Inductive operators on resolvable structures

by

ANDERS M. NYBERG.

1. Introduction.

The relationship between inductive definability and admissible sets is at present well understood and accounted for several places such as in [BGM], Barwise [2], [3], [Gandy] and Moschovakis [8]. We shall concentrate on a few basic facts, so assume $A = \langle A, \in \rangle$ is an admissible set, then:

a) Every Σ_1 positive inductive operator on A has a Σ_1 least fixed point.

b) The length of every Σ_1 positive inductive definition on A does not exceed the ordinal $o(A)$ of A. ($o(A) = A \cap \mathrm{Ord}.$)

As immediate corollaries of a) and b) we mention the apparently weaker a') and b'):

a') There exists a realtion on A which is first order inductive but not Σ_1 inductive.

b') There exists a first order inductive definition on A of length strictly greater than any Σ_1 inductive definition on A.

By an unpublished result of J. Stavi it turns out that there is a transitive (in fact prim. rec. closed) set A satisfying a) and b) which is not admissible. On the other hand if $A = L_\alpha$ for some infinite ordinal α then Barwise (unpublished) has shown that even the properties a') and b') each imply that A is admissible.

In this paper we show that on resolvable structures each of tne four properties are equivalent to admissibility. Our main purpose is, however, not so much to obtain these results as to point out that they really are absolute versions of some properties related to invariant definability.

In the last section we discuss the possibility of obtaining similar results on the relationship between $s-\Pi^1_1$ and Π^1_1, monotone inductive definability on one hand and the $s-\Pi^1_1$ reflection principle on the other. We are able to give some answers, but this area seems to lead to interesting problems for further research.

2. Preliminaries.

We will have to assume familiarity with the basic notions from the theory of positive inductive definability as presented in Moschowakis [8]. Thus we shall by inductive operators, inductive definitions etc. always refer to positive induction. For notions relating to definability, like $s-\Pi^1_1$, $s-\Delta^1_1$ s.i.i.d. etc., the reader should consult Barwise [2] and Kunen [7]. Just note that we will always allow parameters to occure in defining formulas. For example if $\mathfrak{A} = \langle A, \in \rangle$ is a structure, $s-\Pi^1_1$ denotes the class of relations definable over $\mathfrak{A}$ by $s-\Pi^1_1$ formulas allowing parameters from A .

In the following $\mathfrak{A} = \langle A, \in, R_1, \ldots, R_k \rangle$ will be some fixed structure with A a transitive set and $\in$ the membership relation restricted to A .

2.1. Definition. Let $\mathfrak{A}$ be as above. A function τ: $\mathrm{Ord} \cap A \to A$ with $A = \bigcup \mathrm{range}(\tau)$ is called a <u>resolution</u> of A. We call $\mathfrak{A}$ <u>$s-\Delta_1^1$ resolvable</u> (<u>resolvable</u>) if there is a resolution τ of A which is $s-\Delta_1^1$ definable (Δ_1 definable) on $\mathfrak{A}$.

Barwise [2] shows that $s-\Pi_1^1$ and s.i.i.d. coincides on structures $\mathfrak{A}$, at least if A is closed under ordinary set theoretic pairing. Thus it is easily seen that such structures will be $s-\Delta_1^1$ resolvable if and only if they have an i.i.d. hierarchy in the sense of Barwise [2]. This means in particular that theorem 5.4 of that paper applies directly to $s-\Delta_1^1$ resolvable sets.

Kunen [6] shows that when a structure $\mathfrak{A}$ is self definable then s.i.i.d. and Π_1^1 coincide on $\mathfrak{A}$. We shall not use the notion self definability later in this paper, just use the mentioned result of Kunen together with theorem 5.4 of Barwise [2] to make the following observation:

2.2 Theorem. Assume $\mathfrak{A} = \langle A, \in, R_1, \ldots, R_k \rangle$ when A is transitive and closed under pairing and $\mathfrak{A}$ is $s-\Delta_1^1$ resolvable.

If there is a relation R which is Π_1^1 but not $s-\Pi_1^1$ on $\mathfrak{A}$ then $\mathfrak{A}$ satisfies the $s-\Pi_1^1$ reflection principle.

<u>Proof</u>: Since s.i.i.d. and $s-\Pi_1^1$ coincide on $\mathfrak{A}$ we must under the assumption of the theorem have that not every Π_1^1 relation on $\mathfrak{A}$ is s.i.i.d. Hence by Kunen's result $\mathfrak{A}$ can not be self definable, but the theorem of Barwise just mentioned yields that $\mathfrak{A}$ then must satisfy the $s-\Pi_1^1$ reflection principle. $\dashv$

We end this list of preliminaries by quoting a special case of a theorem of Aczel [1].

2.3. Theorem. Assume $\mathfrak{A}$ is countable with A closed under pairing. Then every $s-\Pi_1^1$ relation on $\mathfrak{A}$ is Σ_1 inductively definable.

3. Σ_1 induction and admissibility.

The key to the results of this section is contained in the following lemma.

3.1. Lemma. Let $\mathfrak{A} = \langle A, \in, R_1, \ldots, R_k \rangle$ be a resolvable structure with A transitive and closed under pairing. If there is a relation R on $\mathfrak{A}$ which is inductive but not Σ_1 inductive then $\mathfrak{A}$ must satisfy the Σ reflection principle.

Proof: It suffices to prove the lemma for countable structures , because if it is true for all countable $\mathfrak{A}$ we can just use an absoluteness argument of the type, so successfully employed by Barwise in several contexts, to get the general result. (See for instance Barwise [3].) To sketch the absoluteness argument observe that the statement "$\exists \mathfrak{A}$($\mathfrak{A}$ satisfies the hypothesis of the lemma but not the conclusion)" is a Σ_1^{ZF} statement and hence if true, it must hold in $\langle H(\omega_1), \in \rangle$. But the consepts "inductive", "Σ_1-inductive" and "Σ-reflection" are absolute so this would produce a countable structure $\mathfrak{A}$ which does not satisfy the Σ reflection principle, but satisfies the hypothesis of the lemma.

Assume now that $\mathfrak{A}$ is countable and that R is a relation which is inductive but not Σ_1 inductive on $\mathfrak{A}$. Then R is not $s-\Pi_1^1$ on $\mathfrak{A}$ by theorem 2.3. Since all inductive relations are Π_1^1 we can then conclude that R is Π_1^1 but not $s-\Pi_1^1$ on $\mathfrak{A}$ and hence by theorem 2.2 $\mathfrak{A}$ must satisfy the $s-\Pi_1^1$ reflection principle and in particular the Σ reflection principle. $\dashv$

We should now be ready to prove the main theorem of this section.

3.2. Theorem. Let $\mathfrak{A}$ be as in lemma 3.1 and assume in addition that $\mathfrak{A}$ satisfies Δ_0-separation. The following are then equivalent:

i) $\mathfrak{A}$ is admissible.

ii) Every Σ_1 inductive operator on $\mathfrak{A}$ has a Σ_1 definable least fixed point.

iii) Not every inductive relation is Σ_1 inductive on $\mathfrak{A}$.

iv) The length of Σ_1 induction on $\mathfrak{A}$ does not exceed the ordinal of $\mathfrak{A}$.

v) There exists a first order inductive definition on of length strictly greater than any Σ_1 induction on $\mathfrak{A}$.

Proof: i) => ii) and i) => iv) are both due to Gandy, see for instance theorem 2.4.2 of Gandy [5]. That ii) implies iii) follows from the fact that not all inductive relations are first order definable, hence certainly not Σ_1 definable, on acceptable structures. That iv) implies v) is equally obvious by the fact that the length of first order induction on acceptable structures is the ordinal of the "<u>next</u>" admissible set.
That v) implies iii) is a little more subtle and needs the observation that v) implies that every Σ_1 inductive set on $\mathfrak{A}$ is hyperelementary on $\mathfrak{A}$. (See Moschovakis [8].) Since then exists inductive relations which are not hyperelementary on $\mathfrak{A}$ we can conclude iii):

To complete the proof we can now appeal to lemma 3.1 which yields that iii) implies i). $\dashv$

This theorem will, as mentioned in the introduction, apply to structures $\mathfrak{A}$ of the form $\langle L_\alpha, \in\rangle$. We find it also striking that for "almost" all structures of the form $\mathfrak{A} = \langle V_\alpha, \in, \mathcal{P}, R_1 \ldots, R_k\rangle$ we will have that Σ_1-induction and first order induction coincide. V_α is the set of sets of rank less that the limit ordinal α and $\mathcal{P}$ is the graph on the power set relation on V_α. $R_1, \ldots, R_k$ is an arbitrary list of relations on V_α. (We have to include $\mathcal{P}$ in order to make $\mathfrak{A}$ resolvable and not just $s-\Delta_1^1$ resolvable.) Our excuse for using the term "almost" is that the exceptions will require $\mathfrak{A}$ to be admissible so that α would at least have to be a strong limit cardinal, in fact a fixed point for the beth hierarchy (i.e. $\alpha = \beth_\alpha$).

4. $s-\Pi_1^1$ induction and $s-\Pi_1^1$ reflection.

Theorem 3.2 is really a result on the correspondence between Σ_1 induction and the Σ reflection principle. In this section we show that, at least with respect to the lengths, there is a similar relationship between $s-\Pi_1^1$ induction and the $s-\Pi_1^1$ reflection principle. The main result reads as follows.

4.1. Theorem. Let $\mathfrak{A} = \langle A, \in, R_1, \ldots, R_k\rangle$ where A is a transitive set closed under pairing. If $\mathfrak{A}$ satisfies the $s-\Pi_1^1$ reflection principle then the length of $s-\Pi_1^1$ induction does not exceed the ordinal of $\mathfrak{A}$. If in addition $\mathfrak{A}$ is $s-\Delta_1^1$ resolvable then the converse holds.

Proof: That $s-\Pi_1^1$ reflection yields the restriction on the length of $s-\Pi_1^1$ induction is due to Barwise and mentioned in Barwise [2]. Since, however, no proof of this has been published, this seems to be the right place to give at least an outline of a proof.

So let $\Phi(x_1,\dots,x_n,S)$ $(= \Phi(\bar{x},S))$ be a $s-\Pi^1_1$ formula with S n-ary and occurring positively in Φ. Let $\Phi'(\bar{x},x_{n+1},S') \underset{\text{Def.}}{\Longleftrightarrow} \Phi(\bar{x},\{\bar{y}|\exists\alpha < x_{n+1}(\bar{y},\alpha)\in S'\})$ where S' is n+1-ary.

The fixed point $I_{\Phi'}$ of Φ' can be given a $s-\Pi^1_1$ definition Ψ, thus

$$(\bar{x},\alpha)\in I_{\Phi'} \longleftrightarrow \mathfrak{A}\models\Psi[\bar{x},\alpha] .$$

A straight forward induction on α now shows that

$$(1) \qquad \bar{x}\in I^{\alpha}_{\Phi} \longleftrightarrow (\bar{x},\alpha)\in I_{\Phi'} .$$

Assume now $\bar{x}\in I^{\tau}_{\Phi}$ where $\tau = o(A)$.

Then by (1) $\mathfrak{A}\models\Phi(\bar{x};\{\bar{y}|\exists\alpha\,\Psi(\bar{y},\alpha)\})$.

By $s-\Pi^1_1$ reflection there exists a transitive $w\in A$ such that $\mathfrak{A}\restriction w\models\Phi(\bar{x},\{\bar{y}|\exists\alpha\,\Psi(\bar{y},\alpha)\})$ and since $s-\Pi^1_1$ formulas persist under end-extensions we can conclude that

$$\mathfrak{A}\models\Phi(\bar{x},\{\bar{y}|\exists\alpha<\delta\,\Psi(\bar{y},\alpha)\}) \quad \text{where} \quad \delta = w\cap On .$$

Again by (1) this yields that $\bar{x}\in I^{\delta}_{\Phi}$ and hence the closure ordinal $\|\Phi\|$ of the induction given by Φ must satisfy $\|\Phi\|\le\tau$.

We now turn to the proof of the converse. An alternative proof of this has been obtained independently by Grant [6].

Let Φ be a $s-\Pi^1_1$ sentence and assume $\mathfrak{A}\models\Phi$. Let us also assume that Φ can be written as $\forall T\,\exists x\,\varphi$ where φ is Δ_0. Let Ψ be the $s-\Pi^1_1$ definition of the resolution τ of A.

We define an inductive operator Γ on A as follows:

$$(x_1,x_2) \in \Gamma(S) = \begin{cases} \forall z \in x_2 \exists z' \in x_1((z',z) \in S \wedge \Psi(x_1,x_2)) \vee \\ \forall T \exists v \exists w((w,v) \in S \wedge \exists x \in w\varphi(x,T) \wedge x_1 = x_2 = \langle o,o\rangle) . \end{cases}$$

Γ is clearly $s-\Pi^1_1$ with S occurring positively. Moreover for $\nu < o(A)$ a straight forward induction shows that

(1) $(x,y) \in I^{\nu}_{\Gamma} \leftrightarrow (y \in \mathrm{Ord} \cap A \wedge y \leq \nu \wedge \tau(y) = x) \vee (x = y = \langle o,o\rangle)$

Since $A = \bigcup \mathrm{range}\ \tau$ we have that $\forall T \exists x \exists \nu(x \in \tau(\nu) \wedge \varphi(x,S))$ is true on $\mathfrak{A}$, so by (1) the second clause of the definition of Γ will apply and hence $(\langle o,o\rangle,\langle o,o\rangle) \in I_{\Gamma}$.

Assume now that the length of Γ does not exceed the ordinal of A. That is $I_{\Gamma} = \Gamma(\bigcup_{\nu<o(A)} I^{\nu}_{\Gamma}) = \bigcup_{\nu<o(A)} I^{\nu}_{\Gamma}$, and hence $(\langle o,o\rangle,\langle o,o\rangle) \in I^{\nu}_{\Gamma}$ for some $\nu < o(A)$. Let ν_o be the least such ordinal. Then, by the definition of Γ we get:

$$\forall T \exists v \exists w((v,w) \in \bigcup_{\xi<\nu_o} I^{\xi}_{\Gamma} \wedge \exists x \in w\,\varphi(x,T))$$

which by (1) implies that $\forall T \exists x \in \tau(\nu_o)\varphi(x,T)$. If we take w as the transitive closure of $\tau(\nu_o) \cup \{\text{parameters of } \varphi\}$ it follows that $\mathfrak{A} \upharpoonright w \models \forall T \exists x\varphi(x,T)$. Thus if the length of $s-\Pi^1_1$ induction on $\mathfrak{A}$ does not exceed the ordinal of $\mathfrak{A}$ we can conclude that $\mathfrak{A}$ must satisfy the $s-\Pi^1_1$ reflection principle. $\dashv$

4.2. Corollary. Let $A = \langle A,\in\rangle$ be an admissible set satisfying the $s-\Pi^1_1$ reflection principle and let μ be the ordinal of A. Then the structure $L_{\mu} = (L_{\mu},\in)$ is admissible and satisfy the $s-\Pi^1_1$ reflection principle.

Proof: That L_{μ} is admissible follows just from the fact that A is admissible. To see that L_{μ} satisfy the $s-\Pi^1_1$ reflection principle assume the converse. Then by theorem 4.1 there must be a $s-\Pi_1^1$ inductive operator Γ on L_μ of length $|\Gamma| > \mu$ since L_μ is resolvable.

This Γ will also be $s-\Pi_1^1$ definable on $\langle A,\in\rangle$, using that the predicate "$x \in L_\mu$" is Σ_1 definable on $\langle A,\in\rangle$. This will, however, contradict theorem 4.1 since $\langle A,\in\rangle$ satisfies the $s-\Pi_1^1$ reflection principle and hence $|\Gamma| \leq \mu$. $\dashv$

This corollary has been known to experts, the reason we have mentioned it here is not only because this is the simplest proof we know of this result, but also because we will use it as a base for further discussion.

One of the direct consequences of this corollary is that it enables us to talk about the "next $s-\Pi_1^1$ reflecting admissible set" in analogy with the "next admissible set". When concerned with inductive definability it seems reasonable to ask how far this analogy would work.

At last for a $s-\Delta_1^1$ resolvable structure $\langle A,\in\rangle$, theorem 4.1 tells us that if $\langle A,\in\rangle$ does not satisfy the $s-\Pi_1^1$ reflection principle the closure ordinal $|s-\Pi_1^1|$ of $s-\Pi_1^1$ induction exceeds the ordinal of A. In fact one can prove along the same lines as in theorem 3.2 that in this case the ordinal of Π_1^1 monotone induction $|\Pi_1^1|$ must satisfy $|\Pi_1^1| = |s-\Pi_1^1|$. Hence $s-\Pi_1^1$ induction will take us at least up to the ordinal of the next admissible set. It is tempting to conjecture that $|s-\Pi_1^1|$ would be the ordinal of the next $s-\Pi_1^1$ reflection admissible set. Except that we know this is true when A is countable, we don't know very much about what the situation is like in general.

We feel that results on questions in this direction should make it possible to pin down some of the more recursion theoretic properties of invariant definability.

References.

1. P. Aczel. Implicit and inductive definability. J.S.L. 35 (1970) p. 599 (Abstract).

2. K.J. Barwise Applications of strict Π_1^1 predicates to infinitary logic. J.S.L. 34 (1969) pp. 409-423.

3. K.J. Barwise Admissible sets and structures. Springer Ω series. To appear.

4. Barwise, Gandy Moschovakis The next admissible set. J.S.L. 36 (1971) pp. 108-120 (Here referred to as [BGM]).

5. R. Gandy Inductive definitions. Generalized Recursion Theory, Oslo 72. Ed. by Fenstad and Hinman, North Holland (1974).

6. P.W. Grant Inductive definitions and recursion theory. D. Phil. Dissertation, Oxford (1973).

7. K. Kunen Implicit definability and infinitary languages. J.S.L. 24 (1968) pp. 446-451.

8. Y. Moschovakis Elementary induction on abstract structures. North Holland (1974).