,j_m \in \omega$ and $x_1,\dots,x_m \in X$ such that $y_1 = h(j_1,x_1),\dots,y_m = h(j_m,x_m)$. Let $x = \langle x_1,\dots,x_m\rangle$. By assumption, $x \in X$. But clearly, as h is $\Sigma_n^{\underset{\sim}{M}}(\{p\})$, P is then $\Sigma_n^{\underset{\sim}{M}}(\{p,x\})$, so $P \neq \emptyset \to \exists y P(y) \to (\exists i \in \omega)P(h(i,x)) \to (\exists y \in Y)P(y)$.

Corollary 36

Let $\underset{\sim}{M}$, h be as above. Let $X \subset M$ and suppose $h''(\omega\times X)$ is closed under ordered pairs. Then $X \subset h''(\omega\times X) \prec_{\Sigma_n} \underset{\sim}{M}$.

Proof: Let $Y = h''(\omega\times X)$. Clearly, $Y = h''(\omega\times Y)$, so the result follows by the lemma.

Lemma 37 (Gödel)

There is a bijection $\Phi : OR^2 \leftrightarrow OR$ such that $\Phi(\alpha,\beta) \geq \alpha,\beta$ for all α,β, and $\Phi^{-1}\restriction\omega\alpha$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.

Proof: Define a well-order $<^*$ of OR^2 by

$$(\alpha,\beta) <^* (\gamma,\delta) \leftrightarrow [\max(\alpha,\beta) < \max(\gamma,\delta)] \vee [\max(\alpha,\beta) = \max(\gamma,\delta) \wedge (\alpha < \gamma \vee (\alpha = \gamma \wedge \beta < \delta))].$$

Let $\Phi : <^* \cong OR$. By induction on α, $\Phi^{-1} \upharpoonright \omega\alpha$ is $\Sigma_1^{J_\alpha}$ (uniformly).

Lemma 38

There is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto $(\omega\alpha)^2$ for all α.

Proof: Let $Q = \{\alpha \mid \Phi(0,\alpha) = \alpha\}$. Thus Φ is closed and unbounded in OR. Clearly, $Q = \{\alpha \mid \Phi : \alpha^2 \leftrightarrow \alpha\}$, so $\omega\alpha \in Q \to \omega\alpha = \alpha$. We prove the lemma by induction on α. Assume it is true for all $\nu < \alpha$.

Case 1: $\omega\alpha \in Q$. Then $\Phi^{-1} \upharpoonright \alpha$ suffices.

Case 2: $\alpha = \beta + 1$. If $\beta = 0$, then $\omega\alpha = \omega \in Q$, so we are done by Case 1. Hence we may assume $\beta \geq 1$. Then clearly, there is a $\Sigma_1(J_\alpha)$ map $j : \omega\alpha \leftrightarrow \omega\beta$. By hypothesis, there is a $\Sigma_1(J_\beta)$ map of $\omega\beta$ onto $(\omega\beta)^2$, so there is certainly a $\Sigma_1(J_\beta)$ map g of $(\omega\beta)^2$ one-one into $\omega\beta$. Then $g \in \text{rud}(J_\beta) = J_\alpha$, so for $\nu,\gamma \in \omega\alpha$, define

$$f(\langle \nu,\tau \rangle) = g(\langle j(\nu), j(\tau) \rangle).$$

Then f is $\Sigma_1(J_\alpha)$ and f maps $(\omega\alpha)^2$ one-one into $\omega\beta$. Now $\text{ran}(f) = \text{ran}(y) \in J_\alpha$, so if we define h by (for $\nu \in \omega\alpha$)

$$h(\nu) = \begin{cases} f^{-1}(\nu) & \text{if } \nu \in \text{ran}(f) \\ \langle 0,0 \rangle, & \text{otherwise} \end{cases}$$

we see that h is $\Sigma_1(J_\alpha)$ and $h : \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^2$.

Case 3: $\lim(\alpha) \wedge \omega\alpha \notin Q$. In this case let $\langle \nu,\tau \rangle = \Phi^{-1}(\omega\alpha)$.

Thus $\nu, \tau < \omega\alpha$. Set $c = \{z | z <^* \langle \nu, \tau \rangle\}$ $(\in J_\alpha)$. Thus $\Phi \upharpoonright c$ maps c one-one onto $\omega\alpha$, and is $\Sigma_1(J_\alpha)$. Pick $\gamma < \alpha$ with $\nu, \tau < \omega\gamma$. Then $\Phi^{-1} \upharpoonright \omega\alpha$ is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ one-one into $\omega\gamma$. And by assumption, there is a map $g \in J_\alpha$ mapping $(\omega\gamma)^2$ one-one into $\omega\gamma$. So, setting $f(\langle \iota, \theta \rangle) = g(\langle g(\Phi^{-1}(\iota)), g(\Phi^{-1}(\theta)) \rangle)$, $\iota, \theta < \omega\alpha$, we see that f is a $\Sigma_1(J_\alpha)$ map of $(\omega\alpha)^2$ one-one into $d = g''(g''c)^2$. But $d \in J_\alpha$, so we can define a $\Sigma_1(J_\alpha)$ map h on $\omega\alpha$ by

$$h(\theta) = \begin{cases} f^{-1}(\theta) & \text{, if } \theta \in d \\ \langle 0, 0 \rangle & \text{, otherwise.} \end{cases}$$

Clearly, $h = \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^2$.

The lemma is proved.

Using this lemma, we may now establish the following important result:

<u>Theorem 39</u>

There is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto J_α for all α.

<u>Proof</u>: Let $f : \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^2$ be $\Sigma_1^{J_\alpha}(\{p\})$, where $p \in J_\alpha$ is the $<_J$-least element of J_α for which such an f exists. Define f^0, f^1 by demanding that $f(\nu) = \langle f^0(\nu), f^1(\nu) \rangle$ for all $\nu \in \omega\alpha$. By induction, define $f_n : \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^n$ thus: $f_0 = \text{id} \upharpoonright \omega\alpha$; $f_{n+1}(\nu) = \langle f^0(\nu), f_n \circ f^1(\nu) \rangle$. Hence each f_n is $\Sigma_1^{J_\alpha}(\{p\})$. Let $h = h_\alpha$, the canonical Σ_1 skolem function for J_α. Set $X = h''(\omega \times (\omega\alpha \times \{p\}))$.

<u>Claim 1</u>: X is closed under ordered pairs.

To see this, let $y_1, y_2 \in X$, say $y_1 = h(j_1, \langle \nu_1, p\rangle)$, $y_2 = h(j_2, \langle \nu_2, p\rangle)$. Let $\langle \nu_1, \nu_2\rangle = f_2(\tau)$. Then $\{\langle y_1, y_2\rangle\}$ is a $\Sigma_1^{J_\alpha}(\{\tau, p\})$ predicate, so by definition of h, $\langle y_1, y_2\rangle \in X$, as claimed.

So by corollary 36, $X \prec_{\Sigma_1} J_\alpha$. By the condensation lemma, let $\pi : X \cong J_\beta$, $\beta \leq \alpha$. Since $\omega\alpha \subset X$, we clearly have $\beta = \alpha$ here.

<u>Claim 2</u>: For all $i \in \omega$, $x \in X$, $\pi(h(i,x)) \simeq h(i, \pi(x))$.

To see this, observe first that as h is $\Sigma_1^{J_\alpha}$, there is a rud function H such that $y = h(i,x) \leftrightarrow (\exists t \in J_\alpha)[H(t,i,x,y) = 1]$. Now let $i \in \omega$, $x \in X$. Since $X \prec_{\Sigma_1} J_\alpha$, $y = h(i,x) \in X$ (if defined). Thus, by the above, since $x, y \in X \prec_{\Sigma_1} J_\alpha$, $(\exists t \in X)[H(t,i,x,y) = 1]$. By lemma 12, therefore, $(\exists t \in X)[H(\pi(t), i, \pi(x), \pi(y)) = 1]$. Since $\pi''X = J_\alpha$, this can be rewritten as $(\exists t \in J_\alpha)[H(t,i,\pi(x),\pi(y)) = 1]$. Thus $\pi(y) = h(i, \pi(x))$, as claimed.

Now, $f \subset (\omega\alpha)^3$, so as $\pi \restriction \omega\alpha = \mathrm{id} \restriction \omega\alpha$, $\pi''f = f$. And by isomorphism, $\pi''f$ is $\Sigma_1^{J_\alpha}(\{\pi(p)\})$. So as $\pi(p) \leq_J p$, the choice of p shows that $\pi(p) = p$.
So, by claim 2, if $i \in \omega$, $\nu \in \omega\alpha$, $\pi(h(i, \langle \nu, p\rangle)) \simeq h(i, \langle \nu, p\rangle)$, which is to say $\pi \restriction X = \mathrm{id} \restriction X$. Thus $X = J_\alpha$.

Now define $\tilde{h} : (\omega\alpha)^3 \to J_\alpha$ by setting

$$\tilde{h}(i,\nu,\tau) = \begin{cases} y, & \text{if } (\exists t \in S_\tau)[H(t,i,\langle \nu, p\rangle, y) = 1] \\ \emptyset. & \text{otherwise.} \end{cases}$$

Then $\tilde{h}$ is $\Sigma_1(J_\alpha)$, and clearly $\tilde{h}''(\omega\alpha)^3 = h''(\omega \times (\omega\alpha \times \{p\})) = X = J_\alpha$. Therefore, $\tilde{h} \circ f_3$ is as required by the theorem.

Observe that in lemmas 38, 39, the maps constructed generally have parameters in their definitions. Note also that, being total, these maps are in fact $\Delta_1(J_\alpha)$.

Recalling the results of § 3, we now investigate those ordinals α for which J_α is an admissible set.

Let us call an ordinal α <u>admissible</u> iff $\alpha = \omega\beta$ and J_β is an admissible set.

<u>Theorem 40</u>

$\omega\alpha$ is admissible iff there is no $\Sigma_1(J_\alpha)$ map of any $\gamma = \omega\alpha$ cofinally into $\omega\alpha$. (Note that such a map, having domain $\gamma \in J_\alpha$, would in fact be $\Delta_1(J_\alpha)$.)

<u>Proof</u>: ($\rightarrow$). Let $\gamma < \omega\alpha$ and suppose $f : \gamma \rightarrow \omega\alpha$ is $\Sigma_1(J_\alpha)$. Then $(\forall\xi \in \gamma)(\exists\zeta \in \omega\alpha)(f(\xi) = \zeta)$. If J_α is admissible, then by Σ_1-Replacement, $(\exists\eta \in \omega\alpha)(\forall\xi \in \gamma)(\exists\zeta \in \eta)(f(\xi) = \zeta)$, so f is not cofinal in $\omega\alpha$.

($\leftarrow$) Assume $\omega\alpha$ is not admissible. If $\alpha = \beta + 1$, then the $\Sigma_1(J_\alpha)$ map $\{\langle\omega\beta + n, n\rangle \mid n \in \omega\}$ map ω cofinally into $\omega\alpha$, so we are done. Assume then that $\lim(\alpha)$. Since J_α is not admissible, there must be a $\Sigma_1(J_\alpha)$ relation R and a $u \in J_\alpha$ such that $(\forall x \in u)(\exists y)R(x,y)$ but for all $z \in J_\alpha$, $\neg (\forall x \in u)(\exists y \in z)R(x,y)$. Take $\gamma < \alpha$ with $u \in J_\gamma$. By Theorem 39, let f be a $\Sigma_1(J_\gamma)$ map of $\omega\gamma$ onto J_γ. Thus $f \in J_\alpha$, and $u \subset f''\omega\gamma$.
Define $g : \omega\gamma \rightarrow \omega\alpha$ by

$$g(\nu) = \begin{cases} \text{the least } \tau \text{ such that } (\exists y \in S_\tau)R(f(\nu),y), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } f(\nu) \in u \\ 0 \text{ if } f(\nu) \notin u. \end{cases}$$

Then g is a $\Sigma_1(J_\alpha)$ map of $\omega\gamma$ cofinally into $\omega\alpha$.

Recalling our discussion at the end of § 3, let us call an ordinal α <u>strongly admissible</u> (or <u>non-projectible admissible</u>) iff $\alpha = \omega\beta$ and J_β is strongly admissible. Imitating the proof of Theorem 40, we have:

<u>Theorem 41</u>

$\omega\alpha$ is strongly admissible iff there is no $\Sigma_1(J_\alpha)$ map of a <u>bounded subset</u> of $\omega\alpha$ cofinally into $\omega\alpha$.

The above two results illustrate our earlier remark concerning the difference between a function being partial and being total, and the corresponding difference between a predicate being Σ_n and being Δ_n. The next two results, which strangthen the last two, and are also due to Kripke and Platek, also highlight this distinction.

<u>Theorem 42</u>

The following are equivalent:

(i) $\omega\alpha$ is admissible.

(ii) $\langle J_\alpha A\rangle$ is amenable for all $A \in \Delta_1(J_\alpha)$.

(iii) There is no $\Sigma_1(J_\alpha)$ function mapping a $\gamma < \omega\alpha$ onto J_α. (Of course, any such function would in fact be $\Delta_1(J_\alpha)$.)

<u>Proof</u>: (i) $\rightarrow$ (ii). By lemma 15 (Δ_1-Comprehension)

(ii) $\rightarrow$ (iii). Assume (ii) $\wedge \neg$ (iii). Let $\gamma < \omega\alpha$, and let $f : \gamma \xrightarrow{onto} J_\alpha$ be $\Sigma_1(J_\alpha)$. Then f is $\Delta_1(J_\alpha)$, so $d = \{\nu | \nu \notin f(\nu)\}$ is $\Delta_1(J_\alpha)$. Thus by (ii), $d = d \cap \gamma \in J_\alpha$. So, $d = f(\nu)$ for some $\nu < \gamma$, so $\nu \in f(\nu) \leftrightarrow \nu \in d \leftrightarrow \nu \notin f(\nu)$, a contradiction.

(iii) $\rightarrow$ (i). Assume (iii) $\wedge \neg$ (i). If $\alpha = \beta + 1$, we can easily construct a $\Sigma_1(J_\alpha)$ map of $\omega\beta$ onto $\omega\alpha$, so

Theorem 39 yields the required contradiction. Assume $\lim(\alpha)$. By Theorem 40, there must be a $\tau < \omega\alpha$ and a $\Sigma_1(J_\alpha)$ map f of τ cofinally into $\omega\alpha$. Let f be $\Sigma_1^{J_\alpha}(\{p\})$. Pick $\gamma < \alpha$ with $\tau, p \in J_\gamma$. Let $h = h_\alpha$ be the canonical Σ_1 skolem function for J_α. Set $X = h''(\omega \times J_\gamma)$. As J_γ is closed under ordered pairs, lemma 35 tells us that $X \prec_{\Sigma_1} J_\alpha$. Let $\pi : X \cong J_\beta$. Thus $\pi \upharpoonright J_\gamma = \mathrm{id} \upharpoonright J_\gamma$. By an argument as in Theorem 39, $\pi \upharpoonright X = \mathrm{id} \upharpoonright X$, so $X = J_\beta$. Now, f is $\Sigma_1^{J_\alpha}(\{p\})$ and $p \in X \prec_{\Sigma_1} J_\alpha$, so X is closed under f. But $\tau \subset X$ and so $f''\tau \subset X$, which means, since $f''\tau$ is cofinal in $\omega\alpha$ and $X = J_\beta$ is transitive, that $\omega\alpha \subset J_\beta$. Thus $\beta = \alpha$, and $X = J_\alpha$. Define a $\Sigma_1(J_\alpha)$ map $\tilde{h} : \omega \times \tau \times J_\gamma \to J_\alpha$ as follows. Let H be a $\Sigma_0^{J_\alpha}$ relation such that

$$y = h(i,x) \leftrightarrow (\exists t \in J_\alpha) H(t,i,x,y).$$

Set
$$\tilde{h}(i,\nu,x) = \begin{cases} y, & \text{if } (\exists t \in S_{f(\nu)}) H(t,i,x,y) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\tilde{h}$ is total on $\omega \times \tau \times J_\gamma$, and $\tilde{h}''(\omega \times \tau \times \{x\}) = h''(\omega \times \{x\})$ for any x, as $f''\tau$ is cofinal in $\omega\alpha$. Hence $\tilde{h}''(\omega \times \tau \times J_\gamma) = X = J_\alpha$. By Theorem 39 there is $g \in J_\alpha$, $g : \omega\gamma \xrightarrow{\text{onto}} \omega \times \tau \times J_\gamma$. Then $\tilde{h} \circ g$ is a $\Sigma_1(J_\alpha)$ map of $\omega\gamma$ onto J_α, contrary to (iii).

Theorem 43

The following are equivalent:

(i) $\omega\alpha$ is strongly admissible.

(ii) $\langle J_\alpha, A\rangle$ is amenable for all $A \in \Sigma_1(J_\alpha)$.

(iii) There is no $\Sigma_1(J_\alpha)$ function mapping a bounded subset of $\omega\alpha$ onto J_α.

Proof: (i) $\to$ (ii) $\to$ (iii). Similar to the above.

(iii) → (i). Assume (iii) ∩ ¬ (i) and proceed much as before. So, we assume $\lim(\alpha)$, f is (by Theorem 41) a $\Sigma_1(J_\alpha)$ map of some $a \subset \tau < \omega\alpha$ cofinally into $\omega\alpha$, $f \in \Sigma_1^{J_\alpha}(\{p\})$, and $\tau < \omega\gamma$, $p \in J_\gamma$, $\gamma < \alpha$. As before, if $h = h_\alpha$ and $X = h''(\omega \times J_\gamma)$, then $X = J_\alpha$. Now, since we do not need to bother about functions being total, we can easily contradict (iii). By Theorem 39, let $g \in J_\alpha$, $g : \omega\gamma \xrightarrow{\text{onto}} \omega \times J_\gamma$. Set $\bar{f}(\nu) \simeq h(g(\nu))$. Then $\bar{f}$ is a $\Sigma_1(J_\alpha)$ map of a subset of $\omega\gamma$ onto J_α.

Note that an immediate corollary of Theorem 42 is:

<u>Theorem 44</u>

If $\varkappa$ is a cardinal, then $\varkappa$ is an admissible ordinal.

Using admissibility theory, we can give a quick proof that $L = \bigcup_{\alpha \in OR} J_\alpha$.

<u>Theorem 45</u>

If $\omega\alpha$ is admissible, then $J_\alpha = L_{\omega\alpha}$.

<u>Proof</u>: If $\alpha = 1$, then $J_1 = L_\omega$ = the hereditarily finite sets. Assume $\alpha > 1$. Thus $\omega \in J_\alpha$. Since J_α is admissible, the recursion theorem tells us that $\text{rud}(x) = \bigcup_{n<\omega} \underset{\sim}{S}^n(x)$ is $\Sigma_1(J_\alpha)$. But if u is transitive, then $\Sigma_\omega(u) = \mathcal{P}(u) \cap \text{rud}(u)$. Hence the map $L_\gamma \mapsto \Sigma_\omega(L_\gamma) = L_{\gamma+1}$ is $\Sigma_1(J_\alpha)$ $(\gamma < \omega\alpha)$. So, by the recursion theorem again, we see that $\langle L_\nu \mid \nu < \omega\alpha \rangle$ is $\Sigma_1(J_\alpha)$. Hence $L_{\omega\alpha} = \bigcup_{\nu<\omega\alpha} L_\nu \subset J_\alpha$. For the converse inclusion, it suffices to show that $L_{\omega\alpha}$ is admissible. (For then, by the recursion theorem, $\langle S_\nu \mid \nu < \omega\alpha \rangle$ is $\Sigma_1(L_{\omega\alpha})$, so $J_\alpha = \bigcup_{\nu<\omega\alpha} S_\nu \subset L_{\omega\alpha}$.)

Let R be $\Sigma_0(L_{\omega\alpha})$, $x \in L_{\omega\alpha}$, and assume $(\forall z \in x)\exists y R(y,z)$. Since $\langle L_\nu | \nu < \omega\alpha \rangle$ is $\Sigma_1(J_\alpha)$, we may define a $\Sigma_1(J_\alpha)$ predicate R' by $R'(\nu,z) \leftrightarrow z \in x \wedge (\exists y \in L_\nu)R(y,z)$. Since J_α is admissible, there is $\tau < \omega\alpha$ with $(\forall z \in x)(\exists \nu < \tau)R'(\nu,z)$. Hence $(\forall z \in x)(\exists y \in L_\tau)R(y,z)$. So as $L_\tau \in L_{\omega\alpha}$, $L_{\omega\alpha}$ satisfies the Σ_0-Replacement axiom. Since $\lim(\omega\alpha)$, it follows easily that $L_{\omega\alpha}$ is admissible.

Let $\alpha, n \geq 0$. The <u>Σ_n-projectum</u> of α, ρ_α^n , is the largest $\rho \leq \alpha$ such that $\langle J_\rho, A \rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \mathcal{P}(J_\rho)$.

Roughly speaking, our reason for introducing the Σ_n projectum is this. We have seen that, for example, we can reasonably handle $\Sigma_n(J_\alpha)$ predicates when J_α is Σ_n-admissible. This is because Σ_n-admissibility is a sort of "hardness" condition on J_α for Σ_n predicates. For, if we take an arbitrary J_α, it may be "soft" for $\Sigma_n(J_\alpha)$ predicates; we may, for instance, find $\Sigma_n(J_\alpha)$ subsets of members of J_α which are not themselves members of J_α, or even $\Sigma_n J_\alpha)$ functions which <u>project</u> a subset of a member of J_α onto all of J_α. But if J_α is Σ_n-admissible, none of these situations can arise. Thus, we try to isolate that part of J_α which <u>is</u> "hard" for $\Sigma_n(J_\alpha)$ predicates, a sort of "Σ_n-admissible core" of J_α. One natural way of formalising these ideas is provided by the Σ_n projectum. Clearly, $J_{\rho_\alpha^n}$ is a reasonable interpretation of the notion of a "Σ_n-hard core" of J_α. We shall eventually give two characterisations of the Σ_n-projectum which make it appear even more reasonable - if not inevitable. One of these is that ρ_α^n is the smallest $\rho \leq \alpha$ for which there is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\rho$ onto J_α. Then, since we clearly have, for $\omega\alpha$ admissible, that $\omega\alpha$ is strongly admissible iff $\rho_\alpha^1 = \alpha$, we obtain some justification for our alternative name of "non-projectible admissible" for strong admisibility.

It is convenient, at this point, for us to define an obvious generalisation of the notion of the Σ_n-projectum of an ordinal.

Let $\langle J_\alpha, A\rangle$ be amenable. The <u>Σ_n-projectum</u> of $\langle J_\alpha, A\rangle$, $\rho_{\alpha,A}^n$, is the largest $\rho \leq \alpha$ such that $\langle J_\rho, B\rangle$ is amenable for all $B \in \Sigma_n(\langle J_\alpha, A\rangle)$.

Note that by Theorem 43, $\omega\rho_{\alpha,A}^n$ is always strongly admissible.

We shall make strong use of the Σ_n-projectum in proving that every J_α is Σ_n-uniformisable, all $n \geq 1$. Since most of the following lemmas are directed towards this goal, it is worth indicating briefly our strategy.

We already know that $\langle J_\delta, A\rangle$ is Σ_1-uniformisable for all $\langle J_\delta, A\rangle$. What we shall do is attempt to "reduce" $\Sigma_n(J_\alpha)$ predicates to $\Sigma_1(\langle J_{\rho_\alpha^n}, A\rangle)$ predicates for some $A \subset J_{\rho_\alpha^n}$ which is itself $\Sigma_n(J_\alpha)$. To carry out this reduction, we need to have at our disposal a $\Sigma_n(J_\alpha)$ map of a subset (at least) of $J_{\rho_\alpha^n}$ onto J_α. Thus, what we shall do is to simultaneously prove, by induction on n,α, the following two propositions:

(P 1) J_α is Σ_{n+1}-uniformisable

(P 2) There is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\rho_\alpha^n$ onto J_α.

The proof of (P 1) goes roughly as follows. Let R be a $\Sigma_{n+1}(J_\alpha)$ predicate on J_α. Let $f : \subset \omega\rho_\alpha^n \xrightarrow{\text{onto}} J_\alpha$ be $\Sigma_n(J_\alpha)$. Now, f^{-1} is a $\Sigma_n(J_\alpha)$ relation, so by assuming Σ_n-uniformisability, f^{-1} can be "shrunk" to a $\Sigma_n(J_\alpha)$ map of J_α into $\omega\rho_\alpha^n$. This reduces R to a $\Sigma_{n+1}(J_\alpha)$ predicate R' on $J_{\rho_\alpha^n}$.

Now find a $\Sigma_n(J_\alpha)$ predicate $A \subset J_{\rho_\alpha^n}$ such that R' is in fact $\Sigma_1(\langle J_{\rho_\alpha^n}, A\rangle)$. Uniformise R' by a $\Sigma_1(\langle J_{\rho_\alpha^n}, A\rangle)$ function, and then reverse the procedure to recover a $\Sigma_{n+1}(J_\alpha)$ uniformising function for R. There is one doubtful point in the above outline. Can we in fact find a set A as required. That we can has to be proved as we proceed, so we shall in fact simultaneously prove <u>three</u> propositions, (P 1), (P 2), and a proposition (P 3) to be formulated precisely later.

<u>Lemma 46</u>

Let $n \geq 1$, and assume J_α is Σ_n-uniformisable. Let $\gamma \leq \alpha$ be the least ordinal such that $\mathcal{P}(\omega\gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$. Then there is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\gamma$ onto J_α.

<u>Proof</u>: By lemma 33, J_α has a Σ_n skolem function, h. Let h be $\Sigma_n^{J_\alpha}(\{p\})$. We may assume p is the $<_J$-least element of J_α for which such an h exists.

Let $a \subset \omega\gamma$, $a \in \Sigma_n(J_\alpha)$, $a \notin J_\alpha$. Let q be the $<_J$-least element of J_α such that $a \in \Sigma_n^{J_\alpha}(\{q\})$. Define $\tilde{h}$ by $\tilde{h}(i,x) \simeq h(i,\langle x,p,q\rangle)$. It is easily seen that $\tilde{h}$ is a Σ_n skolem function for J_α and that $\langle p,q\rangle$ is a good parameter for $\tilde{h}$.

Set $X = \tilde{h}''(\omega \times J_\gamma)$. Now, there is a $\Sigma_1(J_\gamma)$ map $g : \omega\gamma \xrightarrow{\text{onto}} J_\gamma$, so $\tilde{h} \circ g$ is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\gamma$ onto X. Hence it suffices to show that $X = J_\alpha$.

Clearly, $X \prec_{\Sigma_n} J_\alpha$. Let $\pi : X \cong J_\beta$, $\beta \leq \alpha$. Then $\pi \upharpoonright J_\gamma = \mathrm{id} \upharpoonright J_\gamma$, so in particular, $\pi''a = a$. Also $\pi''a$ is $\Sigma_n^{J_\beta}(\{\pi(q)\})$. But look, this implies that $a = \pi''a \in J_{\beta+1}$. Hence we must have $\beta = \alpha$ (and here we have used our hypothesis that $\mathcal{P}(\omega\gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$!). Thus, in particular, $a = \pi''a$ is

$\Sigma_n^{J_\alpha}(\{\pi(q)\})$, so by the choice of q, we see that $\pi(q) = q$. Again, it is easy to see that $h' = \pi \circ h \circ \pi^{-1}$ is a $\Sigma_n^{J_\alpha}(\{\pi(p)\})$ Σ_n skolem function for J_α, so by choice of p, $\pi(p) = p$. But then h, h' are both defined by the same Σ_n formula (with parameter p) in J_α, so $h = h'$. It follows immediately that $\pi \circ \tilde{h} \circ \pi^{-1} = \tilde{h}$, of course. So for $i \in \omega$, $x \in J_\gamma$, $\pi \circ \tilde{h}(i,x) \simeq \tilde{h} \circ \pi(i,x) \simeq \tilde{h}(i,x)$. Thus $\pi \restriction X = \mathrm{id} \restriction X$, and $X = J_\alpha$.

Lemma 46 plays a direct part in the proof of (P 1)-(P 3). The next lemma, however, is only used during the proof of the lemma which follows it, and may, on first sight, appear somewhat uninspiring.

Lemma 47

Let $\langle J_\alpha, A\rangle$ be amenable, $\rho = \rho^1_{\alpha,A}$. If $B \subset J_\rho$ is $\Sigma_1(\langle J_\alpha, A\rangle)$, then $\Sigma_1(\langle J_\rho, B\rangle) \subset \Sigma_2(\langle J_\alpha, A\rangle)$.

Proof: Case 1. There is a $\Sigma_1(\langle J_\alpha, A\rangle)$ map of some $\gamma < \omega\rho$ cofinally into $\omega\alpha$.

Let g be such a map, and let $\bar{B}$ be $\Sigma_0(\langle J_\alpha, A\rangle)$ such that $B(x) \leftrightarrow \exists z \bar{B}(z,x)$ for each $x \in J_\rho$. Define B' by $B'(\langle\nu,x\rangle) \leftrightarrow (\exists z \in S_{g(\nu)})\bar{B}(z,x)$, for $\nu \in \gamma$, $x \in J_\rho$. Thus B' is $\Delta_1(\langle J_\alpha, A\rangle)$. And since $B(x) \leftrightarrow (\exists \nu \in \gamma)B'(\langle\nu,x\rangle)$, $\Sigma_1(\langle J_\rho, B\rangle) \subset \Sigma_1(\langle J_\rho, B'\rangle)$. Thus, we need only prove that $\Sigma_1(\langle J_\rho, B'\rangle \subset \Sigma_2(\langle J_\alpha, A\rangle)$. It clearly suffices to prove that $\Sigma_0(\langle J_\rho, B'\rangle) \subset \Sigma_2(\langle J_\alpha, A\rangle)$.
Let R be $\Sigma_0(\langle J_\rho, B'\rangle)$. Thus R is rud in B' and some parameter $p \in J_\rho$. By choice of ρ, $\langle J_\rho, B'\rangle$ is amenable, so by lemmas 3 and 4, there is a $\Sigma_0(J_\rho)$ predicate P and functions $f_1,\ldots,f_{m+k}$, rud in parameter p, such that $R(\vec{x}) \leftrightarrow$ $P(\vec{x}, f_1(\vec{x}),\ldots,f_m(\vec{x}), B' \cap f_{m+1}(\vec{x}),\ldots,B' \cap f_{m+k}(\vec{x}))$. Hence

$R(\vec{x}) \leftrightarrow \exists y_1, \ldots \exists y_k[y_1 = B' \cap f_{m+1}(\vec{x}) \wedge \ldots \wedge y_k = B' \cap f_{m+k}(\vec{x}) \wedge P(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x}), y_1, \ldots, y_k)]$. Now P is certainly $\Sigma_0(J_\alpha)$, and $f_1, \ldots f_{m+k}$ are rud in parameter p, so it suffices to show that the function $b(u) = B' \cap u$ is $\Sigma_2(\langle J_\alpha, A\rangle)$. It is in fact $\Pi_1(\langle J_\alpha, A\rangle)$, because: $y = b(u) \leftrightarrow \forall x[x \in y \leftrightarrow x \in u \wedge B'(x)]$, and B' is $\Delta_1(\langle J_\alpha, A\rangle)$.

Case 2. Otherwise.

As before, we must show that $\Sigma_0(\langle J_\rho, B\rangle) \subset \Sigma_2(\langle J_\alpha, A\rangle)$. Again as before, this reduces, by the amenability of $\langle J_\rho, B\rangle$, to proving that the function $b(u) = B \cap u$ is $\Sigma_2(\langle J_\alpha, A\rangle)$ on J_ρ. Now, we clearly have

$$y = b(u) \leftrightarrow (\forall x \in y)(x \in u \wedge B(x)) \wedge (\forall x \in u)(B(x) \rightarrow x \in y).$$

Now, the second conjunct here is $\Pi_1(\langle J_\alpha, A\rangle)$. We show that the first conjunct is $\Sigma_1(\langle J_\alpha, A\rangle)$, which is sufficient. It reduces to showing that $(\forall x \in y)B(x)$ is $\Sigma_1(\langle J_\alpha, A\rangle)$. But look, we know that Case 1 fails to hold, so this is proved just as in lemma 14.

The next lemma is the key step involved in proving, by induction, the as yet unformulated (P 3).

Lemma 48

Let $\langle J_\alpha, A\rangle$ be amenable, $\rho = \rho^1_{\alpha,A}$. Suppose there is a $\Sigma_1(\langle J_\alpha, A\rangle)$ map of a subset of $\omega\rho$ onto J_α. Then there is a $B \subset J_\rho$, $B \in \Sigma_1(\langle J_\alpha, A\rangle)$, such that $\Sigma_n(\langle J_\rho, B\rangle) = \mathcal{P}(J_\rho) \cap \Sigma_{n+1}(\langle J_\alpha, A\rangle)$ for all $n \geq 1$.

Proof: Let $u \subset \omega\rho$, and let $f : u \xrightarrow{\text{onto}} J_\alpha$ be $\Sigma_1(\langle J_\alpha, A\rangle)$.
Pick $p \in J_\alpha$ such that f is $\Sigma_1^{\langle J_\alpha, A\rangle}(\{p\})$. Let $\langle \varphi_i | i < \omega\rangle$ be a recursive enumeration of Fml^{Σ_1}.

$$B = \{\langle i,x\rangle \mid i \in \omega \wedge x \in J_\rho \wedge \models^{\Sigma_1}_{\langle J_\alpha,A\rangle} \varphi_i[x,p]\}.$$

Now, $\langle J_\alpha,A\rangle$ is amenable, and hence rud closed, so by lemma 9, $B \in \Sigma_1(\langle J_\alpha,A\rangle)$. And of course $B \subset J_\rho$.

Commencing with lemma 47, an easy induction shows that for all $n \geq 1$, $\Sigma_n(\langle J_\rho,B\rangle) \subset \Sigma_{n+1}(\langle J_\alpha,A\rangle)$.

For the converse, let $R(\vec{x})$ be a $\Sigma_{n+1}(\langle J_\alpha,A\rangle)$ relation on J_ρ, $n \geq 1$. Assume, for the sake of argument, that n is even. Let P be a $\Sigma_1(\langle J_\alpha,A\rangle)$ relation such that, for $\vec{x} \in J_\rho$, $R(\vec{x}) \leftrightarrow \exists y_1 \forall y_2 \ldots \forall y_n P(\vec{y},\vec{x})$. Define $\tilde{P}$ by $\tilde{P}(\vec{z},\vec{x}) \leftrightarrow [\vec{z},\vec{x} \in J_\rho \wedge P(f(\vec{z}),\vec{x})]$. By choice of f, any $x \in J_\alpha$ is $\Sigma_1^{\langle J_\alpha,A\rangle}(\{p,\nu\})$ for some $\nu < \omega\rho$, so by definition of B, $\tilde{P}$ is rud in B and some parameter $\nu < \omega\rho$. In particular, $\tilde{P}$ is $\Delta_1(\langle J_\rho,B\rangle)$.
Again, $D = \mathrm{dom}(f)$ is rud in B and some parameter $\tau < \omega\rho$, so D is also $\Delta_1(\langle J_\rho,B\rangle)$.
But for $\vec{x} \in J_\rho$, $R(\vec{x}) \leftrightarrow (\exists z_1 \in D)(\forall z_2 \in D) \ldots (\forall z_n \in D)\tilde{P}(\vec{z},\vec{x})$, which is thus $\Sigma_n(\langle J_\rho,B\rangle)$.

We are now ready to formulate (P 3) and prove our promised uniformisation theorem.

Let α, $n \geq 0$. A <u>Σ_n master code</u> for J_α is a set $A \subset J_{\rho_\alpha^n}$, $A \in \Sigma_n(J_\alpha)$, such that whenever $m \geq 1$, $\Sigma_m(\langle J_{\rho_\alpha^n},A\rangle) = \mathcal{P}(J_{\rho_\alpha^n}) \cap \Sigma_{n+m}(J_\alpha)$.

<u>Theorem 49</u>

Let α, $n \geq 0$. Then:

(P 1) J_α is Σ_{n+1}-uniformisable.

(P 2) There is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\rho_\alpha^n$ onto J_α.

(P 3) J_α has a Σ_n master code.

<u>Proof</u>: We prove the theorem (for all n) by induction on α. For $\alpha = 0$, it is trivial. So assume $\alpha > 0$ and that (P 1)-(P 2) hold (for all n) for all $\beta < \alpha$. We prove (P 1)-(P 3) at α by induction on n.

<u>Case 1</u>: $n = 0$. (P 1) is already proved (Theorem 30)

(P 2) $\rho_\alpha^0 = \alpha$, so (P 2) is already proved (Theorem 39)

(P 3) Since $\rho_\alpha^0 = \alpha$, $A = \emptyset$ is a Σ_0 master code for J_α.

Case 2: $n = m + 1$, $m \geq 0$. Let $\rho = \rho_\alpha^n$ for convenience.

We first prove that ρ is the least ordinal such that some $\Sigma_n(J_\alpha)$ function maps a subset of $\omega\rho$ onto J_α.
To this end, let δ be the least such ordinal. Suppose first that $\delta < \rho$. Then $B = \{\xi \in \omega\delta \mid \xi \notin f(\xi)\}$ is a $\Sigma_n(J_\alpha)$ subset of J_ρ, so by definition of ρ , $\langle J_\rho, B\rangle$ is amenable. Thus, as $\delta < \rho$, $B = B \cap \omega\delta \in J_\rho \subset J_\alpha$. So $B = f(\xi)$ for some $\xi \in \omega\delta$, whence $\xi \in f(\xi) \leftrightarrow \xi \in B \leftrightarrow \xi \notin f(\xi)$, which is absurd. Hence $\rho \leq \delta$. Suppose $\rho < \delta$. By definition of ρ, this means that for some $\Sigma_n(J_\alpha)$ set $B \subset J_\delta$, $\langle J_\delta, B\rangle$ is not amenable. Since $\langle J_1, B\rangle$ must be amenable, $\delta > 1$. If $\delta = \gamma + 1$, then since there is a $\Sigma_1(J_\alpha)$ map of $\omega\gamma$ onto $\omega\delta$, there is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\gamma$ onto J_α, contrary to the choice of δ. Hence $\lim(\delta)$. It follows, since $\langle J_\delta, B\rangle$ is not amenable, that there is $\tau < \delta$ with $B \cap J_\tau \notin J_\delta$. By induction hypothesis, J_α is Σ_n-uniformisable. So as $\tau < \delta$, lemma 46 implies that $\mathcal{P}(\omega\tau) \cap \Sigma_n(J_\alpha) \subset J_\alpha$. But there is $h \in J_\alpha$, $h : \omega\tau \xrightarrow{\text{onto}} J_\tau$, so this implies $\mathcal{P}(J_\tau) \cap \Sigma_n(J_\alpha) \subset J_\alpha$. In particular, $B \cap J_\tau \in J_\alpha$. Hence for some $\beta < \alpha$, $B \cap J_\tau$ is J_β-definable. Let β be the least such, and let r be least such that $B \cap J_\tau$ is $\Sigma_r(J_\beta)$. By definition, $\langle J_{\rho_\beta^r}, B \cap J_\tau\rangle$ is

amenable, so if $\tau < \rho_\beta^r$, then $B \cap J_\tau = (B \cap J_\tau) \cap J_\tau \in J_{\rho_\beta^r} \subset J_\beta$, contrary to the choice of β. Hence $\tau \geq \rho_\beta^r$. By induction hypothesis, there is a $\Sigma_r(J_\beta)$ map g from a subset of $\omega\rho_\beta^r$ onto J_β. And since $B \cap J_\tau \in J_{\beta+1}$ and $B \cap J_\tau \notin J_\delta$, $\beta + 1 > \delta$, or $\beta \geq \delta$. Hence there is a $\Sigma_r(J_\beta)$ map g' from a subset of $\omega\rho_\beta^r$ onto $\omega\delta$. Then $f \circ g'$ is a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\rho_\beta^r$ onto J_α. But we have established that $\rho_\beta^r \leq \tau < \delta$, so this contradicts the choice of δ. Hence $\delta = \rho$.

(P 2) follows immediately from the above result of course.

We turn now to (P 3). By induction hypothesis, let A be a Σ_m master code for J_α. Set $\eta = \rho_\alpha^m$ for convenience.
By the above, let f be a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\rho$ onto J_α. By choice of A, $f' = f \restriction (f^{-1}{}''J_\eta)$ is a $\Sigma_1(\langle J_\eta, A\rangle)$ map of a subset of $\omega\rho$ onto J_η. By choice of A, it is clear that $\rho = \rho_\alpha^n = \rho_{\eta,A}^1$. Finally, of cource, $\langle J_\eta, A\rangle$ is amenable. So, we may apply lemma 48 to $\langle J_\eta, A\rangle$ to obtain a $\Sigma_1(\langle J_\eta, A\rangle)$ set $B \subset J_\rho$ such that $\Sigma_r(\langle J_\rho, B\rangle) = \mathcal{P}(J_\rho) \cap \Sigma_{r+1}(\langle J_\eta, A\rangle)$ for all $r \geq 1$. By choice of A, $B \in \Sigma_n(J_\alpha)$ and $\Sigma_r(\langle J_\rho, B\rangle) = \mathcal{P}(J_\rho) \cap \Sigma_{n+r}(J_\alpha)$ for all $r \geq 1$. Hence B is a Σ_n master code for J_α.

Finally we prove (P 1). Let B be, as above, a Σ_n master code for J_α. Let $R(y,\vec{x})$ be a $\Sigma_{n+1}(J_\alpha)$ relation on J_α. Define, with f as above, a relation $\tilde{R}$ on J_ρ by $\tilde{R}(y,\vec{x}) \leftrightarrow [y,\vec{x} \in J_\rho \wedge R(f(y), f(\vec{x}))]$. Then $\tilde{R}$ is $\Sigma_{n+1}(J_\alpha)$, and hence $\Sigma_1(\langle J_\rho, B\rangle)$. Let $\tilde{r}$ be a $\Sigma_1(\langle J_\rho, B\rangle)$ function uniformising $\tilde{R}$. Since f is $\Sigma_n(J_\alpha)$, so is f^{-1}. But J_α is Σ_n-uniformisable, by induction hypothesis, so we can let $\tilde{f}$ be a $\Sigma_n(J_\alpha)$ function uniformising

f^{-1}. Set $r = f \circ \tilde{r} \circ f^{-1}$. It is clear that r is a $\Sigma_{n+1}(J_\alpha)$ function which uniformises R. The proof is complete.

The above results give us two (intuitive) equivalent formulations of the Σ_n-projectum:

Theorem 50

Let $\alpha, n \geq 0$. Let δ be the least ordinal such that some $\Sigma_n(J_\alpha)$ function maps a subset of $\omega\delta$ onto J_α. Let γ be the least ordinal such that $\mathcal{P}(\omega\gamma) \cap \Sigma_n(J_\alpha) \not\subseteq J_\alpha$. Then $\delta = \gamma = \rho_\alpha^n$.

Proof: That $\delta = \rho_\alpha^n$ was actually proved during the proof of Theorem 49. Since we now know that J_α is Σ_n-uniformisable, lemma 46 tells us that $\delta \leq \gamma$. Assume $\delta < \gamma$. Now by definition, let $u \subset \omega\delta$, and let $f : u \xrightarrow{\text{onto}} J_\alpha$ be $\Sigma_n(J_\alpha)$. Let $Z = \{\xi \mid \xi \notin f(\xi)\}$. Then $Z \subset \omega\delta$ and $Z \in \Sigma_n(J_\alpha)$, so by definition of γ, $Z \in J_\alpha$. Thus $Z = f(\xi)$ for some ξ, so $\xi \in f(\xi) \longleftrightarrow \xi \notin f(\xi)$, which is absurd. Hence $\delta = \gamma$.

There is, of course, a concept which, for Δ_n predicates, plays the role that the Σ_n projectum plays for Σ_n predicates. And, as might be expected, there is a corresponding "total function" or Δ_n equivalent of Theorem 50 for this concept.

Let $\alpha, n \geq 0$. The Δ_n-projectum of α (sometimes called the weak Σ_n-projectum), η_α^n, is the largest $\eta \leq \alpha$ such that $\langle J_\eta, A\rangle$ is amenable for all $\Delta_n(J_\alpha)$ sets $A \subset J_\eta$.

Thus the Δ_n-projectum of α represent the "hard core" of J_α with regards to Δ_n predicates on J_α.
Clearly, $\eta_\alpha^n \geq \rho_\alpha^n$. We do not, however, necessarily have equality

here. For example, let α be the first admissible ordinal $> \omega$. Then it is easily seen that $\eta_\alpha^1 = \alpha$, whereas $\rho_\alpha^1 = \omega$.

Corresponding to lemma 46, we have:

Lemma 51

Let $n \geq 1$, and let γ be the least ordinal such that $\mathcal{P}(\omega\gamma) \cap \Delta_n(J_\alpha) \nsubseteq J_\alpha$. Then there is a $\Sigma_n(J_\alpha)$ (and hence $\Delta_n(J_\alpha)$) map of $\omega\gamma$ onto J_α.

Proof: Let $n = m+1$, $n \geq 0$. Since $\Sigma_m(J_\alpha) \subset \Delta_n(J_\alpha) \subset \Sigma_n(J_\alpha)$, Theorem 50 implies that $\rho_\alpha^n \leq \gamma \leq \rho_\alpha^n$. Theorem 50 also implies that there is a $\Sigma_m(J_\alpha)$ map of a subset of $\omega\rho_\alpha^m$ onto J_α. So, we can clearly define a $\Sigma_n(J_\alpha)$ map of $\omega\rho_\alpha^m$ itself onto J_α. This reduces our problem to showing that there is a $\Sigma_n(J_\alpha)$ map of $\omega\gamma$ onto $\omega\rho_\alpha^m$. As a first step, we have the:

Claim: There is a $\Sigma_n(J_\alpha)$ map g from $\omega\gamma$ cofinally into $\omega\rho_\alpha^m$.

Let A be a Σ_m master code for J_α. By hypothesis, let $b \subset \omega\gamma$, $b \in \Delta_n(J_\alpha)$, $b \notin J_\alpha$. By choice of A, b is $\Delta_1(\langle J_{\rho_\alpha^m}, A\rangle)$. Suppose b is in fact defined by:

$$\nu \in b \leftrightarrow \exists y B_0(y,\nu) \quad , \quad \nu \notin b \leftrightarrow \exists y B_1(y,\nu),$$

where B_0, B_1 are $\Sigma_0(\langle J_{\rho_\alpha^m}, A\rangle)$. Then

$$(\forall \nu \in \omega\gamma)\exists y[B_0(y,\nu) \vee B_1(y,\nu)].$$

But $\langle J_{\rho_\alpha^m}, A\rangle$ is amenable, and hence rud closed, so as $b \notin J_{\rho_\alpha^m}$, there can be no $\tau < \omega\rho_\alpha^m$ such that

$$(\forall \nu \in \omega\gamma)(\exists y \in S_\tau)[B_0(y,\nu) \vee B_1(y,\nu)].$$

Define $g : \omega\gamma \to \omega\rho_\alpha^m$ by

$$g(\nu) = \text{the least } \tau \text{ such that } (\exists y \in S_\tau)[B_0(y,\nu) \vee B_1(y,\nu)].$$

Clearly, g is $\Sigma_n(J_\alpha)$ and cofinal in $\omega\rho_\alpha^m$, proving the claim.

We now prove the lemma. Since $\rho_\alpha^n \leq \gamma$, there must be a $\Sigma_n(J_\alpha)$ map f from a subset of $\omega\gamma$ onto $\omega\rho_\alpha^m$ ($\leq \omega\alpha$), and of course such an f will then be $\Sigma_1(\langle J_{\rho_\alpha^m}, A\rangle)$. Define $\bar{f} : (\omega\gamma)^2 \xrightarrow{\text{onto}} \omega\rho_\alpha^m$ as follows. Let f be given by $f(\nu) = \tau \leftrightarrow \exists y F(y,\tau,\nu)$, where F is $\Sigma_0(\langle J_{\rho_\alpha^m}, A\rangle)$. Set

$$\bar{f}(\nu,\tau) = \begin{cases} \theta, & \text{if } (\exists y \in S_{y(\tau)})F(y,\theta,\nu) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\bar{f}$ is $\Sigma_1(\langle J_{\rho_\alpha^m}, A\rangle)$, and hence $\Sigma_n(J_\alpha)$. And $\bar{f}$ clearly maps $(\omega\gamma)^2$ onto $\omega\rho_\alpha^m$, as g is cofinal in $\omega\rho_\alpha^m$.

Since we have (by lemma 38) a $\Sigma_1(J_\alpha)$ map of $\omega\gamma$ onto $(\omega\gamma)^2$, the lemma follows.

Corresponding to Theorem 50, we have:

<u>Theorem 52</u>

Let $\alpha, n > 0$. Let δ be the least ordinal such that some $\Sigma_n(J_\alpha)$ (and hence $\Delta_n(J_\alpha)$) function maps $\omega\delta$ onto J_α. Let γ be the least ordinal such that $\mathcal{P}(\omega\gamma) \cap \Delta_n(J_\alpha) \not\subset J_\alpha$. Then $\delta = \gamma = \eta_\alpha^n$.

<u>Proof</u>: Suppose $\gamma < \eta_\alpha^n$. Let $B \subset \omega\gamma$, $B \in \Delta_n(J_\alpha)$, $B \notin J_\alpha$. Then $\omega\gamma \cap B = B \notin J_{\eta_\alpha^n}$, contrary to $\langle J_{\eta_\alpha^n}, B\rangle$ being amenable.

Suppose now that $\eta_\alpha^n < \gamma$. Then there is $A \subset J_\gamma$, $A \in \Delta_n(J_\alpha)$, such that $\langle J_\alpha, A\rangle$ is not amenable. In particular, $\gamma > 1$. Suppose $\gamma = \xi + 1$. There is then a $\Sigma_1(J_\gamma)$ map of $\omega\xi$ onto $\omega\gamma$, so by lemma 51 there is a $\Sigma_n(J_\alpha)$ map, f, of $\omega\xi$ onto J_α.

Then $Z = \{\iota \in \omega\xi \mid \iota \notin f(\iota)\}$ is a $\Delta_n(J_\alpha)$ subset of $\omega\xi$. Clearly, $Z \notin J_\alpha$, so this contradicts the choice of γ. Hence $\lim(\gamma)$. Thus as $\langle J_\gamma, A\rangle$ is not amenable, there must be $\tau < \gamma$ with $A \cap J_\tau \notin J_\gamma$. But by choice of γ, $\tau < \gamma \rightarrow A \cap J_\tau \in J_\alpha$, so for some $\theta < \alpha$, $A \cap J_\tau$ is J_θ-definable. Let θ be the least such. Then $A \cap J_\tau \in \mathcal{P}(J_\tau) \cap \Delta_m(J_\theta)$ for some $m \in \omega$, and $A \cap J_\tau \notin J_\theta$. Thus by lemma 51 there is a $\Sigma_m(J_\theta)$ map f of $\omega\tau$ onto J_θ. (Actually the hypotheses of lemma 51 require that we have a $\Delta_m(J_\theta)$ subset of $\omega\tau$ not in J_θ, whereas we have only exhibited a subset of J_τ with these properties. However, since there is available a $\Sigma_1(J_\tau)$ map of $\omega\tau$ onto J_τ, this point causes no problem.) Since $\theta < \alpha$, $f \in J_\alpha$. But $A \cap J_\tau \notin J_\gamma$, and $A \cap J_\tau \in J_{\theta+1}$, so $\theta \geq \gamma$, and there is thus a map $f' \in J_\alpha$ of $\omega\tau$ onto $\omega\gamma$. By lemma 51, again, this gives us a $\Sigma_n(J_\alpha)$ map k of $\omega\tau$ onto J_α. Then, clearly, $K = \{\iota \mid \iota \notin K(\iota)\}$ is a $\Delta_n(J_\alpha)$ subset of $\omega\tau$ not lying in J_α, contrary to $\tau < \gamma$.

Hence $\gamma = \eta^n_\alpha$. Now, by lemma 51, we have $\delta \leq \gamma$. Suppose $\delta < \gamma$. Let $f : \omega\delta \xrightarrow{\text{onto}} J_\alpha$ be $\Sigma_n(J_\alpha)$. Let $Z = \{\nu \mid \nu \notin f(\nu)\}$. Then $Z \in \mathcal{P}(\omega\delta) \cap \Delta_n(J_\alpha) - J_\alpha$. But this contradicts the choice of γ. Hence $\delta = \gamma$.

Remark: Lemmas 46 and 51 can be regarded as much sharper versions of the following, much earlier theorem of Putman:

> Suppose $\mathcal{P}(\gamma) \cap L_{\alpha+1} \not\subseteq L_\alpha$. Then $L_{\alpha+1}$ contains a well-ordering of γ of order type α. (For $\gamma \geq \omega$.)

Putman actually proved this result for the case $\rho = \omega$, but his proof works in the general case.

The methods described above have, of course, many uses. We give just one, very general, example, showing that (in certain cirumstances) it is possible to carry out Löwenheim-Skolem arguments for non-regular ordinals α which can generally only be done when α is actually a regular cardinal.

More precisely, the following theorem, is well-known:

Theorem 53

Let $\varkappa$ be a regular cardinal. Let $\gamma < \varkappa \leq \omega\beta$, and suppose that $Y \subset J_\beta$, $|Y| < \varkappa$. Then there is $X \prec J_\beta$ such that $Y \cup \gamma \subset X$ and $\varkappa \cap X \in \varkappa$.

To prove this, one simply forms an ω-chain $X_0 \prec X_1 \prec \dots \prec X_n \prec \dots \prec J_\beta$ of elementary submodels of J_β, taking X_0 as the skolem hull of $Y \cup \gamma$ in J_β, and X_{n+1} as the skolem hull of $X_n \cup \sup(\varkappa \cap X_n)$ in J_β, and then $X = \bigcup_{n<\omega} X_n$ is the required submodel of J_β. By construction, $\varkappa \cap X$ is transitive, and hence an ordinal, and since $\varkappa$ is regular, $|X| < \varkappa$, so $\varkappa \cap X \in \varkappa$.
It should be observed that $\varkappa$ being regular is a necessary condition for the above procedure to work (in general).
However, providing we can, in some way, ensure that for each n, $\sup(\varkappa \cap X_n) < \varkappa$, then we can, of course, get by with just $\mathrm{cf}(\varkappa) > \omega$. The theorem below shows that, in certain cases we can do just this, providing we relax our demands somewhat.

Let $n \geq 1$, $\alpha \leq \omega\beta$. We say that α is Σ_n-regular at β iff there is no $\Sigma_n(J_\beta)$ map of a bounded subset of α cofinally into α.

For example, by Theorem 43, $\omega\alpha$ is strongly admissible iff $\omega\alpha$ is Σ_1-regular at α.

Theorem 54

Let $n \geq 1$, $\omega\beta \geq \alpha \geq 1$. Suppose α is Σ_n-regular at β. Let $Y \subset J_\beta$, $\omega \leq |Y| < \mathrm{cf}(\alpha)$, and let $\gamma < \alpha$. Then there is an

$X \prec_{\Sigma_n} J_\beta$ such that $Y \cup \gamma \subset X$ and $\alpha \cap X \in \alpha$.

Proof: Since $cf(\alpha) > \omega$, it clearly suffices to prove that, under the stated hypotheses, there is $X \prec_{\Sigma_n} J_\beta$ such that $Y \cup \gamma \subset X$ and $\sup(\alpha \cap X) < \alpha$.

Let h be a Σ_n skolem function for J_β (by Theorem 49 and lemma 33). Since $\omega \leq |Y| < cf(\alpha)$, we may, without loss of generality, assume that Y is closed under ordered pairs. Furthermore, let Φ be the function defined in lemma 37. Since α is Σ_n-regular at β, α is certainly strongly admissible. Hence, by lemma 37, $\{\xi \in \alpha \mid \Phi''\xi^2 \subset \xi\}$ is unbounded in α. It follows that we may also, without loss of generality, assume that $\Phi''\gamma^2 \subset \gamma$. Recall that $\Phi \upharpoonright \gamma^2$ is $\Sigma_1^{J_\beta}$.

Let $X = h''(\omega \times (Y \times \gamma))$. Then we claim that X is closed under ordered pairs. To see this, let $x_1, x_2 \in X$, say $x_1 = h(i_1, \langle y_1, \nu_1\rangle)$, $x_2 = h(i_2, \langle y_2, \nu_2\rangle)$. Let $y = \langle y_1, y_2\rangle \in Y$ and $\nu = \Phi(\langle \nu_1, \nu_2\rangle) \in \gamma$. Clearly $\{\langle x_1, x_2\rangle\}$ is $\Sigma_1^{J_\beta}(\{p, \langle y, \nu\rangle\})$, where p is a good parameter for h. Thus for some $i \in \omega$, $\langle x_1, x_2\rangle = h(i, \langle y, \nu\rangle) \in X$, as required. So, by corollary 36, $X \prec_{\Sigma_n} J_\beta$. And of course, we clearly have $Y \cup \gamma \subset X$. We show that $\sup(\alpha \cap X) < \alpha$.

For $y \in Y$, $i \in \omega$, define $h_{i,y} : \subset \gamma \to \alpha$ by $h_{i,y}(\nu) \simeq h(i, \langle y, \nu\rangle)$. Thus $h_{i,y}$ is $\Sigma_n(J_\beta)$, and so as α is Σ_n-regular at β, $\sup(h_{i,y}''\gamma) \simeq \gamma(i,y) < \alpha$. Since $|Y| < cf(\alpha)$, it follows that $\sup_{y \in Y} \gamma(i,y) \simeq \gamma(i) < \alpha$. Since $cf(\alpha) > \omega$, we conclude finally that $\sup_{i \in \omega} \gamma(i) < \alpha$. But clearly, $\sup_{i \in \omega} \gamma(i) = \sup(\alpha \cap X)$, so we are done.

The above lemma may be used to prove that if $V = L$ and $\varkappa$ is a regular uncountable, non-weakly compact cardinal, then there is a Souslin $\varkappa$-tree. (Jensen.)

Footnote for Page 1

(1) Since we wrote this paper, a slightly revised version of these notes has been published as a research paper. See R.B. Jensen, "The Fine Structure of the Constructible Hierarchy", Annals of Mathematical Logic, Vol 4 [1972], p 227. The present paper represents a lengthy discourse on an expansion of the earlier parts of Jensen's paper, and it is hoped that the somewhat more leisurely pace adopt here (as opposed to Jensen's paper) will be of benefit to those not predominantly interested in the set theoretical consequences of the Fine Structure Theory. For those who are so inclined, the notation we use is almost identical to that of Jensen, so this paper should provide a good introduction to Jensen's.