INTERSECTION PROPERTIES OF BALLS IN COMPLEX BANACH SPACES WHOSE DUALS ARE L_1 SPACES

OTTE HUSTAD

INTRODUCTION. In the paper [10] L. Nachbin discovered and exploited the basic connection that exists between intersection properties of balls and extension properties of linear operators. This connection has been most strikingly revealed in the paper [8] by J. Lindenstrauss. For the aim of the present work, we want to exhibit the following result of that paper: We say with Lindenstrauss that a normed space A has the n,k <u>intersection property</u> if for every collection of n balls in A such that any k of them have a non void intersection, there is a point common to all the n balls. If A has the n,k intersection property for any $n \geq k$, then A has the <u>finite</u> k <u>intersection property</u>. It is then proved in [8, Theorem 6.1 and Theorem 5.5] that for a real Banach space A, the following three properties are equivalent.

(i). The dual A^* of A is isometric to an L_1 space.

(ii) The space A has the $4,2$ intersection property.

(iii) For any 3-dimensional normed space Y and any 4-dimensional normed space $X \supset Y$ such that the unit ball of X is the convex hull of the unit ball in Y and a finite number of additional points, there exists for every linear operator $T: Y \to A$ a norm preserving extension $\tilde{T}: X \to A$.

We remark that it is essential in this characterization that the space A is a <u>real</u> Banach space. Already the space $\mathbb{C}$ of all complex numbers shows that (ii) can not be valid in the complex case.

The starting point of the present work was the observation that it suffices in property (iii) to take just one space Y and just one space X, namely $X = l_1^4(\mathbb{R})$ and $Y = \{(x_j) \in l_1^4(\mathbb{R}): \Sigma x_j = 0\}$. In fact, what we observed was that a normed space A has the n,2 intersection property if and only if every linear operator T from the space

$$H^n(\mathbb{R}) = \{(x_j) \in l_1^n(\mathbb{R}): \sum_{j=1}^{n} x_j = 0\}$$

into A admits a norm preserving extension $\tilde{T}: l_1^n(\mathbb{R}) \to A$. (see Corollary 1.11). With this observation at hand, we define for a given integer $n \geq 1$ that a complex Banach space A is an <u>E(n) space</u> (where E stands for extension) if every linear operator T from the space

$$H^n(\mathbb{C}) = \{(z_j) \in l_1^n(\mathbb{C}): \sum_{j=}^{n} z_j = 0\}$$

into A admits a norm preserving extension $\tilde{T}: l_1^n(\mathbb{C}) \to A$. And if every $\tilde{T}: H^n(\mathbb{C}) \to A$ admits for any $\varepsilon > 0$ an extension $\tilde{T}: l_1^n(\mathbb{C}) \to A$ such that $\|\tilde{T}\| \leq \|T\|(1+\varepsilon)$, then we call A an <u>almost E(n) space.</u> Finally, if A is an E(n) space for any $n \geq 1$, then we say that A is an <u>E space</u>, and similarly we define an <u>almost E space.</u> We can then formulate our main result (see Theorem 4.9) as follows: If A is an almost E(7) space, then the dual A^* of A is isometric to an L_1 space. And conversely, if the dual of A is isometric to an L_1 space, then A is an E space. For the proof of this result, the following intersection property of balls has been very usefull: A finite family $\{B(a_j,r_j)\}$ of balls (we denote with $B(a,r)$ the closed ball with center a and radius r) has the <u>weak intersection property</u> if for any linear functional φ with norm ≤ 1, the family $\{B(\varphi(a_j),r_j)\}$ of balls in $\mathbb{C}$ (or in $\mathbb{R}$) has a non empty intersection. We prove (Theorem 4.9) that the E spaces are just the complex Banach spaces where any finite family of balls with the weak intersection property has a non empty intersection.

Every finite family of balls such that any three of them have a non empty intersection will have the weak intersection property. This is a consequence of the Helly theorem on intersection of convex sets, but it also follows from the description of the extreme points of the unit ball of $H^n(\mathbb{C})$ given in Theorem 3.6. The converse is not valid. In fact, we get the most important example of families with the weak intersection property as follows: Let A, X and Y be normed spaces with $Y \subset X$, let $x \in X \setminus Y$ and let $T: Y \to A$ be a linear operator with norm ≤ 1. Then any finite subfamily of the family $\{B(Ty,\|x-y\|): y \in Y\}$ has the weak intersection property (see Lemma 2.1), whereas it can happen (we give an example in section 5) that three balls from this family have an empty intersection. These facts explain on the one hand why we are able to get extensions of compact operators into an E space (Theorem 2.3). On the other hand, they clearify why such extensions have not been established for spaces that have the finite 3 intersection property. We show (Corollary 4.7) that every E space has the finite 3 intersection property. It is an unsolved problem whether the converse is valid.

The present work leans heavily on the paper [8]. It is a pleasure at this point to acknowledge the great influence of that fundamental memoir on the paper at hand.

NOTATIONS AND PRELIMINARIES. We will use the following notations.

$\mathbb{N}$: the set of all integers $n \geq 1$.

$\mathbb{R}$: the set of all real numbers.

$\mathbb{C}$: the set of all complex numbers.

$\mathbb{K}$: either $\mathbb{C}$ or $\mathbb{R}$.

$\{e_1,\cdots,e_n\}$: the standard base in $\mathbb{K}^n$.

$z = (z_j)$: the generic element of $\mathbb{K}^n$.

$H^n = H^n(\mathbb{K}) = \{z \in \mathbb{K}^n: \sum_{j=}^{n} z_j = 0\}$.

We let $r = (r_j) \in \mathbb{R}^n$ denote a multi-radius which means that $r_j > 0$, $j = 1,\cdots,n$. On $\mathbb{K}^n$ we introduce a norm $\| \ \|_r$ defined by

$$\|z\|_r = \sum_{j=1}^{n} |z_j| r_j ,$$

and we let $(\mathbb{K}^n, \| \ \|_r)$ denote the space $\mathbb{K}^n$ equipped with the norm $\| \ \|_r$. The notation $(H^n, \| \ \|_r)$ has a similar meaning. Observe that if $r = (1,\cdots,1)$, then $(\mathbb{K}^n, \| \ \|_r)$ is just the ordinary $l_1^n(\mathbb{K})$ space. We let A denote a complex or real normed space, and we denote the norm in A with $\| \ \|$. As noted in the introduction, $B(a,R)$ denotes the closed ball in A with center a and radius $R > 0$, that is $B(a,R) = \{p \in A: \|p-a\| \leq R\}$. When deemed necessary, we shall also use the notation $B_A(a,R)$ for this ball. An operator will always be a bounded linear operator. We follow [3,p.94] and say that a Banach space B is a $\mathscr{P}_1$ space if for every normed space Y and every normed space $X \supset Y$ there exists for any operator $T: Y \to B$ a norm preserving extension $\tilde{T}: X \to B$. We say that a Banach space is an L_1 space if it is an $L_1(\mu)$ space for some measure μ. It was shown by A. Grothendieck [6] that if

A is a real Banach space, then the dual A^* of A is isometric to an L_1 space if and only if the bidual A^{**} of A is a $\mathcal{P}_1$ space. It follows from results of S. Sakai [11] that this theorem is also valid in the case of complex Banach spaces.

§ 1 EXTENSION OF OPERATORS DEFINED ON $(H^n, \| \|_r)$.

In the first part of the present section we show how extension properties of a linear operator $T: (H^n, \| \|_r) \to A$ can be expressed by intersection properties of n balls in A. We use this result to give a quantitative criterion for n balls in A to have the weak intersection property (as defined in the introduction). In particular, we get a quantitative condition for n balls in $\mathbb{C}$ to have a non empty intersection. We finish this section with Proposition **1.13**, which states that if A is an almost E(n)space, then any family of n balls in A with the weak intersection property has almost a non empty intersection.

<u>Lemma 1.1.</u> Let A be a normed space over $\mathbb{K}$, let $n \geq 1$ be an integer and let $r = (r_j)$ be a multi-radius. Let $\varepsilon \geq 0$ and let $a_1, \cdots, a_n \in A$. The linear operator

$$T: (H^n(\mathbb{K}), \| \|_r) \to A : (z_j) \to \sum_{j=1}^{n} z_j a_j$$

admits an extension $\tilde{T}: (\mathbb{K}^n, \| \|_r) \to A$ satisfying

$$(1.1) \qquad \|\tilde{T}\| \leq \|T\|(1+\varepsilon) ,$$

if and only if the family $\{B(a_j, \|T\|(1+\varepsilon)r_j\}_{j=1}^{n}$ has a non empty intersection.

Proof. Assume that $a \in A$ satisfies

(1.2) $$\|a - a_j\| \leq \|T\|(1+\varepsilon)r_j \; ; \quad j = 1,\cdots,n.$$

Let the operator $\tilde{T}$ be defined by

$$\tilde{T}: (\mathbb{K}^n, \|\ \|_r) \to A: (z_j) \to \sum_{j=1}^{n} z_j(a_j - a).$$

Then $\tilde{T}$ is an extension of T, and it follows from (1.2) that if $z \in \mathbb{K}^n$, then

$$\|\tilde{T}z\| \leq \sum_{j=1}^{n} |z_j| \ \|a_j - a\| \leq \ \|T\|(1+\varepsilon)\|z\|_r.$$

Hence (1.1) is valid. Assume conversely that T admits an extension $\tilde{T}: (\mathbb{K}^n, \|\ \|_r) \to A$ satisfying (1.1). Put $a = a_1 - \tilde{T}e_1$. Then

(1.3) $$\tilde{T}(z) = \sum_{j=1}^{n} z_j(a_j - a) \; ; \qquad z \in \mathbb{K}^n.$$

For any $k = 1,\cdots,n$, we have $\|r_k^{-1}e_k\|_r = 1$. It therefore follows from (1.1) and (1.3) that

$$\|r_k^{-1}(a_k - a)\| = \|\tilde{T}(r_k^{-1}e_k)\| \leq \|\tilde{T}\| \leq \|T\|(1+\varepsilon).$$

This means that a belongs to the intersection of the family $\{B(a_j, \|T\|(1+\varepsilon)r_j)\}_{j=1}^{n}$.

Proposition 1.2. Let A be a normed space over $\mathbb{K}$ and let $\varepsilon \geq 0$. Let $n \in \mathbb{N}$ and assume that $r = (r_j) \in \mathbb{R}^n$ is a multi-radius. Then the following two properties are equivalent.

(i) Every linear operator $T: (H^n(\mathbb{K}), \|\ \|_r) \to A$ admits an extension $\tilde{T}: (\mathbb{K}^n, \|\ \|_r) \to A$ such that $\|\tilde{T}\| \leq \|T\|(1+\varepsilon)$.

(ii) If $a_1, \cdots, a_n \in A$ satisfy the condition

$$(*) \qquad \| \sum_{j=1}^{n} z_j a_j \| \leq \sum_{j=1}^{n} |z_j| r_j \quad ; \qquad z \in H^n(\mathbb{K}),$$

then

$$(1.4) \qquad \bigcap_{j=1}^{n} B(a_j, (1+\varepsilon) r_j) \neq \emptyset .$$

<u>Proof.</u> (i) $\Rightarrow$ (ii). Assume that $a_1, \cdots, a_n \in A$ satisfy the (*)-condition. This means that the linear operator

$$T: (H^n(\mathbb{K}), \| \ \|_r) \to A : (z_j) \to \sum_{j=1}^{n} z_j a_j$$

has a norm $\|T\| \leq 1$. It therefore follows from Lemma 1.1 that (1.4) is satisfied.

(ii) $\Rightarrow$ (i). Let the linear operator $T: (H^n(\mathbb{K}), \| \ \|_r) \to A$ be given. We can and shall assume that $T \neq 0$. Put $a_j = T(e_j - e_1)$; $j = 1, \cdots, n$ and let $z \in H^n(\mathbb{K})$. From the equation $z = \sum_{j=1}^{n} z_j (e_j - e_1)$ we get $Tz = \sum_{j=1}^{n} z_j a_j$. Hence in order to prove (i), it is, by Lemma 1.1, sufficient to prove that the family $\{B(a_j, \|T\| (1+\varepsilon) r_j)\}_{j=1}^{n}$ has a non empty intersection. Let $z \in H^n(\mathbb{K})$. Then

$$\| \sum_{j=1}^{n} z_j \|T\|^{-1} a_j \| = \|T\|^{-1} \|Tz\| \leq \|z\|_r .$$

This means that the set $\{\|T\|^{-1} a_j : j = 1, \cdots, n\}$ satisfies the (*)-condition. Hence there exists an $a \in A$ such that

$$\| a - \|T\|^{-1} a_j \| \leq (1+\varepsilon) r_j \quad ; \qquad j = 1, \cdots, n.$$

It follows that $a\|T\|$ belongs to the intersection of the family $\{B(a_j, \|T\| (1+\varepsilon) r_j)\}_{j=1}^{n}$.

Comment. If the family $\{B(a_j,r_j)\}_{j=1}^{n}$ has a non empty intersection,then the (*)-condition in Proposition 1.2 is always fulfilled. In fact, if $a \in A$ satisfies $\|a-a_j\| \leq r_j$, $j=1,\cdots,n$, then we get for any $z \in H^n(\mathbb{K})$

$$\|\sum_{j=1}^{n} z_j a_j\| = \|\sum_{j=1}^{n} z_j(a_j-a)\| \leq \sum_{j=1}^{n} |z_j| r_j .$$

Corollary 1.3. A finite family $\{B(u_j r_j)\}_{j=1}^{n}$ of balls in $\mathbb{K}$ has a non empty intersection if and only if

$$(1.5) \qquad |\sum_{j=1}^{n} z_j u_j| \leq \sum_{j=1}^{n} |z_j| r_j \quad , \qquad z \in H^n(\mathbb{K})$$

First proof. By the Hahn-Banach theorem, the property (i) in Proposition 1.2 is fulfilled for any $n \in \mathbb{N}$ and with $\varepsilon = 0$.

Second proof. We think it is of some interest to give a proof independent of the Hahn-Banach theorem. In fact, for the case $\mathbb{K} = \mathbb{C}$, such a proof, combined with the Helly theorem for an infinite family of compact convex sets, can be used to give a direct geometric proof of the complex Hahn-Banach theorem (confer section 2). The case $\mathbb{K} = \mathbb{R}$ is easily handled. Indeed, let $k,l \in \{1,\cdots,n\}$. Then, if we choose $z = e_k - e_l$ in (1.5), we get $|u_k-u_l| \leq r_k + r_l$. Hence any two of the n balls have a non empty intersection. Since $\mathbb{R}$ has the $n,2$ intersection property, it follows that the whole family has a non empty intersection. Let us now assume that $\mathbb{K} = \mathbb{C}$. We have to show that (1.5) implies that the family $\{B(a_j,r_j)\}_{j=1}^{n}$ has a non empty intersection. By the Helly theorem (see e.g.[5]), we can and shall assume that $n = 3$. First we want to verify the following statement: Let $a,b,c \in \mathbb{C}$ be given. Assume that c is <u>between</u> a and b in the sense that $\mathrm{Arg}.a < \mathrm{Arg}.c < \mathrm{Arg}.b$ and $\mathrm{Arg}.b < \pi + \mathrm{Arg}.a$. Then there exist complex numbers u,v such that $u + v = 1$ and such that

$$|ua + vb + c| = |u||a| + |v||b| + |c|.$$

In fact, putting $\alpha = \mathrm{Arg}.a$, $\beta = \mathrm{Arg}.b$, $\gamma = \mathrm{Arg}.c$, it suffices to choose

$$u = \frac{\sin(\beta-\gamma)}{\sin(\beta-\alpha)} e^{i(\gamma-\alpha)} \quad ; \quad v = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)} e^{i(\gamma-\beta)}$$

As above, we get for any $k,l \in \{1,2,3\}$ that $|u_k-u_l| \leq r_k+r_l$. In particular, the intersection $S = B(u_1,r_1) \cap B(u_2,r_2)$ is non empty. We have to prove that $r_3 \geq \mathrm{dist}(u_3,S)$. Let q_1 and q_2 be the two points in $\mathbb{C}$ which satisfy the equations $|u_1-q| = r_1$, $|u_2-q| = r_2$. (The case that no such q exists is trivial). There are two possible cases: (i) For some $j \in \{1,2\}$, $\mathrm{dist}(u_3,S) \leq |u_3-u_j| - r_j$. (ii) For some $j \in \{1\ 2\}$, $\mathrm{dist}(u_3,S) = |u_3-q_j|$. Since $|u_3-u_j| - r_j \leq r_3$, the first case is settled. As for the second case, we observe that then u_3-q_j is between q_j-u_1 and q_j-u_2 in the sense defined above. Hence we can find complex numbers z_1, z_2 such that $z_1 + z_2 = 1$ and such that

$$|z_1(q_j-u_1) + z_2(q_j-u_2) + u_3 - q_j| =$$
$$= |z_1||q_j-u_1| + |z_2||q_j-u_2| + |u_3-q_j|.$$

Using the definition of q_j, we get from this equation and from (1.5)

$$|z_1|r_1 + |z_2|r_2 + |u_3-q_j| = |z_1u_1 + z_2u_2 - u_3|$$
$$\leq |z_1|r_1 + |z_2|r_2 + r_3$$

Hence $\qquad \mathrm{dist}(u_3,S) = |u_3-q_j| \leq r_3.$

<u>Corollary 1.4.</u> A family $\{B(a_j,r_j)\}_{j=1}^{n}$ of n balls in a normed space A has the weak intersection property (as defined in the introduction) if and only if

$$(*) \qquad \|\sum_{j=1}^{n} z_j a_j\| \leq \sum_{j=1}^{n} |z_j| r_j \quad ; \qquad z \in H^n.$$

<u>Proof.</u> Assume that (*) is satisfied. Let $\varphi \in A^*$ and assume that $\|\varphi\| \leq 1$. It follows from (*) that if $z \in H^n$, then

$$|\sum_{j=1}^{n} z_j \varphi(a_j)| \leq \|\varphi\| \, \|\sum_{j=1}^{n} z_j a_j\| \leq \sum_{j=1}^{n} |z_j| r_j .$$

Thus we conclude, by Corollary 1.3, that the family $\{B(\varphi(a_j),r_j)\}_{j=1}^{n}$ has a non empty intersection. Assume conversely that the family $\{B(a_j,r_j)\}_{j=1}^{n}$ has the weak intersection property. It then follows from Corollary 1.3 that for any $\varphi \in A^*$ with $\|\varphi\| \leq 1$, and for any $z \in H^n$

$$|\varphi(\sum_{j=1}^{n} z_j a_j)| = |\sum_{j=1}^{n} z_j \varphi(a_j)| \leq \sum_{j=1}^{n} |z_j| r_j .$$

By the Hahn-Banach theorem, we conclude that (*) is fulfilled.

<u>Definition 1.5.</u> A family $\mathcal{F}=\{B(a_j,r_j)\}_{j\in J}$ of balls in A has the <u>almost intersection property</u> if for any $\varepsilon > 0$ the family $\{B(a_j,r_j+\varepsilon)\}_{j\in J}$ has a non empty intersection. If $\mathcal{F}$ has a non empty intersection, then we say that $\mathcal{F}$ has the <u>intersection property.</u>

The almost intersection property is stronger than the weak intersection property. In fact, we have the following

Lemma 1.6. If a family $\{B(a_j,r_j)\}_{j=1}^{n}$ has the almost intersection property, then it has the weak intersection property.

Proof. It suffices, by Corollary 1.4, to show that the (*)-condition is satisfied. Let $z \in H^n$ and let $\varepsilon > 0$ be given. Choose $a \in A$ such that

$$\|a-a_j\| \leq r_j + \varepsilon \|z\|_r^{-1} \qquad j = 1,\cdots,n.$$

(We can clearly assume that $z \neq 0$). It follows that

$$\|\sum_{j=1}^{n} z_j a_j\| = \|\sum_{j=1}^{n} z_j(a_j-a)\| \leq \sum_{j=1}^{n} |z_j|(r_j+\varepsilon\|z\|_r^{-1}) = \|z\|_r + \varepsilon$$

Since this holds for any $\varepsilon > 0$, we conclude that the (*)-condition is fulfilled.

For a complex Banach space A we defined in the introduction what it means that A is an $E(n)$ space or an almost $E(n)$ space. In the case of a real Banach space we shall adhere to the analogous definitions. We then have the following characterization of an $E(n)$ space.

Proposition 1.7. Let $n \in \mathbb{N}$ be given. Then a Banach space A is an $E(n)$ space if and only if every family $\{B(a_j,R)\}_{j=1}^{n}$ of n balls with common radius R has the intersection property whenever it has the weak intersection property. And A is an almost $E(n)$ space if and only if every family $\{B(a_j,R)\}_{j=1}^{n}$ of n balls with the weak intersection property has the almost intersection property.

Proof. If $R = 1$, this follows immediately from Proposition 1.2 and Corollary 1.4. And since the family $\{B(a_j,R)\}$ has the weak intersection property if and only if the family $\{B(a_j R^{-1},1)\}$ has the same property, the general case follows from the special case $R = 1$.

We shall now show that if the bidual A^{**} of a Banach space A is a $\mathcal{P}_1$ space, then A is an almost E space. In fact, we shall show that A has the following formally stronger property.

Proposition 1.8. Let A be a Banach space such that the bidual A^{**} of A is a $\mathcal{P}_1$ space. Then every finite family of balls in A with the weak intersection property has the almost intersection property.

Proof. Let $\{B(a_j,r_j)\}_{j=1}^n$ be a family of balls in A with the weak intersection property. It then follows from Corollary 1·4 that the operator

$$T: (H^n, \| \ \|_r) \to A: z \to \sum_{j=1}^n z_j a_j$$

has a norm $\|T\| \leq 1$. Since A^{**} is a $\mathcal{P}_1$ space, T admits a norm preserving extension $\overline{T}: (\mathbb{K}^n, \| \ \|_r) \to A^{**}$. Let $\varepsilon > 0$ be given. According to the local reflexivity theorem of Lindenstrauss and Rosenthal [9, Theorem 3·1], there exists an operator S: range $\overline{T} \to A$ such that S is the identity on $A \cap$ range $\overline{T}$ and such that $\|S\| \leq 1 + \varepsilon$. Put $\tilde{T} = S \circ \overline{T}$. Since range $T \subset A$, it follows that $\tilde{T}$ is an extension of T. Furthermore, $\|\tilde{T}\| \leq \|\overline{T}\|(1+\varepsilon) = \|T\|\,(1+\varepsilon)$. Since $\|T\| \leq 1$, we conclude from Lemma 1.1 that the family $\{B(a_j,(1+\varepsilon)r_j)\}_{j=1}^n$ has a non empty intersection.

If K is a convex set, we let Ext K denote the set of all extreme points of K.

Lemma 1·9· A family $\mathcal{F} = \{B(a_j,r_j)\}_{j=1}^n$ of n balls in the normed space A has the weak intersection property if and only if

$$(**)\qquad \|\sum_{j=1}^{n} z_j a_j\| \leq 1; \qquad z \in \mathrm{Ext}\{z \in H^n: \|z\|_r \leq 1\}.$$

Proof. The family $\mathcal{F}$ has, by Corollary 1·4, the weak intersection property if and only if the operator

$$T: (H^n, \| \|_r) \to A: (z_j) \to \sum_{j=1}^{n} z_j a_j$$

has a norm $\|T\| \leq 1$. Now the number $\|T\|$ is the maximum of the function $z \to \|T(z)\|$ on the unit ball of $(H^n, \| \|_r)$. That unit ball is, however, the closed convex hull of its extreme points. Hence it follows that $\|T\| \leq 1$ if and only if the condition (**) is satisfied.

Corollary 1·10· If A is a real normed space, then a family $\{B(a_j, r_j)\}_{j=1}^{n}$ of n balls in A has the weak intersection property if and only if any two of the balls have a non empty intersection.

Proof. It is well known (confer section 3) that the set of extreme points of the unit ball of $(H^n(\mathbb{R}), \| \|_r)$ consists of all points of the form $(r_k + r_l)^{-1}(e_k - e_l)$, where $k \neq l$ and where $k, l \in \{1, \cdots, n\}$. Hence the condition (**) of Lemma 1·9 means that $\|a_k - a_l\| \leq r_k + r_l$ whenever $k \neq l$ and $k, l \in \{1, \cdots, n\}$. But this is just the condition that any two of the n balls have a non empty intersection.

Comment. Another (and even simpler) proof of Corollary 1·10 proceeds as follows: Since $\mathbb{R}$ has the finite 2 intersection property, the family $\{B(\varphi(a_j), r_j)\}_{j=}^{n}$ has for a given $\varphi \in A^*$

a non empty intersection if and only if

$$|\varphi(a_k - a_l)| = |\varphi(a_k) - \varphi(a_l)| \leq r_k + r_l \; ; \quad k,l \in \{1,\dots,n\}.$$

It follows from the Hahn-Banach theorem, that $\{B(a_j,r_j)\}$ has the weak intersection property if and only if $\|a_k - a_l\| \leq r_k + r_l$ whenever $k,l \in \{1,\dots,n\}$.

Corollary 1·11. Let $n \in \mathbb{N}$ be given and let A be a real Banach space. Then A is an E(n) space if and only if A has the n,2 intersection property.

Proof. It follows from Proposition 1·7 and Corollary 1·10 that A is an E(n) space if and only if every family $\{B(a_j,R)\}_{j=1}^{n}$ of n balls in A with common radius R has a non empty intersection whenever any two of the balls have a non empty intersection. This property is what Lindenstrauss has defined as the restricted n,2 intersection property, and he has shown [8, Theorem 4·3] that this property is equivalent with the n,2 intersection property.

The complex analogue of the theorem of Lindenstrausss just referred to would be a theorem stating that in a complex E(n) space every family of n balls with the weak intersection property has the intersection property. The next lemma is the first step toward a result of this kind.

Lemma 1·12. Let A be a complex Banach space and let $a_1,\dots,a_n \in A$. Let $r = (r_j) \in \mathbb{R}^n$ be a multi-radius and let $\varepsilon > 0$.

Assume that

$$(1\cdot 6) \qquad \bigcap_{j=1}^{n} B(a_j, r_j + \varepsilon) = \emptyset .$$

Let R be a number such that $R > \max\{r_j : j = 1,\cdots,n\}$. Then there exist n elements $b_1,\cdots,b_n$ in the unit ball of A such that

$$(1\cdot 7) \qquad \bigcap_{j=1}^{n} B(a_j+(R-r_j)b_j,\ R+\tfrac{\varepsilon}{2}) = \emptyset .$$

<u>Remark.</u> If we discard the $\frac{\varepsilon}{2}$- term in (1·7), then the lemma above is contained (in the case of real Banach spaces) in [8 ,Proof of Theorem 4·3]. As remarked in that paper, the basic idea of the proof is due to O.Hanner [7]. The proof we are going to give is just a modification of that given in [8].

<u>Proof.</u> We shall construct the elements $b_1,\cdots,b_n$ inductively. Let $j \in \{0,1,\cdots,n-1\}$, and let us assume that we have constructed elements $b_1,\cdots,b_j$ in the unit ball of A such that

$$(1\cdot 8) \qquad \Big(\bigcap_{k=j+1}^{n} B(a_k,r_k+\varepsilon)\Big) \cap \bigcap_{k=1}^{j} B(a_k+(R-r_k)b_k, R+\tfrac{\varepsilon}{2}) = \emptyset .$$

(This means, by convention, that if $j = 0$, then (1·8) is the same as the condition (1·6), and if $j = n$, then (1·8) is the same as the equation (1·7).) Starting from (1·8) we shall construct an element b_{j+1} in the unit ball of A such that (1·8) is valid with $j+1$ instead of j. We define

$$(1\cdot 9) \qquad K_j = \Big(\bigcap_{k=1}^{j} B(a_k+(R-r_k)b_k, R+\tfrac{\varepsilon}{2})\Big) \cap \bigcap_{k=j+2}^{n} B(a_k,r_k+\varepsilon) .$$

Thus (1·8) means that K_j and $B(a_{j+1}, r_{j+1}+\varepsilon)$ are disjoint. By the separation theorem, there exists a continuous linear functional f on A with $\operatorname{Re} f \neq 0$ and such that

$$(1\cdot 10) \qquad s \overset{d}{=} \sup\{\operatorname{Re} f(x) : x \in B(a_{j+1}, r_{j+1}+\varepsilon)\} \leq \inf\{\operatorname{Re} f(x) : x \in K_j\}$$

Let S be the supremum of $\operatorname{Re} f$ on the unit ball of A. Then $S > 0$, and for any ball $B(a, r_o)$ in A we have the equation

$$(1.11) \qquad r_o S + \operatorname{Re} f(a) = \sup\{\operatorname{Re} f(x) : x \in B(a, r_o)\}.$$

In particular, the equation

$$(1\cdot 12) \qquad (r_{j+1}+\varepsilon)S = s - \operatorname{Re} f(a_{j+1})$$

is valid. Let $\delta > 0$ be a number to be fixed later. Choose $b \in B(0,1)$ such that

$$(1\cdot 13) \qquad \operatorname{Re} f(-b) \geq S - \delta,$$

and put $y_{j+1} = a_{j+1} + (R - r_{j+1})b$. Let $x \in B(y_{j+1}, R + \frac{\varepsilon}{2})$. By the definition of y_{j+1} and by (1·11) and (1·13), we get
$(R + \frac{\varepsilon}{2})S \geq \operatorname{Re} f(x - y_{j+1}) \geq \operatorname{Re} f(x - a_{j+1}) + (R - r_{j+1})(S - \delta)$.
It follows from these inequalities and from (1.12) that

$$(1\cdot 14) \qquad \begin{aligned} \operatorname{Re} f(x) &\leq (R + \tfrac{\varepsilon}{2})S + \operatorname{Re} f(a_{j+1}) + (R - r_{j+1})(\delta - S) = \\ &= (r_{j+1}+\varepsilon)^{-1}((r_{j+1} + \tfrac{\varepsilon}{2})s + \tfrac{\varepsilon}{2} \operatorname{Re} f(a_{j+1})) + \delta(R - r_{j+1}). \end{aligned}$$

Since it follows from (1·12) that $s > \operatorname{Re} f(a_{j+1})$, we can choose δ so small that the right hand side of (1·14) is less than s. With this choise of δ we put $b_{j+1} = b$. It then follows from (1·10) and (1·14) that

$$B(a_{j+1}+(R-r_{j+1})b_{j+1},\ R+\tfrac{\varepsilon}{2})\cap K_j = \emptyset ,$$

and this is exactly (1·8) with $j+1$ instead of j.

<u>Proposition 1·13·</u> Let $n \in \mathbb{N}$, let A be a complex Banach space and assume that A is an almost $E(n)$ space. Then any family of n balls in A with the weak intersection property has the almost intersection property.

<u>Proof.</u> Let $\{B(a_j,r_j)\}_{j=1}^{n}$ be a family of n balls in A with the weak intersection property. Assume that there exists an $\varepsilon > 0$ such that

$$(1\cdot15)\qquad \bigcap_{j=1}^{n} B(a_j,r_j+\varepsilon) = \emptyset .$$

Put $R = 1 + \max\{ r_j : j = 1,\dots,n\}$, and choose, by Lemma 1·12, elements $b_1,\dots,b_n$ in the unit ball of A such that

$$(1\cdot16)\qquad \bigcap_{j=1}^{n} B(a_j + (R-r_j)b_j, R+\tfrac{\varepsilon}{2}) = \emptyset.$$

We now show that the family $\{B(a_j+(R-r_j)b_j,R)\}_{j=1}^{n}$ has the weak intersection property. In fact, let $z \in H^n(\mathbb{C})$. Then, by Corollary 1·4,

$$\|\sum_{j=1}^{n} z_j(a_j+(R-r_j)b_j)\| \le \|\sum_{j=1}^{n} z_j a_j\| + \|\sum_{j=1}^{n} z_j(R-r_j)b_j\| \le$$

$$\le \sum_{j=1}^{n} |z_j| r_j + \sum_{j=1}^{n} |z_j|(R-r_j) = \sum_{j=1}^{n} |z_j| R.$$

This proves, by Corollary 1·4 , our assertion. It follows from Proposition 1·7 that (1·16) can not be valid. This contradiction shows that (1·15) can not be true.

§ 2 EXTENSION OF COMPACT OPERATORS

From now on, every normed space will be a complex normed space.

We have defined an almost E space as a Banach space A with the property that if $n \in \mathbb{N}$, then every operator $T : (H^n, \| \ \|_1) \to A$ admits for any $\varepsilon > 0$ an extension $\tilde{T} : l_1^n \to A$ such that $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$. Since H^n has co-dimension 1 in l_1^n, we say that $\tilde{T}$ is an immediate extension of T. In the present section we shall show that this immediate extension property remains valid whenever T is a compact operator from an arbitrary Banach space into an almost E space. From this result, together with a theorem of J. Lindenstrauss, we get our first main result, namely that the bidual of an almost E space is a $\mathcal{P}_1$ space.

Lemma 2·1· Let A, X, Y be normed spaces with $Y \subset X$. Let $T: Y \to A$ be an operator, let $x \in X \setminus Y$ and let $y_1, \cdots, y_n \in Y$. Then the family

$$\{B(Ty_j, \|T\| \, \|x-y_j\|): \ j = 1, \cdots, n\}$$

has the weak intersection property.

Proof. Let $z \in H^n$. Then

$$\|\sum_{j=1}^{n} z_j T y_j\| = \|T(\sum_{j=1}^{n} z_j y_j\| \leq \|T\| \ \|\sum_{j=1}^{n} z_j (y_j - x)\| \leq \sum_{j=1}^{n} |z_j| \ \|T\| \ \|x-y_j\| .$$

Hence the desired conclusion follows from Corollary 1.4.

We shall say that a family $\mathcal{F}$ of balls in A has the finite almost intersection property if every finite subfamily of $\mathcal{F}$ has the almost intersection property. Similarly we define the <u>finite intersection property.</u> It was proved in [8, Theorem 4.5] that if A is a real Banach space with the finite 2 intersection property, and if $\mathcal{F}$ is a family of balls in A with the finite intersection property, then $\mathcal{F}$ has the intersection property provided the centre set of $\mathcal{F}$ is relatively compact. In the next lemma we prove that if we are given such a family $\mathcal{F}$ in an arbitrary normed space A, then $\mathcal{F}$ will always have the almost intersection property. We prove this lemma with the same "modification of radii" technique as was used in [8] and in [2].

<u>Lemma 2.2.</u> Let A be a normed space and let $\mathcal{F} = \{B(a_j, r_j)\}_{j\in J}$ be a family of balls in A such that $\mathcal{F}$ has the finite almost intersection property. Assume that the centre set $\{a_j : j \in J\}$ of $\mathcal{F}$ is relatively compact. Then $\mathcal{F}$ has the almost intersection property.

<u>Proof.</u> Let F be a finite, non empty subset of J and let $\varepsilon > 0$. Then, by assumption, the set

$$I_{F,\varepsilon} = \bigcap_{j\in F} B(a_j, r_j + \varepsilon)$$

is non empty. For any $a \in A$, we put

$$r_{F,\varepsilon}(a) = \inf\{\|x-a\| : x \in I_{F,\varepsilon}\}.$$

Then

$$(2.1) \qquad |r_{F,\varepsilon}(a) - r_{F,\varepsilon}(b)| \le \|a-b\| \; ; \qquad a, b \in A$$

and

$$(2.2) \qquad B(a, r_{F,\varepsilon}(a) + \delta) \cap I_{F,\varepsilon} \ne \emptyset \; ; \qquad a \in A; \quad \delta > 0.$$

Let $a \in A$. We then observe that if $\delta \leq \varepsilon$, then $r_{F,\delta}(a) \geq r_{F,\varepsilon}(a)$. Since for any $x \in I_{F,\varepsilon}$ and any $j \in F$

$$r_{F,\varepsilon}(a) \leq \|x-a\| \leq r_j+\varepsilon +\|a_j-a\|,$$

we conclude that the limit

$$r_F(a) = \lim_{\varepsilon\to 0} r_{F,\varepsilon}(a) \tag{2.3}$$

exists. Hence, by (2.1),

$$|r_F(a) - r_F(b)| \leq \|a-b\| \; ; \quad a,b \in A. \tag{2.4}$$

Define

$$r_{\mathcal{F}}(a) = \sup\{r_F(a) : F \text{ a finite subset of } J\}. \tag{2.5}$$

Let $j \in J$ and let $\varepsilon > 0$. Then, by assumption, $B(a_j,r_j+\varepsilon) \cap I_{F,\varepsilon} \neq \emptyset$, and therefore $r_{F,\varepsilon}(a_j) \leq r_j+\varepsilon$. Hence

$$r_F(a_j) \leq r_j \tag{2.6}$$

and so

$$r_{\mathcal{F}}(a_j) \leq r_j. \tag{2.7}$$

We now add the ball $B(a_j,r_{\mathcal{F}}(a_j))$ to the family $\mathcal{F}$, and denote this new family $\mathcal{F}(j)$. We then claim that $\mathcal{F}(j)$ has the finite almost intersection property. Indeed, let F be a finite non empty subset of J, and let $\delta > 0$. Choose $\varepsilon > 0$ such that $\varepsilon < \delta$ and such that $r_{F,\varepsilon}(a_j) \leq r_F(a_j) + \frac{\delta}{2}$. It then follows from (2.2) that

$$\emptyset \neq B(a_j,r_{F,\varepsilon}(a_j)+ \tfrac{\delta}{2}) \cap I_{F,\varepsilon} \subset B(a_j,r_{\mathcal{F}}(a_j)+\delta) \cap I_{F,\delta} \quad ,$$

and this proves our claim. Since the set $\{a_j : j \in J\}$ is relatively compact, we can choose a sequence $\{j_k\}_{k=1}^{\infty} \subset J$ such that

(2.8) $\qquad \overline{\{a_j : j \in J\}} = \overline{\{a_{j_k} : k \in \mathbb{N}\}}$

Let $R_1 = r_{\mathcal{F}}(a_{j_1})$ and let $\mathcal{F}_1 = \mathcal{F}(j_1)$. Then $\mathcal{F}_1$ has the finite almost intersection property, and it follows from (2.7) that

$$R_1 \leq r_{j_1}$$

Inductively, we define for $k \geq 2$

$$R_k = r_{\mathcal{F}_{k-1}}(a_{j_k})$$

and

$$\mathcal{F}_k = \mathcal{F}_{k-1} \cup \{B(a_{j_k}, R_k)\} .$$

Then every $\mathcal{F}_k$ has the finite almost intersection property, and from (2.7) we conclude that

(2.9) $\qquad R_k \leq r_{j_k} , \qquad k = 1,2, \cdots$

Finally, we put

$$\mathcal{F}_\infty = \mathcal{F} \cup \{B(a_{j_k}, R_k) : k \in \mathbb{N}\}.$$

We then note that $\mathcal{F}_\infty$ has the finite almost intersection property. Let $\varepsilon > 0$ be given. By compactness, it follows from (2.8) that there exists a natural number $n(\varepsilon)$ such that

(2.10) $\qquad \{a_j : j \in J\} \subset \bigcup_{k=1}^{n(\varepsilon)} B(a_{j_k}, \frac{\varepsilon}{4}).$

Since $\mathcal{F}_\infty$ has the finite almost intersection property, we can find an

(2.11) $\qquad a \in \bigcap_{k=1}^{n(\varepsilon)} B(a_{j_k}, R_k + \frac{\varepsilon}{4}).$

Let $j \in J$ be given. Choose, by (2.10), $k \leq n(\varepsilon)$ such that

$\|a_j - a_{j_k}\| \leq \frac{\varepsilon}{4}$. We then get from (2.11)

$$(2.12) \qquad \|a-a_j\| \leq \|a-a_{j_k}\| + \|a_{j_k}-a_j\| \leq R_k + \frac{\varepsilon}{2}.$$

By the definition of R_k we can find a finite subset F of the index set of the family $\mathcal{F}_{k-1}$ such that $R_k \leq r_F(a_{j_k}) + \frac{\varepsilon}{4}$. It follows from (2.12), (2.4) and (2.6) that

$$\|a-a_j\| \leq r_F(a_{j_k}) + \frac{3}{4}\varepsilon \leq |r_F(a_{j_k}) - r_F(a_j)| +$$

$$+ r_F(a_j) + \frac{3}{4}\varepsilon \leq \|a_{j_k}-a_j\| + r_j + \frac{3}{4}\varepsilon.$$

However, by the choice of j_k, $\|a_{j_k}-a_j\| \leq \frac{\varepsilon}{4}$. We therefore get

$$\|a-a_j\| \leq r_j + \varepsilon, \qquad j \in J.$$

Theorem 2.3. The bidual A^{**} of an almost E space A is a $\mathcal{P}_1$ space.

Proof. It is sufficient, by [8, Theorem 2.1, proof of (4) $\Rightarrow$ (1)] (this proof is equally valid in a complex Banach space), to prove that A has the following property: For every pair of Banach spaces X,Y such that $Y \subset X$ and $\dim X/Y = 1$, for every compact operator $T: Y \to A$ and for any $\varepsilon > 0$ there exists an extension $\tilde{T}: X \to A$ of T such that $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$. Let then X,Y,T and ε be given as above. We can and shall assume that $\|T\| = 1$, and that $\varepsilon \leq 1$. Choose $x \in X \setminus Y$ such that $\|x\| = 1$. The operator T admits, by a basic lemma of Nachbin (see [8, Lemma 5.2]), an extension $\tilde{T}: X \to A$ satisfying $\|\tilde{T}\| \leq 1 + \varepsilon$ if and only if

$$(2.13) \qquad \bigcap_{y \in Y} B(Ty, \|x-y\|(1+\varepsilon)) \neq \emptyset.$$

The family $\{B(Ty,\|x-y\|)\}_{y\in Y}$ has, by Lemma 2.1 and Proposition 1.13, the finite almost intersection property. Let $M \geq 2$ be given. Since the set $\{Ty: \|y\| \leq M\}$ is relatively compact, the family $\{B(Ty,\|x-y\|):\|y\| \leq M\}$ has, by Lemma 2.2, the almost intersection property. Let $R = \inf\{\|x-y\|: y \in Y\}$. Then $R > 0$. Hence we can find an $a_M \in A$ such that

$$(2.14) \qquad \|a_M - Ty\| \leq \|x-y\| + R\varepsilon \leq \|x-y\|(1+\varepsilon); \quad \forall y \in B_Y(0,M).$$

In particular, if we choose $y = 0$, then $\|a_M\| \leq 2$. Let $y \in Y$ be such that $\|y\| > M$. Then $\|x-y\| \geq \|y\| - 1$ and $\|a_M-Ty\| \leq 2 + \|y\|$. Hence

$$(2.15) \qquad \|x-y\|^{-1}\|a_M - Ty\| \leq (\|y\|-1)^{-1}(\|y\|+2) \leq (M-1)^{-1}(M+2).$$

Therefore, if we choose M so large that $(M-1)^{-1}(M+2) \leq 1 + \varepsilon$, then it follows from (2.14) and (2.15) that (2.13) is valid.

Remark. The final part of the proof above is almost the same as in [8, Theorem 5.4, proof of (a) ⇒ (b)].

§ 3 THE EXTREME POINTS OF THE UNIT BALL OF $(H^n(\mathbb{C}), \| \|_r)$.

The need for finding the extreme points of the unit ball in $(H^n(\mathbb{C}), \| \|_r)$ stems from Lemma 1.9. In clear contrast to the real case, we show in Theorem 3.6 that the set of all extreme points of the unit ball in $(H^3(\mathbb{C}), \| \|_r)$ is "almost" the surface of that ball. In general, roughly said, a point on the surface of the unit ball in $(H^n(\mathbb{C}), \| \|_r)$ is an extreme point if and only if at most three of its coordinates are different from zero. We finish this section with some applications to $E(n)$ spaces.

Fix $n \in \mathbb{N}$. For a given multi-radius $r = (r_j) \in \mathbb{R}^n$, we define the following hyperplane in $\mathbb{C}^n$.

$$(3.1) \qquad H_r = H_r^n = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} z_j r_j = 0\} .$$

Furthermore, we let r^{-1} denote the multi-radius $(r_1^{-1}, \ldots, r_n^{-1})$. The following lemma has an obvious proof.

Lemma 3.1. The linear map

$$S : (H^n_{r^{-1}}, \| \|_1) \to (H^n, \| \|_r) : (z_j) \to (r_j^{-1} z_j)$$

is an isometry onto H^n.

Hence, in order to find the extreme points of the unit ball in $(H^n, \| \|_r)$, it suffices to find the extreme points of the unit ball in $(H^n_{r^{-1}}, \| \|_1)$.

Lemma 3.2. Let $n \geq 2$ and assume that $z \in (\mathbb{C}^n, \| \|_1)$ has a norm $\|z\|_1 = 1$. Assume that $z = \frac{1}{2}(p+q)$, where $p, q \in \mathbb{C}^n$ satisfy $\|p\|_1, \|q\|_1 \leq 1$. Then there exist n real numbers

$t_1,\dots,t_n \in [-1,+1]$ such that

$$(3.2) \qquad \begin{cases} p_j = (1+t_j)z_j \\ q_j = (1-t_j)z_j \end{cases} ; \quad j = 1,\dots,n .$$

<u>Proof</u>. Since $1 = \|z\|_1 \leq \frac{1}{2}(\|p\|_1 + \|q\|_1) \leq 1$, we must have

$$(3.3) \qquad \|q\|_1 = \|p\|_1 = 1 .$$

Put

$$(3.4) \qquad \alpha_j = p_j - z_j ; \quad j = 1,\dots,n .$$

Since $2z = p + q$, we get

$$(3.5) \qquad q_j = z_j - \alpha_j ; \quad j = 1,\dots,n .$$

Hence, by (3.3),

$$\sum_{j=1}^{n} |z_j+\alpha_j| + \sum_{j=1}^{n} |z_j-\alpha_j| = 2 = \sum_{j=1}^{n} 2|z_j| \leq$$

$$\leq \sum_{j=1}^{n} |z_j+\alpha_j| + \sum_{j=1}^{n} |z_j-\alpha_j| .$$

We therefore conclude that

$$|z_j+\alpha_j| + |z_j-\alpha_j| = 2|z_j| ; \quad j = 1,\dots,n .$$

But these equations tell us that every α_j is located on the degenerated ellipse with foci in z_j and $-z_j$. Hence there exist $t_1,\dots,t_n \in [-1,1]$ such that $\alpha_j = t_j z_j$ for any $j = 1,\dots,n$. When we combine this result with (3.4) and (3.5), we get (3.2).

The next lemma is crucial for the development in the present section.

Lemma 3.3. Let $n \geq 3$ and let $r = (r_j) \in \mathbb{R}^n$ be a multiradius. For any $z \in \mathbb{C}^n$ and any $j = 1,\ldots,n$, we define

$$R_j(z) = (|z_j|, r_j \operatorname{Re} z_j, r_j \operatorname{Im} z_j) \in \mathbb{R}^3 ,$$

and we put

$$J(z) = \{j \in \mathbb{N} : j \leq n \text{ and } z_j \neq 0\} .$$

Let $z \in H_r^n$ and assume that $\|z\|_1 = 1$. Then z is an extreme point of the unit ball of $(H_r^n, \| \|_1)$ if and only if the set $\{R_j(z) : j \in J(z)\}$ is linearly independent in $\mathbb{R}^3$.

Proof. Assume that z is not an extreme point. Then there exist $p,q \in H_r^n$ with $p \neq q$ and with $\|p\|_1 = \|q\|_1 = 1$ and such that $z = \frac{1}{2}(p+q)$. By Lemma 3.2, there exist $t_1,\ldots,t_n \in [-1,1]$ such that

$$(3.6) \qquad \begin{cases} p_j = (1+t_j)z_j \\ q_j = (1-t_j)z_j \end{cases} ; \quad j = 1,\ldots,n .$$

Hence

$$1 = \sum_{j=1}^{n} (1+t_j)|z_j| = 1 + \sum_{j=1}^{n} t_j |z_j| .$$

It follows that

$$(3.7) \qquad 0 = \sum_{j=1}^{n} t_j |z_j| = \sum_{j \in J(z)} t_j |z_j| .$$

Furthermore, since $z,p \in H_r^n$, it follows from (3.6) that

$$0 = \sum_{j=1}^{n} r_j p_j = \sum_{j=1}^{n} r_j z_j + \sum_{j=1}^{n} r_j t_j z_j = \sum_{j \in J(z)} t_j r_j z_j .$$

Taking real parts and imaginary parts in this equation, we get

$$(3.8) \qquad 0 = \sum_{j \in J(z)} t_j r_j \operatorname{Re} z_j = \sum_{j \in J(z)} t_j r_j \operatorname{Im} z_j .$$

Since $p \neq q$, we conclude from (3.6) that at least one $t_j \neq 0$. Thus, by (3.7) and (3.8), the set $\{R_j(z) : j \in J(z)\}$ is linearly dependent in $\mathbb{R}^3$. Assume conversely that this set is linearly dependent in $\mathbb{R}^3$, say

$$(3.9) \qquad \sum_{j\in J(z)} t_j R_j(z) = 0 ;$$

where at least one $t_j \neq 0$. By dividing this equation with $\max\{|t_j|\}$, we can and shall assume that each $t_j \in [-1,1]$. Put $t_j = 0$ if $j \in \{1,\ldots,n\} \setminus J(z)$, and define $p = ((1+t_j)z_j)_{j=1}^n$ and $q = ((1-t_j)z_j)_{j=1}^n$. Then $z = \frac{1}{2}(p+q)$, and since it follows from (3.9) that $\sum_{j=1}^n t_j r_j z_j = 0$, we conclude that $p,q \in H_r^n$. Furthermore, since, by (3.9), $\sum_{j=1}^n t_j |z_j| = 0$, we get

$$\|p\|_1 = \sum_{j=1}^n (1+t_j)|z_j| = \|z\|_1 = 1 = \|q\|_1 .$$

Finally, since at least one $t_j \neq 0$, we must have $p \neq q$. Hence z can not be an extreme point of the unit ball in $(H_r^n, \| \ \|_1)$.

Corollary 3.4. If $z \in H_r^n$ is an extreme point of the unit ball in $(H_r^n, \| \ \|_1)$, then the set $J(z) = \{j : z_j \neq 0\}$ can at most contain three elements.

Proof. Obvious.

Lemma 3.5. Let $r = (r_j) \in \mathbb{R}^3$ be a multi-radius and let $z \in H_r^3$. Let $R_j(z)$, $j = 1,2,3$ be defined as in Lemma 3.3. Then the set $\{R_j(z) : j = 1,2,3\}$ is linearly independent in $\mathbb{R}^3$ if and only if z_1 and z_2 are linearly independent in $\mathbb{C}$ (when we consider $\mathbb{C}$ as a linear space over $\mathbb{R}$). And if $r_1 z_1 + r_2 z_2 = 0$ and $|z_1| + |z_2| > 0$, then $R_1(z)$ and $R_2(z)$ are always linearly independent in $\mathbb{R}^3$.

Proof. Since $r_1 z_1 + r_2 z_2 + r_3 z_3 = 0$, we get

$$R_3(z) = (r_3^{-1}|r_1 z_1 + r_2 z_2|, -\mathrm{Re}(r_1 z_1 + r_2 z_2), -\mathrm{Im}(r_1 z_1 + r_2 z_2)) .$$

An easy calculation then shows that if z_1 and z_2 are linearly independent, then so are $R_1(z)$, $R_2(z)$ and $R_3(z)$. And an even easier calculation shows that if $r_1 z_1 + r_2 z_2 = 0$ and $|z_1| + |z_2| > 0$, then $R_1(z)$ and $R_2(z)$ are linearly independent. Conversely, if z_1 and z_2 are linearly dependent, then we can assume that there exists a real number s such that $z_2 = s z_1$. Since $R_3(z) = 0$ if $r_1 + r_2 s = 0$, we can and shall assume that $r_1 + r_2 s \neq 0$.

If $r_1 + r_2 s > 0$, put

$$t_1 = -r_2 s - r_3|s|,\ t_2 = r_1 + r_3,\ t_3 = r_3(r_1 + r_2 s)^{-1}(r_2 s - r_1|s|).$$

If $r_1 + r_2 s < 0$ and $r_1 \neq r_3$, put

$$t_1 = r_3|s| - r_2 s,\ t_2 = r_1 - r_3,\ t_3 = r_3(r_1 + r_2 s)^{-1}(r_1|s| - r_2 s).$$

And if $r_1 + r_2 s < 0$ and $r_1 = r_3$, put

$$t_1 = 1,\ t_2 = 0,\ t_3 = r_3(r_1 + r_2 s)^{-1} .$$

In any of these three cases, we get

$$t_1 R_1(z) + t_2 R_2(z) + t_3 R_3(z) = 0 .$$

Theorem 3.6. Let $n \geq 3$ and let $r = (r_j) \in \mathbb{R}^n$ be a multiradius. Then the set of all extreme points of the unit ball in $(H_r^n, \|\ \|_1)$ consists exactly of all points z of the form

$$z = u_k(r_m e_k - r_k e_m) + u_l(r_m e_l - r_l e_m) , \tag{3.10}$$

where $k,l,m \in \{1,\dots,n\}$ are mutually different, and where the complex numbers u_k and u_l satisfy the equation

$$(3.11) \qquad r_m(|u_k| + |u_l|) + |r_k u_k + r_l u_l| = 1 ,$$

and where furthermore u_k and u_l either are linearly independent or $r_k u_k + r_l u_l = 0$.

Proof. Let $z \in H_r^n$ be an extreme point of the unit ball in $(H_r^n, \| \|_1)$. Then $\|z\|_1 = 1$, and there exist, by Corollary 3.4, three different elements $k, l, m \in \{1, 2, \ldots, n\}$ such that $z_j = 0$ whenever j is different from k, l and m . We can and shall assume that z_k and z_l are different from zero. Then $z_m = -r_m^{-1}(r_k z_k + r_l z_l)$ and hence

$$z = z_k e_k + z_l e_l - r_m^{-1}(r_k z_k + r_l z_l) e_m =$$
$$= r_m^{-1} z_k (r_m e_k - r_k e_m) + r_m^{-1} z_l (r_m e_l - r_l e_m) .$$

If we let $u_k = r_m^{-1} z_k$ and $u_l = r_m^{-1} z_l$, the equation above gives us (3.10), and (3.11) follows from the equations

$$1 = |z_k| + |z_l| + |z_m| = r_m(|u_k| + |u_l|) + |r_k u_k + r_l u_l| .$$

Assume that $r_k u_k + r_l u_l \neq 0$. This means that $z_m \neq 0$, and hence $J(z) = \{j : z_j \neq 0\} = \{k, l, m\}$. It follows from Lemma 3.3 that $R_k(z)$, $R_l(z)$ and $R_m(z)$ are linearly independent, and we therefore conclude, by Lemma 3.5, that z_k and z_l are linearly independent. Hence u_k and u_l are linearly independent.

Assume conversely that z is given by (3.10), and that the requirements following (3.10) are satisfied. Then $z \in H_r^n$, and it follows from (3.11) that $\|z\|_1 = 1$. Therefore, in order to prove that z is an extreme point of the unit ball in $(H_r^n, \| \|_1)$, we have, by Lemma 3,3, to prove that the set $\{R_j(z) : z \in J(z)\}$ is linearly independent in $\mathbb{R}^3$. Now $\{k,l\} \subset J(z) \subset \{k,l,m\}$, and we note that the requirements posed on u_k and u_l imply that z_k and z_l either are linearly independent or $r_k z_k + r_l z_l = 0$. Since this equation is satisfied if and only if $J(z) = \{k,l\}$, we get, by Lemma 3.5, that $\{R_j(z): j \in J(z)\}$ is linearly independent in $\mathbb{R}^3$.

Corollary 3.7. A finite family of at least three balls in a normed space A has the weak intersection property if and only if any subfamily of three balls has the weak intersection property.

Proof. We have only to prove the if-part. Assume therefore that $\{B(a_j,r_j)\}_{j=1}^{n}$ is a family of n balls in A such that any subfamily of three balls has the weak intersection property. By Lemma 1.9 we have to prove that $\|\Sigma\, z_j a_j\| \leq 1$ whenever z is an extreme point of the unit ball in $(H^n, \|\ \|_r)$. But if z is such a point, then it follows from Lemma 3.1 and from Theorem 3.6 that the set $J(z) = \{j: z_j \neq 0\}$ can contain at most three elements. By assumption, we therefore get

$$\|\Sigma\, z_j a_j\| \leq \|z\|_r = 1 .$$

Comment. The Corollary 3.7 can also be given a simple proof with help of the Helly theorem on intersection of convex sets. On the other hand, if we start with Corollary 3.7 and choose $A = \mathbb{C}$, then we get, by Corollary 1.3, a proof of the Helly theorem (but only for closed balls in $\mathbb{C}$). We find this connection between Theorem 3.6 and the Helly theorem to be of some interest.

<u>Corollary 3.8.</u> Let $n \geq 3$ and let A be a Banach space. Then A is an $E(n)$ space if and only if for any $a_1,\dots,a_n \in A$ there exist $a \in A$, $k,l,m \in \{1,\dots,n\}$ and $u,v \in \mathbb{C}$ such that

$$(3.12)\quad \begin{cases} |u| + |v| + |u+v| = 1 \ , \\ \text{and} \\ \max_j\{\|a-a_j\|\} = \|u(a_k-a_m) + v(a_l-a_m)\| \ . \end{cases}$$

If (3.12) holds, then either

$$(3.13)\qquad \max_j\{\|a-a_j\|\} = \max_{i,j}\{\tfrac{1}{2}\|a_i-a_j\|\}$$

or

$$(3.14)\qquad \max_j\{\|a-a_j\|\} = \|a-a_k\| = \|a-a_l\| = \|a-a_m\| \ .$$

<u>Proof</u>. By Proposition 1.7, the space A is an $E(n)$ space if and only if for any $a_1,\dots,a_n \in A$ there exists $a \in A$ such that

$$\max_j\{\|a-a_j\|\} \leq \|T\| \overset{d}{=} \max\{\|\Sigma\ z_ja_j\|:z \in H^n \text{ and } \|z\|_1 \leq 1\} \ .$$

But the maximum on the right hand side of this inequality is attained in an extreme point of the unit ball in $(H^n,\|\ \|_1)$ (confer the proof of Lemma 1.9). Hence it follows from Theorem 3.6 that there exist indices k,l,m and complex numbers u,v with $|u| + |v| + |u+v| = 1$ such that

$$\|T\| = \|u(a_k-a_m) + v(a_l-a_m)\| \ .$$

Now we observe that if $a \in A$ and if $z \in H^n$, then

$$\|\Sigma\ z_ja_j\| = \|\Sigma\ z_j(a-a_j)\| \leq \|z\|_1 \max_j\{\|a-a_j\|\}.$$

Hence we always have

$$(3.15)\qquad \|T\| \leq \max_j\{\|a-a_j\|\} \ ; \qquad a \in A \ .$$

Thus we have proved the first statement of the corollary. As for the second statement, we note that if $u,v \in \mathbb{C}$, then

$$\|u(a_k-a_m)+v(a_1-a_m)\| \leq |u|\,\|a_k-a\| + |v|\,\|a_1-a\| + |u+v|\,\|a_m-a\| \leq$$
$$\leq (|u|+|v|+|u+v|)\max\{\|a-a_j\| : j=k,1,m\} .$$

Hence it follow from (3.12) that if $u\cdot v\cdot(u+v) \neq 0$, then (3.14) must be valid. And if $u\cdot v\cdot(u+v) = 0$, then it follows easily from (3.12) and (3.15) that (3.13) is true.

Comment. The equations (3.13) and (3.14) correspond to classical properties of triangles in the complex plane.

§ 4 THE CHARACTERIZATIONS OF THE E SPACES.

In the present section we show that a Banach space is an E space if and only if its dual is an L_1 space. The main step in order to prove this equivalence is the proof of Lemma 4.3. This lemma says (though we have not stated it in this way) that an almost $E(n+1)$ space is an $E(n)$ space. Once we have established this result, the stated characterization follows from the results of section 1 and section 2.

Let $n \geq 2$ and let $\mathcal{F} = \{B(a_j,r_j)\}_{j=1}^{n}$ be a family of n balls in A with the weak intersection property. If $a \in A$, then there exists $R > 0$ such that the family $\mathcal{F} \cup \{B(a,R)\}$ has the weak intersection property. In fact, if $z \in \mathbb{C}^n$, then it follows from the identity

$$\sum_j z_j a_j = \sum_j (z_j - n^{-1}\sum_k z_k)a_j + n^{-1}(\sum_k z_k)\sum_j a_j$$

and from Corollary 1·4 that

$$\|\sum_j z_j a_j - (\sum_j z_j)a\| \leq \sum_j |z_j| r_j + |\sum_j z_j|(\|a\| + n^{-1}(\sum_j r_j + \|\sum_j a_j\|)).$$

Hence

$$R = \|a\| + n^{-1}(\sum_j r_j + \|\sum_j a_j\|)$$

will have the stated property.

We define

$$(4\cdot1) \qquad R_{\mathscr{F}}(a) = \inf\{R > 0 : \mathscr{F} \cup \{B(a,R)\} \text{ has the w.i.p.}\}$$

(here w.i.p. stands for weak intersection property). We note that if A is a real normed space, then, by Corollary 1.10,

$$R_{\mathscr{F}}(a) = \max_j \{\|a-a_j\| - r_j\}.$$

In the complex case, the function $a \to R_{\mathscr{F}}(a)$ is much more involved. However, in the next lemma we show that it has an important continuity property.

Lemma 4·1. If the family $\mathscr{F} = \{B(a_j, r_j)\}_{j=1}^{n}$ has the weak intersection property, then the function

$$R_{\mathscr{F}} : A \to \mathbb{R} : a \to R_{\mathscr{F}}(a)$$

has the following continuity property : For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $a \in A$ satisfies

$$\|a-a_j\| \leq r_j + \delta, \qquad j = 1,\cdots,n;$$

then $R_{\mathscr{F}}(a) < \varepsilon$.

Proof. It follows from Corollary 3.7 that for any $a \in A$

$$R_{\mathcal{F}}(a) = \max\{R_{\mathcal{H}}(a) : \mathcal{H} \subset \mathcal{F} \text{ and card } \mathcal{H} = 2\}.$$

We can therefore, without loss of generality, assume that $n = 2$. Since $\mathcal{F}$ has the weak intersection property, it follows that

(4.2) $$\|a_1 - a_2\| \leq r_1 + r_2.$$

For any $a \in A$ and any complex numer $a \neq -1$ we define

(4.3) $$f(a,u) = \|a_1 - a + (u+1)^{-1}(a_2 - a_1)\| - |u+1|^{-1}(|u|r_1 + r_2).$$

We then claim that

(4.4) $$R_{\mathcal{F}}(a) = \max\{0, \sup_{u \neq -1} f(a,u)\}.$$

In fact, by Corollary 1.4, the family $\mathcal{F} \cup \{B(a,R)\}$ has the weak intersection property if and only if

(4.5) $$\|z_1(a_1-a) + z_2(a_2-a)\| - |z_1|r_1 - |z_2|r_2 \leq |z_1+z_2|R, \quad z \in \mathbb{C}^2.$$

Therefore, if (4.5) holds and if we choose $z_1 = u \neq -1$ and $z_2 = 1$, then we get

(4.6) $$\sup_{u \neq -1} f(a,u) \leq R,$$

and thus

$$\max\{0, \sup_{u \neq -1} f(a,u)\} \leq R_{\mathcal{F}}(a).$$

Assume conversely that $R > 0$ satisfies (4.6). Letting $|u|$ tend to infinity, we get $\|a_1 - a\| - r_1 \leq R$; and this is the inequality (4.5) with $z_1 = 1$ and $z_2 = 0$. Since (4.2) implies that (4.5) is always satisfied when $z_1 + z_2 = 0$, we conclude that (4.6) will imply (4.5).

Hence, if $\sup_{u \neq -1} f(a,u) > 0$, then $0 \leq R_{\mathcal{F}}(a) \leq \sup_{u \neq -1} f(a,u)$, and if $\sup_{u \neq -1} f(a,u) \leq 0$, then $0 \leq R_{\mathcal{F}}(a) \leq R$ for any $R > 0$. This proves (4.4). Therefore, in order to prove the lemma, we have to verify the following statement.

(U) For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $a \in A$ and $\|a-a_j\| \leq r_j + \delta$, $j = 1,2$; then $f(a,u) \leq \varepsilon$ for any $u \in \mathbb{C} \setminus \{-1\}$.

We note that it follows from (4.2) that

$$f(a,u) \leq \|a-a_1\| + |u+1|^{-1}(1-|u|)r_1, \quad a \in A. \tag{4.7}$$

Now, given $\varepsilon > 0$, there exists a $K > 0$ such that if $|u| > K$, then

$$|u+1|^{-1}(1-|u|) \leq -1 + \frac{\varepsilon}{2r_1} .$$

Therefore, if $a \in A$ satisfies $\|a_1-a\| \leq r_1 + \frac{\varepsilon}{2}$ and if $|u| > K$, then, by (4.7),

$$f(a,u) \leq r_1 + \frac{\varepsilon}{2} + r_1(-1 + \frac{\varepsilon}{2r_1}) = \varepsilon.$$

It is therefore, by a compactness argument, sufficient to prove (U) <u>locally</u>. At this point we observe that if $a \in A$ and if $u \neq -1$, then

$$f(a,u) \leq |u+1|^{-1}(|u|(\|a_1-a\|-r_1) + \|a_2-a\|-r_2). \tag{4.8}$$

Therefore, if $\delta > 0$ is given and if

$$\|a-a_j\| \leq r_j+\delta, \qquad j = 1,2 \; ; \tag{4.9}$$

then for any $u \neq -1$

$$(4.10) \qquad f(a,u) \leq |u+1|^{-1}(|u|+1)\delta \leq \delta(1+2|u+1|^{-1}).$$

Let $u_0 \in \mathbb{C}\setminus\{-1\}$ and let $\varepsilon > 0$ be given. Choose $\delta = \delta(u_0) = \frac{\varepsilon}{2}|u_0+1|(|u_0|+1)^{-1}$. We can then find, by (4.10), a neighbourhood V of u_0 such that if $u \in V$ and if $a \in A$ satisfies (4.9), then $f(a,u) \leq \varepsilon$. It follows that the proof of (U) will be finished, once we have proved the following statement.

(U_1) For any $\varepsilon > 0$ there exists a $\delta > 0$ and a neighbourhood V of -1 such that if $\|a-a_j\| \leq r_j + \delta$, $j = 1,2$; then $f(a,u) < \varepsilon$ whenever $u \in V\setminus\{-1\}$.

This statement shall first be proved in the case where (4.2) is a strict inequality, that is in the case where

$$(4.11) \qquad \|a_1-a_2\| < r_1+r_2.$$

By (4.3), we get for any $a \in A$ and any $u \in \mathbb{C}\setminus\{-1\}$

$$(4.12) \qquad f(a,u) \leq \|a_1-a\| + |u+1|^{-1}(\|a_1-a_2\|-|u|r_1-r_2).$$

Let $t = \frac{1}{2}(r_1+r_2-\|a_1-a_2\|)$. Thus (4.11) means that $t > 0$. Hence there exists a neighbourhood V_1 of -1 such that

$$(4.13) \qquad \|a_1-a_2\|-r_1|u|-r_2 \leq -t, \qquad u \in V_1.$$

Define

$$V = \{u \in V_1 : |u+1| \leq t(r_1+1)^{-1}\}.$$

Then V is a neighbourhood of -1. Let $a \in B(a_1,r_1+1)$ and let $u \in V\setminus\{-1\}$. It then follows from (4.12), (4.13) and the definition of V that

$$f(a,u) \leq r_1 + 1 - t|u+1|^{-1} \leq 0.$$

Therefore, in the case $\|a_1 - a_2\| < r_1 + r_2$, we have proved a much stronger statement than (U_1). Hence it remains to prove (U_1) in the case where we assume that

$$(4.14) \qquad \|a_1 - a_2\| = r_1 + r_2.$$

In this case we notice that the inequality $f(a,u-1) \leq \varepsilon$ is equivalent with the inequality

$$(4.15) \qquad \|a_1 - a - u^{-1}(a_1 - a_2)\| \leq \varepsilon + |u|^{-1}(\|a_1 - a_2\| + r_1(|u-1|-1)).$$

Thus, if $a \in A$ satisfies (4.9), then it follows from (4.10) that for any $u \neq 0$

$$(4.16) \qquad \|a_1 - a - u^{-1}(a_1 - a_2)\| \leq \delta(1 + 2|u|^{-1}) + |u|^{-1}(\|a_1 - a_2\| + r_1(|u-1|-1)).$$

Let $t \in \langle 0,1]$, then we have for any $u \neq 0$ and any $a \in A$

$$\|a_1 - a - (tu)^{-1}(a_1 - a_2)\| \leq \|a_1 - a - u^{-1}(a_1 - a_2)\| + (t^{-1} - 1)|u|^{-1}\|a_1 - a_2\|.$$

Hence, if u satisfies (4.15), then

$$(4.17) \qquad \|a_1 - a - (tu)^{-1}(a_1 - a_2)\| \leq \varepsilon + |tu|^{-1}\|a_1 - a_2\| + |u|^{-1} r_1(|u-1|-1) =$$

$$= \varepsilon + |tu|^{-1}(\|a_1 - a_2\| + r_1(|tu-1|-1)) +$$

$$+ r_1(|u|^{-1}(|u-1|-1) - |tu|^{-1}(|tu-1|-1)).$$

Therefore, if we can make the last term on the right hand side of (4.17) small, then tu will satisfy the inequality (4.15), say with 2ε instead of ε, whenever it is satisfied by u.

We shall therefore have need for the following simple

<u>Lemma 4.2.</u> For every $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that if u is a complex number with $|u| = \delta_1$ and if $t \in \langle 0,1]$, then

$$(4.18) \qquad \left| |u|^{-1}(|u-1|-1) - |tu|^{-1}(|tu-1|-1) \right| \leq \varepsilon_1 .$$

Let us assume that Lemma 4.2 is proved. Let $\varepsilon > 0$ be given. Let $\varepsilon_1 = \frac{1}{2}\varepsilon r_1^{-1}$ and choose δ_1 in accordance with Lemma 4.2. Let $\delta = \frac{1}{2}\varepsilon(1+\frac{2}{\delta_1})^{-1}$, and let $a \in A$ satisfy (4.9). Choose $u \in \mathbb{C}$ such that $|u| = \delta_1$. It then follows from (4.16) that

$$\|a_1-a-u^{-1}(a_1-a_2)\| \leq \delta(1+2\delta_1^{-1}) + |u|^{-1}(\|a_1-a_2\|+r_1(|u-1|-1))$$

$$= \frac{\varepsilon}{2} + |u|^{-1}(\|a_1-a_2\|+r_1(|u-1|-1)).$$

This means that u satisfies (4.15) (with $\frac{\varepsilon}{2}$ instead of ε). Let $t \in \langle 0,1]$. We then get from (4.17) and (4.18)

$$\|a_1-a-(tu)^{-1}(a_1-a_2\| \leq \frac{\varepsilon}{2} + |tu|^{-1}(\|a_1-a_2\|+r_1(|tu-1|-1)) + \frac{\varepsilon}{2} .$$

We have therefore proved the inequality (4.15) for any $u \in \mathbb{C}$ such that $0 < |u| \leq \delta_1$ and for any $a \in A$ satisfying (4.9). Thus we have proved the statement (U_1).

<u>Proof of Lemma 4.2.</u> We define the function h on $[-\pi,\pi] \times \langle 0,1]$ by the formula

$$h(\theta,t) = t^{-1}(|te^{i\theta}-1|-1)$$

It is sufficient to prove that h is uniformly continuous, and hence it will suffice to prove that h admits a continuous extension to $[-\pi,\pi] \times [0,1]$. But we have

$$h(\theta,t) = \frac{\sqrt{1+t(t-2\cos\theta)}-1}{t} = \frac{t-2\cos\theta}{\sqrt{1+t(t-2\cos\theta)}+1},$$

and it is therefore immediate that h admits a continuous extension to $[-\pi,\pi]\times[0,1]$.

The next lemma is crucial for the characterization of the E spaces.

Lemma 4.3. Let $n\in\mathbb{N}$ and let A be a complex Banach space with the property that any family of $n+1$ balls in A with the weak intersection property has the almost intersection property. Then it is true that any family of n balls in A with the weak intersection property has the intersection property.

Comment. The hypothesis of this lemma concerns families of $n+1$ balls, whereas the conclusion is about a family of only n balls. In the real case, Lindenstrauss [8] was able to improve a result of Aronszajn and Panitchpakdi [2] and could show that the conclusion above is valid for a family of $n+1$ balls. It follows from Proposition 4.8 that if $n \geq 6$, then this stronger conclusion is also valid in the complex case. It is probably true that this holds for any $n \geq 1$, but we have not been able to prove this.

Proof. Let $\mathcal{F} = \{B(a_j,r_j)\}_{j=1}^{n}$ be a family of n balls in A with the weak intersection property. If we choose $\varepsilon = \frac{1}{2}$ in Lemma 4.1, we can find a $\delta_1 < \frac{1}{2}$ such that if $a\in A$ satisfies

(4.20) $\quad \|a-a_j\| \leq r_j + \delta_1, \quad j = 1,\cdots,n;$

then $R_{\mathcal{F}}(a) < \frac{1}{2}$. Since $\mathcal{F}$ has the weak intersection property, it follows, by hypothesis, that there exists an element $a^{(1)}\in A$ satisfying (4.20). Hence $R_{\mathcal{F}}(a^{(1)}) < \frac{1}{2}$, and so the family

$\mathcal{F} \cup \{B(a^{(1)}, \frac{1}{2})\}$ has the weak intersection property. Choosing $\varepsilon = 2^{-2}$ in Lemma 4.1, we can find a $\delta_2 < 2^{-2}$ such that if $a \in A$ satisfies

$$(4.21) \qquad \|a - a_j\| \leq r_j + \delta_2, \qquad j = 1, \cdots, n;$$

then $R_{\mathcal{F}}(a) < 2^{-2}$. Since $\mathcal{F} \cup \{B(a^{(1)}, \frac{1}{2})\}$ has the weak intersection property, we can, by hypothesis, find an $a^{(2)} \in A$ such that $\|a^{(2)} - a^{(1)}\| \leq \frac{1}{2}$ and such that $a^{(2)}$ satisfies (4.21). Let us assume that we have constructed $a^{(1)}, \cdots, a^{(k)} \in A$ and positive numbers $\delta_1, \cdots, \delta_k$ such that

$$(4.22) \quad \begin{array}{ll} \text{(i)} & \|a^{(i+1)} - a^{(i)}\| \leq 2^{-i} + \delta_{i+1}; \qquad i = 1, \cdots, k-1 \\ \text{(ii)} & \|a^{(i)} - a_j\| \leq r_j + \delta_i \; ; \quad j = 1, \cdots, n; \quad i = 1, \cdots, k. \end{array}$$

Let us also assume that every δ_i is less than 2^{-i} and that δ_i is chosen such that if $\varepsilon = 2^{-i}$ in Lemma 4.1, then the conclusion of that lemma is valid with $\delta = \delta_i$. In particular, we assume that the family $\mathcal{F} \cup \{B(a^{(k)}, 2^{-k})\}$ has the weak intersection property. Choose $\delta_{k+1} < 2^{-k-1}$ such that the conclusion of Lemma 4.1 is valid when $\varepsilon = 2^{-k-1}$ and with $\delta = \delta_{k+1}$. By hypothesis, there exists an $a^{(k+1)} \in A$ such that

$$\|a^{(k+1)} - a^{(k)}\| \leq 2^{-k} + \delta_{k+1}$$

and

$$\|a^{(k+1)} - a_j\| \leq r_j + \delta_{k+1}, \qquad j = 1, \cdots, n.$$

We have therefore, by induction, constructed a sequence $\{a^{(i)}\}_{i=1}^{\infty} \subset A$ and a sequence $\{\delta_i\}$ of positive numbers such that $\delta_i \leq 2^{-i}$, $i = 1, 2, \cdots$, and such that (4.22) is valid for any $i \in \mathbb{N}$.

In particular, we get, by (4.22) (i), that the sequence $\{a_j\}_{j=1}^{\infty}$ is a Cauchy-sequence. Hence $a = \lim_{i\to\infty} a^{(i)}$ exists in A. From (4.22) (ii) we then get

$$\|a-a_j\| \leq r_j + \lim_{i\to\infty} \delta_i = r_j \ ; \quad j = 1,\cdots,n.$$

This shows that a belongs to every member of the family $\mathcal{F}$.

Corollary 4.4. Assume that A fulfills the hypothesis of Lemma 4.3. Then every family of n balls in A with the almost intersection property has the intersection property.

Proof. This follows at once from Lemma 1.6 and Lemma 4.3.

Corollary 4.5. Let A be a complex Banach space such that the bidual A^{**} of A is a $\mathcal{P}_1$ space. Then every finite family of balls in A with the weak intersection property has the intersection property. In particular, the space A is an E space.

Proof. By Proposition 1.8, the hypothesis of Lemma 4.3 is fulfilled for any $n \in \mathbb{N}$. Hence the desired conclusion follows from Lemma 4.3 and Proposition 1.7.

Corollary 4.6. Let $n \in \mathbb{N}$ and assume that the complex Banach space A is an almost E(n+1) space. Then every family of n balls in A with the weak intersection property has the intersection property. In particular, the space A is an E(n) space.

Proof. By Proposition 1.13, the hypothesis of Lemma 4.3 is fulfilled. Hence the statement follows from Lemma 4.3 and Proposition 1.7.

Corollary 4.7. Let $n \geq 3$ and let A be an almost $E(n+1)$ space. Then A has the $n,3$ intersection property.

Proof. Let $\mathcal{F}$ be a family of n balls in A such that any three members of $\mathcal{F}$ have a non empty intersection. It then follows from Corollary 3.7 that $\mathcal{F}$ has the weak intersection property. Hence $\mathcal{F}$ has, by Corollary 4.6, the intersection property.

Let $k \geq 1$ be an integer. We say that a Banach space A has the C_k property if for any family $\{B(a_j,r_j)\}_{j=1}^{k}$ of k balls in A with a non empty intersection there exists for any $\varepsilon > 0$ a $\delta > 0$ such that if

$$a \in \bigcap_{j=1}^{k} B(a_j, r_j + \delta),$$

then

$$\operatorname{dist}(a, \bigcap_{j=1}^{k} B(a_j,r_j)) < \varepsilon.$$

Every Banach space has trivially the C_1 property. When $k \geq 2$ we do not know if it is true that every Banach space has the C_k property. However, if A is an almost $E(k+2)$ space, then it is true that A has the C_k property. In fact, if A is an almost $E(k+2)$ space, then it follows from Corollary 4.6 that a family of $k+1$ balls in A has a non empty intersection (if and only if it has the weak intersection) property. Hence, if $\mathcal{F} = \{B(a_j r_j)\}_{j=1}^{k}$ is a family of k balls in A with a non empty intersection, then

$$R_{\mathcal{F}}(a) = \operatorname{dist}(a, \bigcap_{j=1}^{k} B(a_j,r_j)), \quad a \in A,$$

where $R_{\mathcal{F}}$ is the function defined by (4.1). It therefore follows from Lemma 4.1 that A has the C_k property.

Let $n,k \in \mathbb{N}$ and assume that $n \geq k$. We say that a Banach space A (real or complex) has the <u>almost</u> n,k <u>intersection property</u> if every family of n balls in A has the almost intersection property whenever any k balls of the family have a non empty intersection. We now observe that almost exactly the same proof as in [8, proof of Theorem 4.1] gives us the following

<u>Lemma 4.7.</u> Let $k \geq 2$ be an integer and let n be an integer such that

$$(4.23) \qquad n > \tfrac{1}{2}(4k-5+\sqrt{8(k-1)^2+1}).$$

Let A be a real or complex Banach space with the C_{k-1} property. If A has the almost n,k intersection property, then A has the finite k intersection property.

<u>Proposition 4.8.</u> If A is an almost E(7) space, then A is an E space.

<u>Proof.</u> It follows from Corollary 3.7 and Proposition 1.13 that A has the almost 7,3 intersection property. Since $7 > \frac{1}{2}(7+\sqrt{33})$ and since A has the C_2 property, we get from Lemma 4.7 that A has the finite 3 intersection property. Now let $\mathcal{F}$ be a finite family of balls in A with the weak intersection property. We then conclude from Corollary 4.6 that any three members of $\mathcal{F}$ have a non empty intersection. Thus $\mathcal{F}$ itself has a non empty intersection.

We summarize the main results of the present paper in the following

<u>Theorem 4.9.</u> Let A be a complex Banach space. Then the following properties are equivalent

(i) The dual A^* of A is an L_1 space.

(ii) The bidual A^{**} of A is a $\mathcal{P}_1$ space.

(iii) A is an E space.

(iv) Every finite family of balls in A with the weak intersection property has the intersection property.

(v) Every family of seven balls in A with the weak intersection property has the intersection property.

(vi) A is an almost E(7) space.

Proof. We remarked in the preliminaries that the equivalence of (i) and (ii) follows from a theorem of S. Sakai [11].

(ii) ⇒ (iii) Corollary 4.5.

(iii) ⇒ (iv) Corollary 4.6.

(iv) ⇒ (v) Trivial.

(v) ⇒ (vi) Proposition 1.7.

(vi) ⇒ (ii) Proposition 4.8 together with Theorem 2.3.

§ 5 SOME EXAMPLES AND OPEN PROBLEMS.

We stated in the introduction that it is possible to find an example of three normed spaces A, X and Y with $Y \subset X$ and of a linear operator $T: Y \to A$ such that for some $x \in X \setminus Y$ there exist $y_1, y_2, y_3 \in Y$ with the property that

$$(5.1) \qquad \bigcap_{j=1}^{3} B_A(Ty_j, \|T\| \cdot \|x-y_j\|) = \emptyset$$

The following example may be considered as the complex analogue of an example in [1,p.125]. We want to thank Erik M. Alfsen for some suggestive remarks on this subject.

<u>Example 5.1.</u> Let $X = l_1^4(\mathbb{C})$, let $A = Y = (H^4(\mathbb{C}), \| \ \|_1)$ and let $T: Y \to A$ be the identity map. Furthermore, let $y_j = e_4 - e_j$, $j = 1,2,3$, and let $x = e_4$. Then (5·1) is satisfied.

<u>Proof.</u> We note that $\|T\| = 1$. Let us assume that for some $z \in H^4$ it is true that

$$(5{\cdot}2) \qquad \|z-y_j\|_1 \le \|e_4-y_j\|_1, \qquad j = 1,2,3.$$

Since $z_4 = -\Sigma_{k=1}^3 z_k$, it follows that

$$(5{\cdot}3) \qquad \sum_{k\neq j,4} |z_k| + |z_j+1| + |1 + \sum_{k=1}^{3} z_k| \le 1, \qquad j = 1,2,3.$$

Adding these inequalities, we obtain

$$(5{\cdot}4) \qquad \sum_{j=1}^{3} |z_j+1| + 2\sum_{j=1}^{3}|z_j| + 3|1 + \sum_{j=1}^{3} z_j| \le 3.$$

However, if $j = 1,2,3$, then

$$1 \le |z_j+1| + |z_j| \le |z_j+1| + 2|z_j| + |1+\sum_{j=1}^{3} z_j|,$$

and the last inequality is a strict one if $z_j \neq 0$. It follows that if some $z_j \neq 0$, then

$$3 < \sum_{j=1}^{3} |z_j+1| + 2\sum_{j=1}^{3} |z_j| + 3|1 + \sum_{j=1}^{3} z_j|.$$

By (5·4) we therefore conclude that $z_1 = z_2 = z_3 = 0$. But (5·4) will not be satisfied with this choice of z_1, z_2 and z_3. Hence (5·2) can not be valid for any $z \in H^4$.

It follows from Corollary 4.7 that an E space always has the finite 3 intersection property. We pose the converse of this as the following

<u>Problem 1.</u> If a complex Banach space A has the finite 3 intersection property, does it follow that A is an E space ?

We remark that it suffices in this problem to show that A is an $E(3)$ space, or to show that A is an almost $E(4)$ space.

We think, at least when A is a finite dimensional space, that the following example gives some weight to a conjecture that Problem 1 has a positive solution.

In what follows, D is the closed unit disc in the complex plane $\mathbb{C}$.

<u>Example 5.2.</u> Let $f: [0,1] \to \mathbb{R}$ be a concave, monotonely decreasing, non negative C^1-function different from 0. Let

$$K = \{(z_1, z_2) : z_1 \in D, z_2 \in \mathbb{C} \quad \text{and} \quad |z_2| \leq f(|z_1|)\}$$

Then K is the unit ball of a norm $\| \ \|$ on $\mathbb{C}^2$, and if the space $(\mathbb{C}^2, \| \ \|)$ has the $4,3$ intersection property, then f is a constant and hence $(\mathbb{C}^2, \| \ \|)$ is isometric to $(\mathbb{C}^2, \| \ \|_\infty)$.

<u>Proof.</u> The first statement follows from the fact that K is a closed convex set with interior points and with the property that $uz \in K$ whenever $z \in K$ and $u \in D$. Assume therefore that $(\mathbb{C}^2, \| \ \|)$ has the $4,3$ intersection property. First of all we remark that if $a = (\alpha_1, \alpha_2)$ and $a_j = (\alpha_{1,j}, \alpha_{2,j})$, $j = 1, \cdots, r$ are given elements of $\mathbb{C}^2$, then

$$a \in \bigcap_{j=1}^{r} \{K + a_j\}$$

if and only if

$$\alpha_1 \in \bigcap_{j=1}^{r} \{D+\alpha_{1,j}\} \quad \text{and} \quad \alpha_2 \in \bigcap_{j=1}^{r} B(\alpha_{2,j}, f(|\alpha_1-\alpha_{1,j}|)).$$

Now let $t \in [0,1]$ and $\theta \in [0,\frac{\pi}{2}]$ be given. We define

$$\alpha_{1,2} = t\sin\theta = -\alpha_{1,1} \ ; \ \alpha_{1,4} = it\sin\theta = -\alpha_{1,3}$$

and

$$\alpha_{2,1} = \alpha_{2,2} = 0 \ ; \ \alpha_{2,4} = \alpha_{2,3} = f(t) + f(t|\sin\theta - \cos\theta|).$$

We note that the point $x = t\cos\theta$ belongs to any of the three balls $D+\alpha_{1,j}$, $j = 1,2,3$, and we find that $|x-\alpha_{1,2}| = t|\sin\theta-\cos\theta|$ whereas $|x-\alpha_{1,3}| = |x-\alpha_{1,4}| = t$. Hence the three balls $B(\alpha_{2,j}, f(|x-\alpha_{1,j}|))$, $j = 1,2,3$ have a non empty intersection. By symmetry we therefore conclude that any three members of the family $\{K+(\alpha_{1,j},\alpha_{2,j})\}_{j=1}^{4}$ have a non empty intersection. Hence there exists, by assumption, a number $p \in \bigcap_{j=1}^{4} \{D+\alpha_{1,j}\}$ such that the family

$$\{B(\alpha_{2,j}, f(|p-\alpha_{1,j}|))\}_{j=1}^{4}$$

has a non empty intersection. This means that

$$(5.5) \qquad f(t) + f(t|\sin\theta - \cos\theta|) \leq$$

$$\leq \min\{f(|p-\alpha_{1,1}|), f(|p-\alpha_{1,2}|)\} + \min\{f(|p-\alpha_{1,3}|), f(|p-\alpha_{1,4}|)\}.$$

Since $p \in \cap \{D+\alpha_{1,j}\}$, it follows by a simple argument that the right hand side of (5.5) is less or equal $2f(t\sin\theta)$. Hence f must satisfy the inequality

$$(5.6) \qquad f(t)+f(t|\sin\theta-\cos\theta|) \leq 2f(t\sin\theta) \ ; \quad t \in [0,1], \ \theta \in [0,\tfrac{\pi}{2}]$$

We shall show that (5.6) implies that f is a constant. Let $t \in \langle 0,1\rangle$

and let $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}\rangle$. It then follows from (5.6) that

$$\frac{f(t) - f(t\sin\theta)}{t - t\sin\theta} \leq \frac{f(t\sin\theta) - f(t(\sin\theta - \cos\theta))}{t - t\sin\theta}$$

Hence we get

$$(5.7) \qquad f'(t) \leq \lim_{\theta \to \frac{\pi}{2}} (-f'(t\sin\theta) + f'(t(\sin\theta - \cos\theta))(1 + \frac{\sin\theta}{\cos\theta}))$$

If $f'(t) < 0$, then the right hand side of (5.7) is $-\infty$. Since $f'(t) > -\infty$, it follows that $f'(t) \geq 0$, and since f is decreasing, we conclude that $f'(t) = 0$. Hence f is a constant. It is then clear that $(\mathbb{C}^2, \| \ \|)$ is isometric to $(\mathbb{C}^2, \| \ \|_\infty)$.

In connection with Lemma 4.3, we remarked that it is probably true that the conclusion of that lemma can be strengthened to a statement about $n+1$ balls. We pose this as the following

<u>Problem 2.</u> Let $n \leq 6$. If A is an almost $E(n)$ space, does it follow that A is an $E(n)$ space ?

This problem is akin to the following

<u>Problem 3.</u> What is the smallest natural number $n \leq 7$ such that if A is an almost $E(n)$ space, then A is an E space ?

We remark that problem 3 is closely connected with a problem raised by Lindenstrauss in [8,p.32], namely the problem whether 7 is the smallest number n with the property that if a Banach space A has the $n,3$ intersection property, then A has the finite 3 intersection property.

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UNIVERSITY OF OSLO, NORWAY