A NOTE ON INTERSECTION MULTIPLICITIES

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Let $R, \mathfrak{m}$ denote a local (noetherian) ring and let M and N be R-modules such that $M \otimes N$ has finite length $l(M \otimes N)$. If R is regular we can define the intersection multiplicity:

$$\chi^R(M,N) = \Sigma_i (-1)^i l(\mathrm{Tor}^R{}_i(M,N))$$

as in Serre [2]. The purpose of this note is to prove the following theorem which shows that there is a natural way to extend the notion of intersection multiplicities to the theory of modules over arbitrary local rings R. The $R/\mathfrak{m}$ - vectorspace dimension of $\mathfrak{m}/\mathfrak{m}^2$ will be called the imbedding dimension of R.

THEOREM. Let C be an arbitrary local (noetherian) ring, and let M and N be C - modules of finite type such $M \otimes N$ has finite length. Assume that $A \to C$ and $B \to C$ are surjective ringhomomorphisms, A and B being regular local rings of minimal dimension, that is the dimension of A and B equals the imbedding dimension of C. Then

$$\chi^A(M,N) = \chi^B(M,N).$$

PROOF. We may assume that A, B and C are complete local rings. Hence so is the fiber-product $A \times_C B$. By Cohen's structure theorem $A \times_C B$ is a homomorphic image of a regular local ring R, thus we have a commutative diagram of surjective ringhomomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

Put $\mathfrak{a} := \operatorname{Ker}\ (R \to A)$ and $\mathfrak{b} := \operatorname{Ker}\ (R \to B)$. Let $\mathfrak{m}$ be the maximal ideal of R. Since A,B and R are all regular, the inclusions

$\mathfrak{a} \subset \mathfrak{m}$ and $\mathfrak{b} \subset \mathfrak{m}$ give rise to injections

$$\mathfrak{a}/\mathfrak{m}\mathfrak{a} \to \mathfrak{m}/\mathfrak{m}^2$$
$$\mathfrak{b}/\mathfrak{m}\mathfrak{b} \to \mathfrak{m}/\mathfrak{m}^2$$

By means of these maps we will consider $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$ and $\mathfrak{b}/\mathfrak{m}\mathfrak{b}$ as subspaces of $\mathfrak{m}/\mathfrak{m}^2$. Put

$$s = \dim\ R - \dim\ A$$

Since $\dim\ A = \dim B$ both $\mathfrak{a}$ and $\mathfrak{b}$ are minimally generated by s elements. Let $\eth_1,\dots,\eth_r$ $(r \leq s)$ be a basis for $\mathfrak{a}/\mathfrak{m}\mathfrak{a} \cap \mathfrak{b}/\mathfrak{m}\mathfrak{b}$. Let $a_1,\dots,a_r$ respectively $b_1,\dots,b_r$ be elements in $\mathfrak{a}$ respectively $\mathfrak{b}$ representing $\eth_1,\dots,\eth_r$. Now extend these two sequences to minimal sets of generators

$$a_1,\dots,\ a_r,\dots,a_s$$
$$b_1,\dots,\ b_r,\dots,b_s$$

for $\mathfrak{a}$ and $\mathfrak{b}$ respectively. For each i $(0 \leq i \leq s)$ the elements

$$a_1,\dots,\ a_i,\ b_{i+1},\dots,b_s$$

represent linearly independent elements in $\mathfrak{m}/\mathfrak{m}^2$. Hence they are part of a regular system of parameters for R. Let $\mathfrak{a}_i$ denote the ideal they generate and put

$$A_i := R/\ \mathfrak{a}_i$$

Then each A_i is a regular local ring. Observe that $A_0 = B$ and $A_s = A$. In the following let $1 \leq i \leq s$. To prove the theorem it clearly suffices to prove

$$\chi^{A_{i-1}}(M,N) = \chi^{A_i}(M,N)$$

Here we will use a technique which was used in [1] for a similar purpose. To simplify the notation we put $P := A_{i-1}$ and $Q := A_i$. Let L be the ring $R/\mathfrak{C}$ where $\mathfrak{C}$ is the ideal generated by

$$a_1, \dots, a_{i-1}, a_i, b_i, b_{i+1}, \dots, b_s$$

Observe that L need not be regular. We have exact sequences:

$$0 \to P \xrightarrow{a_i} P \to L \to 0$$
$$0 \to Q \xrightarrow{b_i} Q \to L \to 0$$

where a_i and b_i denotes multiplication by a_i and b_i respectively. From the sequences above we obtain standard spectralsequences

$$\mathrm{Tor}_p^L(M, \mathrm{Tor}_q^R(N,L)) \Rightarrow \mathrm{Tor}_{p+q}^P(M,N)$$

and

$$\mathrm{Tor}_p^L(M, \mathrm{Tor}_q^Q(N,L)) \Rightarrow \mathrm{Tor}_{p+q}^Q(M,N)$$

where $\mathrm{Tor}_q^P(N,L)$ and $\mathrm{Tor}_q^Q(N,L)$ equals N for $q = 0,1$ and eqnals zero for $q \neq 0,1$.
Hence we obtain exact sequences

$$\dots \mathrm{Tor}_i^L(M,N) \to \mathrm{Tor}_{i+1}^P(M,N) \to \mathrm{Tor}_{i+1}^L(M,N) \to \mathrm{Tor}_{i-1}^L(M,N) \to \dots$$

$$\dots \mathrm{Tor}_i^L(M,N) \to \mathrm{Tor}_{i+1}^Q(M,N) \to \mathrm{Tor}_{i+1}^L(M,N) \to \mathrm{Tor}_{i-1}^L(M,N) \to \dots$$

from which it follows that

$$\chi^P(M,N) = \chi^Q(M,N) \ .$$

References

1. M.-P. Malliavin - Brameret, Une remarque sur les anneaux locaux réguliers, Seminaire Dubreil - Pisot (Algèbre et Théorie des Nombres), 1970/1971 no.13.

2. J.P. Serre, Algèbre Locale Multiplicités, (Lecture Notes in Mathematics 11), Springer - Verlag, 1965.