TRANSLATION INVARIANT SUBSPACES

and

SETS OF UNIQUENESS FOR $L_p(G)$

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1. INTRODUCTION. Let G be a locally compact Abelian topological group and let $L_p(G)$, $1 \leq p \leq \infty$, denote the usual Lebesgue spaces with respect to Haar measure on G . The purpose of this note is two-fold. First, if G is noncompact, we shall establish some elementary necessary and some elementary sufficient conditions for a measurable subset of $\hat{G}$, the dual group of G , to be a set of uniqueness for $L_p(G)$, $1 \leq p < 2$, and utilize these results to obtain some information on the topological structure of sets of uniqueness. Secondly, the results concerning sets of uniqueness will be applied to construct some examples of nonzero closed translation invariant linear subspaces X of $L_2(G)$, G being a noncompact group, such that $X \cap L_1(G) = \{0\}$. Obviously no such subspaces X of $L_2(G)$ can exist when G is compact as in this case $X \cap L_1(G) = X$.

The Fourier transform of f in $L_1(G)$ will, as usual, be denoted by $\hat{f}$, while the Hausdorff-Young or Plancherel transform of f in $L_p(G)$, $1 < p \leq 2$, will be denoted by $\widehat{f}$. If $\tilde{f}$ is defined by $\tilde{f}(t) = f(-t)$, then $\check{f} = (\hat{f})^{\sim}$ and $\breve{f} = (\widehat{f})^{\sim}$. We recall that given $1 \leq p \leq 2$ and $f \in L_p(G)$ such that either $\hat{f}$ or $\widehat{f}$ belongs to $L_r(\hat{G})$ for some r , $1 \leq r \leq 2$, then $f = (\hat{f})^{\vee}$ if $p = r = 1$, $f = (\hat{f})^{\smile}$ if $p = 1$ and $1 < r \leq 2$, $f = (\widehat{f})^{\vee}$ if $1 < p \leq 2$ and $r = 1$, and $f = (\widehat{f})^{\smile}$ if $1 < p \leq 2$ and $1 < r \leq 2$. The formulas are naturally to be interpreted in the sense of equality almost everywhere with respect to Haar measure. The reader is referred to [3,pp.240 and 241] for further details.

The Greek letter η will denote (normalized) Haar measure on $\hat{G}$ and $C_o(G)$ will denote the Banach space of continuous complex-valued functions on G that vanish at infinity. The symbol # indicates the completion of a proof.

2. SETS OF UNIQUENESS FOR $L_p(G)$. If G is a locally compact Abelian topological group and $1 \leq p \leq \infty$, then a measurable subset E of $\hat{G}$ is a <u>set of uniqueness for</u> $L_p(G)$ if there exists no non-zero element g in $L_1(\hat{G})$ such that $g(\gamma) = 0$ for almost all γ in $\hat{G} \sim E$ and $\hat{g} \in L_p(G)$. That is, E is a set of uniqueness for $L_p(G)$ if whenever $g \in L_1(\hat{G})$ is such that $g(\gamma) = 0$ almost everywhere off of E and $\hat{g} \in L_p(G)$, then $g = 0$.

This notion of a set of uniqueness is less restrictive than the classical concept of Cantor uniqueness. A measurable subset E of the unit circle group Γ is a set of <u>Cantor uniqueness</u> if whenever $\Sigma_{n=-\infty}^{\infty} c_n e^{int}$ converges to zero almost everywhere off of E , then $c_n = 0$ for all n . Every set of Cantor uniqueness is of measure zero, but not conversely [4,pp.52 and 53]. In contrast to this, it is easily seen that every measurable subset of $\hat{G}$ with measure zero is a set of uniqueness for $L_p(G)$, $1 \leq p \leq \infty$, and the converse is also true if $2 \leq p \leq \infty$ and G is noncompact, or if $1 \leq p \leq \infty$ and G is compact. Indeed, if G is compact, then the only set of uniqueness for $L_p(G)$ is the empty set. The situation is, however, quite different for noncompact groups and $1 \leq p < 2$ as shown by the following theorem of Figà-Talamanca and Gaudry [1]. In the case that G is the integers, the result was established by Katznelson [5].

THEOREM 1. _Let_ G _be a noncompact locally compact Abelian topological group and let_ F _be a measurable subset of_ $\hat{G}$ _with finite positive Haar measure._ _If_ $\epsilon > 0$, _then there exists a measurable set_ $E \subset F$ _such that_:

(i) $\eta(E) > \eta(F) - \epsilon$.

(ii) _For each_ p , $1 \leq p < 2$, E _is a set of uniqueness for_ $L_p(G)$.

In particular, if G is noncompact, then there exist sets of uniqueness for $L_p(G)$, $1 \leq p < 2$, with finite positive Haar measure. In a moment we shall give an example of a set of uniqueness for $L_1(G)$ with infinite measure.

Let E be a measurable subset of $\hat{G}$. If $1 < p \leq 2$, then X_E^p will denote the linear subspace of $L_p(G)$ consisting of those f in $L_p(G)$ such that $\check{f}(\gamma) = 0$ for almost all γ in $\hat{G} \sim E$. Evidently, X_E^p is a closed translation invariant linear subspace of $L_p(G)$, and $X_E^p = \{0\}$ if E is of measure zero.

The next theorem provides some sufficient conditions for a measurable subset of $\hat{G}$ to be a set of uniqueness.

THEOREM 2. _Let_ G _be a noncompact locally compact Abelian topological group and let_ E _be a measureable subset of_ $\hat{G}$. _If_ $1 \leq p < r \leq 2$ _and_ $X_E^r \cap L_p(G) = \{0\}$, _then_ E _is a set of uniqueness for_ $L_p(G)$.

PROOF. Suppose $1 \leq p < r \leq 2$ and $X_E^r \cap L_p(G) = \{0\}$. Let $g \in L_1(\hat{G})$ be such that $g(\gamma) = 0$ almost everywhere off of E and $\hat{g} \in L_p(G)$. Since $\hat{g} \in C_o(G)$ and $p < r \leq 2$, it follows at once that $\hat{g} \in L_r(G)$ and $(\hat{g})^{\vee} = g$ almost everywhere on $\hat{G}$. Hence $\hat{g} \in X_E^r \cap L_p(G)$, and so $\hat{g}(t) = 0$ almost everywhere on G . Consequently, $\hat{g}(t) = 0$, $t \in G$, because $\hat{g} \in C_o(G)$ and the Haar measure of every nonempty open subset of G is positive [2,pp.193 and 194]. Thus, by the injectivity of the Fourier transformation, we conclude that $g = 0$. Therefore E is a set of uniqueness for $L_p(G)$. #

Obviously the converse of Theorem 2 is valid for any set of uniqueness with measure zero. Furthermore, the converse remains valid for sets of uniqueness with finite positive measure, the existence of which are ensured by Theorem 1. More precisely, we have the next result:

THEOREM 3. _Let_ G _be a noncompact locally compact Abelian topological group_, _let_ $1 \leq p < 2$, _and let_ E _be a measurable subset of_ $\hat{G}$ _with finite positive Haar measure_. _If_ E _is a set of uniqueness for_ $L_p(G)$ _and_ $p < r \leq 2$, _then_ $X_E^r \cap L_p(G) = \{0\}$.

PROOF. Suppose E is a set of uniqueness for $L_p(G)$, $p < r \leq 2$, and let $f \in X_E^r \cap L_p(G)$. Then $f \in L_p(G)$, $\check{f}(\gamma) = 0$ almost everywhere off of E , and $\check{f} \in L_{p'}(\hat{G})$, $\frac{1}{p} + \frac{1}{p'} = 1$, by the Hausdorff-Young Theorem [3,pp.226 and 227]. Moreover, since E has finite measure, it is apparent that the characteristic function of E belongs to $L_p(\hat{G})$, and an easy argument reveals that $\check{f} \in L_1(\hat{G})$. Consequently, $(\check{f})^\wedge = f$ and so $(\check{f})^\wedge \in L_p(G)$. Hence $\check{f} = 0$ as E is a set of uniqueness for $L_p(G)$. Therefore, by the injectivity of the Hausdorff-Young transformation, we conclude that $f = 0$. #

We note that Theorems 2 and 3 are trivially valid when $p = r$, since in this case $X_E^r = \{0\}$. Obviously, if G is discrete, then the measurable subsets of $\hat{G}$ all have finite measure.

Theorems 2 and 3 can be used to shed some light on the topological properties of sets of uniqueness. We observe that every set of uniqueness for $L_p(G)$, $1 < p < 2$, is a set of uniqueness for $L_1(G)$. The proof is elementary. The closure of a subset E of $\hat{G}$ will be denoted by $cl(E)$, and the interior of E by $int(E)$.

THEOREM 4. *Let G be a noncompact locally compact Abelian topological group and let E be a measurable subset of $\hat{G}$ with finite Haar measure. If $1 \leq p < 2$ and E is a set of uniqueness for $L_p(G)$, then $cl(\hat{G} \sim E) = \hat{G}$.*

PROOF. It is easily verified that if E is a set of uniqueness for $L_p(G)$, then $-E$ is a set of uniqueness for $L_p(G)$, and hence $-E$ is a set of uniqueness for $L_1(G)$. Thus, by Theorem 3 and the remarks immediately preceding it, we see that $X_{-E}^2 \cap L_1(G) = \{0\}$. However, we claim that

$$\{f \mid f \in L_1(G), \hat{f}(\gamma) = 0, \gamma \in \hat{G} \sim E\} \subset X_{-E}^2 \cap L_1(G) .$$

To see this it clearly suffices to show that every member of the set on the left hand side of the containment belongs to $L_2(G)$. But if $f \in L_1(G)$ and $\hat{f}(\gamma) = 0$, $\gamma \in \hat{G} \sim E$, then, since $\hat{f} \in C_o(G)$ and E has finite measure, we deduce that $\hat{f} \in L_2(\hat{G})$. Consequently, $f = (\hat{f})^\vee$ belongs to $L_2(G)$.

Denoting, as is customary, the kernel of a set of maximal regular ideals in $L_1(G)$ by $k(F)$, and the hull of a closed ideal I in $L_1(G)$ by $h(I)$, we conclude from the argument of the previous paragraph that $k(\hat{G} \sim E) = \{0\}$, whence $h[k(\hat{G} \sim E)] = \hat{G}$. Since $L_1(G)$ is a regular commutative Banach algebra, it follows at once that $\hat{G} \sim E$ is dense in $\hat{G}$. #

The reader is referred to [6, Chapter 7] for a discussion of regular Banach algebras.

COROLLARY 1. _Let_ G _be a noncompact locally compact Abelian topological group and let_ E _be a measurable subset of_ $\hat{G}$. _If_ $1 \leq p < 2$ _and_ E _is a set of uniqueness for_ $L_p(G)$, _then_ $cl(\hat{G} \sim E) = \hat{G}$.

PROOF. If E has finite measure, then the result is precisely the content of Theorem 4. So assume E has infinite measure and that $cl(\hat{G} \sim E) \neq \hat{G}$, that is, $int(E) \neq \emptyset$. It is easily seen that every measurable subset of a set of uniqueness is again a set of uniqueness. Consequently, since $int(E) \neq \emptyset$, there exists an open set $U \subset E$ of finite measure that is a set of uniqueness for $L_p(G)$, contrary to the conclusion of Theorem 4. Hence $cl(\hat{G} \sim E) = \hat{G}$. #

COROLLARY 2. _Let_ G _be a noncompact locally compact Abelian topological group, let_ $1 \leq p < 2$, _and let_ E _be a set of uniqueness for_ $L_p(G)$. _If_ $\eta[cl(E) \sim E] = 0$, _then_ E _is a nowhere dense subset of_ $\hat{G}$.

PROOF. An elementary argument shows that $cl(E)$ is a set of uniqueness for $L_p(G)$ whenever E is such a set and $\eta[cl(E) \sim E] = 0$.

Thus, by Corollary 1, $cl[\hat{G} \sim cl(E)] = \hat{G}$, that is, E is nowhere dense. #

In particular, Corollary 2 shows that every closed set of uniqueness for $L_p(G)$, $1 \leq p < 2$, must be nowhere dense. The converse assertion is also valid in the case $p = 1$. This will be an immediate consequence of the following theorem:

THEOREM 5. Let G be a noncompact locally compact Abelian topological group and let E be a measurable subset of $\hat{G}$. If E is nowhere dense, then E is a set of uniqueness for $L_1(G)$.

PROOF. In view of Theorem 2 it suffices to show that $X_E^2 \cap L_1(G) = \{0\}$. But if $f \in X_E^2 \cap L_1(G)$, then $f \in L_1(G) \cap L_2(G)$ and $\check{f}(\gamma) = 0$ almost everywhere on $\hat{G} \sim E$. Actually, we claim that $\check{f}(\gamma) = 0$, $\gamma \in \hat{G} \sim E$. Indeed, if $\gamma_0 \in \hat{G} \sim E$ is such that $\check{f}(\gamma_0) \neq 0$, then, since $\check{f}$ is continuous, there exists an open neighborhood U of γ_0 such that $\check{f}(\omega) \neq 0$, $\omega \in U$. However, since E is nowhere dense, there also exists a nonempty open set $V \subset U$ such that $V \cap E = \emptyset$. Thus $V \subset \hat{G} \sim E$, $\eta(V) > 0$, and $\check{f}(\omega) \neq 0$, $\omega \in V$, contradicting the assumption that $\check{f}(\gamma) = 0$ almost everywhere on $\hat{G} \sim E$. Hence $\check{f}(\gamma) = 0$, $\gamma \in \hat{G} \sim E$.

Appealing once more to the fact that E is nowhere dense, we deduce that $\check{f}(\gamma) = 0$, $\gamma \in \hat{G}$, since $cl(\hat{G} \sim E) = \hat{G}$. Consequently $\check{f} = 0$, and $X_E^2 \cap L_1(G) = \{0\}$. #

COROLLARY 3. Let G be a noncompact locally compact Abelian topological group and let E be a closed subset of $\hat{G}$. Then E is a set of uniqueness for $L_1(G)$ if and only if E is a nowhere dense subset of $\hat{G}$.

Theorem 5 also provides us with a means of constructing sets of uniqueness for $L_1(G)$ that have infinite measure. For example, let $G = \mathbb{R}$, the additive group of the real numbers, and for each positive integer n let E_n be a Cantor subset of the closed interval $[2n, 2n+1]$ of Lebesgue measure $\frac{1}{2}$ [7,p.63]. If $E = \cup_{n=1}^{\infty} E_n$, then E is a measurable nowhere dense subset of $\mathbb{R}$ with infinite measure, and so E is a set of uniqueness for $L_1(\mathbb{R})$ with infinite measure.

3. TRANSLATION INVARIANT SUBSPACES. The proof of Theorem 5 actually shows that $X_E^2 \cap L_1(G) = \{0\}$ whenever E is a measurable nowhere dense subset of $\hat{G}$ and G is noncompact. In particular, if E is a nowhere dense subset of $\hat{G}$ with positive measure, then X_E^2 is a nonzero closed translation invariant linear subspace of $L_2(G)$ that contains no nonzero continuous function with compact support. This answers a question posed to the author by Arne Hole. Considerably more can be said as shown by the next result:

THEOREM 6. <u>If</u> G <u>is a noncompact locally compact Abelian topological group, then there exists a nonzero closed translation invariant linear subspace</u> X <u>of</u> $L_2(G)$ <u>such that</u> $X \cap (\cup_{1 \leq p < 2} L_p(G)) = \{0\}$.

PROOF. Let E be a measurable subset of $\hat{G}$ with finite positive Haar measure that is a set of uniqueness for $L_p(G)$ for all p, $1 \leq p < 2$. The existence of such a set is ensured by Theorem 1. Let $X = X_E^2$. Then X is a closed translation invariant linear subspace of $L_2(G)$ and $X \neq \{0\}$. The latter is true since the Plancherel transform of the characteristic function of E, which is not zero, belongs to X. Moreover, by Theorem 3, $X \cap (\cup_{1 \leq p < 2} L_p(G)) = \{0\}$. #

COROLLARY 4. <u>Let</u> G <u>be a noncompact locally compact Abelian topological group.</u> <u>If</u> X <u>is a nonzero closed translation invariant linear subspace of</u> $L_2(G)$, <u>then there exists a nonzero closed translation invariant linear subspace</u> Y <u>of</u> $L_2(G)$ <u>such that</u> $Y \subset X$ <u>and</u> $Y \cap (\cup_{1 \leq p < 2} L_p(G)) = \{0\}$.

PROOF. It is well known that $X = X_F^2$ for some measurable subset F of $\hat{G}$ with positive Haar measure [3,p.237]. By Theorem 1 there exists a measurable subset E of $\hat{G}$ with finite positive measure such that $E \subset F$ and E is a set of uniqueness for $L_p(G)$, $1 \leq p < 2$. Let $Y = X_E^2$. #

Two obvious questions remain unanswered. First, if G is noncompact and $1 < p < 2$, do there exist nonzero closed translation invariant linear subspaces X of $L_p(G)$ such that $X \cap L_1(G) = \{0\}$? The argument utilized in Theorem 6 will not work in this situation, since if E is a set of uniqueness for $L_p(G)$ with finite positive measure, then $X_E^p = \{0\}$. Second, if $1 < p < 2$ and G is noncompact, do there exist nonzero closed translation invariant linear subspaces X of $L_2(G)$ such that $X \cap L_1(G) = \{0\}$, but $X \cap L_p(G) \neq \{0\}$. To prove the existence of such a subspace it would suffice, in view of Theorem 3, to establish the existence of a measurable subset E of $\hat{G}$ with finite positive measure that was a set of uniqueness for $L_1(G)$, but not for $L_p(G)$. Conversely, the existence of sets of uniqueness for $L_1(G)$ that are not sets of uniqueness for $L_p(G)$, $1 < p < 2$, could be established by proving the existence of closed translation invariant linear subspaces X in $L_2(G)$ for which $X \cap L_1(G) = \{0\}$, but $X \cap L_p(G) \neq \{0\}$. We remark that Theorem 1 only ensures the existence of subsets of $\hat{G}$ with finite positive measure that are simultaneously sets of uniqueness for each $L_p(G)$, $1 \leq p < 2$.

REFERENCES

[1] A. Figà-Talamanca and G.I. Gaudry, Multipliers and sets of uniqueness for L^p, Michigan Math. J. 17 (1970), 179-191.

[2] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis. I., Springer, Berlin-Göttingen-Heidelberg, 1963.

[3] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis. II., Springer, Berlin-Heidelberg-New York, 1970.

[4] J.-P. Kahane and R. Salem, Ensembles parfaits et séries trigonométriques, Hermann, Paris, 1963.

[5] I. Katznelson, Sets of uniqueness for some classes of trigonometric series, Bull. Amer. Math. Soc. 70 (1964), 722-723.

[6] R. Larsen, Banach Algebras: An Introduction, Dekker, New York, 1973.

[7] H. Royden, Real Analysis, 2nd edition, Macmillan, New York, 1968.