Contents of Part I

NON-COMMUTATIVE SPECTRAL THEORY
FOR AFFINE FUNCTION SPACES ON CONVEX SETS

Part I

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Introduction.

In this paper we develop a non-commutative spectral theory and functional calculus for a class of partially ordered normed linear spaces. The spaces in question can be represented (isometrically and order-isomorphically) as spaces of affine functions on convex sets, and among them are the following:

(i) The space of all self-adjoint elements of a von Neumann algebra.

(ii) The space of all bounded affine functions on a (Choquet) simplex.

(iii) The space of all continuous affine functions on a rotund compact convex set (e.g. the unit ball of L_p for $1 < p < \infty$).

These particular cases do not exhaust all possibilities. Nevertheless, the class of spaces for which our spectral theory is available, is quite restricted; among affine function spaces those with spectral theory must be considered the exception rather than the rule. The study of particular examples and applications is postponed to Part II, while the general theory is presented in Part I.

The theory presented in this paper concerns an affine function space A defined on a convex set K where A and K are subject to certain requirements (see below). Basic notions are: the collection $\mathcal{U}$ of "projective units" $u \in A$, the collection $\mathcal{F}$ of "projective faces" $F \subset K$, and the collection $\mathcal{P}$ of P-projections $P : A \to A$. Between any two of these there is a canonical bijection; every $u \in \mathcal{U}$ determines a unique $F \in \mathcal{F}$ and a unique $P \in \mathcal{P}$, and so on. In the example (i) above, $\mathcal{U}$ corresponds to the (self-adjoint) projections, $\mathcal{F}$ corresponds to certain faces of the normal state space (the relativization of the annihilators of one-sided ultraweakly closed ideals), and $\mathcal{P}$ corresponds to the maps $a \to pap$ where p is a (self-adjoint) projection. The collections $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ can also be identified in the examples (ii) and (iii); in the former they are "very large" in the latter they are "very small". (This is all treated in Part II, where the precise statements are given.) Note also that the projective faces generalize split faces F (cf. [AA_1]), that the projective units generalize in a similar way the corresponding (affine) envelopes $\widehat{\chi_F}$, and that the P-projections generalize splitting projections [W_1]. (This will also be treated in Part II.) The notions of projective unit, projective face and P-projection admit various equivalent definitions which are presented in § 1-2 together with the basic properties of these notions.

In the following sections, §§ 3-4 , it is assumed that K has "many" projective faces (specifically that every exposed face is projective) and also that A enjoys a completeness property (pointwise monotone σ-completeness). Under these hypotheses it is proved that   is a σ-complete orthomodular lattice in the natural ordering and with the orthocomplementation $u \to e-u$ where e denotes the element of A which takes the constant value 1 on K . In particular it is shown that the center of the orthomodular lattice $\mathcal{U}$ consists of precisely those elements of $\mathcal{U}$ which are in the center of the order-unit space (A,e) (cf. [W_1] and [AA_2]). Important new concepts are those of a projective unit being "compatible" or "bicompatible" with an element of A. These concepts generalize commutation and bicommutation in operator algebras, and they are fundamental for the subsequent development of the theory.

The next section, § 5, is the key section of the paper. Here the "spectral axiom" is introduced and the spectral theorem is proved. The spectral axiom plays a role similar to Stone's axiom in ordinary ("commutative") integration theory. Recall that in the well known Notes on Integration from 1948-49 [S_1] Stone observed that such an axiom was needed to connect the linear functional approach with measure theory. Originally stated in the form $f \in L \Longrightarrow f \wedge 1 \in L$ (where L is the vector-lattice of functions on which the elementary integral is defined), Stone's axiom serves to guarantee that there are "sufficiently many" measurable sets. Specifically, for every $f \in M$ (the class of measurable functions) and for every $\lambda \in \mathbb{R}$ the set $E = \{x \mid f(x) \leq \lambda\}$ shall be measurable i.e. the characteristic function χ_E shall again belong to M. In the present non-commutative setting the characteristic functions χ_E are replaced by projective units. Now the "weak spectral axiom" states that for each $a \in A$ and each $\lambda \in \mathbb{R}$ there shall exist a projective unit h compatible with a such that:

$$\{x \in K \mid h(x) = 1\} \subset \{x \in K \mid a(x) \leq \lambda\} ,$$

whereas the complementary unit $h' = e - h$ shall satisfy:

$$\{x \in K \mid h'(x) = 1\} \subset \{x \in K \mid a(x) > \lambda\} ;$$

and the word "weak" is omitted if h is unique. (We continue to assume A is monotone σ-complete. The assumption made previously of having "many" projective faces is now implied by the weak spectral axiom.) Note that unlike characteristic functions, the projective units can take intermediate values between 0 and 1 (even at the extreme points of K), and that the above inclusions will be strict in general. Assuming the weak spectral axiom, we prove in § 5 that every $a \in A$ admits a spectral integral representation:

$$a(x) = \int \lambda \, de_\lambda(x) \qquad \text{for all } x \text{ in } K .$$

Here $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is an increasing, right continuous family of projective units (a "spectral family"). The representation above is unique if the spectral axiom is assumed.

The next section, § 6, contains a discussion of various properties of spectral families. It is proved that the weak spectral axiom can be stated in an equivalent form based on decomposition of elements of A as differences of mutually orthogonal positive and negative parts. (Compare B.Sz - Nagy's treatment of spectral theory for operators on a Hilbert space in [N_1]. See also [R.N.$_1$].) It is shown that while the weak spectral axiom implies existence of "many" projective faces (in the precise sense explained before), the converse implication does not hold. Also it is proved that with the spectral axiom all "spectral units" of an element a of A will be bicompatible with a, and conversely that one may pass from the weak spectral axiom to the spectral axiom by requiring h to be bicompatible with a rather than by explicitly requiring h to be unique.

The next section, § 7, treats the functional calculus, which is defined by means of the spectral integral representation of elements of A . Here the spectral axiom is assumed, and it is shown that the functional calculus is unique under the natural isomorphism requirements and the additional requirement that it shall take characteristic functions into extreme points of the order interval [0,e] of A . (These extreme points are precisely the projective units.) In standard spectral theory (see e.g. [B_1,Ch.1]) one deals with algebras, and the functional calculus is required to be a multiplicative isomorphism as well. Then the extreme-point-preserving nature will follow since the extreme points in question are precisely the idempotents. In the present setting for the theory, the extreme-point-preserving property is all that remains of multiplicativity, and it is perhaps somewhat surprising that such a property, defined only in terms of linearity and order, will suffice to guarantee uniqueness of the functional calculus.

The last section, § 8, is a study of certain subspaces of A , called "abelian", which are organized to vector lattices and to commutative Banach algebras in a natural way. It is shown how the general spectral theory reduces to Freudenthal's vector lattice theory for (weakly closed) abelian subspaces ([F]; see also [LZ]), and it is also shown how notions like functional calculus and spectrum reduce to the corresponding ones for commutative Banach algebras. However, the relativization to the abelian subspace M(a) generated by a given element a of A , will not provide an alternative approach to the general theory, since the very definition of M(a) seems to require the full strength of the general theory. In particular, it invokes the notion of compatibility in an essential way. At the end of § 8 it is shown that "all possible" definitions of center for A will coincide, and there are some characterizations of spectra in terms of notions familiar from commutative Banach algebras and operator theory.

Throughout Part I there are examples illustrating the general theory and the interrelationship between the various requirements imposed on A and V . Passing to Part II one will find a more systematic investigation of some special cases of intrinsic interest, in particular the application to operator algebras and their state spaces. In this connection it should be noted that the state spaces of C*-algebras are compact convex sets with remarkable properties. In some respects they behave like simplexes (e.g. all Archimedean faces are split [AA_1], [St]). In other respects they behave like rotund balls (in fact, the state space of the 2×2-matrix algebra _is_ a Euclidean ball in $\mathbb{R}^3$). Some of the properties of the state spaces depend essentially on the spectral theorem, others invoke more of the algebraic structure. (An example to this effect is the existence of "sufficiently many" split-face preserving, or "inner", automorphisms, which depends on Kadison's transitivity theorem [AA_1], [K_2], [GK].) It is our purpose to investigate those properties which depend on spectral theory.

We will now turn to a brief discussion of the historical background of the subject matter of the present paper.

The classical works on spectral theory by Hilbert [H], von Neumann [Neu], Stone [S_2] and others focused on the self-adjoint operators on a Hilbert space. During the thirties Freudenthal [F] Riesz [R_1], Nakano [Na] and others proved versions of the spectral theorem for abstract vector lattices satisfying suitable assumptions (cf. also [L-Z]). At about the same time Stone proved a spectral theorem for a class of partially ordered (and necessarily commutative) linear algebras over the reals [S_3].

Segal's 1947 paper on axiomatic quantum mechanics [Se] was the first in a series of works in which a spectral resolution or a functional calculus was postulated in a linear and partially ordered, but non-vector-lattice (or "non-commutative") context.

Recently some finite dimensional versions of a spectral theorem have been obtained in the non-vector-lattice context by Gunson [G] and Ludwig [L] in works on axiomatic quantum mechanics; and the work of Ludwig has been slightly generalized by Ancona [An].

There are some remnants of commutative structure in non-commutative operator algebras, for example the two-sided ideals and the center. The two-sided ideals of a C*-algebra with identity element correspond to the invariant faces of the state space (cf. [St]), and these faces are generalized by the "split-faces" of convexity theory. The notion of a split face of K was independently introduced and studied by Perdrizet and Combes [Pe_1], [Pe_2], [CP] and by Alfsen and Andersen [AA_1]. The center of a C*-algebra with identity element was generalized to the "ideal center" by Dixmier [D]. This notion was in turn generalized to partially ordered vector spaces by Wils [W] and simultaneously to the (somewhat less general) context of order-unit spaces by Alfsen and Andersen [AA_2]. Every central projection p in the enveloping von-Neumann algebra $\mathfrak{A}^{**}$ of a given C*-algebra $\mathfrak{A}$ generates a weak* (or ultraweakly) closed two-sided ideal of $\mathfrak{A}^{**}$, and the maps $a \to pap$ (with p central) can be order theoretically characterized as "splitting projections". These splitting projections form the starting point of Wils' discussion of the ideal center of a partially ordered vector space. In the context of the present paper, every splitting projection of A is a P-projection, and a P-projection is splitting iff it is central.

The center of an order-unit space (or equivalently of an A(K)-space) is a vector lattice. Therefore one can attempt to apply the vector lattice version of the spectral theorem to this center (after a suitable "completion" of the space permitting "spectral units"). Work in this direction has been done by Rogalski [Ro] and C.M. Edwards [Ed].

To achieve a truly non-commutative theory, one needs to work with the notions associated with one-sided ideals. These ideals have been thoroughly investigated by Effros [E] and Prosser [P] and their properties are very relevant to our work. (See also the survey [GR].) Every weak *-closed left ideal J in a von Neumann algebra $\mathfrak{A}$ is generated by a self-adjoint projection p (which will be central precisely when J is two-sided). In the study of such ideals, an important role is played by the maps $a \to pap$ from $\mathfrak{A}_{sa}$ into itself. The annihilators (in the predual of $\mathfrak{A}$) of such ideals are precisely the norm-closed invariant subspaces, whose intersections with the normal state space will be certain faces. These projections, maps, and faces can be characterized in terms of the notions we develop in this paper, as the projective units, P-projections, and projective faces respectively. The results of Effros and Prosser have to a great extent motivated our approach to non-commutative spectral theory in Part I, and we shall return to them in our discussion of the applications to operator algebras in Part II.

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§ 1. Smooth projections.

We shall first recall some definitions from convexity theory. We consider two (real) vector spaces X,Y in separating duality with respect to a bilinear form $\langle\ ,\ \rangle$, and we shall use the terms "weak" and "weakly" to denote the weak topologies defined on X and Y by this duality.

Let K be a convex subset of X. A convex set $F \subset K$ is said to be a face of K if for any $(\lambda, y, z) \in (0,1) \times K \times K$ $\lambda y + (1-\lambda)z \in F$ implies $y, z \in F$. An affine subspace H of X is said to be a supporting subspace for K if $K \cap H \neq \emptyset$ and $K \setminus H$ is convex. It is easily verified that a non-empty subset F of K is a face iff it is of the form $F = K \cap H$ for some supporting subspace H. (One may take $H = \mathrm{aff}(F)$). Note in particular that the whole space X is a supporting subspace for K, and that the whole set K and the empty set $\emptyset$ are both faces of K.

The intersection of all weakly closed supporting hyperplanes containing a given subset F of K, will be denoted by $\tilde{F}$. We shall say that a supporting subspace H of K is smooth if $H = (K \cap H)^{\sim}$, and we shall say that a face F of K is semi-exposed if $F = \tilde{F} \cap K$. Also we shall say that a face F of K is exposed if there exists a closed supporting hyperplane H such that $F = H \cap K$. (Note that these definitions depend on the given duality).

In the pictures below we first show a smooth and a non smooth supporting subspace, and then a semi-exposed and a non-semi-exposed face.

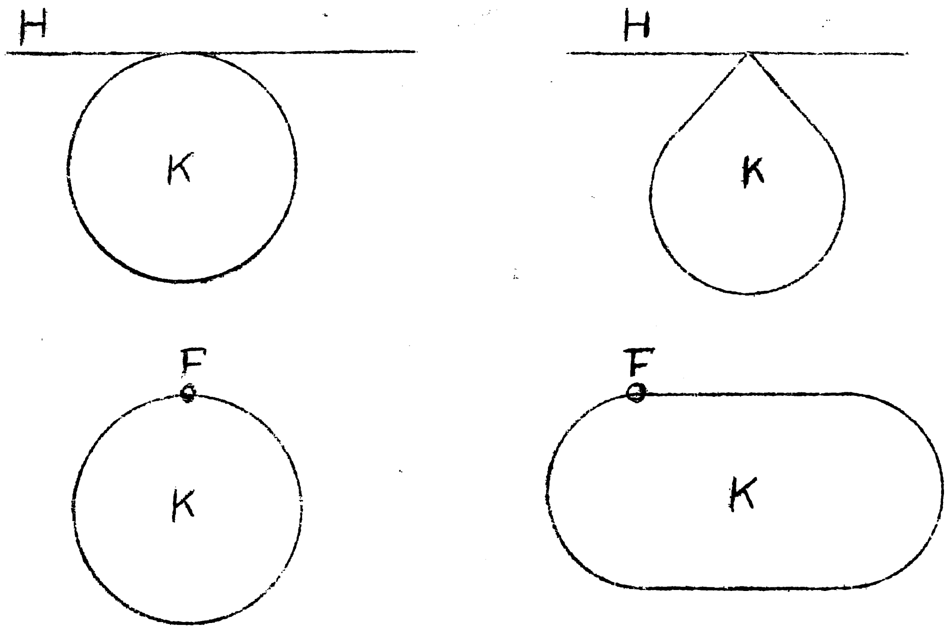

Fig. 1.

Throughout the rest of this section we shall assume that X, Y are two positively generated partially ordered vector spaces in <u>separating ordered duality</u>, i.e. for $x \in X$, $y \in Y$

$$(1.1) \quad \begin{cases} x \geq 0 \iff \langle x,y \rangle \geq 0 \quad \text{all} \quad y \geq 0, \\ y \geq 0 \iff \langle x,y \rangle \geq 0 \quad \text{all} \quad x \geq 0. \end{cases}$$

The supporting subspaces H of the cone X^+ are necessarily linear spaces (i.e. $0 \in H$), and they are in fact exactly the <u>order ideals</u> of X (see e.g. [A_1,p.67]). Correspondingly the faces of X^+ are the <u>hereditary subcones</u> $C = H \cap X^+$ (defined by the requirement that $0 \leq x' \leq x \in C$ shall imply $x' \in C$, see e.g. [A_1,p.82]). The supporting subspaces and faces of Y^+ can of course be characterized in the same way. For the sake of brevity we shall use the term <u>smooth order ideal</u> to denote a smooth supporting subspace for X^+, and likewise for Y^+.

For a given subset B of X we shall use the symbol B^o to denote the <u>annihilator</u> of B, and we shall use the symbol $P^{\perp}$ to denote the <u>positive annihilator</u> of B . Thus we have:

$$(1.2)\quad \begin{cases} B^{o} = \{y \in Y \mid \langle x,y\rangle = 0 \ \text{ all } \ x \in B\}, \\ B^{\perp} = \{y \in Y^{+} \mid \langle x,y\rangle = 0 \ \text{ all } \ x \in B\}. \end{cases}$$

(The notation B^{o} is not likely to cause any misunderstanding since no "polars" will be needed in the sequel.)

Note that for a given subset C of X^{+}

$$(1.3)\quad \tilde{C} = C^{\perp o} = \{x \in X \mid \langle x,y\rangle = 0 \ \text{ when } \ y \in C^{\perp}\}.$$

In the sequel we shall study weakly continuous positive projections $P : X \to X$. (By "projection" we mean any idempotent map). For such projections we define:

$$(1.4)\quad \ker^{+}P = (\ker P) \cap X^{+}, \qquad \text{im}^{+}P = (\text{im}\,P) \cap X^{+}$$

Clearly, $\ker^{+}P$ and $\text{im}^{+}P$ are <u>subcones</u> of X^{+} , and the former is also a <u>face</u> of X^{+} .

By hypothesis X is positively generated, and this implies that $\text{im}\,P$ is positively generated, i.e.

$$(1.5)\qquad \text{im}\,P = \text{im}^{+}P - \text{im}^{+}P .$$

Note, however, that $\ker P$ will not be positively generated in general.

For given $y \in Y$ the linear functional $x \to \langle Px,y\rangle$ on X will be weakly continuous. Hence there is a (unique)element $P^{*}y$ of Y such that

$$(1.6)\qquad \langle Px,y\rangle = \langle x,P^{*}y\rangle ,$$

and $P^{*} : Y \to Y$ is seen to be a weakly continuous positive projection on Y . We say that P^{*} is the <u>dual projection</u> of P .

We note the following basic formulas:

(1.7) $\qquad (\ker P)^{o} = \operatorname{im} P^{*} , \qquad (\operatorname{im} P)^{o} = \ker P^{*} .$

The above discussion is completely symmetric in X and Y. Hence we may give the similar definitions with X and Y interchanged, and obtain the same results. In particular $\operatorname{im} P^{*}$ will be positively generated. Hence by (1.7):

(1.8) $\qquad (\ker P)^{o} = (\ker P)^{\perp} - (\ker P)^{\perp} .$

From this we obtain

$$\ker P = (\ker P)^{oo} = (\ker P)^{\perp o} \supset (\ker^{+} P)^{\perp o},$$

which gives the general formula:

(1.9) $\qquad \ker P \supset \widetilde{\ker^{+} P} .$

<u>Definition</u>. A projection $P : X \to X$ is said to be <u>smooth</u> (with respect to the given duality) if it is weakly continuous and positive and also satisfies the requirement:

(1.10) $\qquad y \in Y^{+} , \; y = 0 \text{ on } \ker^{+} P \implies y = 0 \text{ on } \ker P .$

A smooth projection on Y is defined analogously.

The requirement (1.10) may be restated in the following condensed form:

(1.11) $\qquad (\ker^{+} P)^{\perp} \subset (\ker P)^{o} .$

Clearly, one may write $(\ker P)^{\perp}$ in place of $(\ker P)^{o}$ in (1.11); and since the opposite inclusion is trivial, one shall actually have the following equality for any smooth projection P

(1.12) $\qquad (\ker^{+} P)^{\perp} = (\ker P)^{\perp} .$

The definition of a smooth projection is motivated by the following:

Proposition 1.1. A weakly continuous positive projection $P : X \to X$ is smooth iff $\ker P$ is a smooth order ideal, i.e.

$$(1.13) \qquad \ker P = \widetilde{\ker^+ P}$$

Proof. By virtue of (1.9) the non-trivial half of (1.13) is the inclusion

$$(1.14) \qquad \ker P \subset \widetilde{\ker^+ P} .$$

Assuming (1.11) we obtain

$$\ker P = (\ker P)^{oo} \subset (\ker^+ P)^{\perp o} = \widetilde{\ker^+ P} ,$$

and (1.14) is proved.

Conversely, we assume (1.14) and get

$$(\ker^+ P)^{\perp} \subset [(\ker^+ P)^{\perp}]^{oo} = [\widetilde{\ker^+ P}]^{o} \subset (\ker P)^{o} .$$

Hence we are back to (1.11). □

By virtue of (1.5) and (1.13) a smooth projection P is completely determined by $\operatorname{im}^+ P$ and $\ker^+ P$, and so the dual projection P^* will also be determined by these two cones. We now proceed to give an explicit formula for P^* in terms of $\operatorname{im}^+ P$ and $\ker^+ P$. (One may give a similar formula for P, but it will not be needed in the sequel.)

In this connection we shall need the following restatement of the basic requirement (1.10) for a smooth projection, obtained by the equality $(\ker P)^o = \operatorname{im} P^*$:

(1.15) $$y \in Y^+ , \ y = 0 \ \text{on} \ \ker^+ P \Longrightarrow P^* y = y .$$

Proposition 1.2. If P is a smooth projection, then for $y, y' \in Y^+$:

(1.16) $$\begin{cases} y' \leq y \ \text{on} \ \mathrm{im}^+ P , \ y' = 0 \ \text{on} \ \ker^+ P \Longrightarrow y' \leq P^* y \\ y' \geq y \ \text{on} \ \mathrm{im}^+ P , \ y' = 0 \ \text{on} \ \ker^+ P \Longrightarrow y' \geq P^* y \end{cases}$$

Proof. We assume $y' \leq y$ on $\mathrm{im}^+ P$, $y' = 0$ on $\ker^+ P$. For an arbitrary $x \in X^+$

$$\langle x, P^* y' \rangle = \langle Px, y' \rangle \leq \langle Px, y \rangle = \langle x, P^* y \rangle .$$

By (1.15) $P^* y' = y'$, and so $\langle x, y' \rangle \leq \langle x, P^* y \rangle$. This proves the first implication of (1.16), since $x \in X^+$ was arbitrary. The second implication is proved in the same way. □

From Proposition 1.2 one easily obtains the following:

Corollary 1.3. If $P : X \to X$ is a smooth projection and $y \in Y^+$, then $P^* y$ is the unique positive element of Y which coincides with y on $\mathrm{im}^+ P$ and vanishes on $\ker^+ P$. Moreover one has the explicit formula:

(1.17) $$\begin{aligned} P^* y &= \sup\{ y' \in Y^+ \mid y' \leq y \ \text{on} \ \mathrm{im}^+ P , \ y' = 0 \ \text{on} \ \ker^+ P \} \\ &= \inf\{ y' \in Y^+ \mid y' \geq y \ \text{on} \ \mathrm{im}^+ P , \ y' = 0 \ \text{on} \ \ker^+ P \} . \end{aligned}$$

Note that if the cones $\mathrm{im}^+ P$ and $\ker^+ P$ are replaced by the subspaces $\mathrm{im}\, P$ and $\ker P$, then the uniqueness statement of Corollary 1.3 will subsist for any weakly continuous projection P.

Note also that in the uniqueness statement for smooth projections given in Corollary 1.3, the term "positive" is essential. There may be non-positive elements other than P^*y coinciding with y on im^+P and vanishing on ker^+P. This can be seen from the picture below where P is the (smooth) orthogonal projection onto the z-axis.

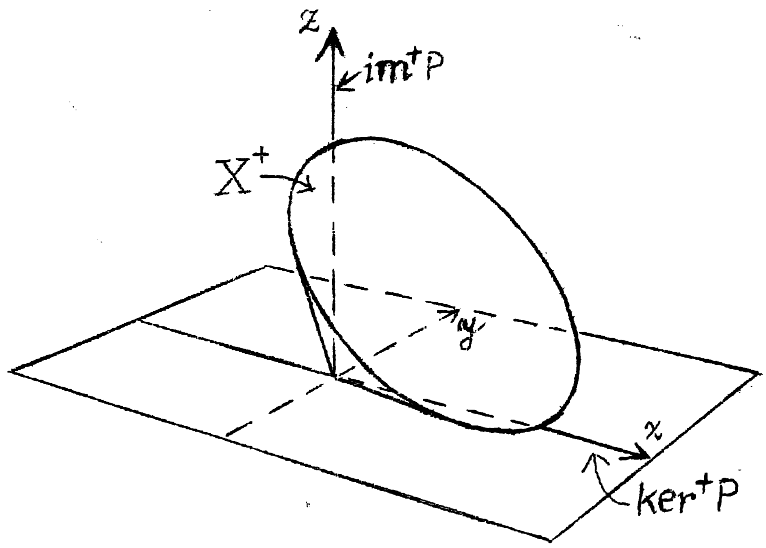

Fig. 2.

We now proceed to characterize projections $P : X \to X$ with smooth dual $P^* : X \to X$. In this connection we shall need a few simple formulas valid for an arbitrary weakly continuous and positive projection P. By (1.5) and (1.7), $\ker P^* = (\operatorname{im} P)^o = (\operatorname{im}^+ P)^o$, and so

(1.18) $$\ker^+ P^* = (\operatorname{im}^+ P)^{\perp} .$$

Passing to annihilators we get:

(1.19) $$(\ker^+ P^*)^o = \widetilde{\operatorname{im}^+ P} .$$

Proposition 1.4. Let $P : X \to X$ be a weakly continuous positive projection. The dual projection $P^* : Y \to Y$ is smooth iff $\mathrm{im}^+ P$ is a semi-exposed face of X^+, i.e.

$$(1.20) \qquad \mathrm{im}^+P = \widetilde{\mathrm{im}^+P} \cap X^+$$

Proof. By (1.11) P^* is smooth iff

$$(1.21) \qquad (\ker^+ P^*)^\perp \subset (\ker P^*)^\circ .$$

The space $\mathrm{im}\,P$ is weakly closed. Hence by (1.7) $(\ker P^*)^\circ = (\mathrm{im}\,P)^{\circ\circ} = \mathrm{im}\,P$, and so we may replace (1.21) by the equivalent formula

$$(\ker^+ P^*)^\perp \subset \mathrm{im}\,P .$$

By (1.19) this is equivalent to

$$\widetilde{(\mathrm{im}^+P)} \cap X^+ \subset \mathrm{im}^+ P ,$$

which is the non-trivial half of (1.20) and the proof is complete. □

For the sake of later references we shall also present the above result in a dual setting where the given projection is defined on Y.

Corollary 1.5. Let R be a positive and weakly continuous projection on Y. Then R is a smooth projection on Y iff $(\ker R)^\perp$ is a semi-exposed face of X^+.

Proof. Let $P = R^*$. Since $\mathrm{im}\,P$ is weakly closed, we have

$$(\ker R)^\perp = (\ker P^*)^\perp = (\mathrm{im}\,P)^{\circ\circ} \cap X^+ = \mathrm{im}^+P ,$$

and the corollary follows from Proposition 1.4. □

Definition. Two weakly continuous positive projections $P, Q : X \to X$ are said to be quasicomplementary if

$$\ker^+ P = \operatorname{im}^+ Q , \qquad \operatorname{im}^+ P = \ker^+ Q . \tag{1.22}$$

We shall also say that Q is a quasicomplement of P, and vice versa.

It is not hard to give examples (in $\mathbb{R}^3$) of a weakly continuous positive projection with no (positive) quasicomplement, and of one with infinitely many quasicomplements. However, our next two lemmas will provide a necessary condition for the existence of a quasicomplement, and a sufficient condition for uniqueness.

In this connection we first observe that for every weakly continuous and positive projection $P : X \to X$ the formula (1.9) entails

$$\ker^+ P = \widetilde{\ker^+ P} \cap X^+ , \tag{1.23}$$

and so $\ker^+ P$ will always be a semi-exposed face of X^+.

Lemma 1.6. If a weakly continuous positive projection $P : X \to X$ admits a quasicomplement Q, then P^* is necessarily smooth.

Proof. By the above remark, $\operatorname{im}^+ P = \ker^+ Q$ is a semi-exposed face of X^+, and by Proposition 1.4 the dual projection P^* must be smooth. □

Lemma 1.7. If a weakly continuous positive projection $P : X \to X$ admits a smooth quasicomplement Q, then Q is the only quasicomplement of P.

Proof. Let $R : X \to X$ be any quasicomplement of P. We shall prove that $\operatorname{im} Q \subset \operatorname{im} R$ and $\ker Q \subset \ker R$, which will give $Q = R$.

By assumption

$$\mathrm{im}^+Q = \ker^+P = \mathrm{im}^+R ,$$

and since $\mathrm{im}\,Q$ and $\mathrm{im}\,R$ are positively generated, they must be equal.

Next we use formula (1.13) for the smooth projection Q and formula (1.9) for R (i.e. the "trivial half" of the same formula), and we obtain

$$\ker Q = \widetilde{\ker^+Q} = \widetilde{\mathrm{im}^+P} = \widetilde{\ker^+R} \subset \ker R$$

This completes the proof. □

<u>Theorem 1.8. Let $P,Q : X \to X$ be two weakly continuous positive projections. Then the following three statements are equivalent:</u>

(i) <u>P,Q are smooth and quasicomplementary</u>

(ii) <u>P^*,Q^* are smooth and quasicomplementary</u>

(iii) <u>P,Q are quasicomplementary, and so are P^*,Q^*</u>.

<u>Proof</u>. It suffices to prove (i) <==> (iii) since the statement (iii) is completely symmetric in X and Y.

1) We first assume (i). Using the general formula(1.18) and the formula (1.12) for the smooth projection Q, we obtain

$$\ker^+P^* = (\mathrm{im}^+P)^\perp = (\ker^+Q)^\perp = (\ker Q)^\perp = \mathrm{im}^+Q^* .$$

Similarly we prove $\ker^+Q^* = \mathrm{im}^+P^*$. Hence P^*,Q^* are quasicomplementary.

2) We next assume (iii). By Lemma 1.6 the quasicomplemented projection P^* will have a smooth dual $P^{**} = P$. Similarly we prove that $Q^{**} = Q$ is smooth, and the proof is complete. □

§2. Projective units and projective faces

Henceforth we shall consider an order-unit space (A,e) and a base-norm space (V,K) (for definitions see e.g. [A_1,Ch.II,§1]), and we assume that they arc in separating order and norm duality, i.e. we shall assume (1.1) together with the following requirement in which $a \in A$, $x \in V$:

$$(2.1) \quad \begin{cases} \|a\| \leq 1 \iff |\langle a,x\rangle| \leq 1 \text{ whenever } \|x\| \leq 1 , \\ \|x\| \leq 1 \iff |\langle a,x\rangle| \leq 1 \text{ whenever } \|a\| \leq 1 . \end{cases}$$

From this it easily follows that $\langle e,x\rangle = 1$ for all $x \in K$, and more generally that $\langle e,x\rangle = \|x\|$ for all $x \in V^+$.

Note that the space A can be identified with a subspace of the space $A(K)$ of all bounded, weakly continuous affine functions on K . Specifically, the restriction map is an isometric, linear- and order- isomorphism of A into $A(K)$, but it need not be surjective. In fact, every affine function a_o on K can be uniquely extended to a linear function a on V satisfying

$$(2.2) \qquad a(\lambda x-\mu y) = \lambda a_o(x) - \mu a_o(y)$$

for $\lambda,\mu \in \mathbb{R}$; but a_o need not be weakly continuous, and hence not in A , even if a_o is bounded and weakly continuous. (If $V = A^*$, then A is a dense subspace of the complete space $A(K)$, and the two spaces will coincide iff A is complete; see e.g. [A_1,p.74]).

We shall often find it convenient to think of the elements of A as affine functions on K , and we shall prefer the notation $a(x)$ for the more "symmetric" notation $\langle a,x\rangle$ used in § 1.

In this section we shall be concerned with weakly continuous positive projections on either A or V and with norm at most 1. For such a projection the dual projection P^* will also be of norm at most 1 by virtue of (2.1). We also note the following simple formula valid for a weakly continuous positive projection P on V:

$$\|Px\| = e(Px) = (P^*e)(x) , \qquad \text{all } x \in V^+ . \tag{2.3}$$

<u>Definition</u>. If P is a projection on either of the two spaces A or V which is smooth with norm at most 1 and admits a smooth quasicomplement with norm at most 1, then P is said to be a <u>P-projection</u>.

By Lemma 1.7 the quasicomplement of a P-projection P is unique; and we shall denote it by P'. Clearly P' is also a P-projection.

It follows from Theorem 1.8 that a weakly continuous positive projection P on one of the two spaces is a P-projection iff the dual projection P^* is a P-projection on the other space. It also follows from the same theorem that a weakly continuous positive projection P of norm at most 1 defined on one of the two spaces will be a P-projection iff P and P^* both admit a positive quasicomplement of norm at most 1 and these quasicomplements are duals of each other. The last mentioned property of P-projections can be stated in a formula:

$$(P^*)' = (P')^* . \tag{2.4}$$

We shall now characterize P-projections on A and V in various ways. In particular we shall see that they are completely determined by their ranges, and in this connection it will be essential that the projections are of norm not exceeding 1 and that there are certain conditions imposed on the spaces to relate ordering and norm.

The following observations will be useful:

If $P: V \to V$ is a weakly continuous positive projection with $\|P\| \leq 1$ and $x \in V^+$, then $(P^*e)(x) = \|Px\| \leq \|x\| = e(x)$; from which it follows that

$$(2.5) \qquad 0 \leq P^*e \leq e .$$

If P is a smooth projection with $\|P\| \leq 1$, then for given $a \in (\ker^+P)^\perp$ with $0 \leq a \leq e$, we can apply formula (1.15) to obtain $a = P^*a \leq P^*e \leq e$.

Hence the following explicit formula is valid for any smooth projection P on V with $\|P\| \leq 1$:

$$(2.6) \qquad P^*e = \sup\{a \in A \mid 0 \leq a \leq e,\ a = 0 \text{ on } \ker^+P\} .$$

Note also that it follows from the results of § 1 that for a P-projection P on V or A the sets im^+P and $\ker^+P$ will be <u>semi-exposed faces</u> of the cone of positive elements.

Finally we note that if P and Q are weakly continuous positive projections on V, then the following three statements are equivalent:

$$(2.7) \qquad P^*e + Q^*e = e ,$$

$$(2.8) \qquad \|Px + Qx\| = \|x\| , \qquad \text{all } x \in V^+ ,$$

$$(2.9) \qquad (P+Q)(K) \subset K .$$

<u>Proposition 2.1</u>. <u>If P, Q are quasicomplementary P-projections on V, then $P^*e + Q^*e = e$</u>.

Proof. By Theorem 1.8 P^*, Q^* are quasicomplementary. By (2.5) $e - P^*e \geq 0$, and clearly $P^*(e-P^*e) = 0$. Hence $e - P^*e \in \ker^+P^* = \mathrm{im}^+Q^*$, so

$$Q^*(e-P^*e) = e - P^*e \; .$$

Also $P^*e \in \mathrm{im}^+P^* = \ker^+Q^*$, so $Q^*P^*e = 0$. Hence $Q^*e = e - P^*e$, and the proof is complete. □

Lemma 2.2. If P is a P-projection on V, then for $x \in V^+$:

$$(2.10) \qquad \|Px\| = \|x\| \Longrightarrow x \in \mathrm{im}^+P$$

Proof. Let $x \in V^+$ and $\|Px\| = \|x\|$. Then $(P^*e)(x) = e(x)$, and by Proposition 2.1 $(Q^*e)(x) = (e-P^*e)(x) = 0$. Hence $\|Qx\| = 0$, and so $x \in \ker^+Q = \mathrm{im}^+P$. □

Clearly the opposite implication of (2.10) is valid, so we have the following formula for a P-projection P on V:

$$(2.11) \qquad \mathrm{im}^+P = \{x \in V^+ \mid \|Px\| = \|x\|\} \; .$$

Definition. A weakly continuous and positive projection P on V is said to be neutral if it is of norm at most 1 and the implication (2.10) is valid when $x \in V^+$.

The term neutral is motivated by physics. The implication (2.10) is a property of physical filters which are "neutral" in the sense that if a beam passes through with intensity undiminished ($\|Px\| = \|x\|$), then the filter is "neutral" to the beam ($Px = x$).

Lemma 2.3. Let P be a weakly continuous positive projection on V. If P is neutral, then P^* is smooth.

Proof. We assume that P is neutral, and by Proposition 1.4 it suffices to prove that im^+P is semi-exposed, i.e. $(\mathrm{im}^+P)^\sim \cap V^+ \subset \mathrm{im}^+P$.

Let $x \in (\mathrm{im}^+P)^\sim \cap V^+$ be arbitrary, and consider the function $b = e - P^*e \geq 0$ (see (2.5)). Clearly $b \in (\mathrm{im}^+P)^\perp$, and so $b(x) = 0$. Hence

$$\|x\| - \|Px\| = e(x) - (P^*e)(x) = 0 ,$$

and this gives $x \in \mathrm{im}^+P$, since P was assumed to be neutral. □

Proposition 2.4. Let P, Q be weakly continuous positive projections on V of norm at most 1. Then P, Q are quasicomplementary P-projections iff P and Q are neutral and P^*, Q^* are quasicomplementary.

Proof. The necessity follows from Theorem 1.8 and Lemma 2.2, and the sufficiency follows from Theorem 1.8 and Lemma 2.3. □

The next result is a characterization of P-projections P on V in terms of "neutrality" and uniqueness of functions in A^+ with prescribed values on $\ker^+P$ and vanishing on im^+P, and likewise for the quasicomplement of P.

Theorem 2.5. Let P, Q be weakly continuous positive projections on V with norm at most 1. Then P, Q are quasicomplementary P-projections iff they are both neutral and for given $a \in A^+$ the functions $b = P^*a$ and $c = Q^*a$ are the only elements of A^+

<u>such that</u>

(2.12) $b = a$ on $\ker^+ Q$, $b = 0$ on $\mathrm{im}^+ Q$

(2.13) $c = a$ on $\ker^+ P$, $c = 0$ on $\mathrm{im}^+ P$.

<u>Proof</u>. 1) If P, Q are quasicomplementary P-projections, then $\ker^+ Q = \mathrm{im}^+ P$ and $\mathrm{im}^+ Q = \ker^+ P$. Hence (2.12) follows from Corollary 1.3. Similarly for (2.13).

2) By Proposition 2.4 it suffices to prove that P^*, Q^* are quasicomplementary.

If $a \in \ker^+ P^*$, then $a \geq 0$ and $P^* a = 0$. Hence $a = 0$ on $\mathrm{im}^+ P$, and since $Q^* a$ is supposed to be the only element of A^+ which vanishes on $\mathrm{im}^+ P$ and coincides with a on $\ker^+ P$, we must have $a = Q^* a$. Thus we have proved $\ker^+ P^* \subset \mathrm{im}^+ Q^*$.

If $a \in \mathrm{im}^+ Q^*$, then $a \geq 0$ and $Q^* a = a$. By hypothesis $a = Q^* a$ will vanish on $\mathrm{im}^+ P$. Hence for any $x \in V^+$, $(P^* a)(x) = a(Px) = 0$. Thus $P^* a = 0$, and we have proved $\mathrm{im}^+ Q^* \subset \ker^+ P^*$.

Combining the results, we get $\ker^+ P^* = \mathrm{im}^+ Q^*$, and in the same way we prove $\ker^+ Q^* = \mathrm{im}^+ P^*$. This completes the proof. $\square$

We shall now see that for a P-projection P on A or V either one of the two cones $\mathrm{im}^+ P$, $\ker^+ P$ will determine the other, and hence the projection P . We have already mentioned that this result will not prevail for arbitrary partially ordered normed spaces in separating order and norm duality and arbitrary pairs of quasicomplementary smooth projections of norm not exceeding 1 . (One may give counterexamples in $\mathbb{R}^3$.)

The clue to this result for order-unit and base-norm spaces is the fact that P-projections on V are neutral.

Lemma 2.6. If P is a smooth neutral projection on V, then the following are equivalent for $x \in V^+$ and $a, b \in A$:

(i) $x \in \mathrm{im}^+P$,

(ii) $(P^*e)(x) = e(x)$,

(iii) $e(x) = \sup\{a(x) \mid 0 \leq a \leq e,\ a = 0 \text{ on } \ker^+P\}$,

(iv) $0 = \inf\{b(x) \mid 0 \leq b \leq e,\ b = e \text{ on } \ker^+P\}$.

Proof. (i) <==> (ii) Application of (2.11).

(ii) <==> (iii) Application of (2.6).

(iii) <==> (iv) Substitution of $b = e - a$. □

Proposition 2.7. If P is a P-projection on V, then $\ker^+P$ consists of those $x \in V^+$ such that for $b \in A$:

$$\inf\{b(x) \mid 0 \leq b \leq e,\ b = e \text{ on } \mathrm{im}^+P\} = 0 . \tag{2.14}$$

Proof. Application of Lemma 2.6 ((i) <==> (iv)) with P' in place of P . □

Corollary 2.8. If P_1, P_2 are two P-projections on V and $\mathrm{im}^+P_1 = \mathrm{im}^+P_2$, then $P_1 = P_2$.

Proof. Apply Proposition 2.7 and remember that by the results of § 1 a smooth projection P is completely determined by im^+P and $\ker^+P$ (cf. (1.5) and (1.13)). □

Note that it follows by passage to quasicomplements that the conclusion of Corollary 2.8 will remain valid if we substitute $\ker^+$ for im^+.

We shall now dualize to obtain similar results for A.

<u>Corollary 2.9. If R_1, R_2 are two P-projections on A and $\operatorname{im}^+R_1 = \operatorname{im}^+R_2$, then $R_1 = R_2$.</u>

<u>Proof</u>. By formula (1.18)

$$\ker^+R_1^* = (\operatorname{im}^+R_1)^\perp = (\operatorname{im}^+R_2)^\perp = \ker^+R_2^* .$$

Since R_1^* and R_2^* are P-projections on V, they must be equal, and so $R_1 = R_2$. ▯

We shall state a few simple formulas valid for a P-projection R on A.

First we note that by Proposition 2.1:

(2.15) $$R'e = e - Re .$$

Next we note that $(\ker R)^\perp = (\ker R)^\circ \cap V^+ = \operatorname{im}^+R^*$, and similarly $(\operatorname{im} R)^\perp = \ker^+R^*$. Applying this and Lemma 2.6 (ii) we get the first of the following two formulas. The second equality of the second formula follows when we apply the first with R' in the place of R and use (2.4):

(2.16) $$\begin{cases} (\ker R)^\perp = \operatorname{im}^+R^* = \{x \in V^+ \mid (Re)(x) = e(x)\} \\ (\operatorname{im} R)^\perp = \ker^+R^* = \{x \in V^+ \mid (Re)(x) = 0\} \end{cases}$$

We shall have $(\operatorname{im} R^*)^\perp = (\operatorname{im}^+R^*)^\perp$ since $\operatorname{im} R^*$ is positively generated, and $(\ker R^*)^\perp = (\ker^+R^*)^\perp$ by (1.12).

Hence by (2.16):

$$(2.17)\qquad \begin{cases} \ker^+ R = (\operatorname{im} R^*)^{\perp} = \{x \in V^+ \mid (Re)(x) = e(x)\}^{\perp} \\ \operatorname{im}^+ R = (\ker R^*)^{\perp} = \{x \in V^+ \mid (Re)(x) = 0\}^{\perp} \end{cases}$$

Definition. For a given P-projection R on A the element Re will be in the order interval $[0,e]$, and such elements Re will be called projective units of A. Moreover, the set $F_R = (\operatorname{im} R^*) \cap K$ will be a face of K, and such faces F_R will be called projective faces of K.

The following two propositions are stated for a P-projection R on A, and they are phrased in terms of its associated projective unit and projective face. But the proofs will only depend on the fact that R is weakly continuous, positive and of norm at most 1, and on the fact that $\operatorname{im}^+ R$ is a face of A^+.

Proposition 2.10. If R is a P-projection on A, then

$$(2.18)\qquad \operatorname{im} R \cap [-e,e] = [-Re,Re],$$

and so $(\operatorname{im} R, Re)$ is an order-unit space with the relativized ordering and norm.

Proof. If a is in the left side of (2.18) then $a = Ra \le Re$ and $a = Ra \ge -Re$, so a is also in the right side.

If a is in the right side of (2.18) then $-e \le -Re \le a \le Re \le e$. The set $\operatorname{im} R$ is an order ideal of A since $\operatorname{im}^+ R$ is a face of A^+. Hence $a \in \operatorname{im} R$, so a belongs to the left side as well. □

Corollary 2.11. <u>If R is a P-projection on A, then $\operatorname{im} R$ is the order ideal of A generated by the projective unit Re.</u>

Corollary 2.12. <u>If R is a P-projection on A, then the projective unit Re is an extreme point of $[0,e]$.</u>

Proof. Suppose $Re = \lambda a + (1-\lambda)b$ where $0 < \lambda < 1$ and $a,b \in [0,e]$. Then $0 \leq \lambda a \leq Re$ and $0 \leq (1-\lambda)b \leq Re$. Hence $a,b \in \operatorname{im} R$. Also $a,b \in [0,e] \subset [-e,e]$, and by (2.18) $a,b \in [-Re,Re]$. But then the relation

$$Re = \lambda a + (1-\lambda)b \leq \lambda Re + (1-\lambda)Re = Re$$

will imply $Re = a = b$. □

Proposition 2.13. <u>If R is a P-projection on A, then</u>

$$\operatorname{im} R^* \cap \operatorname{co}(K \cup -K) = \operatorname{co}(F_R \cup -F_R), \tag{2.19}$$

<u>and so $(\operatorname{im} R^*, F_R)$ is a base-norm space in the ordering and norm relativized from V.</u>

Proof. We only have to show that the left side of (2.19) is contained in the right. Assuming

$$x = \lambda y - (1-\lambda)z \in \operatorname{im} R^*,$$

where $0 \leq \lambda \leq 1$ and $y,z \in K$, we conclude that

$$x = R^*x = \lambda R^*y - (1-\lambda)R^*z \in \operatorname{co}(F_R \cup -F_R). \quad □$$

We noted in § 1 that a face F of K is <u>exposed</u> if there is a weakly closed affine hyperplane H in V such that $F = H \cap K$. This means that there shall exist an $a \in A$ and an $\alpha \in \mathbb{R}$ such that

$$a(x) = \alpha \quad \text{for} \quad x \in F, \qquad a(x) > \alpha \quad \text{for} \quad x \in K \setminus F .$$

If F is a proper face of K (i.e. $F \neq \emptyset$ and $F \neq K$), then it determines a proper face, cone F, of V^+, and every proper face of V^+ other than $\{0\}$ is of this form. Moreover, if F is an exposed face of K and a and α are as above, then the function $b = a - \alpha e \in A^+$ will satisfy

$$\text{cone}\, F = \{x \in V^+ \mid b(x) = 0\} .$$

Hence cone F will be an exposed face of V^+.

Conversely, if cone F is an exposed face of V^+, then it is easily seen that F must be an exposed face of K. Hence $F \to \text{cone}\, F$ maps the proper exposed faces of K biuniquely onto the proper exposed faces of V^+ other than $\{0\}$. (However, $\{0\}$ is always an exposed face of V^+ since $\{0\} = \{x \in V^+ \mid e(x) = 0\}$.)

Note that similar arguments will give the same result for semi-exposed faces.

Proposition 2.14. If R is a P-projection on A, then

(2.20) $$F_R = \{x \in K \mid (Re)(x) = 1\} ;$$

hence every projective face of K is exposed.

Proof. Application of Lemma 2.6 (ii). □

It follows that im^+R^* (and $\ker^+R^*$) are exposed faces of V^+ for every P-projection R on A. However, we only know im^+R (and $\ker^+R$) to be semi-exposed faces of A^+.

It will be an important feature of the spaces we shall consider later on, that every exposed face of K is a projective face and that every extreme point of [0,e] is a projective unit. But these properties will not characterize projective faces and projective units in the general case.

In our next picture we have shown a base norm space (V,K) where $V = \mathbb{R}^3$. The corresponding order-unit space (A,e) shall be the space of all linear functionals on V where e is determined by $K \subset e^{-1}(1)$ (as usual). Here it can be verified that the linear functional a which assigns to every point z of V its z-coordinate, will be extreme in [0,e], but it will not be a projective unit. In fact, a is extreme in [0,e] since it is the only function in A(K) with values in [0,1] which assumes the extreme values 0,1 on the x-axis and z-axis, respectively. If a = Re for a positive projection R, then R^* must leave the z-axis pointwise fixed and vanish on the x,y-plane. Hence R^* is the orthogonal projection onto the z-axis. This projection is smooth, but it will not admit any smooth quasicomplement. (In fact, R^* admits many quasicomplements, but none of them are smooth.) Hence R is not a P-projection.

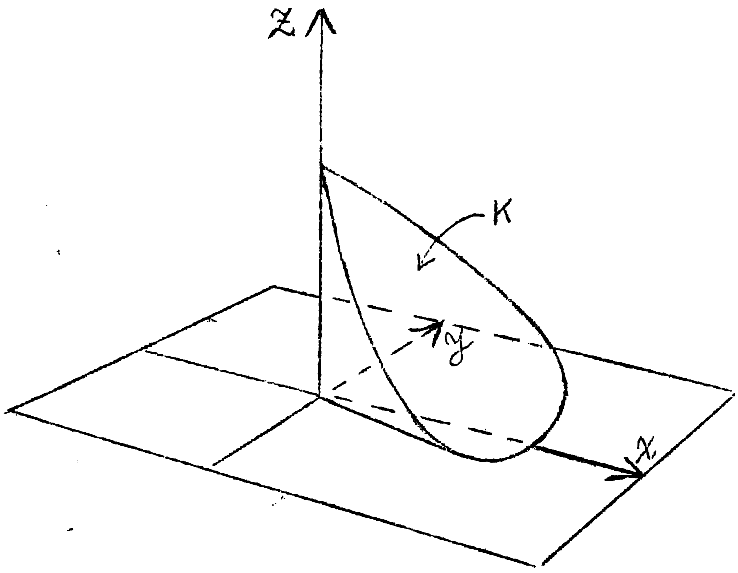

Fig. 3.

If R is a P-projection on A, then we may use Proposition 2.7 to obtain an explicit formula for $F_{R'}$ in terms of F_R. For $x \in K$ and $b \in A$ we shall have

$$(2.21) \qquad x \in F_{R'} \iff \inf\{b(x) \mid \chi_{F_R} \leq b \leq 1 \text{ on } K\} = 0 .$$

This motivates the following:

Definition. To an arbitrary face F of K is associated a set $F^{\#}$, called the quasicomplement of F, consisting of all $x \in K$ such that

$$\inf\{b(x) \mid \chi_F \leq b \leq 1 \text{ on } K\} = 0 .$$

Hence by definition $F_{R'} = (F_R)^{\#}$.

Note that $F^{\#}$ need not be convex for an arbitrary given face F. Hence $F^{\#}$ is not always a face. It is not hard to verify that $F^{\#}$ is a union of faces in the general case; hence it is a face whenever it is convex. But we shall not need these results in the sequel.

Note also that the definition of $F^{\#}$ closely resembles a known characterization of the ordinary complement F' of a closed face F of a compact convex set K [A_1,p.133]. The only difference is the occurence of the upper bound 1 for the variable function $b \in A$, but this difference can be quite essential as shown in the picture below.

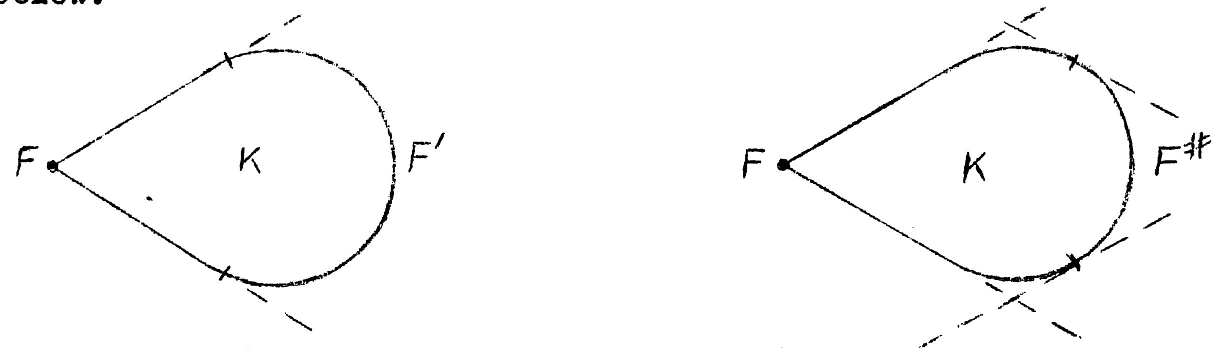

Fig. 4.

It is clear from Corollary 2.8 that a P-projection R on A is completely determined by its associated projective face F_R, and it is clear from Corollary 2.9 and Corollary 2.11 that R will also be determined by its associated projective unit Re. We are going to make these results more explicit, and in this connection we shall need some notation.

The set of all P-projections on A will be denoted by $\mathcal{P}$, the set of all projective units of A will be denoted by $\mathcal{U}$, and the set of all projective faces of K will be denoted by $\mathcal{F}$. Each of these sets is endowed with a natural operation of <u>complementation</u>, respectively $R \to R'$, $Re \to e - Re$, and $F \to F^{\#}$. The two sets $\mathcal{U}$ and $\mathcal{F}$ are also endowed with a natural <u>partial ordering</u>, respectively the ordering relativized from A, and the inclusion ordering of subsets of K. We complete the picture by giving the following:

<u>Definition</u>. If $R, S \in \mathcal{P}$ and $\operatorname{im} R \subset \operatorname{im} S$, then we shall write $R \preccurlyeq S$.

The relation $R \preccurlyeq S$ is antisymmetric since a P-projection is determined by its range, and thus it is a partial ordering.

<u>Lemma 2.15</u>. <u>If $R, S \in \mathcal{P}$ then the following are equivalent:</u>

(i) $R \preccurlyeq S$

(ii) $SR = R$

(iii) $Re \leq Se$

(iv) $\operatorname{im} R^* \subset \operatorname{im} S^*$

(v) $F_R \subset F_S$

(vi) $RS = R$

(vii) $S' \preccurlyeq R'$

Proof. (i) $\Rightarrow$ (ii) If $R \preccurlyeq S$ then $\operatorname{im} R \subset \operatorname{im} S$, so $SR = R$.

(ii) $\Rightarrow$ (iii) Generally $Re \leq e$ and $SRe \leq Se$. If $SR = R$, then $Re \leq Se$.

(iii) $\Rightarrow$ (iv) Application of (2.16).

(iv) $\Leftrightarrow$ (v) By the definition of F_R and F_S.

(iv) $\Rightarrow$ (vi) If $\operatorname{im} R^* \subset \operatorname{im} S^*$ then $S^*R^* = R^*$, which gives $RS = R$.

(vi) $\Rightarrow$ (vii) If $RS = R$ then $S^*R^* = R^*$, and so $\operatorname{im} R^* \subset \operatorname{im} S^*$. By (2.17) $\ker^+ S \subset \ker^+ R$, and so $\operatorname{im}^+ S' \subset \operatorname{im}^+ R'$, which means $S' \preccurlyeq R'$.

(vii) $\Rightarrow$ (i) We have already proved (i) $\Rightarrow$ (vii). Now we use this implication with S', R' in place of R, S and recall that $R'' = R$ and $S'' = S$. $\square$

We shall find it convenient to restate some of our previous results in terms of projective units and projective faces.

If $R \in \mathcal{P}$ then it follows from Corollary 1.3 and the equality $F_R^\# = (\operatorname{im}^+ R^*)' \cap K = (\ker^+ R)' \cap K$, that for a given $a \in A^+$

(2.22) $$Ra = a \quad \text{on} \quad F_R, \qquad Ra = 0 \quad \text{on} \quad F_R^\#,$$

and that Ra is the unique element of A^+ with these properties.

More specifically, we get by (1.17):

(2.23) $$\begin{aligned} Ra &= \sup\{b \in A^+ \mid b \leq a \ \text{on} \ F_R,\ b = 0 \ \text{on} \ F_R^\#\} \\ &= \inf\{b \in A^+ \mid b \geq a \ \text{on} \ F_R,\ b = 0 \ \text{on} \ F_R^\#\} \end{aligned}$$

Applying the above result with $a = e$, we conclude that

(2.24) $$Re = 1 \quad \text{on} \quad F_R, \qquad Re = 0 \quad \text{on} \quad F_R^\#,$$

and that Re is the unique element of A^+ with these properties.

In fact, by (2.6) we get the explicit formula:

$$(2.25)\qquad Re = \sup\{b \in A \mid 0 \le b \le e,\ b = 0 \text{ on } F_R^{\#}\}$$

Note that (2.25) is not a mere specialization of (2.23), since in (2.25) we have assumed $b \le e$ and not only $b \le e$ on F_R .

Applying (2.25) to R' and using $Re = e - R'e$ and $F_R = (F_{R'})^{\#}$, we get the alternative formula:

$$(2.26)\qquad Re = \inf\{c \in A \mid \chi_{F_R} \le c \le 1 \text{ on } K\} .$$

We shall close this section with a theorem. It contains no new information but may be considered a summary of some of the main results of the preceding pages.

Theorem 2.16. The map $R \to F_R$ is an order isomorphism of $\mathcal{P}$ onto $\mathcal{F}$ carrying the map $R \to R'$ into the map $F \to F^{\#}$, and its inverse is given by (2.23). Similarly the map $F_R \to Re$ given by (2.25) is an order isomorphism of $\mathcal{F}$ onto $\mathcal{U}$ carrying the map $F \to F^{\#}$ into the map $Re \to e - Re$, and its inverse is given by (2.20).

In the next section we shall show that under an additional hypothesis $\mathcal{P}$ (and hence also $\mathcal{F}$ and $\mathcal{U}$) is an orthomodular lattice.

§ 3. The lattice of P-projections.

Throughout this section we shall keep the assumptions of § 2, i.e. (A,e) and (V,K) shall be respectively an order-unit space and a base-norm space in separating order and norm duality. In addition we shall impose the following two requirements:

(3.1) A is pointwise monotone σ-complete.

(3.2) Every exposed face of K is projective.

The requirement (3.1) means that if $\{a_n\}$ is an increasing sequence from A which is bounded above, then there exists $a \in A$ such that $a(x) = \sup_n a_n(x)$ for all $x \in K$. In this case we shall write $a = \sup_n a_n$. (Clearly (3.1) implies the same statement for the pointwise infimum $\inf_n a_n$ of a descending sequence).

Note that (3.2) is a strong requirement which imposes severe restrictions on the convex set K. However, it will be implied by the "spectral axiom" we will assume later.

The P-projections mentioned henceforth will be defined on A unless otherwise specified. We have previously endowed the set of P-projections on A with a partial ordering $\preccurlyeq$, and we now agree to write $\vee_\alpha P_\alpha$ and $\wedge_\alpha P_\alpha$ respectively for the least upper bound and the greatest lower bound of a family $\{P_\alpha\}$ from $\mathcal{P}$, when these elements exist.

Lemma 3.1. If $\{P_n\}$ is a sequence from $\mathcal{P}$ then $P = \wedge_n P_n$ exists in $\mathcal{P}$, and its associated projective face is given by

$$F_P = \bigcap_n F_{P_n} \tag{3.3}$$

Proof. It follows from the pointwise monotone σ-completeness of A that $a = \Sigma_n 2^{-n} P_n' e \in A$. The function a takes values in $[0,1]$, and the set $K \cap a^{-1}(0)$ must be an exposed face of K. Hence there exists $P \in \mathcal{P}$ such that $F_P = K \cap a^{-1}(0)$.

For $x \in K$ one has $a(x) = 0$ iff $(P_n' e)(x) = 0$ for all n, which is equivalent to $(P_n e)(x) = 1$ for all n, and in turn to $x \in F_{P_n}$ for all n. Hence (3.3.) is valid.

It remains to prove that P is the greatest lower bound of $\{P_n\}$. Clearly $P \preccurlyeq P_n$ for all n, since $F_P \subset F_{P_n}$ for all n. Also if $Q \preccurlyeq P_n$ for all n, then $F_Q \subset \bigcap_n F_{P_n} = F_P$; and so $Q \preccurlyeq P$. Hence $P = \bigwedge_n P_n$. $\square$

<u>Proposition 3.2. The set $\mathcal{P}$ of P-projections on A ordered by $\preccurlyeq$, is a σ-complete lattice.</u>

Proof. The proposition follows from Lemma 3.1 since $P \to P'$ is an order reversing involution on $\mathcal{P}$. $\square$

We will now extend the notations $\vee$ and $\wedge$ to the lattices $\mathcal{F}$ and $\mathcal{U}$ of projective units and projective faces respectively. (We shall continue to use $\sup_\alpha a_\alpha$ and $\inf_\alpha a_\alpha$ to denote pointwise suprema and infima, when they exist, for families $\{a_\alpha\}$ from A.) For convenience we shall also write $h' \doteq e-h$ when $h \in \mathcal{U}$, but we shall continue to denote the quasicomplement of $F \in \mathcal{F}$ by the symbol $F^\#$ (since F' might be confused with the customary complement of F in K).

Since $F \to F^\#$ is an order reversing involution on $\mathcal{F}$, we have the following general formulas for $F_\alpha \in \mathcal{F}$:

(3.4) $$(\bigvee_\alpha F_\alpha)^\# = \bigwedge_\alpha F_\alpha^\# , \quad (\bigwedge_\alpha F_\alpha)^\# = \bigvee_\alpha F_\alpha^\#$$

By these formulas and (3.3.) we get the following expressions for the lattice operations for a sequence $\{F_n\}$ from $\mathcal{F}$:

(3.5) $$\begin{aligned} \bigwedge_n F_n &= \bigcap_n F_n \\ \bigvee_n F_n &= [\bigcap_n F_n^\#]^\# \end{aligned}$$

<u>Definition</u>. Two P-projections P and Q are said to be <u>orthogonal</u> if $P \preccurlyeq Q'$, and we then write $P \perp Q$.

Note that if $P \preccurlyeq Q'$ then $Q = Q'' \preccurlyeq P'$, so $Q \perp P$. Hence the relation $\perp$ is symmetric. Clearly $P \perp P'$ always holds.

We list some simple conditions for orthogonality, and we note that the last one depends on the equivalence of $P \preccurlyeq Q'$ and $\operatorname{im}^+ P \subset \operatorname{im}^+ Q' = \ker^+ Q$.

(3.6) $$P \perp Q \iff Pe + Qe \leq e$$

(3.7) $$P \perp Q \iff F_P \subset F_Q^\# \iff F_Q \subset F_P^\#$$

(3.8) $$P \perp Q \iff PQ = 0 \iff QP = 0 .$$

The notation $\perp$ will also be extended to $\mathcal{F}$ and $\mathcal{U}$. By (3.6) the relation $g \perp h$ holds for two elements g and h of $\mathcal{U}$ iff $g + h \leq e$, and by (3.7) the relation $F \perp G$ holds for two elements F and G of $\mathcal{F}$ iff $F \subset G^\#$, or equivalently $G \subset F^\#$.

We now record some simple observations which will be useful. If a and b are in A with $a \leq b$, then the set

$$(b-a)^{-1}(0) = \{x \in K \mid a(x) = b(x)\}$$

is an exposed, therefore projective, face.

Thus if F and G are projective faces, we have

(3.9) $$a \leq b\,,\quad a = b \text{ on } F \cup G \Rightarrow a = b \text{ on } F \vee G\,.$$

Immediate consequences of this are:

(3.10) $$0 \leq a\,,\quad a = 0 \text{ on } F \cup G \Rightarrow a = 0 \text{ on } F \vee G$$

and

(3.11) $$a \leq e\,,\quad a = 1 \text{ on } F \cup G \Rightarrow a = 1 \text{ on } F \vee G\,.$$

The results above can be extended to any finite or countably infinite union of projective faces.

<u>Lemma 3.3. Let $P, Q \in \mathcal{P}$ and $P \perp Q$. Then</u>

(3.12) $$Pe + Qe = (P \vee Q)e$$

<u>Proof.</u> By (3.6) $Pe + Qe \leq e$. Clearly $(Pe+Qe)(x) = 1$ for $x \in F_P \cup F_Q$, and by (3.9) $Pe + Qe = e$ on $F_P \vee F_Q$. Also $Pe + Qe = 0$ on the face $F_P^{\#} \cap F_G^{\#} = (F_P \vee F_G)^{\#}$. This implies (see (2.22)) $Pe + Qe = (P \vee Q)e$. □

<u>Proposition 3.4. Let $\{P_i\}$ be a finite sequence from $\mathcal{P}$. Then the following are equivalent</u>

(i) $P_i \perp P_j$ for $i \neq j$

(ii) $\sum_i P_i e = (\bigvee_i P_i)e$

(iii) $\sum_i P_i e \leq e$

<u>Proof.</u> (i) ⇒ (ii) The proof goes by induction on the number n of elements of $\{P_i\}$. For $n = 1$ the statement is trivial. We

assume the statement valid for $n-1$ and consider a finite sequence $\{P_1,\ldots,P_n\}$ such that (i) holds. Let $Q = P_1 \vee \ldots \vee P_{n-1}$. We have $P_n \preccurlyeq P_i'$ for $i = 1,\ldots,n-1$. Hence $P_n \preccurlyeq P_1' \wedge \ldots \wedge P_{n-1}' = Q'$, and so $P_n \perp Q$. By Lemma 3.3 and the induction hypothesis:

$$(P_1 \vee \ldots \vee P_n)e = Qe + P_n e = P_1 e + \ldots + P_n e .$$

This completes the induction.

(ii) ⟹ (iii) Trivial.

(iii) ⟹ (i) Follows from (3.6). ▯

In view of the preceding result we will write $P_1 \dotplus \ldots \dotplus P_n$ in place of $P_1 \vee \ldots \vee P_n$ when $P_1,\ldots,P_n$ are mutually orthogonal. Note however, that in general $P_1 \dotplus \ldots \dotplus P_n \neq P_1 + \ldots + P_n$. (We shall give conditions for equality in §4.)

Turning to a finite sequence $\{g_1,\ldots,g_n\}$ from $\mathcal{U}$, we get the following useful formula:

(3.13) $$g_1 \vee \ldots \vee g_n = g_1 + \ldots + g_n , \quad \text{if } g_i \perp g_j \text{ when } i \neq j .$$

We are now ready to show that $\mathcal{P}$ is in fact an orthomodular lattice. In Theorem 3.5 below (3.14), (3.15), (3.16) state that the map $P \to P'$ is an "orthocomplementation" on $\mathcal{P}$, and (3.17) is the "orthomodular identity".

Theorem 3.5. The σ-complete lattice $\mathcal{P}$ is orthomodular; that is, for P and Q in $\mathcal{P}$:

(3.14) $P'' = P$

(3.15) $P \preccurlyeq Q$ implies $Q' \preccurlyeq P'$

(3.16) $P \wedge P' = 0$ and $P \vee P' = I$

(3.17) $P \preccurlyeq Q$ implies $Q = P \dotplus (Q \wedge P')$

Proof. Statement (3.14) follows at once from the fact that P is the quasicomplement of P', and (3.15) follows from Lemma 2.15.

In order to prove (3.16) we consider $Q \in \mathcal{P}$ such that $Q \preccurlyeq P$ and $Q \preccurlyeq P'$. By Lemma 2.15 $Q = PQ = P'Q$. Since $P \perp P'$ we also have $PP' = 0$ (compare (3.8)). Hence

$$Q = PQ = P(P'Q) = (PP')Q = 0 .$$

Thus $P \wedge P' = 0$, and by complementation also

$$P \vee P' = (P' \wedge P)' = 0' = I .$$

The orthomodular identity (3.17) is most conveniently proved in the lattice $\mathcal{U}$. If $g,h \in \mathcal{U}$ and $g \leq h$ then $g \perp h'$, so by (3.13)

$$h \wedge g' = (h' \vee g)' = (h'+g)' = e - h' - g = h - g .$$

Since $h \wedge g' \leq g'$ we have $(h \wedge g') \perp g$. Hence we may apply (3.13) once more and obtain the desired equality:

$$g \vee (h \wedge g') = g + (h \wedge g') = g + (h-g) = h . \quad \square$$

We close this section by a proposition involving the analogue of the range projection in a von Neumann algebra. For the statement and proof of this proposition it is convenient to use the short notation face(a) to denote the smallest face of A^+ containing a given element a of A^+.

Lemma 3.6. Let $F \in \mathcal{F}$, say $F = F_P$ where $P \in \mathcal{P}$. Then $F^{\perp} = \text{face}(h)$ where h is the projective unit defined by

$$h = P'e = \sup\{a \in A \mid 0 \leq a \leq e,\ a = 0 \text{ on } F\} \tag{3.18}$$

Proof. Writing $Q = P'$ we shall have $F = (\ker^+ Q^*) \cap K$, and by application of (1.12) also

$$F^{\perp} = (\ker^+ Q^*)^{\perp} = (\ker Q^*)^{\perp} = \mathrm{im}^+ Q .$$

By Corollary 2.11, $\mathrm{im}^+ Q$ is a face of A^+ generated by $Qe = P'e$. Hence $F^{\perp} = \mathrm{face}(P'e)$.

The last equality sign of (3.18) is justified by virtue of (2.25). □

We shall also need the following simple equivalence valid for projective units h and k:

(3.19) $$h \leq k \iff \{x \in K \mid k(x) = 0\} \subset \{x \in K \mid h(x) = 0\} .$$

In fact, $h \leq k$ iff $k' \leq h'$, and the projective faces associated with $k' = e - k$ and $h' = e - h$ are $\{x \in K \mid k(x) = 0\}$ and $\{x \in K \mid h(x) = 0\}$, respectively, (see (2.20)).

Proposition 3.7. For each $a \in A^+$ there exists a smallest projective unit h such that $a \in \mathrm{face}(h)$, and h is the unique element of $\mathcal{U}$ such that for $x \in K$:

(3.20) $$h(x) = 0 \iff a(x) = 0 .$$

Moreover, $a < \|a\| h$.

Proof. The set $F = \{x \in K \mid a(x) = 0\}$ is an exposed, hence projective, face of K. Let $F = F_P$ where $P \in \mathcal{P}$, and define $h = P'e$. Then $Pe = e - h$, and so

(3.21) $$F = F_P = \{x \in K \mid h(x) = 0\} .$$

Hence the equivalence (3.20) is valid.

Clearly we may assume $a \neq 0$. Then $0 \leq \|a\|^{-1}a \leq e$, and $\|a\|^{-1}a = 0$ on F. Hence (3.18) gives $\|a\|^{-1}a \leq P'e = h$, and so $a \leq \|a\|h$.

Clearly $a \in F^{\perp}$, and $F^{\perp} = \text{face}(h)$ by Lemma 3.6. Hence $a \in \text{face}(h)$. Now suppose $a \in \text{face}(k)$ for a projective unit k. Then $a \leq \lambda k$ for some $\lambda \in \mathbb{R}^{+}$. Hence

(3.22) $$\{x \in K \mid k(x) = 0\} \subset F .$$

It follows from (3.21) and (3.22) that the inclusion at the right side of (3.19) is valid. Hence $h \leq k$. $\square$

<u>Definition</u>. For given $a \in A^{+}$ we shall denote the projective unit h of Proposition 3.7 by $rp(a)$.

The following consequence of Proposition 3.7 will be useful later:

(3.23) $$0 \leq a \leq e \implies a \leq rp(a) .$$

§4. Compatibility

Our assumptions in this section will be the same as those of the preceding section, i.e. (A,e) and (V,K) shall be order-unit and base-norm spaces in separating order and norm duality, satisfying (3.1) and (3.2).

Definition. A P-projection P on A and an element a of A are said to be compatible if $Pa + P'a = a$. (We shall also say that P is compatible with a and vice versa).

To motivate this definition we will anticipate a result to be proved later: If A is the self-adjoint part of a von Neumann algebra and V its predual, then the P-projections on A are exactly the maps $a \to pap$ where p is a projection in A, and the orthocomplementation $P \to P'$ in $\mathcal{P}$ will correspond to passage to orthogonal complements $p \to p' = I - p$ for projections in A. Now it can be easily checked that a and p commute iff $pap + p'ap' = a$. Hence the notion of compatibility will correspond to the notion of commutation.

Proposition 4.1. A P-projection P on A is compatible with an element a of A^+ iff $Pa \leq a$.

Proof. If P and a are compatible, then $a = Pa + P'a \geq Pa$.

Conversely, if $Pa \leq a$, then $a - Pa \geq 0$ and $P(a-Pa) = Pa - Pa = 0$. Hence $a - Pa \in \ker^+ P = \operatorname{im}^+ P'$.
Thus

$$a - Pa = P'(a-Pa) = P'a,$$

and so $a = Pa + P'a$. □

The next proposition provides the explicit expression $P \wedge Q = PQ$ when P and Q commute. Thus in this case the product of the two $\mathcal{P}$-projections P and Q is again a $\mathcal{P}$-projection; this property also serves to characterize compatibility of P and the projective unit associated with Q and vice versa.

Proposition 4.2. Let P and Q be $\mathcal{P}$-projections; then the following are equivalent:

(i) PQ is a $\mathcal{P}$-projection

(ii) $PQ = P \wedge Q$

(iii) P is compatible with Qe

(iv) Q is compatible with Pe

(v) $PQ = QP$.

Proof. (i) $\Longleftrightarrow$ (ii) Assume PQ is a $\mathcal{P}$-projection, and write $PQ = R$. Then $PR = PQ = R$, so $R \preccurlyeq P$. (Lemma 2.15). Also $RQ = PQ = R$, so $R \preccurlyeq Q$. Hence $R \preccurlyeq P \wedge Q$.

Now suppose $S \in \mathcal{P}$ and $S \preccurlyeq P$ and $S \preccurlyeq Q$. Then $SP = S$ and $SQ = S$. Hence

$$SR = SPQ = SQ = S,$$

and so $S \preccurlyeq R$. This proves $R = P \wedge Q$.

The reverse implication is trivial.

(ii) $\Longleftrightarrow$ (iii) If $PQ = P \wedge Q$, then

$$P(Qe) = (P \wedge Q)e = (Pe) \wedge (Qe) \leq Qe,$$

and by Proposition 4.1, P is compatible with Qe.

Conversely, assume that P is compatible with Qe. Then $0 \leq P(Qe) \leq Qe$ by compatibility, and $0 \leq P(Qe) \leq Pe$ since $Qe \leq e$.

It follows that $rp(PQe)$ is below Qe and Pe, and by (3.23):

$$0 \leq PQe \leq rp(PQe) \leq (Pe) \wedge (Qe) = (P \wedge Q)e .$$

It follows that for every $a \in A$ with $0 \leq a \leq e$, $0 \leq PQa \leq (P \wedge Q)e$, and so PQa is in the order ideal generated by $(P \wedge Q)e$, which is equal to $im(P \wedge Q)$. Combining this with Lemma 2.15 (vi), we get

$$PQa = (P \wedge Q)(PQa) = (P \wedge Q)Qa = (P \wedge Q)a ,$$

and so $PQ = P \wedge Q$.

(iii) <==> (iv) Assume that P is compatible with Qe, and write h in place of Pe and g for Qe. We will show that h is compatible with Q.

Since $h = Pg + (h-Pg)$, we shall have

$$Qh = QPg + Q(h-Pg) . \tag{4.1}$$

By Proposition 4.1. it suffices to prove $Qh \leq h$, and we shall do this by showing that $QPg \leq h$ and $Q(h-Pg) = 0$.

Since P is compatible with g, one has $0 \leq Pg \leq g$. Hence Pg belongs to the order ideal generated by $g = Qe$, and so $Pg \in im\,Q$ (Corollary 2.11). It follows that

$$QPg = Pg \leq Pe = h . \tag{4.2}$$

Since P is compatible with g, one also has $Pg = g - P'g$. By substitution of this expression for Pg and by use of the inequality $P'g \leq P'e = h'$, we get

$$h - Pg = h - g + P'g \leq h + h' - g = e - g = g' .$$

Thus $0 \leq Q(h-Pg) \leq Qg' = 0$; and so we have proved

$$Q(h-Pg) = 0 , \tag{4.3}$$

as needed.

The converse statement follows by interchanging P and Q.

(iv) $\iff$ (v) Assume Q compatible with Pe. Going back to (ii) we conclude $PQ = P \wedge Q$. By the equivalence of (iii) and (iv) we also have compatibility of P and Qe, and the same argument with P and Q interchanged gives $QP = P \wedge Q$. In particular $PQ = QP$.

Conversely, if $PQ = QP$ then $Q(Pe) = P(Qe) \leq Pe$, so Q is compatible with Pe. $\square$

<u>Definition</u>. Two P-projections P and Q are said to be <u>compatible</u> if they satisfy the equivalent conditions (i) - (v) of Proposition 4.2 ; and this notion of compatibility is also transferred from $\mathcal{P}$ to the lattices $\mathcal{U}$ and $\mathcal{F}$ isomorphic with $\mathcal{P}$.

Note that the notion of compatibility for two P-projections P and Q can be considered an extension of the previously defined notion of compatibility for a P-projection P and an element a of A, by virtue of statements (iii) and (iv).

Note also that by Lemma 2.15 and by formula (3.8) the following implication is valid:

(4.4) $\quad P \precsim Q \implies P$ and Q are compatible,

(4.5) $\quad P \perp Q \implies P$ and Q are compatible.

Observe that if P is compatible with a, then

(4.6) $\quad a = Pa + P'a = (P')'a + P'a$,

and thus P' is compatible with a. Now if P and Q are compatible then P is compatible with Qe, so P' is compatible with Qe, and thus P' and Q are compatible. It follows that the following statements are equivalent:

(4.7) $\quad P$ and Q are compatible

(4.8) P and Q' are compatible

(4.9) P' and Q are compatible

(4.10) P' and Q' are compatible.

We next state a condition characterizing compatibility of P and Q in terms of a decomposition property of the lattice $\mathcal{P}$. In this connection we shall need the following simple consequences of Proposition 3.4:

(4.11) $$P_1 \perp P_2 \implies P_1 \preccurlyeq P_1 \dot{+} P_2 ,$$

and

(4.12) $$P_1 \perp P_2 , P_1 \perp P_3 , P_2 \perp P_3 \implies P_1 \perp (P_2 \dot{+} P_3) .$$

<u>Proposition 4.3. Two P-projections P and Q are compatible iff there exist mutually orthogonal P-projections R, S, T such that</u>

(4.13) $$P = R \dot{+} S , \quad Q = S \dot{+} T .$$

<u>If such a decomposition exists it is unique, in fact</u>

(4.14) $$R = P \wedge Q' , \quad S = P \wedge Q , \quad T = Q \wedge P' .$$

<u>Proof</u>. 1.) Assume first that (4.13) holds. By (4.11) $S \preccurlyeq P$ and $S \preccurlyeq Q$, and by (4.12) $P \perp T$ and $Q \perp R$. By Lemma 2.15 and formula (3.8):

(4.15) $$PS = SP = S , \quad PT = TP = 0 ,$$

(4.16) $$QS = SQ = S , \quad QR = RQ = 0 .$$

This implies $P(Qe) = Se \leq Qe$, and by Proposition 4.1 P and Q must be compatible.

Now that P and Q are known to be compatible, we can write

$$(P\wedge Q)e = PQe = P(Se+Te) = Se ,$$

and so $P \wedge Q = S$.

We also know that P and Q' are compatible (see (4.8)). Hence

$$(P\wedge Q')e = PQ'e = P(e-Se-Te) = Pe - Se = Re ,$$

and so $P \wedge Q' = R$. Similarly we prove $Q \wedge P' = T$.

2.) Conversely, assume P and Q compatible, and define R,S,T by (4.14). Since P and Q' also are compatible, we shall have

$$Re = (P\wedge Q')e = PQ'e = Pe - PQe = Pe - (P\wedge Q)e = Pe - Se .$$

Hence $Pe = Re + Se$. By Proposition 3.4, R and S are orthogonal and $P = R \dotplus S$. Similarly we prove $S \perp T$ and $Q = S \dotplus T$. Finally

$$Re + Te = (P\wedge Q')e + (Q\wedge P')e \leq Pe + P'e = e ,$$

which proves $R \perp T$. ∎

Observe that if P and Q are compatible P-projections, then

(4.17) $$P \vee Q = P \dotplus Q \wedge P' = P \dotplus QP' .$$

In fact, the relation $P \vee Q \geq P \dotplus Q \wedge P'$ will hold for any two P-projections, and if P and Q are compatible then by Proposition 4.3

$$P \vee Q = R \vee S \vee T = P \dotplus Q \wedge P' .$$

The last equality of (4.17) follows since P' and Q also are compatible.

We will now study the connection between compatibility and Boolean algebras, and we recall that the notion of a Boolean algebra may be defined as a distributive orthocomplemented lattice (cfr.§3).

Proposition 4.4. Let L be a subset of $\mathcal{P}$ containing 0 and I and assume that L is closed under the operations of $\mathcal{P}$, that is $P,Q \in L$ shall imply $P \wedge Q \in L$, $P \vee Q \in L$ and $P' \in L$. Then with these operations L is a Boolean algebra iff every pair of projections in L are compatible.

Proof. 1.) Assume first that every pair of elements of L are compatible. Since $P \to P'$ is an order reversing involution on L, it suffices to prove the distributive law

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) . \tag{4.18}$$

The inequality

$$P \wedge (Q \vee R) \geq (P \wedge Q) \vee (P \wedge R)$$

is valid in any lattice. We will show that the opposite inequality also holds in the present case. By (4.17)

$$P \wedge (Q \vee R)e = P(Q+RQ')e = PQe + PRQ'e .$$

Hence (by Proposition 3.4):

$$P \wedge (Q \vee R) = P \wedge Q \dot{+} P \wedge R \wedge Q' \leq (P \wedge Q) \vee (P \wedge R) .$$

2.) Assume next that L is a Boolean algebra, and let P and Q be any two elements of L. The decomposition (4.13) with R,S,T defined as in (4.14) follows at once from the distributive law. It is also easily checked that $R \preccurlyeq S'$, $R \preccurlyeq T'$ and $S \preccurlyeq T'$. Hence R,S,T are mutually orthogonal. Then it follows by Proposition 4.3 that P and Q are compatible. $\square$

We shall now prove that if P and Q are orthogonal P-projections, then $P \dot{+} Q$ and $P + Q$ will agree on elements $a \in A$ which are compatible with P and Q. (This will generalize Proposition

3.4 where the same result is proved with $a = e$).

We first observe that the set of elements of A compatible with a given P-projection P, is a linear subspace of A containing the order unit e (in fact it is just $\ker(I-P-P')$). It follows that for $P \in \mathcal{P}$ and $a \in A$:

(4.19) $\left\{ \begin{array}{l} P \text{ and } a \text{ are compatible iff } P \text{ and } a + \lambda e \\ \text{are compatible for one (hence all) } \lambda \in \mathbb{R} . \end{array} \right.$

We shall also need the following lemma which is independent of any orthogonality requirement for the occuring P-projections.

<u>Lemma 4.5. Let P and Q be compatible P-projections. If $a \in A$ is compatible with P and Q then a is compatible with $P \vee Q$ and $P \wedge Q$.</u>

<u>Proof</u>. By (4.19) we can assume $a \geq 0$. Since a is compatible with P and Q we shall have $Pa \leq a$ and $Qa \leq a$ (Proposition 4.1), and since P and Q are compatible we shall have $P \wedge Q = PQ$ (Proposition 4.2). Hence

$$(P \wedge Q)a = P(Qa) \leq Pa \leq a .$$

Thus $P \wedge Q$ is compatible with a .

By (4.6) a is compatible with P' and Q' , and also with $P \vee Q = (P' \wedge Q')'$. ▯

<u>Proposition 4.6. If $P_1, \ldots, P_n$ are mutually orthogonal P-projections and $a \in A$ is compatible with each of them, then</u>

(4.20) $$(P_1 \dotplus \ldots \dotplus P_n)a = P_1 a + \ldots + P_n a .$$

Proof. 1.) We first assume $n = 2$. The P-projections P_1' and P_2' are compatible with a by (4.5) and (4.6), and with each other by (4.10). Then it follows from Lemma 4.5 that $P_1' \wedge P_2'$ is compatible with a, and this means that $a = (P_1' \wedge P_2')a + (P_1' \wedge P_2')'a$. Hence

$$(P_1 \dot{+} P_2)a = (P_1 \vee P_2)a = (P_1' \wedge P_2')'a = a - (P_1' \wedge P_2')a =$$

$$= a - P_1'P_2'a = a - P_1'(a - P_2 a) = (a - P_1'a) + P_1'P_2 a = P_1 a + P_2 a .$$

2.) The proposition for $n > 2$ follows by induction. (Note that the relation $P_n \perp (P_1 \dot{+} \ldots \dot{+} P_{n-1})$ follows from (4.12)).

Definition. A projection P on A will be said to be central if it is a P-projection compatible with all elements a of A.

Now the following corollary will be an immediate consequence of Proposition 4.6:

Corollary 4.7. If $P_1, \ldots, P_n$ are mutually orthogonal central projections on A, then

$$(4.21) \qquad P_1 \dot{+} \ldots \dot{+} P_n = P_1 + \ldots + P_n .$$

Clearly 0 and I are central projections, and any two central projections are compatible. By (4.6) P' is a central projection whenever P is, and by Lemma 4.5 $P \vee Q$ and $P \wedge Q$ are central projections whenever P and Q are. It then follows from Proposition 4.4 that the collection of central projections is a Boolean algebra.

Definition. The Boolean algebra of central projections on A will be called the Boolean center of A.

It will follow from the next proposition that this concept is (up to a canonical isomorphism) the same as Wils' Boolean center [W_1]. Note, however, that Wils' definition applies to much more general spaces than those of the present section.

Proposition 4.8. If P is a weakly continuous projection on A, then the following are equivalent:

(i) P is central.

(ii) P is a P-projection such that $P' = I - P$.

(iii) $0 \leq Pa \leq a$ for all $a \in A^+$.

Proof. (i) ==> (ii) Assume $P \in \mathcal{P}$ and P compatible with all $a \in A$. Then $P'a = a - Pa$ for all $a \in A$, so $P' = I - P$.

(ii) ==> (iii) If $P \in \mathcal{P}$ and $P' = I - P$, then P and $I - P$ are both positive. Hence (iii) follows.

(iii) ==> (i) Assume (iii), and note first that $P \geq 0$. Also $Pe \leq e$, and so $-e \leq Pa \leq e$ when $-e \leq a \leq e$. Hence $\|P\| \leq 1$. Similarly $(I-P) \geq 0$ and $\|I-P\| \leq 1$.

Clearly P and $I - P$ are quasicomplementary, as are P^* and $(I-P)^* = I - P^*$. Thus by Theorem 1.8 P is a P-projection. Finally, by Proposition 4.1, (iii) implies that P is central. ∎

The notion of centrality has a lattice theoretic analogue. In an orthomodular lattice L one says that two elements are compatible if they admit a decomposition into orthogonal parts as described in (4.13) (cf. [M,p.70]). The center of the lattice L then is defined to be the set of those elements of L which are compatible with all elements of L; this is always a Boolean algebra.

Observe that the central projections we have defined are always contained in the lattice center of $\mathcal{P}$. It will follow from the spectral theorem (proved in § 5 under an additional hypothesis) that the converse holds, i.e. that every projection in the lattice center of $\mathcal{P}$ is central.

We next define a concept which will play an important role in the spectral theory.

<u>Definition</u>. A P-projection P is said to be <u>bicompatible</u> with an element a of A if it is compatible with a and with all P-projections compatible with a. The collection of all P-projections bicompatible with a is called the <u>$\mathcal{P}$-bicommutant</u> of a and is denoted $\mathcal{B}(a)$.

The term "$\mathcal{P}$-bicommutant" is motivated by the application to von Neumann algebras. Here the $\mathcal{P}$-bicommutant will be (canonically isomorphic to) the Boolean algebra of projections in the customary bicommutant. (See the motivating remarks for the definition of compatibility at the beginning of this section.) A partial justification for the term "$\mathcal{P}$-bicommutant" is also provided by the fact that an element P of $\mathcal{B}(a)$ will actually <u>commute</u> with all P-projections compatible with a.

We will show that $\mathcal{B}(a)$ is a σ-complete Boolean algebra for every $a \in A$, and we shall need the following lemma which is of some independent interest.

<u>Lemma 4.9</u>. <u>If $\{P_n\}$ is an increasing sequence of P-projections all of which are compatible with $a \in A$, then $P = \vee_n P_n$ is compatible with a. If $a \geq 0$, then</u>

(4.22) $$Pa = \sup_n P_n a .$$

<u>Proof</u>. Note first that by observation (4.19) we may assume $a \geq 0$ throughout the proof.

Let F_n be the projective face associated with P_n and F the projective face associated with P. By (2.22):

(4.23) $$P_n a = a \quad \text{on } F_n , \qquad P_n a = 0 \quad \text{on } F_n^{\#} .$$

Note that $P_n = P_{n+1} P_n$ since $P_n \preccurlyeq P_{n+1}$, and $P_n a \leq a$ since P_n is compatible with a.
Hence

$$P_n a = P_{n+1} P_n a \leq P_{n+1} a ,$$

and it follows that $\{P_n a\}$ is increasing and bounded above by a.
We write

$$b = \sup_n P_n a \leq a .$$

By Lemma 3.1, $F^{\#} = (\vee_n F_n)^{\#} = \cap_n F_n^{\#}$. Hence $x \in F^{\#}$ implies $x \in F_n^{\#}$ for all n and by (4.23) also $(P_n a)(x) = 0$ for all n. Thus

(4.24) $$b = 0 \quad \text{on } F^{\#} .$$

For fixed n we consider an arbitrary point $y \in F_n$. If $m \geq n$ then $y \in F_n \subset F_m$, and by (4.23) $(P_m a)(y) = a(y)$. Hence $b = a$ on F_n. Since $b \leq a$, the set $\{x \in K \mid b(x) = a(x)\}$ is an exposed, hence projective, face. We have just seen that this face must contain all F_n, and therefore also $F = \vee_n F_n$. Thus

(4.25) $$b = a \quad \text{on } F .$$

By (4.24) and (4.25) (and by the uniqueness statement concerning (2.22)), we shall have $b = Pa$, and (4.22) is proved.

By the inequality $Pa = b \leq a$ (and by Proposition 4.1) the P-projection P is compatible with a . $\square$

<u>Theorem 4.10. For each $a \in A$ the $\mathcal{P}$-bicommutant $\mathcal{B}(a)$ contains 0 and I, it is closed under the map $P \to P'$, and it is closed under finite and countable lattice operations. Furthermore, every pair of elements of $\mathcal{B}(a)$ is compatible, and thus $\mathcal{B}(a)$ is a σ-complete Boolean algebra.</u>

<u>Proof</u>. Clearly 0 and I are in $\mathcal{B}(a)$.

Assume $P \in \mathcal{B}(a)$. Then P is compatible with a , and it follows that P' is compatible with a (see (4.6)). If Q is compatible with a , then P is compatible with Q , and it follows that P' is compatible with Q (see (4.9)). Hence $P' \in \mathcal{B}(a)$.

Assume next that $P \in \mathcal{B}(a)$ and $Q \in \mathcal{B}(a)$. Since Q is compatible with a and P is compatible with all P-projections compatible with a , P must be compatible with Q . By Lemma 4.5 , $P \vee Q$ and $P \wedge Q$ are compatible with a . If $R \in \mathcal{P}$ is compatible with a , then P and Q will be compatible with R and hence with Re . By Lemma 4.5 , $P \vee Q$ and $P \wedge Q$ are compatible with Re and hence with R . This shows that $\mathcal{B}(a)$ is closed under finite lattice operations. By Proposition 4.4 , $\mathcal{B}(a)$ is a Boolean algebra.

Finally we consider a sequence $\{P_n\}$ in $\mathcal{B}(a)$. We shall prove that $P = \bigvee_n P_n$ is in $\mathcal{B}(a)$. By the preceding part of the proof we can assume $\{P_n\}$ increasing, and it follows by application of Lemma 4.9 that P is compatible with a , and that P is compatible with Re for all $R \in \mathcal{P}$ compatible with a . Hence $P \in \mathcal{B}(a)$. Since $\bigwedge_n P_n = (\bigvee_n P_n')'$, this completes the proof. $\square$

We will also transfer the definition of compatibility from the lattice $\mathcal{P}$ to the lattices $\mathcal{U}$ and $\mathcal{F}$. We define two projective units $h = Pe$ and $g = Qe$, respectively two projective faces $F = F_P$ and $G = F_Q$, to be <u>compatible</u> if P and Q are compatible. Similarly we shall say that an element a of A is <u>compatible</u> with a projective unit $h = Pe$, respectively with a projective face $F = F_P$, if a is compatible with P. (Note that the two definitions above are consistent if a happens to be a projective unit, say $a = Qe$). Finally we shall say that a projective unit Pe, respectively a projective face F_P, <u>is bicompatible</u> with a if $P \in \mathcal{B}(a)$.

It is not difficult to give alternative expressions for compatibility in terms of projective units and projective faces. For $b \in A^+$ we denote the order ideal generated by b by $[b]$, and we recall that for $P \in \mathcal{P}$ one has $\operatorname{im} P = [Pe]$, (Corollary 2.11); and since $\operatorname{im} P' \subset \ker P$ and $\operatorname{im} P \subset \ker P'$ we conclude that an element a of A is compatible with a projective unit $h = Pe$ iff

$$a \in [h] + [h'] . \tag{4.26}$$

Next we note that an element a of A^+ is compatible with a projective face $F = F_P$ iff it admits a decomposition into positive elements a_1 and a_2 such that

$$a = a_1 + a_2 , \quad a_1 = 0 \text{ on } F^{\#}, \quad a_2 = 0 \text{ on } F . \tag{4.27}$$

In fact, the necessity of this condition follows from (2.22), and the sufficiency follows from the uniqueness statement accompanying the same formula (2.22).

Finally we note that by (4.19) an arbitrary element of A will be compatible with $F = F_P$ iff $a + \lambda e$ admits a decomposition of the type (4.27) for some λ such that $a + \lambda e \geq 0$.

We shall give several concrete examples later, but we feel that at least one simple example should be presented here to illustrate the notions studied in the last two sections. For this purpose we return to the circular cone in the second figure of § 1 . Let (V,K) be the base-norm space shown in this picture, and let (A,e) be the order-unit space of all linear functionals on V , with $e(x) = 1$ for all $x \in K$. Now K is a plane circular disk, and A can be identified with the (3 - dimensional) space of all affine functions on K with pointwise ordering and uniform norm.

It is easily verified that the requirements (3.1) and (3.2) are satisfied in this case. In fact, the only proper faces of K are the extreme points, and each extreme point is a projective face whose quasicomplement is the diametrically opposite extreme point. Applying the definitions and results of the last two section, one will observe that the only projective faces compatible with a proper projective face F , are $\emptyset, K, F$ itself, and $F^{\#}$. (One way to see this is to note that for all other projective faces G , $F \neq (F \wedge G) \dot{+} (F \wedge G') = \emptyset$; hence one does not have a decomposition of the type (4.13)). One will also observe that a non-constant function $a \in A(K)$ will be compatible with F iff the lines $a(x) = \text{const}$ are parallel with the tangent to K at F . (One way to see this is to note that a decomposition of the type (4.27) into positive components is possible in this case only.)

In the picture below we have shown a projective face $F = F_P$, its quasicomplement $F^{\#} = F_{P'}$, and the corresponding projective unit $h = Pe$.

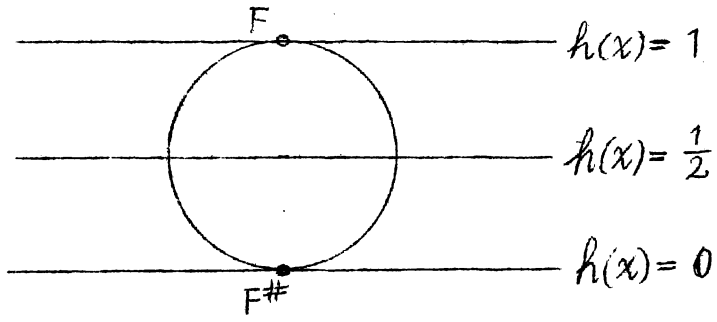

Fig. 5.

In this example every element $a \in A(K)$ is of the form $\lambda_o e + \lambda h$ for some $\lambda_o, \lambda \in \mathbb{R}$ and some projective unit h (and trivially $h \in \mathcal{B}(a)$). This is due to the extreme simplicity of the present example, and it will not hold for more general cases. However, in the next section on spectral theory we shall give conditions such that every $a \in A$ can be uniformly approximated by linear combinations of projective units in $\mathcal{B}(a)$.

§ 5. The spectral theorem

We shall keep the assumptions of the preceding sections unless otherwise stated. Thus (A,e) and (V,K) shall be order-unit and base-norm spaces in separating order and norm duality satisfying (3.1) and (3.2).

We begin by proving a general result (based on pointwise monotone σ- completeness) which will be needed in the treatment of the spectral theorem.

Proposition 5.1. The space A is norm complete.

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be a Cauchy sequence from A, and assume without loss of generality that $\|a_n - a_{n-1}\| \leq 2^{-n}$ for $n = 1,2,\ldots$. Writing

$$(5.1) \quad b_n = a_0 + \sum_{j=1}^{n} [(a_j - a_{j-1}) + 2^{-j}e] = a_n + (1-2^{-n})e ,$$

we get an increasing sequence $\{b_n\}_{n=1}^{\infty}$ such that $\|b_n - b_{n-1}\| \leq 2^{-n+1}$. In particular, $\{b_n\}$ is bounded above, so it has a pointwise limit b in A. In fact, $\{b_n\}$ is norm-Cauchy, and

$$(5.2) \qquad \|b - b_n\| \leq 2^{-n+1} .$$

By (5.1) and (5.2)

$$\|(b-e) - a_n\| = \|(b-b_n) - 2^{-n}e\| \leq 3 \cdot 2^{-n} .$$

Hence $\{a_n\}$ converges in norm to the limit $b - e$. □

As was discussed in the introduction, to achieve the spectral theorem we shall impose a condition which will play a role similar to that of Stone in ordinary ("commutative") integration theory.

Definition. The spaces A and V will be said to be in weak spectral duality if for every $a \in A$ and every $\lambda \in \mathbb{R}$ there exists a projective face F compatible with a such that

(5.3) $$a \leq \lambda \text{ on } F, \qquad a > \lambda \text{ on } F^{\#}.$$

If in addition F is unique, then A and V are said to be in spectral duality.

Note that weak spectral duality makes the requirement (3.2) redundant. More specifically, one has:

Proposition 5.2. If (A,e) and (V,K) are order-unit and base norm spaces in separating order and norm duality and if every $a \in A^+$ admits an $F \in \mathcal{F}$ such that (5.3) holds with $\lambda = 0$, then every exposed face of K is projective.

Proof. Let G be an exposed face of K, say $G = a^{-1}(0) \cap K$ for some $a \in A^+$. By hypothesis there exists an $F \in \mathcal{F}$ such that $F \subset G$ and $a(y) > 0$ for $y \in F^{\#}$. We claim $F = G$.

Let P be the P-projection corresponding to F, so $F = (\operatorname{im} P^*) \cap K$. Observe that $a = 0$ on F implies $P'a = a$ (cf. e.g. (2.22)), so for $x \in G$ we have

$$0 = a(x) = (P'a)(x) = a((P')^*x) = a((P^*)'x).$$

Since $\operatorname{im}^+(P^*)' = \operatorname{cone}(F^{\#})$, and since by assumption $a(y) > 0$ for $y \in \operatorname{cone}(F^{\#}) \setminus \{0\}$, we conclude that $(P^*)'x = 0$ and thus $x \in F$. Hence we have shown $G \subset F$, and so $G = F$. $\square$

It will be useful to have the first inequality of the definition (5.3) stated in a slightly different form. If F is a projective face corresponding to a P-projection P, i.e. if $F = (\operatorname{im} P^*) \cap K$, then $a \leq \lambda$ on F iff $a(x) \leq \lambda e(x)$ for all $x \in \text{cone}\, F$, which is equivalent to $a(P^*x) \leq \lambda e(P^*x)$ for all $x \in V^+$, and this in turn is equivalent to $Pa \leq \lambda Pe$. Hence

$$(5.4) \qquad a \leq \lambda \text{ on } F \iff Pa \leq \lambda Pe .$$

Applying this inequality to $-\lambda, -a$ and P', we also get

$$(5.5) \qquad a \geq \lambda \text{ on } F^{\#} \iff P'a \geq \lambda P'e .$$

(Note that the left side of (5.5) is not the same as the right side of (5.3). However, one may change to strict inequality at the left side of (5.5) if the inequality at the right is required to be strict on $\text{cone}(F^{\#}) \setminus \{0\}$.)

We now proceed to prove the existence and uniqueness of spectral decompositions of elements of A under the assumption of spectral duality. In fact, weak spectral duality will suffice for the existence, and the proof is based on the following crucial lemma.

Lemma 5.3. Let A and V be in weak spectral duality, let $a \in A$ and let $\lambda_1 \leq \lambda_2 \leq \lambda_3$. If P_1, P_3 are P-projections compatible with a such that $P_1 \preccurlyeq P_3$ and

$$(5.6) \qquad P_i a \leq \lambda_i P_i e , \quad P_i' a \geq \lambda_i P_i' e ,$$

for $i = 1,3$, then one can choose a P-projection P_2 compatible with a such that (5.6) holds for $i = 2$, and such that $P_1 \preccurlyeq P_2 \preccurlyeq P_3$.

Proof. By considering $a + \gamma e$ for large γ we can assume without loss of generality that $a \geq 0$ and $0 < \lambda_i$ for $i = 1,2,3$. Since $P_1 \preccurlyeq P_3$, these two P-projections are compatible, and so $P_3 P_1' \in \mathcal{P}$ (Proposition 4.2).

We now consider the element $b = P_3 P_1' a \in A^+$. By weak spectral duality there exists a P-projection Q compatible with b such that

(5.7) $$Qb \leq \lambda_2 Qe , \quad Q'b \geq \lambda_2 Q'e$$

By compatibility $Q'b \leq b$, and thus (5.7) implies

$$\lambda_2 Q'e \leq Q'b \leq b = P_3P_1'a \leq \|a\| P_3P_1'e .$$

Since $\lambda_2 > 0$, then $Q'e \in [P_3P_1'e]$, which implies

(5.8) $$Q' \preccurlyeq P_3P_1' = P_3 \wedge P_1'$$

Thus $Q' \preccurlyeq P_3$ and $Q' \preccurlyeq P_1'$; from this it follows that P_1, P_3, and Q are compatible. We now define $P_2 = QP_3$, and obtain

$$P_1 \preccurlyeq P_2 \preccurlyeq P_3$$

Using (5.8) and compatibility of Q' with b, and compatibility of a with P_1' and P_3:

$$Q'a = Q'(P_3P_1'a) = Q'b \leq b = P_3P_1'a \leq a .$$

Thus Q' (and therefore Q) is compatible with a. By Lemma 4.5, $P_2 = Q \wedge P_3$ is also compatible with a.

There remains to prove that

(5.9) $$P_2a \leq \lambda_2 P_2 e , \quad P_2'a \geq \lambda_2 P_2'e .$$

Observe that by (5.7)

$$P_2P_1'a = QP_3P_1'a = Qb \leq \lambda_2 Qe ,$$

and so since $P_2P_1' \preccurlyeq P_3P_1'$

(5.10) $$P_2P_1'a = (P_3P_1')(P_2P_1'a) \leq \lambda_2 P_3P_1'Qe = \lambda_2 P_2P_1'e .$$

Since $P_1 \preccurlyeq P_2$ then $P_2 = P_1 \dotplus P_2 \wedge P_1'$, and so by Proposition 4.6 and (5.10)

$$P_2a = (P_1 \dotplus P_2P_1')a = P_1a + P_2P_1'a$$
$$\leq \lambda_1 P_1 e + \lambda_2 P_2 P_1' e \leq \lambda_2(P_1e + P_2P_1'e) = \lambda_2 P_2 e ,$$

which establishes the first half of (5.9).

Observe that by (5.8), $Q'a = Q'(P_3P_1'a) = Q'b$, and that $P_2' = (Q \wedge P_3)' = Q' \dotplus P_3'$. Using these facts together with Proposition 4.6 and (5.7), we have

$$P_2'a = Q'a + P_3'a \quad = Q'b + P_3'a$$
$$\geq \lambda_2 Q'e + \lambda_3 P_3'e \geq \lambda_2(Q'e + P_3'e) = \lambda_2 P_2'e .$$

This establishes the last half of (5.9) and completes the proof. ∎

<u>Lemma 5.4. Let $\{F_n\}$ be an increasing sequence of projective faces such that each F_n is compatible with $a \in A$. If $a > \lambda$ on each F_n, then $a > \lambda$ on $\bigvee_n F_n$.</u>

<u>Proof</u>. We assume without loss of generality that $\lambda = 0$. Let $F = \bigvee_n F_n$ and let $\{P_n\}$ be the sequence of P-projections corresponding to $\{F_n\}$ and P the P-projection corresponding to F.

Observe that $P_n a \geq 0$ for all n. By Proposition 4.6 for $m \leq n$:

$$(5.11) \qquad P_n a = (P_m \dotplus P_m' P_n)a = P_m a + P_m' P_n a \geq P_m a .$$

We now apply Lemma 4.9. (Note that although Lemma 4.9 was stated with the restriction $a \geq 0$, it remains valid for all $a \in A$ if we replace "sup" with pointwise limit.) Thus for $x \in K$, $(Pa)(x) = \lim_{n\to\infty} (P_n a)(x)$, and by (5.11)

$$(Pa)(x) = \sup_n (P_n a)(x) \geq 0 .$$

Here the equality sign is valid iff $a(P_n^* x) = 0$ for all n, i.e. iff $x \in \bigcap_n F_n^\# = F^\#$. For $x \in F$ we therefore have $a(x) = (Pa)(x) > 0$. ∎

<u>Lemma 5.5. Assume $a \in A$ and $\lambda_n \searrow \lambda$. If $\{P_n\}$ is a descending sequence of P-projections compatible with a such that</u>

(5.12) $P_n a \leq \lambda_n P_n e$, $P_n' a \geq \lambda_n P_n' e$, for $n = 1,2,\ldots$,

<u>then $P = \bigwedge_n P_n$ will be compatible with a and satisfy</u>

(5.13) $Pa \leq \lambda Pe$, $P'a \geq \lambda Pe$.

<u>If $\lambda_n > \lambda$ for $n = 1,2,\ldots,$ then we shall have strict inequality $a > \lambda$ on $F^{\#}$, where $F = (\operatorname{im} P^*) \cap K$.</u>

<u>Proof</u>. By Lemma 4.9, Pa and $P'a$ are pointwise limits (on K) for the sequences $\{P_n a\}$ and $\{P_n' a\}$, and likewise with e in place of a.

By compatibility of P_n and a:

$$Pa + P'a = \lim_n P_n a + \lim_n P_n' a = \lim_n (P_n a + P_n' a) = a .$$

Hence P and a are compatible.

By (5.12):

$$Pa = \lim_n P_n a \leq \lim_n \lambda_n P_n e = \lambda Pe$$

and

$$P'a = \lim_n P_n' a \geq \lim_n \lambda_n P_n' e = \lambda P'e .$$

Hence (5.13) is satisfied.

Observe that each P_n will be compatible, not only with a, but with Pa and with $P'a$, since

$$Pa = P(P_n a + P_n' a) = P_n(Pa) + P_n'(Pa) ,$$

and likewise with P' in place of P.

For each n we denote by F_n the projective face corresponding to P_n (i.e. $F_n = \operatorname{im}^+ P_n^*$). Then $\{F_n^{\#}\}$ will be an increasing sequence of projective faces compatible with $P'a$. For each n and each $x \in F_n^{\#}$:

$$(P'a)(x) = (P_n'a)(x) \geq \lambda_n > \lambda .$$

By Lemma 5.4, $P'a > \lambda$ on $\bigvee_n F_n^{\#} = (\bigwedge_n F_n)^{\#}$, and thus $a = P'a > \lambda$ on $F^{\#}$. □

Theorem 5.6. Assume A and V are in weak spectral duality. Then for each $a \in A$ there exists a family $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ of P-projections compatible with a such that for $\lambda, \mu \in \mathbb{R}$:

(i) $P_\lambda a \leq \lambda P_\lambda e$, $P_\lambda' a \geq \lambda P_\lambda' e$,

(ii) $P_\lambda \preccurlyeq P_\mu$ when $\lambda < \mu$

(iii) $P_\lambda = \bigwedge_{\lambda < \mu} P_\mu$

If A and V are in spectral duality, then $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ is uniquely determined by the requirements (i), (ii), (iii).

Proof. Without loss of generality we assume $0 \leq a \leq e$, and we denote by Δ the set of dyadic fractions in $[0,1]$.

By Lemma 5.3 we can find a family $\{R_\rho\}_{\rho \in \Delta}$ of P-projections compatible with a such that $R_\rho \preccurlyeq R_\sigma$ when $\rho < \sigma$ and such that

(5.14) $R_\rho a \leq \rho R_\rho e$, $R_\rho' a \geq \rho R_\rho' e$, for $\rho \in \Delta$.

Since $\mathcal{P}$ is a σ-complete lattice we can define a P-projection P_λ for each $\lambda \in [0,1]$ by writing $P_1 = I$ and

(5.15) $$P_\lambda = \bigwedge_{\lambda < \rho \in \Delta} R_\rho \qquad \text{for } \lambda \in [0,1\rangle .$$

We also define $P_\lambda = 0$ for $\lambda < 0$ and $P_\lambda = I$ for $\lambda > 1$.

It is clear from this definition that (ii) and (iii) are satisfied.

If $\lambda \geq 1$ or $\lambda < 0$, then (i) is trivially satisfied. For given $\lambda \in [0,1\rangle$ we extract a sequence $\{\rho_n\}$ from Δ such that $\rho_n \searrow \lambda$ and $\rho_n > \lambda$ for all $n = 1,2,\ldots$. Then $\{\rho_n\}$ is cofinal (to the left) in $\{\rho \in \Delta \mid \rho > \lambda\}$, and so $P_\lambda = \bigwedge_n R_{\rho_n}$. By Lemma 5.5, we conclude that P_λ is compatible with a and that (i) is satisfied.

We also conclude from Lemma 5.5 that $a > \lambda$ on $F^\#$ where $F = (\operatorname{im} P_\lambda^*) \cap K$. By (5.4) and (5.5) this means

$$(5.16) \qquad a \leq \lambda \text{ on } F, \quad a > \lambda \text{ on } F^\#.$$

If A and V are in spectral duality, then there is just one $F \in \mathcal{F}$ which is compatible with a and satisfies (5.16), and then P_λ must be the unique P-projection corresponding to this projective face. □

The following definition is motivated by the preceding theorem.

Definition. Assume A and V are in weak spectral duality. A family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of projective units is said to be a spectral family if for $\lambda, \mu \in \mathbb{R}$

(i) $e_\lambda \leq e_\mu$ when $\lambda < \mu$

(ii) $e_\lambda = \bigwedge_{\mu > \lambda} e_\mu$

(iii) $\bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0, \quad \bigvee_{\lambda \in \mathbb{R}} e_\lambda = e$

We shall say that such a family has compact support if there exist $\alpha, \beta \in \mathbb{R}$ such that $e_\lambda = 0$ for all $\lambda \leq \alpha$ and $e_\lambda = e$ for all $\lambda \geq \beta$.

If $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ is a family of P-projections compatible with an element $a \in A$ such that (i), (ii), (iii) of Theorem 5.6 are satisfied, then the family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ of projective units $e_\lambda = P_\lambda e$ is said to be <u>a spectral family for a</u>. If A and V are in spectral duality, then the elements of the unique spectral family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ for a will be termed <u>spectral units</u> for a. More specifically, we shall call e_λ <u>the spectral unit for a corresponding to the value λ</u>, or briefly the <u>spectral λ-unit</u> for a.

Note that in the proof of Theorem 5.6 there was proved slightly more than stated in the theorem. If A and V are in weak spectral duality and $\{e_\lambda\}$ is a spectral family for $a \in A$, then by the argument leading up to (5.16):

$$(5.17) \qquad a \leq \lambda \text{ on } F = e_\lambda^{-1}(1), \quad a > \lambda \text{ on } F^\# = e_\lambda^{-1}(0).$$

Here the non-trivial part of the statement is the strict inequality at the right side, which depends in an essential way on requirement (iii) ("right-continuity").

Note also that under the same hypotheses:

$$(5.18) \qquad e_\lambda = 0 \text{ for } \lambda < -\|a\|, \quad e_\lambda = e \text{ for } \lambda \geq \|a\|.$$

We shall now prove some simple, but useful, facts on approximation of elements of A by linear combinations of projective units. In this connection it is convenient to use the term <u>partition of $[\alpha,\beta]$</u> to denote a finite sequence $\gamma = \{\lambda_i\}_{i=0}^n$ such that

$$(5.19) \qquad \alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta.$$

Also we shall use the symbol $\|\gamma\|$ to denote the <u>norm</u> of the partition, i.e. $\|\gamma\| = \max_{1\leq i\leq n} |\lambda_i - \lambda_{i-1}|$.

Proposition 5.7. *Assume A and V are in weak spectral duality and that $\{e_\lambda\}$ is a spectral family for $a \in A$. For a given partition $\gamma = \{\lambda_i\}_{i=0}^{n}$ of $[-\|a\|-\varepsilon, \|a\|]$, $\varepsilon > 0$, the elements*

(5.20) $$\underline{s}_\gamma = \sum_{i=1}^{n} \lambda_{i-1}(e_{\lambda_i} - e_{\lambda_{i-1}}), \quad \bar{s}_\gamma = \sum_{i=1}^{n} \lambda_i(e_{\lambda_i} - e_{\lambda_{i-1}}),$$

will satisfy

(5.21) $$\underline{s}_\gamma \leq a \leq \bar{s}_\gamma$$

and

(5.22) $$\|\bar{s}_\gamma - \underline{s}_\gamma\| \leq \|\gamma\| .$$

Proof. Let $e_\lambda = P_\lambda e$ where the P-projections P_λ are compatible with a and satisfy (i), (ii), (iii) of Theorem 5.6. Then we have the following two inequalities

$$\lambda_{i-1} P'_{\lambda_{i-1}} e \leq P'_{\lambda_{i-1}} a , \quad P_{\lambda_i} a \leq \lambda_i P_{\lambda_i} e .$$

Applying P_{λ_i} to the first and $P'_{\lambda_{i-1}}$ to the second, we find

(5.23) $$\lambda_{i-1} P_{\lambda_i} P'_{\lambda_{i-1}} e \leq P_{\lambda_i} P'_{\lambda_{i-1}} a \leq \lambda_i P_{\lambda_i} P'_{\lambda_{i-1}} e .$$

Since $P_{\lambda_i} = P_{\lambda_{i-1}} \dotplus P_{\lambda_i} \wedge P'_{\lambda_{i-1}}$ and since the occuring P-projections are compatible with a, we may apply Proposition 4.6 to get $P_{\lambda_i} a = P_{\lambda_{i-1}} a + P_{\lambda_i} P'_{\lambda_{i-1}} a$. Clearly also $P_{\lambda_i} e = P_{\lambda_{i-1}} e + P_{\lambda_i} P'_{\lambda_{i-1}} e$. Hence by (5.23):

$$\lambda_{i-1}(P_{\lambda_i} e - P_{\lambda_{i-1}} e) \leq P_{\lambda_i} a - P_{\lambda_{i-1}} a \leq \lambda_i (P_{\lambda_i} e - P_{\lambda_{i-1}} e)$$

Adding, we obtain (5.21).

Finally by (5.20) and (5.18):

$$\bar{s}_\gamma - \underline{s}_\gamma = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(e_{\lambda_i} - e_{\lambda_{i-1}}) \leq \|\gamma\| \sum_{i=1}^{n} (e_{\lambda_i} - e_{\lambda_{i-1}}) = \|\gamma\| e ,$$

and the proof is complete. $\square$

By Proposition 5.7 , $\|a-\underline{s}_\gamma\| \to 0$ and $\|a-\bar{s}_\gamma\| \to 0$ when $\|\gamma\| \to 0$. Hence it is natural to express a as a Riemann-Stieltjes integral with respect to the given spectral family. Thus we shall write

(5.24) $$a = \int \lambda \, de_\lambda .$$

This formula can be interprated as an ordinary Riemann-Stieltjes integral with respect to a real valued increasing function when the occuring elements of A are evaluated at a given point x of K .

Proposition 5.8. *Assume A and V are in weak spectral duality and that $\{e_\lambda\}$ is a spectral family for a. For each $x \in K$ the function $\lambda \to e_\lambda(x)$ is increasing, right continuous, $e_\lambda(x) = 0$ for $\lambda < -\|a\|$, and $e_\lambda(x) = 1$ for $\lambda > \|a\|$. Moreover*

(5.25) $$a(x) = \int \lambda \, de_\lambda(x) .$$

Proof. As in the preceding proof, we write $e_\lambda = P_\lambda e$. Then it follows by (ii) of Theorem 5.6 that $\lambda \to e_\lambda(x)$ is increasing, and it follows by (iii) together with Lemma 4.9 that this function is right continuous. By (5.18) $e_\lambda(x) = 0$ when $\lambda < -\|a\|$ and $e_\lambda(x) = 1$ when $\lambda \geq \|a\|$.

As before we define $\underline{s}_\gamma$ and $\bar{s}_\gamma$ by (5.20). By the definition of the Riemann-Stieltjes integral, $\underline{s}_\gamma(x)$ and $\bar{s}_\gamma(x)$ will both converge to $\int \lambda \, de_\lambda(x)$ when $\|\gamma\| \to 0$. Hence (5.25) will follow from (5.21).

Proposition 5.9. *Assume A and V are in weak spectral duality. Let $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ be a spectral family of compact support. Then*

<u>there exists a unique element $a \in A$ such that $\{e_\lambda\}$ is a spectral family for a</u>.

<u>Proof</u>. If there exists such an a, then it is unique by (5.25).

Let $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ be the family of P-projections corresponding to the family $\{e_\lambda\}$ of projective units $\{e_\lambda\}_{\lambda \in \mathbb{R}}$. For each partition γ of a fixed interval $[\alpha, \beta]$ with $e_\lambda = 0$ for $\lambda \leq \alpha$ and $e_\lambda = e$ for $\lambda \geq \beta$, we define $\underline{s}_\gamma$ and $\bar{s}_\gamma$ as in (5.20). We note that formula (5.22) will be valid since the proof of this formula only depended on those properties of $\{e_\lambda\}$ which are assumed as hypotheses in the present proposition. We now define a real valued function a on K by

$$a(x) = \lim_{\|\gamma\| \to 0} \underline{s}_\gamma(x) = \lim_{\|\gamma\| \to 0} \bar{s}_\gamma(x) ,$$

By well known results on the Riemann-Stieltjes integral, these limits exist and for each partition γ the common limit $a(x)$ will satisfy

$$\underline{s}_\gamma(x) \leq a(x) \leq \bar{s}_\gamma(x) . \tag{5.26}$$

Note also that a is bounded and affine. By (5.22) and (5.26)

$$\|\bar{s}_\gamma - a\| \leq \|\bar{s}_\gamma - \underline{s}_\gamma\| \leq \|\gamma\| ,$$

and so $\bar{s}_\gamma \to a$ when $\|\gamma\| \to 0$ (norm convergence). By norm completeness of A we conclude that $a \in A$.

It is easily verified that $\underline{s}_\gamma = P_\lambda \underline{s}_\gamma + P'_\lambda \underline{s}_\gamma$ for every $\lambda \notin [\alpha, \beta]$, and that this equality also holds for $\lambda \in [\alpha, \beta]$ if λ is included among the "dividing points" for γ. Passing to the limit we obtain $a = P_\lambda a + P'_\lambda a$. Hence P_λ is compatible with a.

Finally we observe that if γ is a partition including λ

among its "dividing points", then $P_\lambda \underline{s}_\gamma \leq \lambda P_\lambda e$ and $P'_\lambda \bar{s}_\gamma \geq \lambda P'_\lambda e$. Passing to the limit, we obtain $P_\lambda a \leq \lambda P_\lambda e$ and $P'_\lambda a \geq \lambda P'_\lambda e$ for all $\lambda \in [\alpha,\beta]$. These inequalities hold trivially for $\lambda \notin [\alpha,\beta]$.

This shows that $\{e_\lambda\}$ is the spectral family for a. □

Combining Theorem 5.6 and Proposition 5.9, we get

<u>Corollary 5.10.</u> <u>If A and V are in spectral duality, then there is a 1 - 1 correspondence of spectral families $\{e_\lambda\}$ of compact support and elements $a \in A$, given by</u>:

$$a = \int \lambda \, de_\lambda \; .$$

§ 6. Properties of spectral families

In the present section we shall study spectral families for spaces A and V in (weak) spectral duality, and we shall also give various alternative definitions of weak spectral duality and of spectral duality. Unless otherwise is stated, we shall assume that (A,e) and (V,K) are order-unit and base-norm spaces in separating order and norm duality satisfying (3.1) and (3.2).

Definition. Two elements $a,b \in A^+$ are said to be orthogonal, in symbols $a \perp b$, if $rp(a) \perp rp(b)$.

Observe that for $a,b,a_o,b_o \in A^+$ one has the implication

(6.1) $\quad a \leq a_o\,,\ b \leq b_o\,,\ a_o \perp b_o \Longrightarrow a \perp b$

From this one can easily obtain the following implication valid for $a \in A^+$ and $P,Q \in \mathcal{P}$:

(6.2) $\quad P \perp Q \Longrightarrow Pa \perp Qa\,.$

In fact, we may assume $a \leq e$ without loss of generality and then apply the previous inequality with $a_o = Pe$ and $b_o = Qe$.

Proposition 6.1. *A and V will be in weak spectral duality iff every element a of A admits a decomposition $a = a_1 - a_2$ with $a_1, a_2 \in A^+$ and $a_1 \perp a_2$.*

Proof. 1.) Assume first that A and V are in weak spectral duality. For given $a \in A$ we choose a P-projection P compatible with a such that (5.3) holds with $F = (\operatorname{im} P^*) \cap K$ and $\lambda = 0$. By (5.4) and (5.5), $Pa \leq 0$ and $P'a \geq 0$. By compatibility

$a = Pa + P'a$, and clearly $(-Pa) \perp P'a$. Writing $a_1 = P'a$ and $a_2 = -Pa$, we get a decomposition of the desired type.

2.) Assume next that every $a \in A$ can be decomposed as a difference of two orthogonal elements of A^+. We shall prove that for a given $a \in A$ and $\lambda \in \mathbb{R}$ there exists a projective face F compatible with a such that (5.3) holds.

Without loss of generality we assume $a \geq 0$ and $\lambda \geq 0$. By assumption we may decompose

$$(6.3) \qquad a - \lambda e = b_1 - b_2$$

where $b_1, b_2 \in A^+$ and $b_1 \perp b_2$.

Now we consider the projective face

$$(6.4) \qquad F = \{x \in K \mid b_1(x) = 0\}$$

together with the corresponding P-projection P, i.e. $F = (\operatorname{im} P^*) \cap K$. By Proposition 3.7 (especially (3.20)), one has $b_1(x) = 0$ iff $rp(b_1)(x) = 0$. Hence $F = \{x \in K \mid rp(b_1)(x) = 0\}$, and after passage to quasicomplements

$$(6.5) \qquad F^{\#} = \{x \in K \mid rp(b_1)(x) = 1\} .$$

Thus we have $P'e = rp(b_1)$. By assumption $rp(b_1) \perp rp(b_2)$, and so $rp(b_2) \leq rp(b_1)' = Pe$. Hence

$$(6.6) \qquad P'b_1 = b_1 , \quad Pb_2 = b_2 , \quad P'b_2 = Pb_1 = 0 .$$

Thus, $Pa = \lambda Pe - b_2$ and $P'a = \lambda P'e + b_1$, which implies

$$Pa + P'a = \lambda e + b_1 - b_2 = a .$$

This proves that a is compatible with P, and then also with F.

By (6.6)

$$a - \lambda e = P(a - \lambda e) = -b_2 \leq 0 \qquad \text{on } F ,$$

and

$$a - \lambda e = P'(a-\lambda e) = b_1 > 0 \qquad \text{on} \quad F^{\#} .$$

(Observe that $b_1(x) > 0$ for $x \in F^{\#} \subset K \setminus F$). Combining these two inequalities, we get the desired formula (5.3). □

Note that unlike the original definition of weak spectral duality, the existence of an orthogonal decomposition of every $a \in A$ into positive components does not make the requirement (3.2) redundant. In fact, we used this assumption in an essential way to conclude that the face F of the above proof was projective.

The following lemma will be useful later.

<u>Lemma 6.2. If F and G are two orthogonal faces compatible with $a \in A$ and if $a > \lambda \in \mathbb{R}$ on $F \cup G$, then $a > \lambda$ on $F \dotplus G$.</u>

<u>Proof.</u> Let F and G correspond to the P-projections P and Q. We assume without loss of generality that $\lambda = 0$. Then for $x \in F \dotplus G$ Proposition 4.6 yields

$$a(x) = (P \dotplus Q)(a)(x) = (Pa+Qa)(x) = a(P^*x) + a(Q^*x) .$$

Since $a > 0$ on $F \cup G$, the rightmost expression is ≥ 0 and equals zero only if $P^*x = Q^*x = 0$, i.e. only if $x \in F^{\#} \cap G^{\#} = (F \dotplus G)^{\#}$. Therefore since $x \in F \dotplus G$, $a(x) > 0$. □

We shall now investigate the spectral family $\{e_\lambda\}$ of an element $a \in A$ in the case where A and V are in spectral duality. When specification of the element a is needed, we shall indicate it by a superscript attached to the spectral units. Thus e_λ^a shall denote the spectral λ-unit of $a \in A$. For the sake of convenience we shall also use the symbol $r_\lambda(a)$ to denote the complement of e^a_λ in $\mathcal{U}$. Thus

$$(6.7) \qquad r_\lambda(a) = (e^a_\lambda)' = e - e^a_\lambda$$

If $a \in A^+$, let $F = \{x \in K \mid rp(a)(x) = 0\}$. Then $a = 0$ on F and $a > 0$ on $F^\#$. Since A and V are in spectral duality it follows that $F = (e^a_o)^{-1}(1) \cap K$, and so $rp(a) = r_o(a)$. For all $\lambda > 0$ we also have $r_\lambda(a) \leq rp(a)$.

<u>Lemma 6.3. Assume A and V in spectral duality. If $a, b \in A^+$ and $a \perp b$, then $r_\lambda(a) + r_\lambda(b) = r_\lambda(a+b)$ for every $\lambda > 0$.</u>

<u>Proof</u>. Let P and Q be the P-projections corresponding to the projective units e^a_λ and e^b_λ, so $P'e = r_\lambda(a)$ and $Q'e = r_\lambda(b)$. Also we denote by F and G the projective faces corresponding to P and Q, i.e. $F = (\mathrm{im}P^*) \cap K$ and $G = (\mathrm{im}Q^*) \cap K$. Since $a \perp b$, we shall have $rp(a) \perp rp(b)$, and since $r_\lambda(a) \leq rp(a)$ and $r_\lambda(b) \leq rp(b)$, it follows that $r_\lambda(a) \perp r_\lambda(b)$. Hence $F^\# \perp G^\#$.

By definition of spectral units, F and G are both compatible with a, and $a > \lambda$ on $F^\#$ and $b > \lambda$ on $G^\#$. Then by application of Lemma 6.2:

$$(6.8) \qquad a + b > \lambda \quad \text{on} \quad F^\# \ddagger G^\# = (F \cap G)^\#.$$

Next we make the following observation of a rather general nature

$$(6.9) \qquad Pa \leq \lambda P\, rp(a), \quad Qb \leq \lambda Q\, rp(b).$$

To verify the first of these inequalities, we consider the P-projection R corresponding to $rp(a)$, i.e. $Re = rp(a)$. Note that e^a_o and e^a_λ are compatible since $e^a_o \leq e^a_\lambda$. It follows that

$Re = rp(a) = r_o(a) = e - e_o^a$ is compatible with $Pe = e_\lambda^a$, and thus R and P commute. Applying R to both sides of the inequality $Pa \leq \lambda Pe$ (cf. (5.4)), we obtain the first inequality of (6.9). The proof of the second inequality is similar.

Since $P' \perp Q'$ the P-projections P and Q commute, and since $rp(a) \perp rp(b)$ we have $rp(a) + rp(b) \leq e$. Hence by (6.9):

$$PQ(a+b) = QPa + PQb$$
$$\leq \lambda\, QP\, rp(a) + \lambda\, PQ\, rp(b) \leq \lambda\, PQe ,$$

which gives

$$\text{(6.10)} \qquad a + b \leq \lambda \quad \text{on} \quad F \cap G .$$

Since

$$Re = rp(a) \leq rp(b)' \leq r_\lambda(b)' = Qe ,$$

then $Qa = Q(Ra) = Ra = a$, and similarly $Pb = b$. Thus we have

$$PQ(a+b) = PQa + QPb = Pa + Qb \leq a + b ,$$

which proves compatibility of $a + b$ with the P-projection PQ, and hence also with the projective face $F \cap G$.

It is now seen from (6.8) and (6.10) that $F \cap G$ has all the properties characterizing the projective face associated with e_λ^{a+b}. Hence $F^\# \dotplus G^\#$ is the face corresponding to $r_\lambda(a+b)$, and it follows that $r_\lambda(a) + r_\lambda(b) = r_\lambda(a+b)$. □

<u>Lemma 6.4. Assume A and V in spectral duality. If $a \in A^+$ and Q is a P-projection compatible with a, then $r_\lambda(Qa) = Qr_\lambda(a)$ for every $\lambda > 0$.</u>

<u>Proof.</u> Applying Lemma 6.3 with Qa and Q'a in place of a and b, we obtain

(6.11) $$r_\lambda(Qa) + r_\lambda(Q'a) = r_\lambda(a) .$$

Now $Qa \in \mathrm{im}^+Q = \mathrm{face}(Qe)$, and so $r_\lambda(Qa) \leq rp(Qa) \leq Qe$. Hence $r_\lambda(Qa) \in \mathrm{face}(Qe) = \mathrm{im}^+Q$. Similarly $r_\lambda(Q'a) \in \mathrm{im}^+Q' = \ker^+Q$. By application of Q to both sides of the equation (6.11), we now obtain $r_\lambda(Qa) = Qr_\lambda(a)$. ▯

<u>Theorem 6.5. If A and V are in spectral duality, then for every $a \in A$ the spectral units of a will be bicompatible with a.</u>

<u>Proof</u>. Without loss of generality we assume $a \geq 0$ and consider λ-spectral units for $\lambda \geq 0$ only. By definition the spectral units of a are compatible with a. To prove bicompatibility, we consider an arbitrary P-projection Q compatible with a. By Lemma 6.3 and Lemma 6.4:

$$r_\lambda(a) = r_\lambda(Qa) + r_\lambda(Q'a) = Qr_\lambda(a) + Q'r_\lambda(a) .$$

Hence Q is compatible with $r_\lambda(a)$, and then also with $e_\lambda^a = r_\lambda(a)'$ for every $\lambda \geq 0$. ▯

By Theorem 6.5 the uniqueness of spectral units implies bicompatibility with the given element $a \in A$. We shall now establish an opposite result to the effect that bicompatibility with a implies uniqueness.

<u>Proposition 6.6. Let $a \in A$ and $\lambda \in \mathbb{R}$, and assume that there exists a projective face F bicompatible with a such that</u>

(6.12) $$a \leq \lambda \quad \text{on } F, \qquad a > \lambda \quad \text{on } F^\# .$$

<u>Then for every G compatible with a and such that $a \leq \lambda$ on G, we shall have $G \subset F$. If in addition $a > \lambda$ on $G^\#$, then $G = F$.</u>

Proof. 1.) Assume first that G is compatible with a and that $a \leq \lambda$ on G. Since F is bicompatible with a, it will be compatible with G. Hence by Proposition 4.3:

$$G = F \cap G \dotplus F^{\#} \cap G .$$

However, $F^{\#} \cap G = \emptyset$, since $x \in F^{\#}$ implies $a(x) > \lambda$ and $x \in G$ implies $a(x) \leq \lambda$. Hence $G = F \cap G \subset F$, as claimed.

2.) Assume next that G satisfies the same requirements and in addition $a > \lambda$ on $G^{\#}$. Now we consider the decomposition

$$F = F \cap G \dotplus F \cap G^{\#} .$$

Here $F \cap G^{\#} = \emptyset$, since $x \in F$ implies $a(x) \leq \lambda$ and $x \in G^{\#}$ implies $a(x) > \lambda$. Hence $F = F \cap G \subset G$, and we are done. $\square$

From Theorem 6.5 and Proposition 6.6 we obtain the following two corollaries:

<u>Corollary 6.7.</u> <u>A and V are in spectral duality iff for every $a \in A$ and every $\lambda \in \mathbb{R}$ there exists a projective face F bicompatible with a such that $a \leq \lambda$ on F and $a > \lambda$ on $F^{\#}$.</u>

<u>Corollary 6.8.</u> <u>If A and V are in spectral duality, then the spectral λ-unit e_λ of an element $a \in A$ is determined by the fact that the corresponding P-projection P is the supremum of all $Q \in \mathcal{P}$ which are compatible with a and satisfy the inequality $Qa \leq \lambda Qe$.</u>

We will now give an example showing that the assumptions (3.1) and (3.2) will not guarantee spectral duality.

Proposition 6.9. Let A consist of all sequences $a = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ with $\sum_i \alpha_i^2 < \infty$, and let V consist of all sequences $x = \{\xi_0, \xi_1, \ldots, \xi_n, 0, 0, \ldots\}$ which are eventually zero. Also let $e \in A$ be defined by $e = \{1, 0, 0, \ldots\}$, and let $K \subset V$ consist of all $x = \{\xi_i\} \in V$ such that $\xi_0 = 1$ and $\sum_{i \geq 1} \xi_i^2 \leq 1$. Then (A,e) is an order unit space with positive cone:

$$A^+ = \{a \in A \mid \alpha_0 \geq 0, \sum_{i \geq 1} \alpha_i^2 \leq \alpha_0^2\}$$

and norm:

$$\|a\| = |\alpha_0| + (\sum_{i \geq 1} \alpha_i^2)^{\frac{1}{2}}, \qquad \text{for } a \in A.$$

Also (V,K) is a base norm space with positive cone:

$$V^+ = \{x \in V \mid \xi_0 \geq 0, \sum_{i \geq 1} \xi_i^2 \leq \xi_0^2\}$$

and norm:

$$\|x\| = \max\{|\xi_0|, (\sum_{i \geq 1} \xi_i^2)^{\frac{1}{2}}\} \qquad \text{for } x \in V.$$

The spaces (A,e) and (V,K) are in separating order and norm duality under the form:

$$\langle a,x \rangle = \sum_i \alpha_i \xi_i .$$

(In fact A can be identified with V^*). Now the requirements (3.1) and (3.2) are satisfied, but A and V are not in weak spectral duality.

Proof. It is routine to verify that (A,e) and (V,K) are order-unit and base-norm spaces, and that A can be identified with V^*. It follows that A and V are in separating order and norm duality, and that (3.1) holds: A is pointwise monotone σ-complete.

By considering the natural affine embedding of K in the unit ball of l^2 we observe that K has no proper faces other than the extreme points, and they are exactly the points satisfying

$$\xi_o = 1 , \quad \sum_{i\geq 1} \xi_i^2 = 1 .$$

For a given extreme point $y = \{1,\eta_1,\dots\eta_n,0,0,\dots\}$ we also consider the "antipodal" extreme point $y' = \{1,-\eta_1,\dots,-\eta_n,0,0,\dots\}$ and the elements $h,h' \in A$ defined by $h = \frac{1}{2}\{1,\eta_1,\dots,\eta_n,0,0,\dots\}$ and $h' = \frac{1}{2}\{1,-\eta_1,\dots,-\eta_n,0,0,\dots\}$. Now the formulas

$$Px = \langle h,x\rangle y , \quad P'x = \langle h',x\rangle y'$$

are seen to define weakly continuous positive projections of norm 1 with

$$\mathrm{im}^+P = \ker^+P' = \{\lambda y \mid \lambda \in \mathbb{R}^+\} ,$$

$$\mathrm{im}^+P' = \ker^+P = \{\lambda y' \mid \lambda \in \mathbb{R}^+\} .$$

Thus P and P' are quasicomplementary projections.

To prove that P is smooth we consider $a \in A^+$ such that $\langle a,y'\rangle = 0$ and we shall show $\langle a,Px\rangle = \langle a,x\rangle$ for all $x \in V$ (cf. (1.15)).

By assumption $a = \{\alpha_i\}$ satisfies

$$\alpha_o - \sum_{i=1} \alpha_i\eta_i = \langle a,y'\rangle = 0 ,$$

and so

$$\alpha_o = \sum_{i=1}^{n} \alpha_i\eta_i \leq (\sum_{i=1}^{n} \alpha_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} \eta_i^2)^{\frac{1}{2}} \leq \alpha_o$$

Thus the sign of equality holds in Schwartz' inequality. Therefore $\{\alpha_1,\dots,\alpha_n\} = \gamma\{\eta_1,\dots,\eta_n\}$ for some $\gamma \in \mathbb{R}$. By the equality $\alpha_o = \sum_{i=1}^{n} \alpha_i\eta_i$, we must have $\gamma = \alpha_o$, and so $a = 2\alpha_o h$. Using the definition of P and observing that $\langle h,y\rangle = 1$, we get

$$\langle a,Px\rangle = \langle h,x\rangle\langle a,y\rangle = 2\alpha_o\langle h,x\rangle\langle h,y\rangle = \langle a,x\rangle$$

as claimed.

The same argument shows that P' is smooth. Hence P, P' is a pair of quasicomplementary P-projections. Thus we have proved that every proper face of K is projective, and (3.2) is satisfied.

It remains to prove that A and V are not in spectral duality, and we shall do this by showing that for $a \in A$ with infinitely many non-zero components, there can not be any proper projective face compatible with a.

Let $a = \{\alpha_i\} \in A$ with $\alpha_i \neq 0$ for infinitely many indices i, and let F be a proper face of K. We shall prove that F can not be compatible with a.

By the above remarks the P-projection P corresponding to F is of the form $Px = \langle h,x\rangle y$ where $y \in K$ and $h \in A$ are as above; in particular at most the first $n+1$ components of h are non zero.

For arbitrary $x \in V$

$$\langle P^*a,x\rangle = \langle a,Px\rangle = \langle h,x\rangle\langle a,y\rangle = \langle\langle a,y\rangle h,x\rangle ,$$

and so $P^*a = \langle a,y\rangle h$. Hence only the first $n+1$ components of P^*a can be non-zero.

Similarly we prove that only the first $n+1$ components of $(P')^*a$ can be non-zero. Therefore

$$P^*a + (P^*)'a \neq a ,$$

and P is not compatible with a. $\square$

We now proceed to prove that the assumptions (3.1) and (3.2) will suffice for spectral duality in the finite dimensional case. In this connection we shall need a general result of some independent interest: For every $a \in A^+$, $rp(a)$ is bicompatible with a.

The key point in the proof of this result is the observation that Lemma 6.3 can be stated and proved without spectral duality. More specifically we have the following lemma in which A and V only are supposed to satisfy the standing requirements of this section (i.e. we assume separating order and norm duality together with (3.1) and (3.2), but we do not assume spectral duality or weak spectral duality, and we do not yet assume A and V finite dimensional).

Lemma 6.10 If $a,b \in A^+$ and $a \perp b$, then $rp(a)+rp(b) = rp(a+b)$.

Proof. Clearly $rp(a) \leq rp(a+b)$ and $rp(b) \leq rp(a+b)$. Since $rp(a)$ and $rp(b)$ are two orthogonal projective units, we shall have

$$rp(a)+rp(b) = rp(a) \vee rp(b) \leq rp(a+b).$$

On the orher hand, $a \in face(rp(a))$ and $b \in face(rp(b))$, so $a+b \in face(rp(a)+rp(b))$. Now $rp(a)+rp(b)$ is a projective unit which generates a face of A^+ containing $a+b$, and then by definition

$$rp(a+b) \leq rp(a)+rp(b).$$

This completes the proof. $\square$

Lemma 6.11. If $a \in A^+$ and Q is a P-projection compatible with a, then $rp(Qa) = Q(rp(a))$.

Proof. Similar to the proof of Lemma 6.4, with rp(a) in place of $r_\lambda(a)$. ∎

Proposition 6.12. If $a \in A^+$ then rp(a) is bicompatible with a.

Proof. Similar to the proof of Theorem 6.5. ∎

Proposition 6.13. Let (A,e) and (V,K) be finite dimensional spaces in separating order and norm duality. Then K is compact (in the unique Hausdorff vector space topology for V), and the condition (3.1) is satisfied. Moreover, for every increasing net $\{a_\alpha\}$ from A bounded above there exists a sequence $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$ such that $\sup_n a_{\alpha_n} = \sup_\alpha a_\alpha \in A$.

Proof. The first part of the proposition is easily verified. (See e.g. [A_1, Ch. II. § 1].) To prove the last statement of the proposition we consider an upper bounded increasing net $\{a_\alpha\}$ from A. Clearly the pointwise supremum $a = \sup_\alpha a_\alpha$ is an affine function. Hence $a \in A$. By finite dimensionality we can find points $x_1, \dots, x_m \in K$ such that K is contained in their affine span. Now we choose α_n inductively such that $\alpha_n \leq \alpha_{n+1}$ and

$$a(x_k) - a_{\alpha_n}(x_k) \leq \frac{1}{n} \qquad \text{for } k = 1, \dots, m.$$

Then $a(x) = \sup_n a_{\alpha_n}(x)$ for all $x \in K$. ∎

Theorem 6.14. Let (A,e) and (V,K) be finite dimensional spaces in separating order and norm duality and assume that every exposed face of K is projective. Then A and V will be in spectral duality.

<u>Proof</u>. We shall prove that for given $a \in A$ and $\lambda \in \mathbb{R}$ there exists a projective face F bicompatible with a such that $a \leq F$ on F and $a > \lambda$ on $F^{\#}$. Without loss of generality we assume $\lambda = 0$, and we denote by $\mathcal{G}$ the collection of all $F \in \mathcal{F}$ such that F is bicompatible with a and $a \leq \lambda$ on F.

We claim that the collection $\mathcal{G}$ has a largest member.

If $F, G \in \mathcal{G}$ and if P, Q are the corresponding P-projections, then $P, Q \in \mathcal{B}(a)$. Hence P and Q are compatible (Theorem 4.10), and by (4.17) $P \vee Q = P \dotplus P'Q$. By (5.4) we shall have $Pa \leq 0$ and $Qa \leq 0$, and hence by Proposition 4.6

$$(P \vee Q)(a) = Pa + P'Qa \leq 0 .$$

This implies $a \leq 0$ on $F \vee G$, and since $F \vee G$ is bicompatible with a (by Theorem 4.10), we shall have $F \vee G \in \mathcal{G}$. It follows that $\mathcal{G}$ is directed, and by Proposition 6.13 (applied to the corresponding projective units) and by Lemma 4.9 there exists a projective face F_o bicompatible with a such that

$$F_o = \bigvee_{G \in \mathcal{G}} G \qquad \text{and} \quad a \leq 0 \quad \text{on} \quad F_o .$$

Now $F_o \in \mathcal{G}$, and by definition F_o must be the largest member of $\mathcal{G}$.

It remains to prove that $a > 0$ on $F_o^{\#}$.

We assume the contrary and define

$$\beta = \inf_{x \in F_o^{\#}} a(x) \leq 0 . \tag{6.13}$$

Let P_o be the P-projection corresponding to F_o and note that (6.13) gives $a - \beta e \geq 0$ on $F_o^{\#}$, which will imply the following relation on all of K:

$$P_o'(a-\beta e) \geq 0 . \tag{6.14}$$

In fact, if $x \in F_o = K \cap \ker(P_o'^* x)$ then $P_o'(a-\beta e)(x) = 0$, and if $x \notin F_o$ then $\lambda = \|P_o'^* x\| \neq 0$, $y = \lambda^{-1} P_o'^* x \in F_o^\#$, and $P_o'(a-\beta e)(x) = \lambda(a-\beta e)(y) \geq 0$.

By (6.14) $P_o'a \geq \beta P_o'e \geq \beta e$, which gives the non-trivial part of the equality

$$\beta = \inf_{x \in K} (P_o'a)(x) .$$

By the compactness of K and the continuity of all the functions in A the set

$$H = \{x \in K \mid (P_o'a)(x) = \beta\}$$

is a non-empty exposed, therefore projective, face of K. Since the continuous function a will attain its minimum on the (necessarily compact) face $F_o^\#$, the definition (6.13) of β will give

(6.15) $$H \cap F_o^\# \neq \emptyset .$$

By Proposition 6.12 H is bicompatible with $P_o'a$. We will show that $H \cap F_o^\#$ is bicompatible with a.

Let R be the P-projection corresponding to H. Since P_o' is compatible with $P_o'a$ and R is bicompatible with $P_o'a$, R must be compatible with P_o'. Hence $RP_o' = R \wedge P_o'$ is the P-projection corresponding to $H \cap F_o^\#$.

Since $P_o a \leq 0$, the equality $(RP_o')(P_o a) = 0$ will imply RP_o' compatible with $P_o a$. Since H and $F_o^\#$ both are compatible with $P_o'a$, their intersection will be so (Lemma 4.5); hence RP_o' is compatible with $P_o'a$. It follows that RP_o' is compatible with $a = P_o a + P_o'a$.

Now let Q be compatible with a. Then Q is compatible with $P_o' \in \mathcal{B}(a)$, and so

$$Q(P_o'a) + Q'(P_o'a) = P_o'(Qa + Q'a) = P_o'a .$$

Hence Q is compatible with $P_o'a$. Since R is bicompatible with $P_o'a$, R and Q must be compatible. Thus we obtain

$$RP_o'Q = RQP_o' = QRP_o' .$$

Hence RP_o' is compatible with Q, and so we have proved $RP_o' \in \mathcal{B}(a)$.

Since $F_o \in \mathcal{G}$ we have $P_oa \leq 0$. Also we have $P_o'a = \beta \leq 0$ on $H \cap F_o^{\#}$. It follows that

$$a = P_oa + P_o'a \leq 0 \qquad \text{on} \quad H \cap F_o^{\#}$$

Hence $H \cap F_o^{\#} \in \mathcal{G}$. This gives the desired contradiction since $H \cap F_o^{\#}$ is non-empty and disjoint from F_o. $\square$

Remarks. Theorem 6.14 can be stated in more general terms, the essential requirements being:

(i) The members of A attain their maximum on K.

(ii) A is pointwise monotone complete (not only σ-complete).

Note also that in Theorem 6.14 the two spaces A and V are shown to be in spectral duality and not only in weak spectral duality.

The general question if weak spectral duality implies spectral duality is still up in the air. We do not know of any counterexample.

§7. Functional calculus

In this section we will assume that (A,e) and (V,K) are order-unit and base-norm spaces in spectral duality.

Proposition 7.1 If a is an element of A with spectral family $\{e_\lambda\}$ and if φ is a bounded Borel function of a real variable, then there exists a unique element b of A such that for all $x \in K$:

$$b(x) = \int \varphi(\lambda) de_\lambda(x) . \tag{7.1}$$

Proof. Assume first that φ is continuous. Then (7.1) is a Riemann-Stieltjes integral with respect to a probability measure on $\mathbb{R}$ for every $x \in K$. In fact, the Riemann-sums will converge uniformly with respect to x. By Proposition 5.1, A is norm complete. Hence there exists $b \in A$ satisfying (7.1), and clearly b is unique.

Next, denote by $\mathcal{B}_o$ the set of all bounded Borel functions φ for which there exists $b \in A$ satisfying (7.1). By the monotone convergence theorem and the pointwise monotone σ-completeness of A, $\mathcal{B}_o$ is closed under pointwise limits of bounded monotone sequences. Since $\mathcal{B}_o$ contains all bounded continuous functions, it must contain all bounded Borel functions. $\square$

For the formulation of our next proposition we recall some elementary facts concerning the σ-complete orthomodular lattice of projective units in A. If $\{g_n\}$ is an orthogonal sequence from $\mathcal{U}$

(i.e. $g_n \perp g_m$ for $n \neq m$), then by formula (3.13) and Lemma 4.9:

$$(7.2) \qquad (\bigvee_{n=1}^{\infty} g_n)(x) = \sum_{n=1}^{\infty} g_n(x) \qquad \text{for } x \in K.$$

Thus, if we interpret the elements of $\mathcal{U}$ as functions on K, then $\bigvee_n g_n$ becomes the ordinary pointwise sum of the functions g_n. Accordingly, we shall use the symbols $\bigvee_n g_n$ and $\sum_n g_n$ interchangeably when $\{g_n\}$ is an orthogonal sequence.

Proposition 7.2. Let a be an element of A with spectral family $\{e_\lambda\}$. Then for every Borel set $E \subset \mathbb{R}$ the element p_E of A defined by

$$(7.3) \qquad p_E(x) = \int_E de_\lambda(x) \qquad \text{for } x \in K,$$

is a projective unit bicompatible with a. Moreover, $E \to p_E$ is a mapping from the Borel sets into $\mathcal{U}$ satisfying:

$$(7.4) \qquad p_{\mathbb{R}} = e,$$

$$(7.5) \qquad p_E = \sum_n p_{E_n} \quad \text{for a disjoint sequence } \{E_n\} \text{ with } E = \bigcup_n E_n.$$

Proof. Let $\mathcal{S}$ be the collection of all Borel sets $E \subset \mathbb{R}$ for which p_E is a projective unit bicompatible with a.

By definition $p_{\langle -\infty, \lambda]} = e_\lambda$, and so $\mathcal{S}$ contains all half-open intervals of the form $\langle -\infty, \lambda]$. Also $p_{\mathbb{R}} = e$, and so $\mathbb{R} \in \mathcal{S}$.

By definition $p_{\mathbb{R}\setminus E} = e - p_E = p_E'$ for every Borel set E, and so $\mathbb{R}\setminus E \in \mathcal{S}$ for all $E \in \mathcal{S}$. Hence $\mathcal{S}$ is closed under complementation.

Now assume E_1 and E_2 are in $\mathcal{S}$ and let F_1 and F_2 be the projective faces corresponding to p_{E_1} and p_{E_2}. Observe that $0 \leq p_{E_1 \cup E_2} \leq e$ and $p_{E_1 \cup E_2}(x) = 1$ for $x \in F_1 \cup F_2$. Therefore by (3.11)

$$p_{E_1 \cup E_2}(x) = 1 \qquad \text{for } x \in F_1 \vee F_2 .$$

On the other hand if $x \in (F_1 \vee F_2)^\# = F_1^\# \cap F_2^\#$ then $p_{E_1}(x) = p_{E_2}(x) = 0$. Since $E \to p_E(x)$ is a probability measure for every given x, then

$$p_{E_1 \cup E_2}(x) = 0 \qquad \text{for } x \in (F_1 \vee F_2)^\#$$

By (2.24) $p_{E_1} \vee p_{E_2}$ is the unique element of A^+ which is 1 on $F_1 \vee F_2$ and 0 on $(F_1 \vee F_2)^\#$; therefore $p_{E_1 \cup E_2} = p_{E_1} \vee p_{E_2}$. By Theorem 4.10 we conclude that $E_1 \cup E_2$ is in $\mathcal{S}$.

Assume next that $\{E_n\}$ is a disjoint sequence from $\mathcal{S}$, and let $E = \bigcup_n E_n$. For every $x \in K$, we obtain

$$(7.6) \qquad p_E(x) = \int_E de_\lambda(x) = \sum_n \int_{E_n} de_\lambda(x) = \sum_n p_{E_n}(x) .$$

From this we first conclude that $p_{E_n} + p_{E_m} \leq p_E \leq e$ when $m \neq n$, and so by Proposition 3.4 $\{p_{E_n}\}$ is an orthogonal sequence from $\mathcal{U}$. Next we conclude by means of (7.2) that $p_E = \bigvee_n p_{E_n}$, and so p_E is a projective unit bicompatible with a. Hence $E \in \mathcal{S}$.

Now $\mathcal{S}$ is a σ-algebra containing all intervals $\langle -\infty, \lambda]$. Hence it contains all Borel sets.

The statements (7.4) and (7.5) are proved above. $\square$

A mapping $E \to p_E$ from the Borel sets of $\mathbb{R}$ into the orthmodular lattice $\mathcal{U}$ of projective units satisfying (7.4) and (7.5), will be called a $\mathcal{U}$-valued measure. The particular $\mathcal{U}$-valued measure studied in Proposition 7.2, will be called the ($\mathcal{U}$-valued) spectral measure for the element $a \in A$; and we shall denote it p_E^a when we want to specify the element a.

For every $x \in K$ the mapping $E \to p_E^a(x)$ will be an ordinary (i.e. regular Borel) measure. We shall call this measure the (scalar valued) spectral measure for a at the point x, and we shall denote it by μ_x^a or simply by μ_x. Thus, by definition

$$(7.7) \qquad \mu_x^a(E) = p_E^a(x) = \int_E de_\lambda^a(x) .$$

In particular $\mu_x^a(\langle -\infty, \lambda]) = e_\lambda^a(x)$. Hence the spectral measure for a at the point x will have the distribution function $\lambda \to e_\lambda^a(x)$.

The spectral integral formula can now be restated in the form:

$$(7.8) \qquad a(x) = \int \lambda \, d\mu_x^a(\lambda) \qquad \text{for all } x \in K .$$

We shall see that we can also restate the uniqueness property of spectral families in terms of a uniqueness statement for representations of the form (7.8).

Proposition 7.3. *Let $a \in A$ and let $E \to p_E$ be a $\mathcal{U}$-valued measure such that with $\mu_x(E) = p_E(x)$ we shall have*

$$(7.9) \qquad a(x) = \int \lambda \, d\mu_x(\lambda) \qquad \text{for all } x \in K .$$

Then $E \to p_E$ must be the spectral measure for a.

Proof. Without lack of generality we assume $a \geq 0$. For an arbitrary $\lambda_0 \in \mathbb{R}$ we write $E_0 = \langle -\infty, \lambda_0]$ and we consider the decomposition $a = a_1 + a_2$ where

$$a_1(x) = \int_{E_o} \lambda \, d\mu_x(\lambda) , \qquad a_2(x) = \int_{\mathbb{R}\setminus E_o} \lambda \, d\mu_x(\lambda) .$$

By hypothesis p_{E_o} is a projective unit. Let the corresponding face of K be F. Then $x \in F$ means $\mu_x(E_o) = p_{E_o}(x) = 1$. Hence μ_x lives on E_o when $x \in F$. Therefore a_2 must vanish on E. On the other hand, $x \in F^{\#}$ means $\mu_x(E_o) = p_{E_o}(x) = 0$. Hence μ_x lives on $\mathbb{R}\setminus E_o$ when $x \in F^{\#}$. Therefore a_1 must vanish on $F^{\#}$. By the criterion (4.27), a is compatible with F.

For $x \in F$ we have

$$a(x) = \int \lambda \, d\mu_x(\lambda) = \int_{E_o} \lambda \, d\mu_x(\lambda) \leq \lambda_o ,$$

and for $x \in F^{\#}$

$$a(x) = \int \lambda \, d\mu_x(\lambda) = \int_{\mathbb{R}\setminus E_o} \lambda \, d\mu_x(\lambda) > \lambda_o .$$

Hence F is the unique projective face compatible with a satisfying (5.3). Therefore $p_{E_o} = e^a_{\lambda_o}$, and so $p_E(x) = p^a_E(x)$ for all $x \in K$ and all Borel sets E. $\square$

If $E \to p_E$ is a $\mathcal{U}$-valued measure, then the intersection of all closed $F \subset \mathbb{R}$ for which $p_{\mathbb{R}\setminus F} = 0$ will be called its <u>support</u>. By means of this notion one can define the general concept of spectrum:

<u>Definition</u>. The support of the spectral measure for an element a of A will be called the <u>spectrum</u> of a, and it will be denoted by $\sigma(a)$.

Note that $\sigma(a)$ is the intersection of all closed sets $F \subset \mathbb{R}$ such that $\mu^a_x(\mathbb{R}\setminus F) = 0$ for all $x \in K$.

Hence

(7.10) $$\sigma(a) = \overline{\bigcup_{x\in K} \mathrm{Supp}(\mu_x^a)} .$$

By the definition of the spectral units e_λ^a we shall have $e_\lambda^a(x) = 0$ for $\lambda < \alpha = \inf_{x\in K} a(x)$ and $e_\lambda^a(x) = 1$ for $\lambda \geq \beta = \sup_{x\in K} a(x)$. (Cf. the argument leading up to (5.18).) Hence $\mu_x^a(\mathbb{R}\setminus[\alpha,\beta]) = 0$ for all $x \in K$, and so $\sigma(a) \subset \overline{a(K)}$. It follows that <u>$\sigma(a)$ is compact for every $a \in A$</u>.

By virtue of (7.8), $a(x)$ is the barycenter of the probability measure μ_x^a for every $x \in K$. Since $\mathrm{Supp}(\mu_x^a) \subset \sigma(a)$ for all $x \in K$, we can replace the inclusion $\sigma(a) \subset \overline{a(K)}$ by the equality

(7.11) $$\overline{a(K)} = \mathrm{co}(\sigma(a)) ,$$

and from this we obtain

(7.12) $$\|a\| = \sup_{\lambda\in\sigma(a)} |\lambda| , \qquad \text{for all } a \in A .$$

Returning to formula (7.1) we note that the integral only depends on the values of φ on the compact set $\sigma(a)$. In fact, φ need only be defined on $\sigma(a)$.

<u>Definition</u>. For every $a \in A$ and every φ in the class $\mathcal{B}(\sigma(a))$ of bounded Borel functions on $\sigma(a)$ we shall denote by $\varphi(a)$ the element b of (7.1), i.e.

(7.13) $$\varphi(a)(x) = \mu_x^a(\varphi) = \int \varphi(\lambda)\, de_\lambda^a(x) \qquad \text{for } x \in K ;$$

or briefly

(7.14) $$\varphi(a) = \int \varphi(\lambda)\, de_\lambda^a .$$

<u>Lemma 7.4</u>. <u>If $a \in A$ and $\varphi \in \mathcal{B}(\sigma(a))$, then the spectral family for $b = \varphi(a)$ is given by</u>

(7.15) $$e^b_\lambda = p^a_{\varphi^{-1}(\langle -\infty, \lambda])}$$

Proof. The mapping $E \to p^a_{\varphi^{-1}(E)}$ is seen to be a $\mathcal{U}$-valued measure such that for every $x \in K$:

(7.16) $$p^a_{\varphi^{-1}(E)}(x) = \mu^a_x(\varphi^{-1}(E)) = (\varphi\mu^a_x)(E) .$$

(Here $\varphi\mu^a_x$ denotes the "transported measure" defined by the equality at the right side of (7.16).)

By the definition of b, we have

(7.17) $$b(x) = \int \varphi(\lambda) d\mu^a_x(\lambda) = \int \lambda \, d(\varphi\mu^a_x)(\lambda) .$$

Now (7.15) follows from the uniqueness statement of Proposition 7.3. ▯

It follows from Lemma 7.4 that under the same hypotheses

(7.18) $$p^{\varphi(a)}_E = p^a_{\varphi^{-1}(E)}$$

for all Borel sets E. This in turn gives the equality $\mu^{\varphi(a)}_x(E) = \mu^a_x(\varphi^{-1}(E) = (\varphi\mu^a_x)(E)$ for all Borel sets E and all $x \in K$. Hence

(7.19) $$\mu^{\varphi(a)}_x = \varphi\mu^a_x$$

for every $\varphi \in \mathcal{B}(\sigma(a))$ and $x \in K$.

We shall now prove the following "spectral mapping theorem":

Proposition 7.5. For every $a \in A$ and every $\varphi \in \mathcal{B}(\sigma(a))$ one has

(7.20) $$\sigma(\varphi(a)) \subseteq \overline{\varphi(\sigma(a))} ,$$

and for φ in the class $\mathcal{C}(\sigma(a))$ of all continuous functions on $\sigma(a)$ the equality $\sigma(\varphi(a)) = \varphi(\sigma(a))$ holds.

Proof. Let $\lambda \in \sigma(\varphi(a))$. For every natural number n the open set $U_n = \langle \lambda - \frac{1}{n}, \lambda + \frac{1}{n} \rangle$ must satisfy $\varphi^{-1}(U_n) \cap \sigma(a) \neq \emptyset$, for otherwise $\varphi^{-1}(U_n)$ would be a Borel set disjoint from $\sigma(a)$ and then by (7.18)

$$0 = p^{a}_{\varphi^{-1}(U_n)} = p^{\varphi(a)}_{U_n} ,$$

which in turn would give $U_n \cap \sigma(\varphi(a)) = \emptyset$, which is impossible since this intersection contains λ.

For every n we choose $\xi_n \in \varphi^{-1}(U_n) \cap \sigma(a)$, and we note that by the definition of U_n:

(7.21) $$\lim_{n \to \infty} \varphi(\xi_n) = \lambda ,$$

and thus $\lambda \in \overline{\varphi(\sigma(a))}$.

Now assume $\varphi \in \mathcal{C}(\sigma(a))$. Since $\sigma(a)$ is compact, the sequence $\{\xi_n\}$ will have an accumulation point $\xi \in \sigma(a)$. By (7.21) and the continuity of φ, $\varphi(\xi) = \lambda$. Hence $\lambda \in \varphi(\sigma(a))$.

Assume next $\lambda \notin \sigma(\varphi(a))$. By the definition of spectrum there is an open set U containing λ such that $p^{\varphi(a)}_{U} = 0$. Then it follows from (7.18) that $p^{a}_{\varphi^{-1}(U)} = 0$. Since $\varphi^{-1}(U)$ is open, we must have $\varphi^{-1}(U) \cap \sigma(a) = \emptyset$. Then $\varphi^{-1}[\lambda] \cap \sigma(a) = \emptyset$, and so $\lambda \notin \varphi(\sigma(a))$. $\square$

We are now in the position to list all the basic properties of the functional calculus given by (7.14). For convenience we shall denote by ι and γ the unit function and the identity function on $\mathbb{R}$, i.e. $\iota(\lambda) = 1$ and $\gamma(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$.

<u>Proposition 7.6. For given $a \in A$ the mapping $\varphi \to \varphi(a)$ from $\mathcal{B}(\sigma(a))$ into A will have the following properties:</u>

(7.22) $\quad \iota(a) = e\,, \qquad \gamma(a) = a\,,$

(7.23) $\quad (\alpha\varphi+\beta\psi)(a) = \alpha(a) + \beta\psi(a) \qquad$ for $\alpha,\beta \in \mathbb{R}$,

(7.24) $\quad \varphi \geq 0 \Rightarrow \varphi(a) \geq 0\,,$

(7.25) $\quad \|\varphi(a)\| \leq \sup_{\lambda\in\sigma(a)} |\varphi(\lambda)|$ with equality if $\varphi \in \mathcal{C}(\sigma(a))$,

(7.26) $\quad \varphi_n \searrow 0 \Rightarrow \inf_n \varphi_n(a) = 0$

<u>Moreover, if $\varphi \in \mathcal{B}(\sigma(a))$ and $\psi \in \mathcal{B}(\overline{\varphi(\sigma(a))})$ then</u>

(7.27) $\quad (\psi\circ\varphi)(a) = \psi(\varphi(a))\,.$

<u>Proof</u>. The statements (7.22), (7.23) and (7.24) follow at once from the definitions.

Since μ_x^a is a probability measure with no mass outside $\sigma(a)$, we have

$$|\varphi(a)(x)| = |\mu_x^a(\varphi)| \leq \sup_{\lambda\in\sigma(a)} |\varphi(\lambda)| \qquad \text{for all } x \in K\,.$$

This gives the general inequality of (7.25).

If $\varphi \in \mathcal{C}(\sigma(a))$, then $\sigma(\varphi(a)) = \varphi(\sigma(a))$ by Proposition 7.5. Hence formula (7.12) will give the desired equality:

$$\|\varphi(a)\| = \sup_{\lambda\in\sigma(\varphi(a))} |\lambda| = \sup_{\lambda\in\sigma(a)} |\varphi(\lambda)|\,.$$

Statement (7.26) will follow from the definition (7.13) by the monotone convergence theorem.

Finally by (7.13) and (7.19)

$$(\psi\circ\varphi)(a) = \mu_x^a(\psi\circ\varphi) = (\varphi\mu_x^a)(\psi) = \mu_x^{\varphi(a)}(\psi) = \psi(\varphi(a))$$

for all $\varphi \in \mathcal{B}(\sigma(a))$ and $\psi \in \mathcal{B}(\overline{\varphi(\sigma(a))})$. $\square$

For given $a \in A$ we shall often have to study the element $\varphi(a)$ with $\varphi(\lambda) = \lambda^2$, and we shall denote this element by $a^{(2)}$. Thus

(7.28) $$a^{(2)} = \int \lambda^2 \, de_\lambda^a \, .$$

Note that $a^{(2)}(x)$ is different from $a(x)^2$ in general. In fact, since $a(x)$ is the barycenter (or "mean value") of μ_x^a, one has for every $x \in K$:

(7.29) $$a^{(2)}(x) - a(x)^2 = \int \lambda^2 d\mu_x^a(\lambda) - \left(\int \lambda \, d\mu_x^a(\lambda)\right)^2 = \int (\lambda - a(x))^2 d\mu_x^a(\lambda)$$

Thus, $a^{(2)}(x) - a(x)^2$ is the dispersion (or "variance") of the probability measure μ_x^a. In particular $a^{(2)}(x) \geq a(x)^2$ for all $x \in K$, with equality iff the measure μ_x^a has all the mass in the barycenter.

In other words:

(7.30) $$a^{(2)}(x) = a(x)^2 \quad \text{iff} \quad \mu_x^a = \varepsilon_{a(x)} \, .$$

We saw in § 2 that every projective unit is an extreme point of the order interval $[0,e]$ (Corollary 2.12), and we shall now prove that the opposite statement also holds when A and V are in spectral duality.

<u>Proposition 7.7</u>. <u>Let $a \in A$. Then the following are equivalent</u>:

(7.31) a is a projective unit

(7.32) a is an extreme point of $[0,e]$

(7.33) $a^{(2)} = a$.

<u>Proof</u>. (7.31) => (7.32) is already proved.

(7.32) => (7.33) Let a be an extreme point of $[0,e]$, and consider the two functions $\varphi(\lambda) = \lambda^2$ and $\psi(\lambda) = 2\lambda - \lambda^2$ defined for $\lambda \in [0,1]$. These functions both take values in $[0,1]$ and they satisfy

$$\gamma = \tfrac{1}{2}\varphi + \tfrac{1}{2}\psi .$$

Since $\sigma(a) \subset [0,1]$ we can form $\varphi(a)$ and $\psi(a)$, and by Proposition 7.6, $\varphi(a) \in [0,e]$, $\psi(a) \in [0,e]$, and

$$a = \tfrac{1}{2}\varphi(a) + \tfrac{1}{2}\psi(a) .$$

Since a is an extreme point of $[0,e]$, we must have $a = \varphi(a) = \psi(a)$. From this (7.33) follows.

(7.33) => (7.31) Let $a = a^{(2)}$. We claim that $\sigma(a) \subset \{0,1\}$, which will complete the proof since it implies $a = p^a_{\{1\}} \in \mathcal{U}$.

Let $\varphi(\lambda) = \lambda^2 - \lambda$ for $\lambda \in \mathbb{R}$, and observe that by hypothesis $\varphi(a) = 0$. Also we define $E = \langle -\infty, 0\rangle \cup \langle 1, \infty\rangle$ and $F = \langle 0,1\rangle$. We claim $p^a_E = 0$; for contradiction assume not. Then there exists $x \in K$ such that $p^a_E(x) = 1$. Hence μ^a_x lives on E, so we have

$$0 = \varphi(a) = \int \varphi(\lambda)\, d\mu^a_x(\lambda) = \int_E \varphi(\lambda)\, d\mu^a_x(\lambda)$$

Since φ is strictly positive on E, this gives the desired contradiction. Similarly it follows that $p^a_F = 0$. Thus

$$p^a_{E\cup F} = p^a_E + p^a_F = 0 ,$$

which shows that $\sigma(a) \subset R \setminus (E \cup F) = \{0,1\}$. $\square$

Corollary 7.8. For each $a \in A$ there exists a unique family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of extreme points of $[0,e]$ such that:

(i) $\lambda \to e_\lambda(x)$ is increasing and right continuous for every $x \in K$.

(ii) There exist α and β such that $e_\lambda = 0$ for all $\lambda < \alpha$ and $e_\lambda = e$ for all $\lambda > \beta$.

(iii) $a(x) = \int \lambda \, de_\lambda(x)$ for all $x \in K$.

Proof. By Proposition 7.7 each e_λ is a projective unit; thus $\{e_\lambda\}$ is a spectral family of compact support. The corollary now follows from Corollary 5.10. □

Let E be a Borel set of $\mathbb{R}$. Then a mapping $\Theta : \mathcal{B}(E) \to A$ is said to be a morphism if it is linear and positive with $\Theta(\iota) = e$ and satisfies the requirement

(7.34) $$\varphi_n \searrow 0 \text{ on } E \Rightarrow \inf_n \Theta(\varphi_n) = 0 .$$

The main theorem on functional calculus can now be stated as follows:

Theorem 7.9. There exists one and only one mapping which assigns to every $a \in A$ a morphism $\Theta_a : \mathcal{B}(\sigma(a)) \to A$ such that

(i) $\Theta_a(\gamma) = a$ with $\gamma(\lambda) = \lambda$ for all $\lambda \in \sigma(a)$.

(ii) $\Theta_a(\chi_E)$ is an extreme point of $[0,e]$ for every Borel subset E of $\sigma(a)$.

Specifically, for fixed $a \in A$ the morphism Θ_a is given by

(iii) $\Theta_a(\varphi) = \varphi(a) = \int \varphi(\lambda) de_\lambda^a$.

This mapping will also satisfy

(iv) $\|\Theta(\varphi)\| \leq \|\varphi\|_{\sigma(a)}$ with equality if $\varphi \in \mathcal{C}(\sigma(a))$

(v) $\Theta_a(\psi \circ \varphi) = \Theta_{\Theta_a(\varphi)}(\psi)$ for $\varphi \in \mathcal{B}(\sigma(a))$ and $\psi \in \mathcal{B}(\overline{\varphi(\sigma(a))})$

Proof. It follows from Proposition 7.6 and Proposition 7.7 that the mapping $a \to \Theta_a$ defined by (iii) will satisfy (i), (ii), (iv) and (v). Hence it only remains to prove the uniqueness.

Let $a \in A$ and let $\Theta : \mathcal{B}(\sigma(a)) \to A$ be a morphism such that

$\Theta(\gamma) = a$ and $\Theta(\chi_E)$ is an extreme point for every Borel set $E \subset \sigma(a)$. By Proposition 7.7, $\Theta(\chi_E) \in \mathcal{U}$ for every Borel set $E \subset \sigma(a)$.

If $E = \underset{E}{\cup} E_n$ where $\{E_n\}$ is a disjoint sequence of Borel subsets of $\sigma(a)$, then $\chi_E = \sum_n \chi_{E_n}$. Since Θ is a morphism, we shall have

$$(7.35) \qquad \Theta(\chi_E) = \sum_n \Theta(\chi_{E_n})$$

Hence $E \to \Theta(\chi_E)$ is a $\mathcal{U}$-valued measure.

For every $x \in K$ we consider the corresponding (scalar valued) measure ϑ_x defined by $\vartheta_x(E) = \Theta(\chi_E)(x)$, and we claim that

$$(7.36) \qquad a(x) = \int \lambda \, d\vartheta_x(\lambda) ,$$

which will complete the proof in virtue of the uniqueness statement of Proposition 7.3.

To verify (7.36) we consider a partition $\{\lambda_i\}_{i=0}^n$ of an interval $[\alpha,\beta]$ where $\alpha < -\|a\|$ and $\beta > \|a\|$. Then

$$\sum_{i=1}^n \lambda_{i-1} \chi_{\langle\lambda_{i-1},\lambda_i]} \leq \gamma \leq \sum_{i=1}^n \lambda_i \chi_{\langle\lambda_{i-1},\lambda_i]} \text{ on } \sigma(a) ,$$

and since Θ is a morphism with $\Theta(\gamma) = a$, we also get

$$\sum_{i=1}^n \lambda_{i-1}\Theta(\chi_{\langle\lambda_{i-1},\lambda_i]}) \leq a \leq \sum_{i=1}^n \lambda_i \Theta(\chi_{\langle\lambda_{i-1},\lambda_i]}) .$$

Hence for every $x \in K$, if $E_i = \langle -\infty, \lambda_i]$ then:

$$\sum_{i=1}^n \lambda_{i-1} \vartheta_x(E_i \setminus E_{i-1}) \leq a(x) \leq \sum_{i=1}^n \lambda_i \vartheta_x(E_i \setminus E_{i-1}) .$$

Passing to the limit as $\|\{\lambda_i\}\| \to 0$ and using the definition of a Riemann-Stieltjes integral, we obtain (7.36). $\square$

In the spectral theory of van Neumann algebras the morphisms involved also preserve multiplication; a fortiori they preserve idempotence, hence they take "extremal" Borel functions (i.e. indicator functions χ_E) into extremal elements of $[0,e]$ (i.e. projections). However, it is of interest to note that in the general case, statement (ii) of Theorem 7.9 is all that remains of multiplicative structure, and that the uniqueness now follows from conditions involving only linearity and order.

Note also that condition (ii) is essential. One can always define

$$(7.37)\qquad \Theta_a(\varphi)(x) = (\beta-\alpha)^{-1}[(\beta-a(x))\varphi(\alpha)+(a(x)-\alpha)\varphi(\beta)]$$

where $\alpha = \inf_{y\in K} a(y)$ and $\beta = \sup_{y\in K} a(y)$. That is, one can apply φ to the extremal values and interpolate linearly in between.

Now the map $a \to \Theta_a$ will satisfy (i), the inequality of (iv), and (v) of Theorem 7.9. To see that (ii) can fail in a specific example, one may take K to be the standard 2-simplex in $V = \mathbb{R}^3$, and A to be the space $A(K)$ of all affine functions on K with e the unit function. Then $V \cong l_3^1$, and it is easy to verify that (A,e) and (V,K) are in spectral duality, and to determine the spectral families of elements of A .

In this case the functional calculus defined by spectral theory consists in evaluating the given function φ at all the three extreme points and interpolating linearly in between, which is different from the functional calculus given by (7.37).

It can also be seen directly that statement (ii) will fail for $a \to \Theta_a$ unless the lines $a(x)$ = constant are parallel to one of the edges of the triangle K .

§ 8. Abelian subspaces

In this section we define a notion of compatibility for arbitrary elements of A . We then pick out certain "abelian" subspaces, which inherit from A a vector lattice ordering, and on which a commutative multiplication can be defined in a natural way. We assume throughout that A and V are in spectral duality.

Definition. Two elements a,b of A are said to be *compatible* if the spectral units e_λ^a, e_μ^b are compatible for every pair of values $\lambda,\mu \in \mathbb{R}$.

Clearly this definition is consistent with our previous definition of compatibility for projective units, and it also conforms with operator theory since two bounded self-adjoint operators on a Hilbert space will commute iff any two members of their spectral families commute. We also make the following observation, which we state as a proposition for later references:

Proposition 8.1. *If a,b are two compatible elements of A with spectral measures $E \to p_E^a$, $E \to p_E^b$, then the projective units p_E^a, p_B^b are compatible for every pair E,B of Borel sets of $\mathbb{R}$* .

Proof. Let E,B be arbitrary Borel sets of $\mathbb{R}$, and consider first a fixed $\mu \in \mathbb{R}$. By compatibility $e_\lambda^a = e_\lambda^a \wedge e_\mu^b + e_\lambda^a \wedge (e_\mu^b)'$ for every $\lambda \in \mathbb{R}$. It is easily verified that

$$p_E^a = \int_E de_\lambda^a = \int_E d(e_\lambda^a \wedge e_\mu^b) + \int_E d(e_\lambda^a \wedge (e_\mu^b)') \in [e_\mu^b] + [(e_\mu^b)'] .$$

Thus p_E^a is compatible with e_μ^b , and so $e_\mu^b = e_\mu^b \wedge p_E^a + e_\mu^b \wedge (p_E^a)'$. Arguing as above, we obtain

$$p_B^b = \int_B de_\mu^b = \int_B d(e_\mu^b \wedge p_E^a) + \int_B d(e_\mu^b \wedge (p_E^a)') \in [p_E^a] + [(p_E^a)'] .$$

Hence p_B^b is compatible with p_E^a . □

Corollary 8.2. If a,b are two compatible elements of A and if $\varphi \in \mathcal{B}(\sigma(a))$, $\psi \in \mathcal{B}(\sigma(b))$, then $\varphi(a)$ and $\psi(b)$ are also compatible. In particular $p_E^a = \chi_E(a)$ is compatible with $b = \gamma(b)$ for every Borel set E .

Proof. By (7.15) $e_\lambda^{\varphi(a)} = p^a_{\varphi^{-1}(\langle -\infty,\lambda])}$ and $e_\mu^{\psi(b)} = p^b_{\psi^{-1}(\langle -\infty,\mu])}$ for arbitrary $\lambda,\mu \in \mathbb{R}$. Now the corollary follows from Proposition 8.1. □

We now make an observation of a rather general nature based on the same argument as in the proof of Proposition 7.3: If $a \in A$ and if E is a Borel set of $\mathbb{R}$, then the P-projection P corresponding to p_E^a will satisfy

$$(8.1) \qquad Pa = (\chi_E \cdot \gamma)(a) = \int_E \lambda \, de_\lambda^a .$$

Definition. For every $a \in A$ the positive- and negative-parts of a are given by the formulas:

$$(8.2) \qquad a^+ = \gamma^+(a) = \int_{\mathbb{R}^+} \lambda \, de_\lambda^a ,$$

$$(8.3) \qquad a^- = \gamma^-(a) = -\int_{\mathbb{R}^-} \lambda \, de_\lambda^a .$$

Clearly $a^+ \geq 0$ and $a^+ \geq a$, and similarly $a^- \geq 0$ $a^- \geq -a$. Clearly also $a = a^+ - a^-$, and it is easily verified that $a^+ = a$

iff $a \geq 0$.

Writing $E = \mathbb{R}^+ = [0,\infty\rangle$ (we could equally well take $E = \langle 0,\infty\rangle$) and denoting by P and Q the P-projections corresponding to p^a_E and $p^a_{\mathbb{R}\setminus E}$, we obtain by (8.1)

$$(8.4) \qquad a^+ = Pa , \quad a^- = -Qa .$$

Since $Q = P'$, it follows by (6.2) that $a^+ \perp (-a^-)$. (This information was also implicit in the proof of Proposition 6.1.)

Note that for $x \in K$ the value $a^+(x)$ is not the same as $a(x)^+$ in general. (In fact $x \to a(x)^+$ is not even an affine function on K unless $a \geq 0$ or $a \leq 0$.) Neither will a^+ be the least upper bound of a and 0 in the partially ordered set A in general. (If A is the self-adjoint part of the 2×2-matrix algebra, then $\sup(a,0)$ is non-existent unless $a \geq 0$ or $a \leq 0$ [K_1]). However, we do have the following result:

Proposition 8.3. If $a \in A$, then a^+ is the least upper bound of a and 0 among all elements compatible with a.

Proof. It suffices to prove that $b \geq a$ and $b \in A^+$ implies $b \geq a^+$.

As above, we denote by P the P-projection corresponding to $p^a_{\mathbb{R}^+}$. By Corollary 8.2, P is compatible with b, and since $b \in A^+$ we have $Pb \leq b$. Hence by (8.4) and the hypothesis $a \leq b$, we obtain

$$a^+ = Pa \leq Pb \leq b ,$$

which completes the proof. $\square$

<u>Corollary 8.4. If M is a subspace of A such that all pairs of elements in M are compatible and such that $a \in M$ implies $a^+ \in M$, then M is a vector lattice in the ordering induced from A.</u>

<u>Proof</u>. A partially ordered linear space is a vector lattice iff the least upper bound of every element with zero exists. By Proposition 8.3 this requirement is satisfied for M. □

<u>Corollary 8.5. A is a vector lattice iff all elements of A are mutually compatible</u>.

<u>Proof</u>. By Corollary 8.4 we only have to prove that if A is a vector lattice, then all elements of A are compatible.

Since A is also a norm complete order-unit space, then A is order isomorphic to some C(X) (see e.g. [A_1,CorII.1.11]). The projective units of A are the extreme points of [0,e] (Prop.7.7), and they will correspond to the characteristic functions in C(X). The latter form a Boolean algebra, and therefore it follows that all projective units in A are compatible. This in turn implies that all pairs of elements in A are compatible. □

Observe that Corollary 8.5 does not hold for subspaces. A subspace M of A may consist of mutually compatible elements without being a lattice in the induced ordering. (E.g. consider $M = \{\lambda a \mid \lambda \in \mathbb{R}\}$ with $a \notin A^+ \cup (-A^+)$.) Conversely, a subspace M may be a vector lattice in the induced ordering while containing pairs of elements which are not compatible. (E.g. choose two non-compatible elements $a,b \in A^+$, then lin{a,b} is a vector lattice in the induced ordering.)

Finally, let M be a norm closed subspace which contains e and is a lattice in the induced ordering. Then M is isomorphic to $C(X)$ for suitable X, so one can define a functional calculus (for continuous functions) on M. We shall now explore conditions guaranteeing that this functional calculus ehall agree with the functional calculus defined on A.

<u>Proposition 8.6. Let M be a norm closed subspace of A containing the order unit e. Then the following are equivalent:</u>

(i) M is closed under the map $a \to a^+$

(ii) M is closed under the map $a \to \varphi(a)$ for $\varphi \in C(\sigma(a))$

(iii) M is closed under the map $a \to a^{(2)}$.

<u>Proof</u>. For arbitrary $a \in M$ we define

$$\mathcal{F}_a = \{\varphi \in C(\sigma(a)) \mid \varphi(a) \in M\}, \tag{8.5}$$

and we observe that $\mathcal{F}_a$ is a norm closed linear subspace of $C(\sigma(a))$ containing all linear functions $\xi \to \alpha\xi + \beta$.

Now assume (i). For fixed $a \in M$ and arbitrary $\varphi \in \mathcal{F}_a$, we have

$$\varphi^+(a) = (\gamma^+ \circ \varphi)(a) = \gamma^+(\varphi(a)) = \varphi(a)^+ \in M.$$

Hence $\varphi \in \mathcal{F}_a$ implies $\varphi^+ \in \mathcal{F}_a$, and so $\mathcal{F}_a$ is a vector lattice. By Stone-Weierstrass (lattice version) $\mathcal{F}_a = C(\sigma(a))$, and this proves that (ii) holds.

Trivially (ii) implies (iii).

Finally we assume (iii). Since

$$\varphi \cdot \psi = \tfrac{1}{2}[(\varphi+\psi)^2 - \varphi^2 - \psi^2],$$

it follows that for fixed $a \in M$, $\mathcal{F}_a$ will be a subalgebra of $C(\sigma(a))$. By Stone-Weierstrass (algebra version) $\mathcal{F}_a = C(\sigma(a))$.

In particular $\gamma^+ \in \mathcal{F}_a$, and so $a^+ = \gamma^+(a) \in M$. Thus (i) is valid. $\square$

We shall now prove that for a norm closed subspace closed under the map $a \to a^+$, the implication of Corollary 8.4 can be reversed.

<u>Proposition 8.7. Let M be a norm closed subspace of A containing the order unit e and closed under the map $a \to a^+$. Then the following are equivalent:</u>

(i) All elements of M are mutually compatible.

(ii) M is a vector lattice in the induced ordering.

(iii) M has the Riesz decomposition property.

<u>Proof</u>. Only the implication (iii) => (i) requires proof, so assume M has the Riesz decomposition property. We consider two elements $a,b \in A$, and we shall prove that they are compatible. Since M is closed under the map $a \to a^+$, then M is positively generated. Hence we can (and shall) assume $0 \leq a \leq e$ without loss of generality. Observe that it is sufficient to prove that every spectral unit e_μ^b is compatible with a, since this will give compatibility of e_μ^b with $e_\lambda^a = \chi_{\langle -\infty,\lambda]}(a)$ for every $\lambda \in \mathbb{R}$.

For fixed μ let Q be the P-projection corresponding to the projective unit e_μ^b and let $\{\varphi_n\}$ be a sequence of continuous functions with values in $[0,1]$ such that $\varphi_n \searrow \chi_{\langle -\infty,\mu]}$. Then $\inf_n \varphi_n(b) = e_\mu^b$, and this also means that $\varphi_n(b)$ converges to e_μ^b in the weak topology defined by the duality of A and V.

Clearly $0 \leq \varphi_n(b) \leq e$, and by Proposition 8.6 $\varphi_n(b) \in M$ for $n = 1,2,\ldots$. For every n we consider the decomposition $e = \varphi_n(b) + (e-\varphi_n(b))$, and since $0 \leq a \leq e$ we can use the Riesz decomposition property to find elements $a_n, a_n' \in M$ with $0 \leq a_n \leq \varphi_n(b)$ and $0 \leq a_n' \leq e - \varphi_n(b)$ such that $a = a_n + a_n'$. Since $a_n' \leq e - \varphi_n(b) \leq e - e_\mu^b$, then $a_n' \in [(e_\mu^b)'] = \operatorname{im} Q'$, and so $Q'a_n' = a_n'$. We therefore have

$$(8.6) \qquad Q'a = Q'a_n + Q'a_n' = Q'a_n + a_n' \leq Q'a_n + a.$$

By weak continuity of Q', $Q'(\varphi_n(b))$ converges to $Q'(e_\mu^b)$. Since $0 \leq a_n \leq \varphi_n(b)$ for all n, it follows that $Q'a_n$ converges weakly to 0. By (8.6) this gives $Q'a \leq a$. Thus Q' is compatible with a, and it follows that e_μ^b is compatible with a. $\square$

<u>Definition</u>. A norm closed subspace M of A containing e is said to be an <u>abelian subspace</u> if it is closed under the map $a \to a^+$ and if all pairs of elements are compatible.

It follows from Proposition 8.6 (statement (ii)) that an abelian subspace M is closed under the functional calculus of A. On the other hand it follows from Proposition 8.7 that M is a vector lattice in the ordering induced from A. Hence M is isometrically order isomorphic to some $C(X)$, and the compact Hausdorff space X is unique up to homeomorphisms. (One can take X to be the set of extreme points of the state space of (M,e); see e.g. [A_1;Cor.II.1.11]). This isomorphism induces a functional calculus on M. We now verify that the two functional calculi agree.

<u>Proposition 8.8. Let M be an abelian subspace of A and $\Phi : M \to C(X)$ an isometric order isomorphism for a compact Hausdorff space X. Then the functional calculus induced on M from $C(X)$ coincides with that induced from A, i.e. for $a \in M$ and $\varphi \in C(\mathbb{R})$:</u>

(8.7) $$\Phi^{-1}[\varphi\circ(\Phi a)] = \varphi(a).$$

Proof. Fix $a \in M$, and let J be a compact interval in $\mathbb{R}$ comtaining $\sigma(a) \cup (\Phi a)(X)$. It suffices to prove (8.7) for $\varphi \in C(J)$. Let

(8.8) $$\mathcal{F}_a = \{\varphi \in C(J) \mid \Phi^{-1}[\varphi\circ(\Phi a)] = \varphi(a)\}.$$

Note that Φ must take e into the function identically 1 on X. It follows that $\mathcal{F}_a$ contains all the linear functions $\xi \to \alpha\xi + \beta$. $\mathcal{F}_a$ is also closed under the map $\varphi \to \varphi^+$ since for $\varphi \in \mathcal{F}_a$

$$\Phi(\varphi^+(a)) = \Phi(\varphi(a)\vee 0) = \Phi(\varphi(a)) \vee 0$$
$$= [\varphi\circ(\Phi a)] \vee 0 = [\varphi\circ(\Phi a)]^+ = \varphi^+\circ(\Phi a).$$

It follows that $\mathcal{F}_a$ is a norm closed vector sublattice of $C(J)$ containing the constants and separating points. By Stone-Weierstrass $\mathcal{F}_a = C(J)$, and the proof is complete. □

Corollary 8.9. An abelian subspace M of A is a commutative Banach algebra under the product

(8.9) $$ab = \tfrac{1}{2}\{(a+b)^{(2)} - a^{(2)} - b^{(2)}\}$$

Proof. Consider an isometric order isomorphism $\Phi : M \to C(X)$ and use Proposition 8.8. □

Proposition 8.10. If $a \in A$ then the least abelian subspace containing a is

(8.10) $$M(a) = \{\varphi(a) \mid \varphi \in C(\sigma(a))\},$$

and $\Phi_a : \varphi \to \varphi(a)$ is an isometric order- and algebra- isomorphism of $C(\sigma(a))$ onto $M(a)$.

Proof. By Theorem 7.9, Θ_a is an isometric order isomorphism, and it is multiplicative in virtue of the definition (8.9). It follows from Proposition 8.6 that Θ_a will map $C(\sigma(a))$ onto the least abelian subspace of A which contains a. □

Finally we shall give a characterization of compatibility of elements of A which is most easily obtained from a theorem of Varadarajan on orthomodular lattices [V,Th.6.9]. By this theorem, for a given sequence of $\mathcal{U}$-valued measures $E \to p_E^i$ with mutually commuting ranges there exists a single $\mathcal{U}$-valued measure $E \to p_E$ and a sequence of Borel functions φ_i such that $p_E^i = p_{\varphi_i^{-1}(E)}$ for every Borel set E. If all the given $\mathcal{U}$-valued measures are of compact support, then one can also choose the new $\mathcal{U}$-valued measure to be of compact support and all the functions φ_i to be bounded. (This can be proved from the original statement by application of a "finitizing transform" like $\xi \to \arctan \xi$).

Proposition 8.11. A sequence $\{a_i\}$ of elements of A consists of mutually compatible elements iff there exists $c \in A$ and Borel functions φ_i bounded on $\sigma(c)$ such that $a_i = \varphi_i(c)$ for all i.

Proof. The sufficiency follows from Corollary 8.2 and the necessity from the theorem of Varadarajan just quoted. □

We now pass to the study of weakly closed abelian subspaces.

Proposition 8.12. If M is a weakly closed abelian subspace, then for each $a \in M$ and each $\varphi \in \mathcal{B}(\sigma(a))$, $\varphi(a)$ is in M.

Proof. Fix $a \in M$ and define

$$\mathcal{F}_a = \{\varphi \in \mathcal{B}(\sigma(a)) \mid \varphi(a) \in M\} .$$

Then $C(\sigma(a)) \subset \mathcal{F}_a$, and $\mathcal{F}_a$ is a subalgebra of $\mathcal{B}(\sigma(a))$ closed under bounded, pointwise, monotone, sequential limits (Prop. 7.6). Hence $\mathcal{F}_a = \mathcal{B}(\sigma(a))$. □

Corollary 8.13. If a is an element of a weakly closed abelian subspace M and if $E \to p_E^a$ is the spectral measure of a, then p_E^a is in M for all Borel sets $E \subset \mathbb{R}$. In particular, the spectral λ-unit e_λ^a is in M for all $\lambda \in \mathbb{R}$.

Proposition 8.14. Every weakly closed abelian subspace M is a Dedekind σ-complete vector lattice in the order induced from A.

Proof. Let $\{a_n\}$ be a sequence in M bounded above by $a \in M$. We consider the elements $b_k = \bigvee_{n=1}^{k} a_n$ (least upper bound in the vector lattice M). Then $\{b_k\}$ is an increasing sequence bounded above by a, and by monotone σ-completeness there exists $b \in A$ such that $b = \sup_k b_k$ (i.e. b is pointwise supremum of the sequence $\{b_k\}$). The sequence $\{b_k\}$ will also converge to b in the weak topology, and so $b \in M$. It is now evident that b is the least upper bound of $\{a_n\}$ in M. This proves that M is Dedekind σ-complete. □

At this point we are in the position to clarify the relationship with Freudenthal's spectral theorem for Dedekind σ-complete vector lattices. (See [F]; we shall use the terminology of [L-Z; pp. 249-269].) For a fixed weakly closed abelian subspace M one can apply Freudenthal's theorem [L-Z; Th. 40.2], by which each element $a \in M$ is approximated by linear combinations of "components" $p \in M$ satisfying $0 \leq p \leq e$ and $p \wedge (e-p) = 0$. From the isomorphism of

M with C(X) it is easy to see that the "components" of M coincide with the idempotents, i.e. the elements u such that $u^{(2)} = u$, and they are in turn the projective units of M (Prop. 7.7). From this it follows that the "spectral system of components" $\{p_\alpha\}$ associated with a, is in our terminology a spectral family for a. (See [L - Z; Th.38.4]), and that Freudenthal's theorem [L - Z; Th.40.2] coincides with our theorem relativized to M.

We will next study properties of the set of all elements of A compatible with a given subset of A. Observe that if P is a P-projection, then by the definition of compatibility the set of all elements of A compatible with P is just ker(I - P - P'); hence it is a weakly closed subspace of A. It follows by the definition of compatibility for arbitrary elements of A that the set of all elements of A compatible with all elements of a given subset B of A, will be a weakly closed subspace of A.

<u>Definition</u>. For every subset B of A we denote by B' the weakly closed linear subspace of A consisting of all elements of A compatible with all elements of B. The space (B')', which we will write as B", is called the <u>bicommutant</u> of B.

The connection with the previously defined concept of $\mathcal{P}$-bicommutant is given in the following proposition.

<u>Proposition 8.15. Let $a \in A$ and let P be a P-projection with associated projective unit u = Pe. Then $u \in \{a\}''$ iff $P \in \mathcal{B}(a)$.</u>

<u>Proof</u>. Assume first $u \in \{a\}''$. Then u, and hence also P, is compatible with $a \in \{a\}'$. If Q, or equivalently Qe, is

compatible with a, then u, and hence also P, is compatible with $Qe \in \{a\}'$. This proves $P \in \mathcal{B}(a)$.

Assume next $P \in \mathcal{B}(a)$, and consider $b \in \{a\}'$. Now all spectral projections of b are compatible with a, and therefore they must be compatible with P. Hence P is compatible with b, and so $u \in \{b\}'$. This proves $u \in \{a\}''$. $\square$

Proposition 8.16. If B is a subset of A consisting of mutually compatible elements, then B'' is a weakly closed abelian subspace of A containing B.

Proof. We only have to prove that B'' is an abelian subspace. Clearly $e \in B''$. If a is compatible with an element b, then so is any function of a (Cor.8.2); it follows that B'' is closed under the map $a \to a^+$. Since all pairs of elements of B are compatible, then $B \subset B'$, and therefore $B'' \subset B'$. Now if $a \in B''$ and $b \in B'' \subset B'$, then a and b are compatible. Thus all pairs of elements of B'' are compatible. $\square$

Corollary 8.17. If $B \subset A$ consists of mutually compatible elements, then there exists a smallest abelian subspace (and a smallest weakly closed abelian subspace) containing B.

Definition. The center of A, written $Z(A)$, consists of all those elements of A which are compatible with all elements of A, i.e. $Z(A) = A'$.

Observe that $Z(A)$ is a weakly closed abelian subspace of A since $Z(A) = \{e\}''$. Note also that our previous definition of central P-projection (in §4) conforms with this new definition of center. In fact, a P-projection P was said to be central exactly when P, or equivalently the corresponding projective unit $u = Pe$, was compatible with all $a \in A$. Hence the Boolean center of A consists of all P-projections corresponding to projective units in Z(A).

Recall that by Proposition 4.8 the central P-projections P are those weakly continuous projections $P : A \to A$ such that

$$0 \leq Pa \leq a, \qquad \text{for all } a \in A^+ \tag{8.11}$$

This result was proved under the general assumptions of § 4 (i.e. (3.1) and (3.2)). In the present section we are assuming spectral duality, and we shall see that this makes the requirement to weak continuity redundant.

Lemma 8.18. If P is a projection on A such that $0 \leq Pa \leq a$ for all $a \in A^+$, then P is a central P-projection.

Proof. Let P be a projection such that $0 \leq Pa \leq a$ for all $a \in A^+$. We claim that Pe is an extreme point of $[0,e]$. Suppose a,b are in $[0,e]$, $0 < \lambda < 1$ and $Pe = \lambda a + (1-\lambda)b$. By assumption:

$$Pa \leq a, \quad Pb \leq b, \tag{8.12}$$

and since $Pe = P(Pe)$ we shall have

$$\lambda a + (1-\lambda)b = Pe = \lambda Pa + (1-\lambda)Pb \leq \lambda a + (1-\lambda)b;$$

but this is possible only if the equality signs are valid in (8.12).

Since $a \leq e$ and $b \leq e$, we also have

$$Pa \leq Pe, \quad Pb \leq Pe, \tag{8.13}$$

and so

$$Pe = \lambda Pa + (1-\lambda)Pb \leq \lambda Pe + (1-\lambda)Pe = Pe;$$

but this is possible only if the equality signs are valid in (8.13). Hence $a = Pa = Pe$ and $b = Pb = Pe$, and this shows that Pe is an extreme point of $[0,e]$.

By Proposition 7.7 there exists a P-projection Q such that $Pe = Qe$; we will show that $P = Q$. For this purpose it suffices to prove $\operatorname{im} P \subset \operatorname{im} Q$ and $\ker P \subset \ker Q$, and these inclusions can be obtained as follows:

$$\operatorname{im} P = [Pe] = [Qe] = \operatorname{im} Q$$

and

$$\ker P = \operatorname{im}(I-P) = [e-Pe] = [e-Qe] = \operatorname{im} Q' \subset \ker Q .$$

Since $P = Q$ satisfies (8.11), it is a central P-projection, and the proof is complete. □

We will show that $Z(A)$ is canonically isomorphic to Wils' "ideal center" $Z_i(A)$ [W_1], and that it coincides with the "center" $Z(A,e)$ of the order-unit space (A,e) as defined by Alfsen and Andersen [AA_2]. Recall that $Z_i(A)$ is a subspace of the space of linear operators on A, consisting of all operators T admitting an "order bound" $\lambda \in \mathbb{R}$ such that

$$(8.14) \qquad -\lambda a \leq Ta \leq \lambda a \qquad \text{for all} \quad a \in A^+ ;$$

whereas $Z(A,e)$ is a subspace of A, consisting of all elements which "act multiplicatively on the pure states". (See [AA_2] for details.) Although formally different, these two spaces are closely related. In fact, the map $T \to Te$ is a bijection of $Z_i(A)$ onto $Z(A,e)$ preserving linearity, order and norm. (Cf. [W_1] and [AA_2]. See also [A, Ch. II. §7] for a detailed exposition and [AE] for a more general theory of "centralizer" and "multipliers" for arbitrary (not necessarily ordered) Banach spaces).

Theorem 8.19. The image of the ideal center $Z_i(A)$ under the map $T \to Te$ is the center $Z(A)$; otherwise stated $Z(A,e) = Z(A)$.

Proof. The ideal center $Z_i(A)$ is endowed with a rich structure. It is an order-unit space in the natural order (i.e. $S \leq T$ if $Sa \leq Ta$ for all $a \in A^+$) whose order-unit norm (i.e. $\|T\| = \inf\{\lambda \in \mathbb{R} \mid -\lambda I \leq T \leq \lambda I\}$) coincides with the operator norm; also it is a vector lattice, and a commutative Banach algebra under operator multiplication. (See e.g. [A_1,Ch.II.§7].) By the monotone σ-completeness of A, $Z_i(A)$ will also be monotone σ-complete. Now it follows by standard arguments that $Z_i(A)$ is the norm closed linear hull of projections P satisfying $0 \leq P \leq I$. (E.g. one can apply Freudenthal's spectral theorem [L-Z, Th.40.2] to show that $Z_i(A)$ is the norm closed linear hull of those elements $P \in Z_i(A)$ which satisfy $0 \leq P \leq I$ and $P \wedge (I-P) = 0$, and then use functional representation to show that these elements are idempotent.) By Lemma 8.18, a projection P satisfying $0 \leq P \leq I$ will be a central P-projection, and so $Pe \in Z(A)$.so $Pe \in Z(A)$. It follows that the map $T \to Te$ will map $Z_i(A)$ into $Z(A)$.

Since $Z(A)$ is a weakly closed abelian subspace of A, all spectral units of elements of $Z(A)$ are in $Z(A)$. Hence $Z(A)$ is the norm closed linear hull of the projective units Pe with P in the Boolean center. Now every P-projection in the Boolean center satisfies (8.11), and thus it must belong to $Z_i(A)$. Hence $Z(A)$ is the norm closed linear hull of elements Pe with $P \in Z_i(A)$, and the surjectivity follows.

The last assertion of the proposition is obvious since $Z(A,e) = \{Te \mid T \in Z_i(A)\}$. $\square$

We close this section by various characterizations of spectra. In this connection we agree to write:

$$(8.15) \qquad ab = \tfrac{1}{2}[(a+b)^{(2)} - a^{(2)} - b^{(2)}]$$

if a and b are compatible elements of A. Under this hypothesis a and b will generate an abelian subspace M (Cor.8.17), and ab is simply the product of a and b in the commutative Banach algebra M (Cor.8.9).

Definition. An element $a \in A$ is said to be invertible if there exists $b \in A$ compatible with a such that $ab = e$.

Proposition 8.20. Let $a \in A$ and $\lambda \in \mathbb{R}$. Then $a - \lambda e$ is invertible iff $\lambda \notin \sigma(a)$; in this case the inverse of $a - \lambda e$ is unique and is in $M(a)$.

Proof. Note that by Proposition 7.5 $\sigma(a-\lambda e) = \sigma(a) - \lambda$, and by definition $M(a-\lambda e) = M(a)$. It therefore suffices to consider $\lambda = 0$.

1.) Assume first a invertible, say $ab = e$ with a and b compatible. Let M be the smallest weakly closed abelian subspace containing a and b (Cor.8.17). By Proposition 8.12 and Proposition 7.6 the mapping $\varphi \to \varphi(a)$ is a norm-decreasing homomorphism of the Banach algebra $\mathcal{B}(\sigma(a))$ into M. (By definition it preserves squares, hence also products.)

Let $E = \langle -\beta, \beta \rangle$ where $\beta < \|b\|^{-1}$, and write $u = p_E^a$. We shall verify that $u = 0$, which will give $\sigma(a) \cap E = \emptyset$ and then $0 \notin \sigma(a)$.

The element $u = \chi_E(a)$ will be an idempotent element of M, and the following inequalities will hold:

$$\|ua\| = \|(\chi_E \circ \gamma)(a)\| \leq \sup_{\lambda \in \sigma(a)} |(\chi_E \circ \gamma)(\lambda)| \leq \beta ,$$

$$\|ub\| \leq \|u\| \cdot \|b\| \leq \|b\| .$$

Since $u = ue = (ua)(ub)$, this gives

$$\|u\| \leq \|ua\| \cdot \|ub\| \leq \beta \cdot \|b\| < 1 .$$

Since u is a projective unit, it is either zero or it takes the value 1 at the corresponding projective face. By the above inequality the second alternative is impossible, and so $u = 0$ as claimed.

2.) Assume next $0 \notin \sigma(a)$, and define $\varphi \in \mathscr{C}(\sigma(a))$ by $\varphi(\xi) = \frac{1}{\xi}$. Then $b = \varphi(a) \in M(a)$ by Proposition 8.12, and

$$ab = (\gamma \cdot \varphi)(a) = \iota(a) = e .$$

Thus a is invertible with inverse $b \in M(a)$.

Finally we assume that c is any other inverse of a compatible with a, and we consider the abelian subspace N generated by a and c (Cor.8.17). Then $M(a) \subset N$, and N is itself a commutative Banach algebra (Cor.8.9). But then there can not be more than one inverse of a in N, and so $c = b$. Hence the inverse is unique. $\square$

<u>Definition</u>. A point $x \in K$ is said to be a <u>characteristic point</u> for an element $a \in A$ if the (scalar valued) spectral measure for a at x has all mass concentrated in one point, i.e. if $\mu_x^a = \epsilon_{a(x)}$. A real number λ is said to be a <u>characteristic value</u> for $a \in A$ if there exists a characteristic point $x \in K$ such that $a(x) = \lambda$. The set of characteristic values for a is called the <u>point spectrum</u> for a.

The result (7.30) can now be restated as follows: For $a \in A$ and $x \in K$ one has

(8.16) $$a^{(2)}(x) = a(x)^2$$

iff x is a characteristic point for a.

By definition the point spectrum is contained in the spectrum. The opposite does not hold in general, but it is of some interest to observe that every point of the spectrum "almost" has the properties of a characteristic value.

<u>Proposition 8.21.</u> <u>Let $a \in A$ and $\lambda \in \mathbb{R}$. Then the following are equivalent</u>

(i) $\lambda \in \sigma(a)$

(ii) For every open neighbourhood V of λ there exists $x \in K$ with $\mu_x^a(V) = 1$.

<u>Proof.</u> (i) => (ii) Let $\lambda \in \sigma(a)$ and let V be an open neighbourhood of λ. By definition of $\sigma(a)$, $p_V^a \neq 0$. Since p_V^a is a non-zero projective unit, the corresponding projective face F is non-empty. For x in F we shall have $\mu_x^a(V) = p_V^a(x) = 1$.

(ii) => (i) Follows from the definition of spectrum. $\square$

<u>Corollary 8.22.</u> <u>Let $a \in A$ and $\lambda \in \sigma(a)$. Then for every pair $\delta, \varepsilon > 0$ there exists $x \in K$ such that</u>

(8.17) $$|a(x) - \lambda| < \delta \quad \text{and} \quad |a^{(2)}(x) - \lambda^2| < \varepsilon .$$

<u>Proof.</u> Choose an open neighbourhood V of λ such that $|\xi - \lambda| < \delta$ and $|\xi^2 - \lambda^2| < \varepsilon$ for $\xi \in V$. Then select an element

$x \in K$ such that $\mu_x^a(V) = 1$, and observe that the values

$$a(x) = \int \lambda \, d\mu_x^a(\lambda), \qquad a^{(2)}(x) = \int \lambda^2 \, d\mu_x^a(\lambda)$$

will satisfy (8.17). $\square$

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