Kurepa's Hypothesis and the Continuum

by

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Abstract

Silver [5] proved that Con(ZFC + "there is an inaccessible cardinal") implies Con(ZFC + CH + "there are no Kurepa trees"). In order to obtain this result, he generically collapses an inaccessible cardinal to ω_2. Hence CH necessarily holds in his final model. In this paper we sketch Silver's proof, and then show how it can be modified to obtain a model in which there are no Kurepa trees and the continuum is anything we wish.

Introduction

We work in ZFC and use the usual notation and conventions. For details concerning the forcing theory we require, see Jech [3] or Shoenfield [4]. A tree is a poset $\underset{\sim}{T} = \langle T, \leq_T \rangle$ such that $\hat{x} = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$ for any $x \in T$. The order-type of $\hat{x}$ is the height of x in $\underset{\sim}{T}$, $ht(x)$. The α'th level of $\underset{\sim}{T}$ is the set $T_\alpha = \{x \in T \mid ht(x) = \alpha\}$. $\underset{\sim}{T}$ is an ω_1-tree iff : (i) $(\forall \alpha < \omega_1)(T_\alpha \neq \emptyset)$ & $(T_{\omega_1} = \emptyset)$; (ii) $(\forall \alpha < \beta < \omega_1)(\forall x \in T_\alpha)(\exists y_1, y_2 \in T_\beta)(x <_T y_1, y_2 \ \& \ y_1 \neq y_2)$; (iii) $(\forall \alpha < \omega_1)(\forall x, y \in T_\alpha)(\lim(\alpha) \rightarrow [x = y \leftrightarrow \hat{x} = \hat{y}])$; (iv) $(\forall \alpha < \omega_1)(|T_\alpha| \leq \omega)$ & $|T_0| = 1$. For further details of ω_1-trees, see Jech [2].

If $\underset{\sim}{T}$ is an ω_1-tree, a branch of $\underset{\sim}{T}$ is a maximal totally ordered subset of $\underset{\sim}{T}$. A branch b of $\underset{\sim}{T}$ is cofinal if $(\forall\alpha < \omega_1)(T_\alpha \cap b \neq \emptyset)$. $\underset{\sim}{T}$ is Kurepa if it has at least ω_2 cofinal branches. If $V = L$, then there is a Kurepa tree. This result is due to Solovay. For a proof, see Devlin [1] or Jech [2]. More generally, if $V = L[A]$, where $A \subseteq \omega_1$, then there is a Kurepa tree, from which it follows that if there are no Kurepa trees, then ω_2 is inaccessible in L. (All of this is still due to Solovay, and is proved in [1] and [2].). Hence, in order to establish Con(ZFC + K), where K denotes the statement "there are no Kurepa trees", one must at least assume Con(ZFC + I), where I denotes the statement "there is an inaccessible cardinal".

Now, if M is any cardinal absolute extension of L, and if $\underset{\sim}{T}$ is a Kurepa tree in L, then $\underset{\sim}{T}$ will clearly be a Kurepa tree in M. Hence, if $\varkappa$ is any cardinal of cofinality greater than ω, we can, by standard arguments, find a generic extension of L, with the same cardinals as L, such that, in the extension, there is a Kurepa tree and $2^\omega = \varkappa$. Johnsbråten has pointed out that the consistency of $K + 2^\omega = \varkappa$ (for such $\varkappa$) is not so easily obtained. Now, Silver [5] has shown that Con(ZFC + I) $\rightarrow$ Con(ZFC + $2^\omega = \omega_1$ + K). (And by Solovay's result above, the hypothesis here is as weak as possible). However, the method Silver employs necessarily makes $2^\omega = \omega_1$ hold, so as it stands the only hope to obtain $K + 2^\omega = \varkappa$ would seem to be to take Silver's model and blow-up the continuum generically to $\varkappa$. In fact this procedure does work (i.e. K is preserved), but the proof that it does is fairly delicate, as opposed to the corresponding argument for $\neg K$. Since we shall need all of the tricks employed by Silver in his proof of Con(ZFC + K), we may as well commence by describing his argument.

Silver's Model.

We shall use M to denote an arbitrary countable transitive model (c.t.m.) of ZFC throughout. By poset, we mean, as usual in forcing, a poset P, with a maximum element $\mathbb{1}$, such that every $p \in P$ has at least two incompatible extensions in P, where $p,q \in P$ are compatible, written $p \sim q$, if there is $r \in P$ such that $r \leq p,q$. We say P satisfies the κ chain condition (κ-c.c.), for κ an uncountable cardinal, if there is no pairwise incompatible subset of P of cardinality κ. P is σ-closed if whenever $\langle p_\alpha \mid \alpha < \lambda < \omega_1 \rangle$ is a decreasing sequence from P there is $p \in P$ such that $p \leq p_\alpha$ for all $\alpha < \lambda$. The following lemmas are standard. (See Shoenfield [4] for example.)

Lemma 1 (Cohen; Solovay)

Let P be a poset in M, κ an uncountable regular cardinal in M. Let G be M-generic for P.

(i) If $M \models$ "P satisfies the κ-c.c." then $\lambda \geq \kappa$ is a cardinal in M[G] iff λ is a cardinal in M.

(ii) If $M \models$ " P is σ-closed", then for all $\lambda < \omega_1$, $(M^\lambda)^M = (M^\lambda)^{M[G]}$, so in particular, $\omega_1^M = \omega_1^{M[G]}$ and $\mathcal{P}^M(\omega) = \mathcal{P}^{M[G]}(\omega)$.

<u>Lemma 2</u> (Lévy)

Let $\varkappa$ be an inaccessible cardinal in M , P a poset in M such that $M \models "|P| < \varkappa"$. If G is M-generic for P , then $\varkappa$ is still inaccessible in $M[G]$.

<u>Lemma 3</u> (Solovay)

Let P_1, P_2 be posets in M. If G_1 is M-generic for P_1 and G_2 is $M[G_1]$-generic for P_2 , then G_1 is $M[G_2]$-generic for P_1, G_2 is M-generic for P_2 , $G_1 \times G_2$ is M-generic for $P_1 \times P_2$, and $M[G_1][G_2] = M[G_2][G_1] = M[G_1, G_2] = M[G_1 \times G_2]$, where $P_1 \times P_2$ is the cartesian product of P_1 and P_2 with the partial ordering $\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle \leftrightarrow p_1 \leq_1 q_1 \ \& \ p_2 \leq_2 q_2$. Conversely, if G is M-generic for $P_1 \times P_2$, then $G_1 = \{p \mid \langle p, \mathbb{1}\rangle \in G\}$ is M-generic for P_1, $G_2 = \{q \mid \langle \mathbb{1}, q\rangle \in G\}$ is $M[G_1]$-generic for P_2 , and $G = G_1 \times G_2$.

Let $\varkappa$ be an uncountable cardinal. The poset $P(\varkappa)$ is defined as follows. An element p of $P(\varkappa)$ is a countable function such that $dom(p) \subseteq \omega_1 \times \varkappa$ and $ran(p) \subseteq \varkappa$, and if $\langle \alpha, \delta\rangle \in dom(p)$, then $p(\alpha, \delta) \in \delta$. The ordering on $P(\varkappa)$ is defined by $p \leq q \leftrightarrow p \supseteq q$. If $P = P(\varkappa)$ and $\lambda < \varkappa$, we set $P_\lambda = \{p \upharpoonright (\omega_1 \times \lambda) \mid p \in P\}$, $P^\lambda = \{p - p \upharpoonright (\omega_1 \times \lambda) \mid p \in P\}$, and regard P_λ, P^λ as posets in the obvious manner. Clearly, $P \cong P_\lambda \times P^\lambda$, by a canonical isomorphism.

<u>Lemma 4</u> (Lévy)

Let $\varkappa$ be an inaccessible cardinal in M_1 , and set $P = [P(\varkappa)]^M$. Then, $M \models "P$ is σ-closed and satisfies the $\varkappa$-c.c.". If G is M-generic for P , then $\omega_1^M = \omega_1^{M[G]}$ and $\varkappa = \omega_2^{M[G]}$. Furthermore, if $\lambda < \varkappa$ is an uncountable regular cardinal in M , then $M[G \cap P_\lambda] \models " P^\lambda$ is σ-closed and satisfies $\varkappa$-c.c.".

Proof: See Jech [3] or Silver [5]. For the last part, notice that as P_λ is σ-closed in M, $M[G \cap P_\lambda]$ has no new countable sequences from P^λ, whence P^λ is still σ-closed in $M[G \cap P_\lambda]$. Also, as we clearly have $P^\lambda \cong [P(\varkappa)]^{M[G \cap P_\lambda]}$, lemma 2 will ensure that P^λ has the $\varkappa$-c.c. in $M[G \cap P_\lambda]$. □

For later use, we shall give the proof of the next lemma in full.

Lemma 5 (Silver)

Let P be a poset in M such that $M \models$"P is σ-closed". Let $\underset{\sim}{T}$ be an ω_1-tree in M. Let G be M-generic for P. If b is a cofinal branch of $\underset{\sim}{T}$ in $M[G]$, then in fact $b \in M$.

Proof: We may assume $\underset{\sim}{T} = \langle \omega_1, \leq_T \rangle$. Suppose that, in fact $b \notin M$. Working in M, we define sequences $\langle p_s \mid s \in 2^{\underset{\smile}{\omega}} \rangle$, $\langle x_s \mid s \in 2^{\underset{\smile}{\omega}} \rangle$ so that $p_s \in P$; $t \subseteq s \to p_s \leq p_t$; $x_s \in T$; $t \subset s \to x_t <_T x_s$; $|s| = |t| \to ht(x_s) = ht(x_t)$; and $x_{s^\frown\langle 0 \rangle} \nleq x_{s^\frown\langle 1 \rangle}$. The definition is by induction on $|s|$. Pick $p_\emptyset \in P$ so that $p_\emptyset \Vdash$"$\mathring{b}$ is a cofinal branch of $\check{\underset{\sim}{T}}$ & $\mathring{b} \notin \check{M}$". Let $x_\emptyset$ be the minimal element of $\underset{\sim}{T}$. Suppose p_s, x_s are defined for all $s \in 2^n$, and that $p_s \Vdash$"$\check{x}_s \in \mathring{b}$", where $p_s \leq p_\emptyset$ in particular. Since $p_\emptyset \Vdash$"$\mathring{b} \notin \check{M}$", we can clearly find $p_{s^\frown\langle 0 \rangle}, p_{s^\frown\langle 1 \rangle} \leq p_s$ (each $s \in 2^n$) and points $x_{s^\frown\langle 0 \rangle}, x_{s^\frown\langle 1 \rangle} >_T x_s$ such that $ht(x_{s^\frown\langle 0 \rangle}) = ht(x_{s^\frown\langle 1 \rangle})$ and $x_{s^\frown\langle 0 \rangle} \nleq x_{s^\frown\langle 1 \rangle}$, for which $p_{s^\frown\langle i \rangle} \Vdash$"$\check{x}_{s^\frown\langle i \rangle} \in \mathring{b}$", $i = 0,1$. Furthermore, we may clearly do this in such a way that for any $s,t \in 2^{n+1}$, $ht(x_s) = ht(x_t)$. Since P is σ-closed, for each $f \in 2^\omega$ we may pick $p_f \in P$ such that $p_f \leq p_{f \restriction n}$ for all $n < \omega$. Also, as $|2^{\underset{\smile}{\omega}}| = \omega$, we may pick $\alpha < \omega_1$

such that $ht(x_s) < \alpha$ for all $s \in 2^{\omega}$. Since $p_f \leq p_\emptyset$ (each $f \in 2^\omega$), we can find $p_f' \leq p_f$ such that for some $x_f \in T_\alpha$, $p_f' \Vdash$ "$\check{x}_f \in \mathring{b}$". But, clearly, $p_f' \Vdash$ "$\check{x}_{f\restriction n} <_T \check{x}_f$" for all $n < \omega$, so by our construction, $f \neq g \rightarrow x_f \neq x_g$. (There are just two remarks called for here. Firstly, since $\underset{\sim}{T} \in M$, if $p_f' \Vdash$ "$\check{x}_{f\restriction n} <_T \check{x}_f$" then in fact $x_f <_T x_{f\restriction n}$. Secondly, if $f \neq g$ then for some $n < \omega$, $f\restriction n \neq g\restriction n$.). Thus $\{x_f \mid f \in 2^\omega\}$ is an uncountable subset of T_α, which is absurd. □

<u>Theorem 6</u> (Silver)

Let $\varkappa$ be an inaccessible cardinal in M. Let $P = [P(\varkappa)]^M$. Let G be M-generic for P. Then $M[G] \models$ "$2^\omega = \omega_1 + K$".

Proof: By lemmas 4 and 1, $M[G] \models$ "$2^\omega = \omega_1$" and $\omega_2^{M[G]} = \varkappa$. Also, $\omega_1^{M[G]} = \omega_1^M$, so the notion of an "$\omega_1$-tree" is absolute here. Let $\underset{\sim}{T}$ be an ω_1-tree in $M[G]$. We may assume $\underset{\sim}{T} = \langle \omega_1, \leq_T \rangle$. By the truth lemma, we can find an uncountable regular cardinal $\lambda < \varkappa$ of M such that $\underset{\sim}{T} \in M[G \cap P_\lambda]$. By lemma 2, $\underset{\sim}{T}$ has fewer than $\varkappa$ cofinal branches in $M[G \cap P_\lambda]$. But by lemma 4, P^λ is σ-closed in $M[G \cap P_\lambda]$, and by lemma 3, $G \cap P^\lambda$ is $M[G \cap P_\lambda]$-generic for P^λ, so by lemma 5, $\underset{\sim}{T}$ has no cofinal branches in $M[G \cap P_\lambda][G \cap P^\lambda]$ other than those in $M[G \cap P_\lambda]$. Again by lemma 3, $M[G \cap P_\lambda][G \cap P^\lambda] = M[G]$, so we see that $\underset{\sim}{T}$ has fewer than $\varkappa$ cofinal branches in $M[G]$.

Q.E.D.

The New Model

We shall require the following well-known result, proved in Jech [3].

Lemma 7 (Marczewski)

Let λ be a limit ordinal, $cf(\lambda) = \omega_1$. Let J be a collection of ω_1 finite subsets of λ. There is a finite subset X of λ and an uncountable subfamily J' of J such that $Y, Z \in J' \rightarrow Y \cap Z = X$.

Let κ be an ordinal. The poset $C(\kappa)$ is defined as follows. An element of $C(\kappa)$ is a finite function p such that $dom(p) \subseteq \kappa$ and $ran(p) \subseteq 2$. The partial ordering on $C(\kappa)$ is defined by $p \leq q \iff p \supseteq q$. Thus, if κ is an uncountable regular cardinal in M, $[C(\kappa)]^M$ is the usual poset for adding κ Cohen generic subsets of ω to M. Note that in this case, $[C(\kappa)]^M = C(\kappa)$, both of these being defined by the same, absolute formula of set theory.

It is well known that if κ is an uncountable regular cardinal in M and G is M-generic for $C = [C(\kappa)]^M$, then M and $M[G]$ have the same cardinals, by virtue of the fact that $M \models$ "C satisfies the countable chain condition", and $M[G] \models 2^\omega \geq \kappa$. For our purposes, however, it will be useful to regard the procedure of forcing with C over M here as an iteration of length κ. Accordingly, we make the following definitions.

Let U be the poset consisting of all maps p such that $dom(p) = n$ for some $n \in \omega$ and $ran(p) \subseteq 2$, ordered by $p \leq q \iff p \supseteq q$. Thus $U \in M$ and U is the usual poset for adding one Cohen generic subset of ω to M.

Let $\varkappa \in On$. Set $C^*(\varkappa) = \{\varphi \mid \varphi : \varkappa \to U$ & for some finite set $X \subseteq \varkappa$, $\varphi(\alpha) \neq \emptyset \leftrightarrow \alpha \in X$ (we call X the <u>support</u> of φ , $supp(\varphi))\}$, and partially order $C^*(\varkappa)$ by $\varphi \leq \psi \leftrightarrow (\forall \alpha \in \varkappa)(\varphi(\alpha) \supseteq \psi(\alpha))$. It is easily seen that forcing with $C^*(\varkappa)$ is equivalent to forcing with $C(\varkappa)$. In fact, the complete boolean algebra associated with both of these posets is the Borel algebra on $2^\varkappa$ factored by the ideal of all meager Borel subsets of $2^\varkappa$, where $2^\varkappa$ is given the product topology for the discrete topology on 2. Note also that the definition of $C^*(\varkappa)$ is, like $C(\varkappa)$, absolute for transitive models of ZFC containing $\varkappa$. The point of all of this is that forcing with $C^*(\varkappa)$ can be regarded as a process of forcing with U $\varkappa$ times, successively, using lemma 3.

<u>Lemma 8</u>.

Let $\varkappa$ be an uncountable cardinal in M , $cf^M(\varkappa) > \omega$. Let $C = [C(\varkappa)]^M$. If G is M-generic for C , then $M[G] \models 2^\omega \geq \varkappa$, M and $M[G]$ have the same cardinals and cofinality function, and if $M \models 2^\omega \leq \varkappa$, then $M[G] \models 2^\omega = \varkappa$. Furthermore, if $\underset{\sim}{T} = \langle \omega_1{}^M, \leq_T \rangle$ is an ω_1-tree in M , and b is a cofinal branch of $\underset{\sim}{T}$ in $M[G]$, then $b \in M$.

Proof: The last part of the lemma is the only non-standard part. Let $C^* = [C^*(\varkappa)]^M$. We may assume, by virtue of our above remarks, that G is M-generic for C^* rather than C . Let $\underset{\sim}{T} = \langle \omega_1{}^M, \leq_T \rangle$ be an ω_1-tree in M . We may assume that $\nu <_T \tau \to \nu < \tau$. Note that as $\omega_1{}^{M[G]} = \omega_1{}^M$, $\underset{\sim}{T}$ is still an ω_1-tree in $M[G]$.

If $\gamma < \varkappa$, then clearly $C^*(\gamma) = \{\varphi \restriction \gamma \mid \varphi \in C^*\}$. Set

$G_\gamma = \{\varphi \restriction \gamma \mid \varphi \in G\}$. By lemma 3, G_γ is M-generic for $C^*(\gamma)$ and $M[G]$ is a generic extension of $M_\gamma = M[G_\gamma]$. Clearly, $M_\varkappa = M[G]$, so it suffices to prove, by induction on $\gamma \leq \varkappa$, that if b is a cofinal branch of $\underset{\sim}{T}$ in M_γ, then $b \in M$.

For $\gamma = 0$ there is nothing to prove. Suppose the result holds for $\gamma < \varkappa$. If $H = \{\varphi(\gamma) \mid \varphi \in G\}$, then by lemma 3, H is M_γ-generic for U and $M_{\gamma+1} = M_\gamma[H]$. Let b be a cofinal branch of $\underset{\sim}{T}$ in $M_{\gamma+1}$. It suffices, by virtue of the induction hypothesis, to show that $b \in M_\gamma$. This will be so if, whenever $p \in U$ and $p \Vdash$ " $\mathring{b}$ is a cofinal branch of $\check{\underset{\sim}{T}}$" , there is $q \leq p$ such that $q \Vdash$ "$\mathring{b} \in \check{V}$" . We work in M_γ . Let such a p be given. For each $q \leq p$, let $\alpha(q)$ be the supremum of all ordinals $\xi < \omega_1$ such that $q \Vdash$ " $\check{\nu} \in \mathring{b}$ " for some ν on level ξ of $\underset{\sim}{T}$. Set $\alpha = \sup\{\alpha(q) \mid q \leq p\}$. By the truth lemma for forcing with U over M_γ , $\alpha = \omega_1$. Hence, as $|U| = \omega$, $\alpha(q) = \omega_1$ for some $q \leq p$. Set $b' = \{\nu \in T \mid q \Vdash$ " $\check{\nu} \in \mathring{b}$"$\}$. Then $b' \in M_\gamma$, and clearly $q \Vdash$ " $\mathring{b} = \check{b}'$" , so we are done. Finally, suppose $\gamma \leq \varkappa$, $\lim(\gamma)$, and the result holds for all $\delta < \gamma$. There are three cases to consider.

<u>Case 1</u> $cf^M(\gamma) = \omega$.

Let b be a cofinal branch of $\underset{\sim}{T}$ in M_γ . In M, let $\langle \gamma_n \mid n < \omega \rangle$ be cofinal in γ . Work in M_γ . By the truth lemma for forcing with $C^*(\gamma)$ over M , for each $\nu \in b$ we can find $p_\nu \in G_\gamma$ such that $p_\nu \Vdash$ " $\check{\nu} \in \mathring{b}$ ". Let $X_\nu = \mathrm{supp}(p_\nu)$. Since each X_ν is finite, and $cf(\omega_1) > \omega$, we can find an uncountable set $b' \subseteq b$ such that $\nu \in b' \rightarrow X_\nu \subseteq \gamma_n$ for some fixed $n < \omega$. But clearly,

$b = \{\nu \in T \mid (\exists p \in G_{\gamma_n}) [p \Vdash "\check{\nu} \in \mathring{b}"]\} \in M_{\gamma_n}$. Hence, by induction hypothesis, $b \in M$.

<u>Case 2</u> $\quad cf^M(\gamma) = \omega_1^M$.

Let b be a cofinal branch of $\underset{\sim}{T}$ in M_γ . Suppose, by way of contradiction, that $b \notin M$. By induction hypothesis, therefore, $\delta < \gamma \to b \notin M_\delta$, also. Work in M_γ . For each $\nu \in b$, pick $p_\nu \in G_\gamma$ such that $p_\nu \Vdash_{C^*(\gamma)} "\check{\nu} \in \mathring{b}"$, and let $X_\nu = \text{supp}(p_\nu)$. If $\sup\{\max(X_\nu) \mid \nu \in b\} < \gamma$, then arguing as in case 1 we see that $b \in M_\delta$ for $\delta = \sup\{\max(X_\nu) \mid \nu \in b\}$, and we are done. Hence we may assume $\sup\{\max(X_\nu) \mid \nu \in b\} = \gamma$. It follows, by lemma 6, that we can find an uncountable set $b' \subseteq b$ and a finite set $X \subseteq \gamma$ such that $\nu, \tau \in b'$ and $\nu < \tau$ implies $X_\nu \cap X_\tau = X$ and such that $\nu \in b'$ implies $X_\nu \neq X$. Since $|U| = \omega$, we can find an uncountable set $b'' \subseteq b'$ such that $\nu, \tau \in b''$ implies $p_\nu \upharpoonright X = p_\tau \upharpoonright X = p$, say. From now on $\Vdash$ refers to the forcing relation for $C^*(\gamma)$ over M .

<u>Claim</u>. There is $q \in C^*(\gamma)$, $\text{supp}(q) \cap X = \emptyset$, and $\nu < \omega_1$ such that $\nu \notin b$ but $p \cup q \Vdash "\check{\nu} \in \mathring{b}"$, where $p \cup q \in C^*(\gamma)$ is defined from p and q in the obvious manner.

Suppose the claim is false. In M, set $d = \{\nu \in T \mid (\exists q \in C^*(\gamma)) [\text{supp}(q) \cap X = \emptyset \;\&\; p \cup q \Vdash "\check{\nu} \in \mathring{b}"]\}$. Since the claim fails, $d \subseteq b$. But for each $\nu \in b''$, if $q = p_\nu \upharpoonright (X_\nu - X)$, then $\text{supp}(q) \cap X = \emptyset$ and $p \cup q = p_\nu$ and $p_\nu \Vdash "\check{\nu} \in \mathring{b}"$, so $b \subseteq d$. Hence $b = d \in M$, a contradiction. This proves the claim.

Pick $q \in C^*(\gamma)$ as in the claim and let $\nu < \omega_1$ be such that $\nu \notin b$ and $p \cup q \Vdash "\check{\nu} \in \mathring{b}"$. Pick $\tau \in b''$, $\tau > \nu$, such that $X_\tau \cap \text{supp}(q) = \emptyset$. (This is clearly possible). Clearly, $p_\tau \cup q = p \cup q \cup [p_\tau \upharpoonright (X_\tau - X)] \in C^*(\gamma)$.

But, look, $p_\tau \cup q \leq p_\tau$, so $p_\tau \cup q \Vdash$ " $\check{\tau} \in \mathring{b}$ " , and $p_\tau \cup q \leq p \cup q$, so $p_\tau \cup q \Vdash$ " $\check{\nu} \in \mathring{b}$ ". Hence, as $\nu < \tau$, $p_\tau \cup q \Vdash$ " $\check{\nu} <_T \check{\tau}$ " , which means $\nu <_T \tau$, of course. Thus, as $\tau \in b''$, $\nu \in b$, a contradiction.

Case 3 $cf^M(\gamma) > \omega_1^M$.

This case is trivial by the truth lemma for forcing with $C^*(\gamma)$ over M.

The lemma is proved. □

The following is an analogue of lemma 5.

Lemma 9

Let C, P be posets in M such that $M \models$ "C satisfies c.c.c. and P is σ-closed". Let G be M-generic for $C \times P$. (Thus $\omega_1^M = \omega_1^{M[G]}$.) Let $G_C = \{p \in C \mid \langle p, \mathbb{1} \rangle \in G\}$, $G_P = \{q \in P \mid \langle \mathbb{1}, q \rangle \in G\}$. (Thus G_C is M-generic for C , G_P is $M[G_C]$-generic for P , and $M[G_C][G_P] = M[G]$.) Let $\underset{\sim}{T}$ be an ω_1-tree in $M[G_C]$. If b is a cofinal branch of $\underset{\sim}{T}$ in M[G] , then $b \in M[G_C]$.

Proof: Notice that as P is not necessarily σ-closed in the sense of $M[G_C]$, we cannot argue exactly as in lemma 5. However, with a little extra work, we can carry through an argument parallel to that of lemma 5. We shall assume that $\underset{\sim}{T} = \langle \omega_1, \leq_T \rangle$, as before. In $M[G_C]$, for each $\alpha < \omega_1$, let $D_\alpha = \{q \in P \mid q \Vdash_P$ " $\check{x} \in \mathring{b} \cap \check{T}_{\check{\alpha}}$ " for some $x \in T\}$, where $\Vdash_P$ denotes P-forcing over $M[G_C]$. Clearly each D_α is a dense open subset of P . Suppose that $b \notin M[G_C]$. To cut down on notation, let us suppose that, in fact $\emptyset \Vdash_P$ " $\mathring{b}$ is a cofinal branch of $\check{\underset{\sim}{T}}$ and $\mathring{b} \notin \widecheck{M[G_C]}$ " . (In the general case, we simply work beneath some q_0 in P , of course.). Pick $p^* \in G_C$ so that $p^* \Vdash_C$ "$[\mathring{\underset{\sim}{T}}$ is an ω_1-tree with domain $\check{\omega}_1]$ and $[\emptyset \Vdash_{\check{P}}$ "$\mathring{b}$ is a cofinal branch of $\mathring{\underset{\sim}{T}}$ not in $\check{M}[\mathring{G}_C]$"$]$ and $[\langle \mathring{D}_\alpha \mid \alpha < \check{\omega}_1 \rangle$ is a sequence of dense open subsets of $\check{P}]$".

<u>Claim</u>. Let $\alpha < \omega_1$, $q \in P$. There is $q' \leq_P q$ and $x \in T$ such that $\langle p^*, q' \rangle \Vdash_{C \times P}$ $\check{x} \in \mathring{b} \cap \mathring{T}_{\check{\alpha}}$" .

By induction, we define in M a sequence $\langle \langle p_\nu, q_\nu \rangle \mid \nu < \delta \rangle$, some $\delta < \omega_1$, so that $\nu < \delta \to p_\nu \leq_C p^*$ & $q_\nu \leq_P q$, $\nu < \tau < \delta \to p_\nu \not\sim p_\tau$ & $q_\tau \leq_P q_\nu$, and $p_\nu \Vdash_C$ "$\check{q}_\nu \in \mathring{D}_{\check{\alpha}}$" . The ordinal δ will be determined by the failure of the definition. Since C satisfies c.c.c. , the incompatibility condition on the p_ν's will ensure that $\delta < \omega_1$; and in fact, the definition will stop precisely when $\{p_\nu \mid \nu < \delta\}$ is a maximal pairwise incompatible subset of $\{p \in C \mid p \leq_C p^*\}$. Let $\langle p_0, q_0 \rangle$ be chosen so that $p_0 \leq_C p^*$, $q_0 \leq_P q$, and $p_0 \Vdash_C$ "$\check{q}_0 \in \mathring{D}_{\check{\alpha}}$" . This clearly causes no problems. Suppose $\langle \langle p_\tau, q_\tau \rangle \mid \tau < \nu \rangle$ is defined. Thus $\nu < \omega_1$, so we can find $q_\nu' \in P$, $q_\nu' \leq_P q_\tau$ for all $\tau < \nu$, by the σ-closed nature of P . (Remember that we are working in M here!) Pick (if possible) $p_\nu \in C$, $p_\nu \leq_C p^*$, and $q_\nu \in P$, $q_\nu \leq_P q_\nu'$, so that $\tau < \nu \to p_\tau \not\sim p_\nu$ and $p_\nu \Vdash_C$ "$\check{q}_\nu \in \mathring{D}_{\check{\alpha}}$" . Clearly, if we can find p_ν such that $p_\nu \leq_C p^*$ and $\tau < \nu \to p_\tau \not\sim p_\nu$, then the choice of q_ν causes no trouble. That completes the definition. Some P is σ-closed, let $q' \in P$ be such that $q' \leq_P q_\nu$ for all $\nu < \delta$. Then q' is as required. It suffices to show that $p^* \Vdash_C$ "$\check{q}' \in \mathring{D}_{\check{\alpha}}$" , and for this it is enough to show that $\{p \in C \mid p \Vdash_C$ "$\check{q}' \in \mathring{D}_{\check{\alpha}}$"$\}$ is dense below p^* in C . So let $p <_C p^*$. Then for some $\alpha < \delta$, $p \sim p_\alpha$. Pick $p' \leq_C p, p_\alpha$. Thus $p' \Vdash_C$ "$\check{q}_\alpha \in \mathring{D}_{\check{\alpha}}$" . But $q' \leq_P q_\alpha$. Hence $p' \Vdash_C$ "$\check{q}' \in \mathring{D}_{\check{\alpha}}$" , as required. The claim is proved.

Using the claim, we can now argue as in lemma 5. By induction, pick sequences $\langle q_s \mid s \in 2^{\underset{\smile}{\omega}}\rangle$, $\langle x_s \mid s \in 2^{\underset{\smile}{\omega}}\rangle$ so that, in particular, $\langle p^*, q_s\rangle \Vdash_{C\times P}$ " $\check{x}_s \in \mathring{b}$ " , and so that $\langle q_s \mid s \in 2^{\underset{\smile}{\omega}}\rangle$ decreases along branches in $2^{\underset{\smile}{\omega}}$, etc. Since $\langle p^*, \emptyset\rangle \Vdash_{C\times P}$ " $\mathring{b} \notin \check{M}[\mathring{G}_C]$ " , this follows from the claim just as it followed in lemma 5. Since all of this is done in M, where P is σ-closed, we obtain a contradiction exactly as before. The lemma is proved.□

<u>Theorem 10</u>

Let $\varkappa$ be an inaccessible cardinal in M , and let λ be an arbitrary cardinal in M such that $\lambda \geq \varkappa$ and $cf^M(\lambda) > \omega$. Let $P = [P(\varkappa)]^M$, $C = [C(\lambda)]^M$. Let G be M-generic for $P \times C$. Then ${\omega_1}^M = {\omega_1}^{M[G]}$, $\varkappa = {\omega_2}^{M[G]}$, λ and all other cardinals of M above $\varkappa$ are cardinals in M[G] (so if $\lambda = {\omega_{\varkappa+\gamma}}^M$ then $\lambda = {\omega_{2+\gamma}}^{M[G]}$) , $cf^{M[G]}(\lambda) > \omega$, $M[G] \models$ " $2^\omega = \lambda$ " , and $M[G] \models$ "K".

Proof: Let G_P, G_C be as above. Let $\underset{\sim}{T} = \langle \omega_1, \leq_T\rangle$ be an ω_1-tree in M[G] . By the truth lemma, pick $\gamma < \varkappa$ an uncountable regular cardinal of M such that $\underset{\sim}{T} \in M[G_P \cap P_\gamma][G_C]$. Let $N = M[G_P \cap P_\gamma]$. Notice that by lemma 4, P^γ is σ-closed in the sense of M . Also, by absoluteness, $C = [C(\lambda)]^N$, so C satisfies c.c.c. in N . Now, by lemma 3, G_C is N-generic for C , so by the truth lemma for C-forcing over N we can find, in N , a set $X \subseteq \lambda$, $|X| = \omega_1$, such that $\underset{\sim}{T} \in N[G_C \cap C_X]$, where $C_X = \{p \restriction X \mid p \in C\}$. Now, $X \in N$, so in N there is a canonical isomorphism $C \cong C_X \times C^X$, where $C^X = \{p - p \restriction X \mid p \in C\}$. Thus, by lemma 3 (applied to N) , $G_C \cap C_X$ is N-generic for C_X, $G_C \cap C^X$ is $N[G_C \cap C_X]$-generic for C^X, and $N[G_C \cap C_X][G_C \cap C^X] = N[G_C]$. By lemma 2, $\varkappa$ is inaccessible in $N[G_C \cap C_X] =$

$M[G_P \cap P_Y][G_C \cap C_X]$. Hence $\underset{\sim}{T}$ has fewer than κ cofinal branches in $N[G_C \cap C_X]$. In $N[G_C \cap C_X]$, there is a canonical isomorphism $C^X \cong [C(\lambda)]^{N[G_C \cap C_X]}$. Hence, by lemma 8 applied to $N[G_C \cap C_X]$, $\underset{\sim}{T}$ has no extra cofinal branches in $N[G_C] = N[G_C \cap C_X][G_C \cap C^X]$. But by lemma 3 again, $M[G] = M[G_P][G_C] = M[G_P \cap P_Y][G_P \cap P^Y][G_C] = N[G_C][G_P \cap P^Y]$ and $G_P \cap P^Y$ is $N[G_C]$-generic for P^Y . So, applying lemma 9 to N and the posets C, P^Y , we see that $\underset{\sim}{T}$ has no extra cofinal branches in $M[G]$. Hence $\underset{\sim}{T}$ is not Kurepa in $M[G]$. Q.E.D.

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