NON-COMMUTATIVE SPECTRAL THEORY
FOR AFFINE FUNCTION SPACES ON CONVEX SETS

Part II

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Introduction

This is a continuation of: "Non-commutative spectral theory for affine function spaces on convex sets I", henceforth referred to as "Part I". In the present "Part II" we shall introduce a few new concepts although our main concern will be application of previous results. The key new concepts are "transversality" of an affine retraction of a convex set, a compact convex set being "spectral" or "strongly spectral", and a geometric notion of a "trace" which will generalize the corresponding notion in operator theory. The most important examples studied in Part II are (Choquet) simplexes, unit balls of $L^p(\mu)$-spaces where $1 < p < \infty$ (or more general "rotund" convex compact sets), and operator algebras for which the convex sets in question will be either the normal state space of a von Neumann algebra or the state space of a C^*-algebra. In addition we shall present a few (low dimensional) geometric examples which may help to visualize the concepts of the general theory.

We will now discuss each section in some detail. The general context of § 1 is that of an order-unit space (A,e) and a base-norm space (V,K) in separating order and norm duality (see Part I for definitions). The main results of this section are geometric characterizations of projective faces depending upon the notion of "transversality". A positive projection R of V which leaves K invariant, is said to "transversal" at a convex subset F of $K \cap \operatorname{im} R$ if $\ker R \subset \tilde{F}$; this definition is then relativized to an affine retraction ρ of K in a natural way. (To fix the ideas we recall that by the definition given in § 1 of Part I, $\tilde{F}$ may be thought of as a closed linear subspace "tangent" to K at F, so "transversality of R at F" means that the last term of the direct sum $V = \operatorname{im} R \oplus \ker R$ is contained in the "tangent space" $\tilde{F}$ to K at F). The precise statement of the results is given in Theorems 1.3 and 1.7; of these the former characterizes projective faces via transversal projections of V, the latter via transversal affine retractions of K (for the latter we assume $A \cong V^*$) . In Theorem 1.7 we also show that if P is the P-projection of V corresponding to a projective face F, then $(P+P')|_K$ is the unique affine retraction of K onto $\operatorname{co}(F \cup F^{\#})$.

In § 2 we suppose that K is a convex compact set and that $V = A(K)^*$ (with $A(K)$ the continuous affine functions on K) and $A = V^* \cong A^b(K)$ (the bounded affine functions on K). We begin by showing that a split face of K is always projective, and conversely that a projective face is a split face iff the corresponding P-projection is central. Then we define K to be a "spectral" convex compact set if (A,e) and (V,K) are in spectral duality, and we define K to be "strongly spectral" if in addition the spectral units e_λ^a of all $a \in A(K)$ are upper semi-continuous in the given compact topology on K . The latter concept is of interest because it provides a necessary and sufficient condition that $A(K)$ be closed under the functional calculus by continuous functions (Theorem 2.6). It is proved in § 2 that the unit balls of L^p-spaces $(1 < p < \infty)$ are strongly spectral (Theorem 2.5 and Proposition 2.10; see the remarks after Theorem 2.5 for extension to more general "rotund" convex compact sets). Also it is proved that every (Choquet) simplex is spectral, and that a simplex is strongly spectral iff it is a Bauer simplex, i.e. iff its extreme boundary is closed (Theorem 2.4 and Proposition 2.9). For a simplex K one can also define a functional calculus by means of representing boundary measures μ_x for points $x \in K$ (cf. [A_1; Th.II. 3.6]). Specifically, for $a \in A(K)$ and for a bounded Borel function φ one can define $\varphi(a) \in A^b(K)$ by $\varphi(a)(x) = \mu_x(\varphi \circ a)$. This functional calculus is shown to coincide with the one given by our spectral theory (Proposition 2.8).

In § 3 we begin with a von Neumann algebra $\mathfrak{A}$, and we define A to be the self-adjoint part of $\mathfrak{A}$ and V to be the self-adjoint part of the pre-dual $\mathfrak{A}_*$. If e is the identity of $\mathfrak{A}$ and $K \subset V$ the normal state space of $\mathfrak{A}$, then (A,e) and (V,K) are shown to be in spectral duality, and the functional calculus defined by our spectral theory will coincide with the customary functional calculus for self-adjoint operators (Proposition 3.4). It is also shown that many of the concepts from our general spectral theory can be identified with familiar ones from operator theory: the projective units in A are the (self-adjoint) projections in $\mathfrak{A}$, the P-projections on A are the maps $a \to pap$ where $p \in \mathfrak{A}$ is a projection, and the projective faces of K are exactly the norm closed faces. Moreover, two elements of A are compatible iff they commute (as elements of $\mathfrak{A}$), and it follows from this that the center $Z(A)$ will coincide with the self-adjoint part of the (algebraic) center of $\mathfrak{A}$. In addition we remark that by the general results of § 1 there is associated a unique affine retraction $\rho: K \to co(F \cup F^{\#})$ to every projective (= norm closed) face F of K . This retraction (or rather its dual) gives an example of a conditional expectation in $\mathfrak{A}$, and the uniqueness of ρ can also be derived from general uniqueness theorems for conditional expectations in von Neumann algebras (see the remark after Theorem 3.5).

In the second half of § 3 we treat a C^*-algebra $\mathfrak{A}$ with identity e by means of the results from the first half of this section. Now we are working in the spectral duality of (A,e) and (V,K) where A is the self-adjoint part of the enveloping von Neumann algebra $\mathfrak{A}^{**}$ and V is the self-adjoint part of the dual space $\mathfrak{A}^*$ of $\mathfrak{A}$ with $K \subset V$ the state space of $\mathfrak{A}$. (Recall that $\mathfrak{A}^*$ can be identified with the pre-dual of $\mathfrak{A}^{**}$ and K with the normal state space of $\mathfrak{A}^{**}$, cf. [D_2; § 12]). Note also that the self-adjoint part of $\mathfrak{A}$ can be identified with A(K) and the self-adjoint part of $\mathfrak{A}^{**}$ with $A^b(K)$, so the results of § 2 will also apply in this case. We show first that with the weak* topology the state space K of the given C^*-algebra is a strongly spectral convex compact set (Theorem 3.6). Using the result (due to Effros [E] and Prosser [P]) that the weak*-closed faces of K are semi-exposed, we prove that among the norm closed (hence projective) faces of K , the weak* closed ones will be exactly those for which the corresponding projective unit (alias the "carrier projection" in $\mathfrak{A}^{**}$) is a weak* upper semi-continuous function on K (Theorem 3.7). In § 3 there is also a brief discussion of the connection between concepts of our general spectral theory and one-sided ideals in $\mathfrak{A}^{**}$ and $\mathfrak{A}$. We also include a proof of the simple fact that the state space K is a simplex (and then necessarily a Bauer simplex) iff the given C^*-algebra is commutative. The section closes by a discussion of the geometry of state spaces for finite dimensional C^*-algebras.

In § 4 we assume that (A,e) and (V,K) are in spectral duality and that $A \doteq V^*$. Under this hypothesis we prove that every face of A^+ which is closed in the weak topology determined by the duality with V (the weak* topology), will be of the form im^+P for a P-projection P on A. As a corollary we conclude that every semi-exposed face of K is exposed, hence projective. This in turn yields completeness of the lattice of projective units (previously shown to be σ-complete), generalizing the completeness of the projection lattice of a von Neumann algebra. We next prove that each $x \in V$ admits a unique "orthogonal" decomposition into positive and negative parts, i.e. a decomposition $x = y - z$ where $y,z \geq 0$ and $\|x\| = \|y\| + \|z\|$. For the dual of a C^*-algebra this was first proved by Grothendieck [Gr]). Finally we give a general definition of a "trace", and we give a geometric proof that the traces always will form a linearly compact simplex (Theorem 4.7). It is then proved that our "traces" coincide with the usual ones for the important special cases of von Neumann algebras and C^*-algebras (Theorem 4.10). As a consequence we obtain a new proof of the fact (first proved by Thoma [Th] and Effros-Hahn [EH]) that the traces of a C^*-algebra form a weak* compact simplex.

§ 1. Geometric properties of projective faces.

Throughout this section we assume that (A,e) and (V,K) are order-unit and base-norm spaces in separating order and norm duality. As in Part I we shall use the terms "weak" and "weakly" to denote the weak topologies defined on A and V by this duality.

Definition. A subspace M of V splits into subspaces M_1 and M_2 if M is the direct ordered sum of M_1 and M_2, i.e. if $M = M_1 \oplus M_2$ and the corresponding projections $P_i : M \to M_i$ $(i = 1,2)$ are positive. M splits continuously into M_1 and M_2 if the projections P_i $(i = 1,2)$ are also weakly continuous.

Lemma 1.1. Let M be a subspace of V which splits into subspaces M_1 and M_2, and let the corresponding projections be P_1 and P_2. Then

$$(1.1) \qquad \|P_i x\| \leq \|x\| \qquad \text{for } x \in M^+, \quad i = 1,2.$$

If there exists a bounded positive projection R of V onto M, then P_1, P_2 are bounded with $\|P_i\| \leq \|R\|$ for $i = 1,2$.

Proof. If $x \in M^+$, then $x = P_1x + P_2x$ where $P_i x \in M^+$ for $i = 1,2$. Now

$$\|x\| = e(x) = e(P_1x) + e(P_2x) = \|P_1x\| + \|P_2x\| ,$$

and (1.1) follows.

If $x \in M$ is arbitrary, then we can decompose $x = y - z$ with $y,z \in V^+$ and $\|x\| = \|y\| + \|z\|$. (Note that we can not always choose $y,z \in M$. There are simple counterexamples with $V = l_1^3$) . Assuming that there exists a bounded positive projection R from V onto M , we obtain for i = 1,2:

$$\|P_i x\| = \|P_i Rx\| = \|P_1 Ry - P_i Rz\|$$
$$\leq \|P_i Ry\| + \|P_i Rz\| \leq \|Ry\| + \|Rz\| \leq \|R\|\|x\| .$$

Hence $\|P_i\| \leq \|R\|$ for i = 1,2. □

Note in particular that if the whole space V splits into subspaces M_1 and M_2 and the corresponding projections are P_1 and P_2 , then $\|P_i\| \leq 1$ for i = 1,2. Note also that in this case $M_1^+ = P_2^{-1}(0) \cap V^+$ is a face of V^+ , and so $M_1 \cap K$ is a face of K ; similarly $M_2 \cap K$ is a face of K .

<u>Proposition 1.2.</u> <u>If V splits into M_1 and M_2 , then $F_1 = M_1 \cap K$ and $F_2 = M_2 \cap K$ are complementary split faces of K. Conversely, if F_1 and F_2 are complementary split faces of K , then V splits into $M_1 = \text{lin } F_1$ and $M_2 = \text{lin } F_2$</u> .

<u>Proof</u>. Routine verification (cf. [A_1; Prop.II.6.1]). □

We shall now define some notions which will be used to characterize projections R = P + Q for pairs P,Q of quasicomplementary P-projections of V . (Recall that this problem is of interest only in the non-central case, since by Proposition I.4.8., R = I when P,Q are central).

<u>Definition</u>. A weakly continuous positive projection $R: V \to V$ is said to be an <u>R-projection</u> if

$$(1.2) \qquad \|Rx\| = \|x\| \qquad \text{for } x \in V^+ .$$

It is easily verified that a weakly continuous linear map

$R: V \to V$ with range M is an R-projection iff it maps K onto $M \cap K$ and leaves $M \cap K$ pointwise invariant. It is also easily verified that every R-projection is bounded with $\|R\| \leq 1$.

Note that (1.2) can be rewritten as $e(Rx) = e(x)$ for $x \in V^+$. Since $V = V^+ - V^+$ it follows that $e \circ R = e$. Hence for every R-projection R of V one has

(1.3) $$\ker R \subset e^{-1}(0) .$$

Definition. An R-projection R with range M is said to be transversal at a convex subset F of $M \cap K$ if

(1.4) $$\ker R \subset \tilde{F} .$$

The geometric meaning of the requirement (1.4) can best be seen by considering the affine hyperplane $H = e^{-1}(1)$ which contains K and is invariant under R (since $e \circ R = e$). Recall that by definition $\tilde{F} \cap H$ is the intersection of all supporting hyperplanes of K at F. Hence we may think of $\tilde{F} \cap H$ as the affine "tangent space" to K at F. It follows by means of (1.3) that the requirement (1.4) is equivalent to

(1.5) $$F + \ker R \subset \tilde{F} \cap H .$$

In other words: R is transversal at F iff $R^{-1}(F)$ is contained in the "tangent space" $\tilde{F} \cap H$.

Theorem 1.3. Let $F, G \subset K$ and $M = \operatorname{lin}(F \cup G)$. Then F and G are quasicomplementary projective faces of K iff:

(i) F and G are semi-exposed faces of K

(ii) M splits continuously into $\operatorname{lin} F$ and $\operatorname{lin} G$

(iii) There is an R-projection R of V onto M which is transversal at F and G.

<u>Moreover, if these conditions are satisfied, then there exists just one R-projection R with range M, namely R = P + Q where P and Q are the P-projections corresponding to the projective faces F and G.</u>

<u>Proof</u>. 1.) Assume first that F and G are quasicomplementary projective faces, and let the corresponding P-projections be P and Q. Then

$$(1.6) \qquad (\text{im } P) \cap K = F \, , \quad (\text{im } Q) \cap K = G$$

$$(1.7) \qquad (\ker Q) \cap K = F \, , \quad (\ker P) \cap K = G$$

By known properties of P-projections (see Part I) F and G will be semi-exposed. Also it is easily verified that M will split continuously into lin F and lin G, the "splitting" being performed by the (restriction to M of) the two orthogonal projections P and Q.

Defining R = P + Q we get an R-projection (cf. Part I, (2.8)). For an arbitrary $x \in \ker R$ we also have $x \in \ker Q$. Hence by the smoothness of Q (Part I, (1.13)) and by (1.7):

$$x \in \ker Q = \widehat{\ker^{+} Q} = \tilde{F}$$

This proves that R is transversal at F. Similarly we prove that R is transversal at G.

2.) Assume next (i), (ii), (iii). Let P_o and Q_o be the two weakly continuous projections of M determined by the decomposition $M = (\text{lin } F) \oplus (\text{lin } G)$, and define $P = P_oR$, $Q = Q_oR$. By assumption P_o, Q_o and R are positive and weakly continuous. Hence P, Q are positive and weakly continuous as well.

It follows from Lemma 1.1 that $\|P_o\| \leq 1$ and $\|Q_o\| \leq 1$.

Hence $\|P\| \leq 1$ and $\|Q\| \leq 1$.

It remains to prove that P and Q are smooth projections satisfying (1.6) and (1.7).

By definition $\operatorname{im}^+ P = (\operatorname{lin} F) \cap V^+ = \operatorname{cone} F$, and similarly for Q . From this (1.6) follows.

Clearly $F \subset (\ker Q) \cap K$. To prove the opposite relation we consider an arbitrary element $x \in (\ker Q) \cap K$, which we decompose as follows:

(1.8) $$x = Rx + (x - Rx) .$$

Since $P_o + Q_o$ is the identity operator on $M = \operatorname{im} R$, we have $R = (P_o + Q_o)R = P + Q$. Now the assumption $x \in \ker Q$ entails $Rx = Px$, and so $Rx \in (\operatorname{im} P) \cap K = F$. Clearly also $x - Rx \in \ker R$. Hence by (1.8) and by transversality of R at F

$$(\ker Q) \cap K \subset F + \ker R \subset \tilde{F} .$$

Since F is semi-exposed, this gives

$$(\ker Q) \cap K \subset \tilde{F} \cap K = F .$$

This proves the first equality of (1.7). The second is similar.

To prove that P is a smooth projection, it suffices to show

(1.9) $$\ker P \subset \widetilde{\ker^+ P}$$

(cf. Part I, (1.14)). To this end we consider an arbitrary $x \in \ker P$ and decompose as follows:

(1.10) $$x = Qx + (x - Qx) .$$

Now $Qx \in \operatorname{im} Q \subset \operatorname{lin} G$. Also $x - Qx \in (\ker P) \cap (\ker Q)$, and since $R = P + Q$ we obtain $x - Qx \in \ker R$. Hence by (1.9) and by the transversality of R at G:

$$x \in \operatorname{lin} G + \ker R \subset \operatorname{lin} G + \tilde{G} \subset \tilde{G} .$$

By (1.7) $\tilde{G} = (\ker^+ P)^\sim$, and so $x \in (\ker^+ P)^\sim$. This proves (1.9), and P is shown to be smooth. Similarly we prove that Q is smooth.

3.) It remains to prove that $P+Q$ is the only R-projection of V onto M . To this end we consider an arbitrary R-projection S of V such that $\operatorname{im} S = M$.

Since $\operatorname{im}(P+Q) = M$, we have

$$(P+Q)S = S .$$

We claim that we also have

$$(1.11) \qquad (P+Q)S = P+Q ,$$

from which the equality $S = P+Q$ will follow.

This claim is most easily proved if we pass from V to A . We consider an arbitrary $a \in A^+$. For $x \in \operatorname{im}^+ P \subset \operatorname{im} S$, we have

$$(S^* P^* a)(x) = a(PSx) = a(Px) = (P^* a)(x) ;$$

and for $x \in \ker^+ P = \operatorname{im}^+ Q \subset \operatorname{im} S$, we have

$$(S^* P^* a)(x) = a(PSx) = a(Px) = 0 .$$

Hence $S^* P^* a$ coincides with $P^* a$ on $\operatorname{im}^+ P$ and vanishes on $\ker^+ P$. But $P^* a$ is the only element of A^+ with this property. (Cf. the uniqueness statement of Part I, (2.22)). Thus $S^* P^* a = P^* a$, and since $a \in A^+$ was arbitrary and $A = A^+ - A^+$, this gives $S^* P^* = P^*$. Hence $PS = P$. Similarly we prove $QS = Q$. Now (1.11) follows, and the proof is complete. □

<u>Corollary 1.4</u>. <u>Let F and G be semi-exposed faces of K. Then F and G are quasicomplementary projective faces iff $\tilde{F} \cap \tilde{G} \subset e^{-1}(0)$ and</u>

$$V = (\text{lin } F) \oplus (\text{lin } G) \oplus (\tilde{F} \cap \tilde{G}) \tag{1.12}$$

<u>with weakly continuous projections $P: V \to \text{lin } F$, $Q: V \to \text{lin } G$, $S: V \to \tilde{F} \cap \tilde{G}$, of which P and Q are also positive</u>.

<u>Proof</u>. 1.) Assume first that F and G are quasicomplementary projective faces corresponding to P-projections P and Q.

We shall first prove that $R = P + Q$ will satisfy:

$$\ker R = \tilde{F} \cap \tilde{G} . \tag{1.13}$$

The relation $\ker R \subset \tilde{F} \cap \tilde{G}$ will follow since R is transversal at F and G. If $x \in \tilde{F} \cap \tilde{G}$, then by the smoothness of Q (cf. Part I, (1.13)):

$$x \in \tilde{F} = \widetilde{\text{im}^+ P} = \widetilde{\ker^+ Q} = \ker Q .$$

Hence $Qx = 0$. Similarly we find $Px = 0$, and so $Rx = 0$. This proves (1.13).

By (1.13) and (1.3) we have $\tilde{F} \cap \tilde{G} \subset e^{-1}(0)$.

Defining $S = I - R$ and using (1.13), we get $\text{im } S = \tilde{F} \cap \tilde{G}$. Hence P, Q, S will determine a decomposition (1.12) as desired.

2.) Assume next that the hypotheses of the Corollary are satisfied. We define $R = P + Q = I - S$ and observe that

$$\ker R = \text{im } S = \tilde{F} \cap \tilde{G} \subset e^{-1}(0) . \tag{1.14}$$

From this we conclude that

$$e(Rx) = e(x - Sx) = e(x) \qquad \text{for } x \in V .$$

Since $R = P+Q$ is positive, we obtain $\|Rx\| = \|x\|$ for $x \in V^+$. Hence R is an R-projection. Also it follows from (1.14) that R is transversal at F and G.

Now the conditions (i), (ii), (iii) of Theorem 1.3 will be satisfied, and the proof is complete. □

The following result will be useful later.

Proposition 1.5. *If F and G are quasicomplementary projective faces of K, then*

$$(1.15) \qquad (\text{lin } F \oplus \text{lin } G) \cap K = \text{co}(F \cup G) .$$

Proof. To prove the non-trivial part of (1.15) we consider an arbitrary $x \in (\text{lin } F \oplus \text{lin } G) \cap K$. If P and Q are the P-projections of V corresponding to F and G, then $P+Q$ is an R-projection onto $\text{lin } F \oplus \text{lin } G$; hence $(P+Q)x = x$.

If $Px = 0$, then $x = Qx \in (\text{im } Q) \cap K = G$. Similarly, $Qx = 0$ implies $x \in F$. Thus, if $Px = 0$ or if $Qx = 0$, then $x \in \text{co}(F \cup G)$.

If $Px \neq 0$ and $Qx \neq 0$, then $\|Px\|$ and $\|Qx\|$ are non-zero positive numbers which by formula (2.7) of Part I will satisfy the equation

$$\|Px\| + \|Qx\| = (P^*e + Q^*e)(x) = 1 .$$

Writing $\|Px\| = \lambda$, $\|Qx\| = 1-\lambda$, $y = \|Px\|^{-1}Px$, $z = \|Qx\|^{-1}Qx$, we get a convex combination

$$x = \lambda y + (1-\lambda)z ,$$

where $y \in (\text{im } P) \cap K = F$ and $z \in (\text{im } Q) \cap K$. □

Corollary 1.6. *The R-projection $R = P + Q$ associated with a pair F, G of quasicomplementary projective faces will map K onto $co(F \cup G)$* .

Definitions. A map ρ of K onto a convex subset K' is said to be an affine retraction of K onto K' if it is affine and leaves K' pointwise invariant, i.e. if

$$(1.16) \qquad \rho(\lambda y + (1-\lambda)z) = \lambda\rho(y) + (1-\lambda)\rho(z) ,$$

for $y, z \in K$, $0 \leq \lambda \leq 1$, and if $\rho(x) = x$ for $x \in K'$. An affine retraction $\rho : K \to K'$ is said to be transversal at a convex subset F of K' if for $y, z \in K$, $\rho(y) = \rho(z)$ implies $y - z \in \tilde{F}$.

By definiton, an R-projection R of V onto a subspace M will determine an affine retraction $\rho = R|_K$ of K onto $M \cap K$; and transversality of R at $F \subset M \cap K$ will imply transversality of ρ at F . In particular, if F and G are quasicomplementary projective faces of K , then the corresponding R-projection $R = P + Q$ (cf. Theorem 1.3) will determine an affine retraction ρ of K onto $co(F \cup G)$ (cf. Corollary 1.6); and the affine retraction ρ will be transversal at F and G .

We shall now turn to an important special case where it is possible to extend affine retractions of K to R-projections of the surrounding space V and thereby obtain an intrinsic characterization of pairs of quasicomplementary projective faces in terms of geometric properties of the convex set K .

Theorem 1.7. *Let $F, G \subset K$ and suppose that $A \cong V^*$. Then F and G are quasicomplementary projective faces of K iff:*

(i) F and G are semi-exposed faces of K .

(ii) F and G are affinely independent.

(iii) There is an affine retraction ρ of K onto co(F ∪ G) which is transversal at F and G .

Moreover, if these conditions are satisfied, then there exists just one affine retraction ρ of K onto co(F ∪ G) ; specifically $\rho = (P+Q)|_K$ where P and Q are the P-projections corresponding to F and G .

Proof. 1.) The necessity of the above conditions (i), (ii), (iii) follows immediately from the corresponding statements of Theorem 1.3.

2.) To prove sufficiency, we assume that the requirements (i), (ii), (iii) above are satisfied, and we shall verify statement (iii) and then statement (ii) of Theorem 1.3.

We claim that if

(1.17) $$\lambda y - \mu z = \lambda' y' - \mu' z'$$

where $\lambda, \lambda', \mu, \mu' \geq 0$ and $y, y', z, z' \in K$, then

(1.18) $$\lambda\rho(y) - \mu\rho(z) = \lambda'\rho(y') - \mu'\rho(z') .$$

To prove this implication, we evaluate e at both sides of (1.17) and obtain $\lambda + \mu' = \lambda' + \mu$. We denote this common value by α and divide through by it in (1.17). Then we obtain an equality of two convex combinations:

$$\frac{\lambda}{\alpha} y + \frac{\mu'}{\alpha} z' = \frac{\lambda'}{\alpha} y' + \frac{\mu}{\alpha} z .$$

Since ρ is an affine map, we get

$$\frac{\lambda}{\alpha} \rho(y) + \frac{\mu'}{\alpha} \rho(z') = \frac{\lambda'}{\alpha} \rho(y') + \frac{\mu}{\alpha} \rho(z) ,$$

from which (1.18) follows.

An arbitrary $x \in V$ admits a decomposition $x = \lambda y - \mu z$ with $\lambda, \mu \geq 0$ and $y, z \in K$, and we write

$$Rx = \lambda\rho(y) - \mu\rho(z) . \tag{1.19}$$

Note that the map $R: V \to \mathrm{lin}(F \cup G)$ is well defined by virtue of (1.18), and also that R is an extension of ρ from K to the whole linear space V.

Clearly R is linear. Also $\|R\| \leq 1$, since we can choose the above decomposition $x = \lambda y - \mu z$ such that $\lambda + \mu = \|x\|$ and since $\|\rho(y)\| = \|\rho(z)\| = 1$. By the assumption $A \cong V^*$, a bounded linear operator on V is also continuous with respect to the weak topology determined by the duality of V and A. Hence R is weakly continuous.

If $x \in \mathrm{lin}(F \cup G) = \mathrm{lin}\, F + \mathrm{lin}\, G$, then we can write

$$x = \lambda y - \mu z + \lambda' y' - \mu' z' , \tag{1.20}$$

where $\lambda, \lambda', \mu, \mu' \geq 0$; $y, z \in F$; $y', z' \in G$. Now by linearity of R we get $Rx = x$, and we have thus shown that R is a projection of V onto $\mathrm{lin}(F \cup G)$.

Clearly $x \in V^+$ implies $Rx \in V^+$ and also

$$\|Rx\| = \|x\| \cdot \|\rho(\frac{x}{\|x\|})\| = \|x\| .$$

Thus, R is an R-projection.

In order to prove that R is transversal at F, we consider an arbitrary $x \in \ker R$ and decompose $x = \lambda y - \mu z$ where $\lambda, \mu \geq 0$ and $y, z \in K$. Now $0 = \lambda\rho(y) - \mu\rho(z)$. Evaluating e at the right hand side of this equation, we find $\lambda = \mu$. If $\lambda = \mu = 0$, then $x = 0 \in \tilde{F}$. Otherwise we conclude that $\rho(y) = \rho(z)$.

Since ρ is supposed to be transversal at F , this implies $y-z \in \tilde{F}$. Hence $x = \lambda(y-z) \in \tilde{F}$, and so we have proved that R is transversal at F . Similarly we prove that R is transversal at G .

We now turn to statement (ii) of Theorem 1.3. By the affine independence of F and G one has $(\text{lin } F) \cap (\text{lin } G) = \{0\}$ (cf. e.g. [A_1; Prop. II.6.1]), and hence

$$(1.21) \qquad \text{lin}(F \cup G) = (\text{lin } F) \oplus (\text{lin } G) .$$

Let the corresponding projections of $\text{lin}(F \cup G)$ onto $\text{lin } F$ and $\text{lin } G$ be P_o and Q_o , respectively. We shall prove that P_o and Q_o are positive.

To this end we consider an arbitrary $x \in K \cap \text{lin}(F \cup G)$. Then $x = Rx = \rho(x)$. Therefore $x \in \text{co}(F \cup G)$, say that

$$x = \lambda y + (1-\lambda)z ,$$

where $y \in F$, $z \in G$ and $0 \leq \lambda \leq 1$. By the uniqueness of decompositions with respect to $\text{lin } F$ and $\text{lin } G$ (cf. (1.21)), we must have $P_o x = \lambda y \in V^+$ and $Q_o x = (1-\lambda)z \in V^+$. This proves the positivity of P_o and Q_o .

Next observe that P_o and Q_o are bounded (by Lemma 1.1), and so they are weakly continuous. Now we have proved that $\text{lin}(F \cup G)$ splits continuously into $\text{lin } F$ and $\text{lin } G$.

3.) Finally we assume conditions (i) (ii) (iii) of the theorem and we consider an affine retraction σ of K onto $\text{co}(F \cup G)$. As above we extend σ to an R-projection $S : V \to \text{lin}(F \cup G)$, and we use Theorem 1.3 to conclude that $S = R$. Hence $\sigma = \rho$, and the proof is complete.

Theorem 1.7 is useful for identification of projective faces in special cases. As examples we shall determine pairs of quasi-complementary projective faces for a few 3-dimensional convex sets K . We shall think of K as base of a cone V^+ of positive elements in the (4-dimensional) space V , and we shall take A to be the (4-dimensional) space of affine functions on K with $e(x) = 1$ for all $x \in K$. (Cf. the example shown in Part I, Fig. 3). Now $A \cong V^*$ and Theorem 1.7 applies.

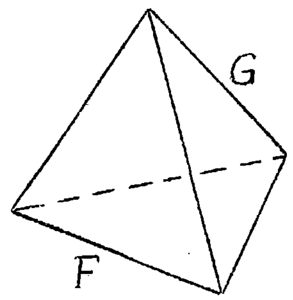

Fig. 1

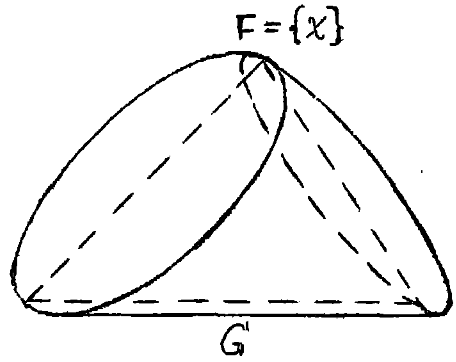

Fig. 2

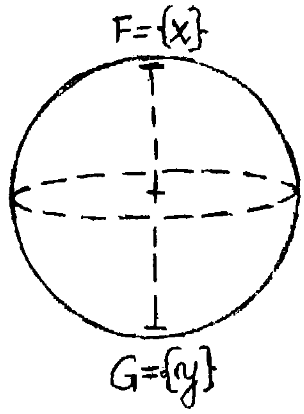

Fig. 3

Fig 1. shows a tetrahedron with two opposite edges F and G. They are quasicomplementary projective faces. (In fact, they are even complementary split faces). Here $co(F \cup G)$ is the whole set K , and the unique affine retraction of K onto $co(F \cup G)$ is the identity mapping.

Fig. 2 shows a section of a circular cylinder, determined by two planes intersecting the surface of the cylinder in two ellipses which at a point x have a common tangent perpendicular to the axis. Now the singleton $F = \{x\}$ and the opposite line segment G on the cylindrical surface will be quasicomplementary projective faces. Here $co(F \cup G)$ is the triangle spanned by F and G , and the unique affine retraction of K onto $co(F \cup G)$ is the orthogonal projection onto this triangle.

Fig. 3 shows a sphere with two antipodal points x and y . Now $F = \{x\}$ and $G = \{y\}$ will be quasicomplementary projective faces. Here $co(F \cup G)$ is the diameter between x and y , and the unique affine retraction of K onto $co(F \cup G)$ is the orthogonal projection onto this diameter.

§ 2. Spectral convex sets.

In this section we shall assume that K is a compact convex subset of a locally convex Hausdorff space E . We shall use the customary symbol A(K) to denote the space of all continuous affine functions on K and the symbol $A^b(K)$ to denote the space of all bounded affine functions on K . Also we shall denote by $A^{\uparrow}(K)$ (respectively $A^{\downarrow}(K)$) the subspace of $A^b(K)$ consisting of pointwise limits of increasing (decreasing) nets from A(K) . Note that these spaces are respectively the l.s.c (u.s.c.) affine functions on K : cf. e.g. [A_1, Cor. I.1.4]. The function $e \in A(K)$ is defined by $e(x) = 1$ for all $x \in K$.

We assume without lack of generality that K is regularly embedded in E (see [A_1; Ch II, § 2]). Then E can be identified with $A(K)^*$ endowed with the weak* topology, and every $x \in E$ can be written in the form

$$(2.1) \qquad x = \lambda y - \mu z ,$$

where $y,z \in K$ and $\lambda,\mu \in \mathbb{R}$. Also (E,K) will be a base-norm space and the identification of E with $A(K)^*$ will be an isometry.

Now every $a \in A^b(K)$ can be uniquely extended to a bounded linear functional on E , which we shall also denote by a . In fact, for every $x \in E$

$$(2.2) \qquad a(x) = \lambda a(y) - \mu a(z)$$

where x is given by (2.1), and it follows by an argument similar to the proof of the implication (1.17) => (1.18) that the extended function is well defined. Clearly every bounded linear functional on E restricts to a bounded affine function on K . Hence we can identify $A^b(K)$ with E^* and then in turn with $A(K)^{**}$.

The spaces $(A(K),e)$ and $(A^b(K),e)$ are seen to be order-unit spaces whose norms will coincide with the sup-norm for functions on K. Since the unit ball of the base-norm space E is given by $E_1 = co(K \cup -K)$, we obtain from (2.2) that for every $a \in A^b(K)$

$$(2.3) \qquad \|a\| = \sup_{x\in K}|a(x)| = \sup_{x\in E_1}|a(x)| \ .$$

From this it follows that the identification of the order-unit space $(A^b(K),e)$ with the dual of the base-norm space (E,K) will be an isometry.

The interrelationship between the various spaces can be summarized in the following diagram where $\cong$ denotes isometric isomorphism:

$$\underset{\text{(order-unit space)}}{e \in A(K)} \quad , \quad \underset{\text{(base-norm space)}}{K \subset E \cong A(K)^*} \quad , \quad \underset{\text{(order-unit-space)}}{e \in A^b(K) \cong E^* \cong A(K)^{**}}$$

The order-unit space $(A(K),e)$ and the base-norm space (E,K) will be in separating order and norm duality, and likewise for the order-unit space $(A^b(K),e)$ and the base-norm space (E,K). Of these two dualities the latter will be the most relevant for our investigations since $A^b(K)$ is pointwise monotone complete (and so requirement (3.1) of Part I is satisfied).

<u>Definition</u>. A convex compact set K is said to be <u>spectral</u> if $(A^b(K),e)$ and (E,K) are in spectral duality.

We now proceed to give examples of spectral convex sets, and we start with some auxiliary results.

Proposition 2.1. *The space E is order complete. More specifically, every descending net $\{x_\alpha\}$ from E^+ has a greatest lower bound x in E^+, and x is the limit of $\{x_\alpha\}$ in the norm of E.*

Proof. It follows by the identification of E with $A(K)^*$ that $\{x_\alpha\}$ has a greatest lower bound $x \in E^+$, which is also the weak* limit of $\{x_\alpha\}$. Since $e \in A(K)$ and $x_\alpha \geq x$ for all α, we obtain

$$\lim_\alpha \|x_\alpha - x\| = \lim_\alpha e(x_\alpha - x) = 0 ,$$

and the proposition is proved. □

Proposition 2.2. *If F is a split face of K, then F is a projective face of K in the duality of $(A^b(K),e)$ and (E,K); the quasicomplement $F^\#$ is equal to the customary complement F' (see e.g. [A_1; p. 132] for definition); the projective unit u associated with F is given by $u(x) = \lambda$ where $x \in K$ and λ is determined by the unique decomposition*

$$(2.4) \qquad x = \lambda y + (1-\lambda)z , \qquad \lambda \in [0,1], \; y \in F, \; z \in F' ;$$

and the P-projection $P : A^b(K) \to A^b(K)$ associated with F is given by $(Pa)(x) = \lambda a(y)$ for $a \in A^b(K)$.

Proof. Since $A^b(K) \cong E^*$, we can apply Theorem 1.7 to the faces F and F'. Defining $u(x) = \lambda$ where λ is given by (2.4), we obtain a function in $A^b(K)$ attaining its extreme values 1 and 0 at F and F' respectively. Hence F and F' are exposed faces, and requirement (i) of Theorem 1.7 is satisfied. By the definition of a split face, F and F' are affinely independent; hence (ii) is satisfied. Also $K = co(F \cup F')$, and clearly the identity mapping $\rho: K \to K$ is transversal at F and F'. Hence (iii) is satisfied. It follows that F and F' are quasicomplementary projective faces, and it is easily verified that the function u and the projection P defined in the proposition, will have the properties defining the projective unit and the P-projection associated with F. □

Remark. It is perhaps of interest to note that a projective face F is a split face of K iff the corresponding P-projection P of $A^b(K)$ is central (i.e. compatible with all $a \in A^b(K)$, cf. Part I, § 4). For if F is a split face, then Proposition 2.2 shows that P satisfies $P + P' = I$. Then $Pa + P'a = a$ for all $a \in A^b(K)$, so P is central. Conversely, if P is central then $P + P' = I$ follows, and this implies that K is the direct convex sum of $F = K \cap \operatorname{im} P^*$ and $F^\# = K \cap \operatorname{im} P'^*$, so F is a split face of K.

We recall that K is a Choquet simplex iff E is a vector lattice (Cf. e.g. [A_1; Ch II, § 3]). If K is a simplex, then $E \cong A(K)^*$ is known to be an L-space (cf. e.g. [Sem]), and so $A^b(K) = E^*$ is an M-space. In particular, $A^b(K)$ is a vector lattice, and for every $a \in A^b(K)$ the positive and negative parts of a in the vector lattice $A^b(K)$ are given by the following formulas for $x \in E^+$:

$$(2.5) \qquad a^+(x) = \sup\{a(y) \mid 0 \leq y \leq x\},$$

$$(2.6) \qquad a^-(x) = -\inf\{a(z) \mid 0 \leq z \leq x\}.$$

(Cf, e.g. [KN; Prop. 23.9]).

Lemma 2.3. Suppose that K is a Choquet simplex. If $a \in A^b(K)$, then for every $x \in E^+$ there exists a decomposition $x = y + z$ where $y, z \in E^+$, such that

(2.7) $$a^+(x) = a(y)\ ,\quad a^-(x) = -a(z)$$

Proof. By (2.5) we can find a sequence $\{y_n\}$ such that $0 \leq y_n \leq x$ and

(2.8) $$a^+(x) - a(y_n) < 2^{-n}\ .$$

Now we observe that for any two elements $y, y' \in E^+$ such that $y \leq x$ and $y' \leq x$:

(2.9) $$a(y) - a(y \wedge y') = a(y \vee y') - a(y') \leq a^+(x) - a(y')$$

Applying (2.9) with $y = y_n$ and $y' = y_{n+1}$ we obtain from (2.8)

$$a(y_n) - a(y_n \wedge y_{n+1}) \leq 2^{-n-1}\ ,$$

and then in turn

(2.10) $$a^+(x) - a(y_n \wedge y_{n+1}) \leq 2^{-n} + 2^{-n-1}\ .$$

Next we may apply (2.9) with $y = y_n \wedge y_{n+1}$ and $y' = y_{n+2}$, and then proceed by induction to get

(2.11) $$a^+(x) - a(y_n \wedge \ldots \wedge y_{n+k}) \leq 2^{-n} + 2^{-n-1} + \ldots + 2^{-n-k}$$

for $k = 1,2,\ldots$.

We denote by u_n the greatest lower bound of $\{y_n \wedge \ldots \wedge y_{n+k}\}_{k=1,2,\ldots}$. By Proposition 2.1 this sequence converges to u_n in the norm of E , and since $a \in A^b(K)$ is norm continuous, we get from (2.11)

(2.12) $$a^+(x) - a(u_n) \leq 2^{-n+1}\ , \qquad n = 1,2,\ldots\ .$$

Finally we denote by y the least upper bound of the increasing sequence $\{u_n\}$. Then $0 \leq y \leq x$, and by (2.12) $a^+(x) = a(y)$.

Now we write $z = x - y \geq 0$. Then by (2.6)

$$a(x) = a^+(x) - a^-(x) \leq a(y) + a(z) = a(x) .$$

Here the equality sign must hold throughout. Hence $-a^-(x) = a(z)$, and the lemma is proved. □

We remark for later reference that if we assume $x \in K$ in Lemma 2.3 and if we normalize y and z , i.e. if we replace the original vectors by $\lambda^{-1}y'$ and $(1-\lambda)z'$ where $\lambda = \|y\| = e(y)$, then we obtain a convex combination

$$x = \lambda y' + (1-\lambda)z' \tag{2.13}$$

where $y',z' \in K$, and where by (2.5), (2.6):

$$\begin{cases} a^+(y') = a(y') \ , & a^-(y') = 0 \ , \\ a^+(z') = 0 & , \ a^-(z') = -a(z') \ . \end{cases} \tag{2.14}$$

We also recall that a face of a Choquet simplex K is split iff it is norm closed. [As-El] . From this it follows by Proposition 2.2 that the projective faces of a Choquet simplex K (in the duality of $(A^b(K),e)$ and (E,K)) are exactly the norm closed ones.

<u>Theorem 2.4</u>. <u>Every Choquet simplex is spectral</u>.

<u>Proof</u>. Assuming K to be a Choquet simplex we shall prove that $(A^b(K),e)$ and (E,K) are in spectral duality. The space $A^b(K)$ is pointwise monotone complete, so requirement (3.1) of Part I is satisfied. Clearly every exposed face of K is norm closed, hence projective, so requirement (3.2) of Part I is also satisfied.

We shall prove that $a^+ \perp a^-$ for every $a \in A^b(K)$ which will guarantee weak spectral duality by virtue of Proposition 6.1 of Part I.

For given $a \in A^b(K)$ we define the two norm closed faces

$$(2.15) \qquad F = \{x \in K \mid a^+(x) = 0\} \ , \quad G = \{x \in K \mid a^-(x) = 0\}$$

By the above remarks F and G are projective (in fact "split"). Now it follows from the equivalence (3.20) of Part I that

$$F = \{x \in K \mid rp(a^+)(x) = 0\} \ , \quad G = \{x \in K \mid rp(a^-)(x) = 0\} \ .$$

Hence $rp(a^+)$ is the projective unit corresponding to the projective face $F^\# = F'$. Similarly $rp(a^-)$ is the projective unit corresponding to $G^\# = G'$.

By (2.13) and (2.14) $K = co(F \cup G)$. This implies $F' \cap G' = \emptyset$ (cf. e.g. [A_1; Prop. II. 6.7]) . By the definition of the complement of a face of a convex set, we get $F' \subset (G')' = (G')^\#$. Hence $F' \perp G'$, and so $rp(a^+) \perp rp(a^-)$. Thus we have proved $a^+ \perp a^-$.

It remains to prove that $(A^b(K),e)$ and (E,K) are in spectral duality and not only in weak spectral duality. In this connection we observe that it follows from the explicit form of the P-projection associated with a given projective (and "split") face F of K that every P-projection on $A^b(K)$ is compatible with every element of $A^b(K)$ (cf. Proposition 2.2). This implies that every P-projection belongs to the $\mathcal{P}$-bicommutant of a given element of $A^b(K)$. By Corollary 6.7 of Part I, this completes the proof. □

<u>Theorem 2.5. If K is affinely isomorphic to the unit ball of $L^p(\mu)$ for $1 < p < \infty$, then K is spectral.</u>

Proof. Let K be as announced above. In particular K is centrally symmetric, and we shall denote the point which is opposite to a given point x by x'. Again the space $A^b(K)$ is pointwise monotone complete, so requirement (3.1) of Part I is satisfied.

By elementary properties of $L^p(\mu)$ the only faces of K are the ones of the form $\{y\}$ where y is an extreme point (K is "strictly convex"). Also we note that for every extreme point y of K there exists a bounded affine function a on K attaining its supremum-value at y and that a is unique up to a constant factor and an additive constant; or otherwise stated that K admits a unique supporting hyperplane at y (K is "smooth"). (For proofs, see e.g. [Kø; p. 351]).

Now it is seen by elementary arguments similar to the ones used in the proof of Proposition 6.9 of Part I, that every face $F = \{y\}$ of K is projective with $F^{\#} = \{y'\}$, that the associated projective unit u is the unique element of $A^b(K)$ which attains its supremum-value 1 at y and its infimum-value 0 at y', and that the associated P-projection is given by

$$P^*x = u(x)y\,, \qquad \text{all } x \in E\,. \tag{2.16}$$

From this it follows that an element of $A^b(K)$ is compatible with P iff it is of the form $\alpha u + \beta$ for $\alpha, \beta \in \mathbb{R}$.

We now consider an arbitrary $a \in A^b(K)$ and define $\alpha = \inf_{x\in K} a(x)$ and $\beta = \sup_{\alpha \in K} a(x)$. We know that there exist points $y, y' \in K$ such that $a(y) = \alpha$ and $a(y') = \beta$ (weak compactness of the unit ball of $L^p(\mu)$; note also that y and y' are unique and that y' is the opposite of y). By the above remarks $\emptyset, \{y\}, \{y'\}, K$ are the only projective faces of K compatible with a. Therefore there exists for every $\lambda \in \mathbb{R}$ a unique projective face F compatible with a such that $a \leq \lambda$ on F and $a > \lambda$ on $F^{\#}$. In fact, $F = \emptyset$ if $\lambda < \alpha$, $F = \{y\}$ if $\alpha \leq \lambda < \beta$, and $F = K$ if $\lambda \geq \beta$. This shows that $(A^b(K),e)$ and (E,K) are in spectral duality, and the proof is complete. □

Note that the above proof will go through under much more general hypotheses. The essential requirements are that K shall be strictly convex and smooth, and that every $a \in A^b(K)$ shall attain its infimum-value. (By James' Theorem [Ja] the latter requirement is equivalent to compactness of K in the weak topology $\sigma(E,E^*) = \sigma(E,A^b(K)))$.

Throughout this section we have assumed that E is endowed with a locally convex Hausdorff topology in which K is compact (the weak*-topology when E is identified with $A(K)^*$). So far we have made no use of this assumption since we have only had to work in the norm-topology and in the weak topology defined on E by the duality with $A^b(K) \cong E^*$. But in the remaining part of this section we shall study properties related to the given (weak*-) topology on K. In this connection we shall need the spaces $A^{\uparrow}(K)$ and $A^{\downarrow}(K)$ defined in the beginning of this section.

<u>Definition</u>. A spectral convex compact set is said to be <u>strongly spectral</u> if for every $a \in A(K)$ the spectral units e_λ^a all satisfy the requirement $e_\lambda^a \in A^{\downarrow}(K)$.

For the proof of the next theorem we remark that $A^{\downarrow}(K)$ is closed under addition and under multiplication by positive real numbers, and also that it is closed under pointwise limits of bounded descending nets. The same statement will hold for $A^{\uparrow}(K)$ if the word "descending" is replaced by "ascending".

Theorem 2.6. Let K be a spectral convex compact convex set. Then the following statements are equivalent:

(i) K is strongly spectral.

(ii) $\chi_U(a) \in A^{\uparrow}(K)$ for all open sets $U \subset \mathbb{R}$ and all $a \in A(K)$.

(iii) $\varphi(a) \in A^{\uparrow}(K)$ for all bounded lower semi-continuous functions φ on $\mathbb{R}$ and all $a \in A(K)$.

(iv) $\varphi(a) \in A(K)$ for all $\varphi \in C(\sigma(a))$ and all $a \in A(K)$.

Proof. (i) => (ii) We first consider an open interval $\langle \alpha, \beta \rangle$ and a function $a \in A(K)$. By Proposition 7.6 of Part I

$$\chi_{\langle \alpha, \infty \rangle}(a) = (\iota - \chi_{\langle -\infty, \alpha]})(a) = e - e_{\lambda}^{a} \in A^{\uparrow}(K) ,$$

and

$$\chi_{[\beta, \infty\rangle}(a) = [\chi_{\langle -\infty, 0]} \circ (\beta\iota - \gamma)](a) = \chi_{\langle -\infty, 0]}(\beta e - a) = e^{\beta e - a} \in A^{\downarrow}(K) .$$

Hence

$$\chi_{\langle \alpha, \beta \rangle}(a) = \chi_{\langle \alpha, \infty \rangle}(a) - \chi_{[\beta, \infty\rangle}(a) \in A^{\uparrow}(K) .$$

An arbitrary open set $U \subset \mathbb{R}$ can be written as a disjoint union $\bigcup_{i=1}^{\infty} \langle \alpha_i, \beta_i \rangle$, and by writing $\varphi_n = \sum_{i=1}^{n} \chi_{\langle \alpha_i, \beta_i \rangle}$, we obtain $\varphi_n \nearrow \chi_U$.

It follows that $\varphi_n(a) \nearrow \chi_U(a)$. By the result just proved, $\varphi_n(a) \in A^{\uparrow}(K)$. Hence also $\chi_U(a) \in A^{\uparrow}(K)$.

(ii) => (iii) We next consider a bounded l.s.c. function φ on $\mathbb{R}$ and a function $a \in A(K)$.

Let $\alpha < \inf_{\xi \in \mathbb{R}} \varphi(\xi) < \beta$, and let $\gamma = \{\lambda_i\}_{i=0}^{n}$ be a partition of $[\alpha, \beta]$, i.e. $\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_n = \beta$. By lower semi-continuity the sets $U_i = \varphi^{-1}(\langle \lambda_i, \infty \rangle)$ are open for $i = 1, \dots, n$.

We now define

$$\psi_\gamma = \alpha\iota + \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})\chi_{U_i} .$$

By virtue of (i), $\psi_\gamma(a) \in A^\uparrow(K)$. Clearly $\psi_\gamma \nearrow \varphi$ (even uniformly) as the partition is being refined. Hence $\varphi(a) \in A^\uparrow(K)$.

(iii) => (iv) If φ is a bounded continuous function on $\mathbb{R}$ and $a \in A(K)$, then we may apply (iii) to φ and $-\varphi$, and obtain $\varphi(a) \in A^\uparrow(K) \cap A^\downarrow(K)$. Therefore $\varphi(a)$ is both upper and lower semi-continuous, and so $\varphi(a) \in A(K)$. Clearly this result will remain valid if φ is defined only on $\sigma(a)$.

(iv) => (i) For given $\lambda \in \mathbb{R}$ and $a \in A(K)$ we consider a sequence $\{\varphi\}$ of continuous functions on $\mathbb{R}$ such that $\varphi_n \searrow \chi_{\langle-\infty,\lambda]}$. Then $\varphi_n(a) \in A(K)$ by virtue of (iv), and it follows that $e_\lambda^a = \chi_{\langle-\infty,\lambda]}(a) \in A^\downarrow(K)$. This completes the proof. $\square$

We now return to spectral convex sets of the two types discussed in Theorem 2.4 and Theorem 2.5. In this connection we recall that if K is a Choquet simplex, then every point x of K is barycenter of a unique positive (in fact, probability-) "boundary" measure μ_x ("boundary" means "maximal in the Choquet ordering"). If K is metrizable, then the "extreme boundary" $\mathscr{E}(K)$ (i.e. the set of extreme points) is a G_δ-subset of K , and a positive measure μ on K is a boundary measure iff $\mu(K \setminus \mathscr{E}(K)) = 0$. Recall also that the mapping $x \to \mu_x$ is Borel with respect to the given compact topology of K and the weak* (or "vague") topology of the space M(K) of measures on K , and that this mapping is continuous iff $\mathscr{E}(K)$ is closed. (See [A_1; Ch II, §§ 3,4] for proofs).

We shall also need the following elementary result, which we state as a lemma for later references.

Lemma 2.7. *If B is a Borel subset of a Choquet simplex K, then $F = \{x \in K \mid \mu_x(B) = 1\}$ and $G = \{x \in K \mid \mu_x(B) = 0\}$ are complementary split faces of K.*

Proof. Clearly F and G are faces of K. For every $x \in K \setminus (F \cup G)$ we write $\lambda = \mu_x(B) \in \langle 0,1 \rangle$, and we denote by y the barycenter of $\lambda^{-1}\mu_x|_B$ and by z the barycenter of $(1-\lambda)^{-1}\mu_x|_{(K \setminus B)}$. Then it is easily verified that

$$x = \lambda y + (1-\lambda)z$$

is the unique decomposition of x as a convex combination of a point in F and a point in G. This completes the proof. $\square$

Proposition 2.8. *Let K be a Choquet simplex and let $a \in A(K)$. Then for every bounded Borel function φ and every $x \in K$:*

$$(2.17) \qquad \varphi(a)(x) = \int (\varphi \circ a) d\mu_x = \int \varphi \, d(a\mu_x)$$

Otherwise stated: We obtain the (scalar valued) spectral measure for a at the point x by transporting the measure μ_x to the real line by means of a.

Proof. The last equality of (2.17) is merely the definition of $a\mu_x$.

To prove the first equality of (2.17), we consider the mapping $\theta : \mathcal{B}(\sigma(a)) \to A^b(K)$ defined by

$$(2.18) \qquad (\theta\varphi)(x) = \int \varphi \, d(a\mu_x) \qquad \text{for } x \in K.$$

Clearly θ is a morphism, and we note that

$$(2.19) \qquad (\theta\gamma)(x) = \int \gamma \, d(a\mu_x) = \int a \, d\mu_x = a(x) \qquad \text{for } x \in K .$$

Hence θ satisfies requirement (i) of Theorem 7.9 of Part I.

To verify requirement (ii) of this theorem, we consider an arbitrary Borel set $E \subset \mathbb{R}$ and evaluate $\theta\chi_E$ for an arbitrary $x \in K$:

$$(\theta\chi_E)(x) = \int \chi_E \, d(a\mu_x) = (a\mu_x)(E) = \mu_x(a^{-1}(E)) .$$

Now it follows by Proposition 2.2 and Lemma 2.7 that $\theta\chi_E$ is the projective unit corresponding to the projective (and in fact "split") face $F = \{x \in K \mid \mu_x(a^{-1}(E)) = 1\}$. Thus, $\theta\chi_E$ is an extreme point of $[0,e]$ (by Proposition 7.7 of Part I), and requirement (ii) of Theorem 7.9 of Part I is satisfied. By the uniqueness statement of this theorem, $\theta\varphi = \varphi(a)$ for every $\varphi \in \mathcal{B}(\sigma(a))$, and the proof is complete. $\square$

Note that in the above proof the assumption $a \in A(K)$ was used only once, to permit the transition from $\int a \, d\mu_x$ to $a(x)$ in (2.19). Hence (2.17) will remain valid for all Borel functions in $A^b(K)$ satisfying the barycentric calculus.

Note also that by (2.17) we obtain for $a \in A(K)$:

$$(2.20) \qquad e^a_\lambda(x) = (\chi_{\langle-\infty,\lambda]}a)(x) = (a\mu_x)(\langle-\infty,\lambda]) = \mu_x(a^{-1}(\langle-\infty,\lambda])),$$

from which it follows that the corresponding projective face F_λ is given by:

$$(2.21) \qquad x \in F_\lambda \iff \mu_x(a^{-1}(\langle-\infty,\lambda])) = 1$$

<u>Proposition 2.9.</u> <u>A simplex K is strongly spectral iff $\mathcal{E}(K)$ is closed. (K is a "Bauer simplex" in the terminology of $[A_1]$).</u>

Proof. 1.) We assume first that $\mathcal{E}(K)$ is closed. Then the mapping $x \to \mu_x$ is continuous from the given compact topology of K to the weak* topology of measures [A_1; Th. II. 4.1]. If $a \in A(K)$ and if φ is a continuous real function, then $\varphi \circ a$ is continuous, and therefore $\varphi(a) : x \to \mu_x(\varphi \circ a)$ will be continuous. Hence, K is strongly spectral.

2.) We next assume that K is strongly spectral, and we note that for an arbitrary simplex K the extreme points x are exactly those for which:

$$a^{(2)}(x) = a(x)^2 \qquad \text{for all } a \in A(K) \tag{2.22}$$

In fact, if $x \in \mathcal{E}(K)$, then by (2.17):

$$a^{(2)}(x) = \int a(\lambda)^2 \, d\varepsilon_x(\lambda) = a(x)^2 .$$

On the other hand, if $x \notin \mathcal{E}(K)$, then the support of μ_x will consist of more than one point. Since A(K) separates the points of K, the support of $a\mu_x$ will also consist of more than one point for a suitable $a \in A(K)$. Hence

$$a^{(2)}(x) = \int \lambda^2 \, d(a\mu_x) > \left(\int \lambda \, d(a\mu_x)\right)^2 = a(x)^2 .$$

By (2.22) we can express $\mathcal{E}(K)$ by the formula:

$$\mathcal{E}(K) = \bigcap_{a \in A(K)} \{x \in K \mid a^{(2)}(x) - a(x)^2 = 0\} ,$$

and it follows from the continuity of $a^{(2)}$ that $\mathcal{E}(K)$ is closed. □

Proposition 2.10. If K is affinely isomorphic to the unit ball of $L^p(\mu)$ for $1 < p < \infty$, then K is strongly spectral.

Proof. It follows from the reflexivity of $L^p(\mu)$ that $A(K) = A^b(K)$ in the present case. Therefore $e_\lambda^a \in A(K) = A^{\downarrow}(K)$ for every $a \in A(K)$. □

We close this section by some examples of two and three dimensional spectral convex compact sets. The two convex sets shown in Fig. 1 and Fig. 3, both have the property that all their faces are projective. By Theorem 6.14 of Part I, they are spectral. (This will of course also follow directly from the general results of Theorem 2.4 and Theorem 2.5 of this section). But the convex set shown in Fig. 2 will not be spectral since it has plenty of exposed non-projective faces.

As a new example, consider any smooth strictly convex two dimensional compact convex set K (e.g. Fig. 4).

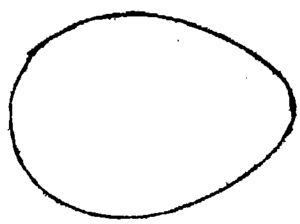

Fig. 4

By the remarks after Theorem 2.5 (or else by direct application of Theorem 1.7) K is seen to be spectral. In fact, it is not difficult to show that the only spectral two dimensional compact convex sets are the 2-simplex (the triangle) and the smooth strictly convex sets.

Fig. 5 below shows a circular cone K. The only proper faces of this set are:

(i) The top-point and the base. These are quasicomplementary projective faces (in fact complementary split faces).

(ii) The extreme points of the base and the extreme rays of the cone. When "diametrically opposite" one such point and one such ray will form a quasicomplementary pair of projective faces. (The properties (i)-(iii) of Theorem 1.7 are verified in the same way as for the faces F,G of Fig. 2).

By Theorem 6.14 of Part I the circular cone K of Fig. 2 is a spectral convex set.

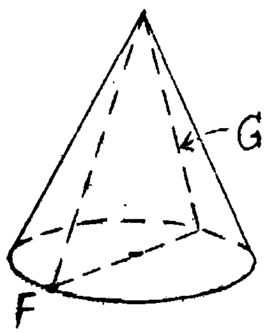

Fig. 5

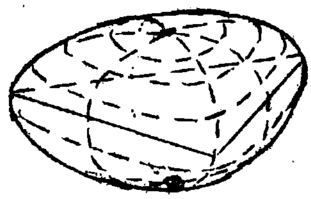

Fig. 6

Fig. 6 shows a convex set which combines "simplicial" and "rotund" features in a slightly less trivial way than the cone. This set may be thought of as a "compressed ball" with a "triangular equator", and a concrete model may be obtained by inflating a balloon which is initially spanned over a triangular frame. (We will not attempt here to give an analytical expression for such a surface).

The proper faces of this convex set K are:

(i) The extreme points off the triangle. They are all smooth points and will admit unique antipodal points, which are also off the triangle. (Antipodal points have parallel tangent planes). Two antipodal extreme points P,P' off the triangle will form a quasicomplementary pair of projective faces. (Use Theorem 1.7).

(ii) The edges and the vertices of the triangle. They will also form pairs of quasicomplementary projective faces. Specifically, the quasicomplement of an edge is the opposite vertex and vice versa. (Use Theorem 1.7 once more).

Again it follows from Theorem 6.14 of Part I that the "compressed ball" of Fig. 6 is a spectral convex compact set.

Finally we note that finite dimensional spectral compact convex sets are necessarily strongly spectral, since reflexivity yields $A(K) = A^b(K) = A^{\downarrow}(K)$.

§ 3. Applications to operator algebras.

In this section we shall specialize the general theory to the context of operator algebras. There will not be any new results on operator algebras in this section, and the presentation will be based on standard reference works such as $[D_2]$ and [Sa].

We will begin with von Neumann algebras and then pass to C^*-algebras which will be studied by means of their enveloping von Neumann algebras.

For the moment we assume that (A,e) is the self-adjoint part of a von Neumann algebra $\mathfrak{A}$ with identity e organized to an order-unit space in the usual way. Also we shall denote by (V,K) the self-adjoint part of the predual space $\mathfrak{A}_*$ organized to a base-norm space in the usual way. (Recall that $\mathfrak{A}_*$ consists of ultraweakly continuous linear functionals and that K consists of ultraweakly continuous, or "normal", states). By elementary properties of von Neumann algebras the spaces (A,e) and (V,K) will be in separating order and norm duality, and note that (A,e) will be isomorphic to $(V,K)^* \cong A^b(K)$. Note also that the weak* topology on A determined by the duality with V will be the same as the ultraweak topology, and recall that multiplication in $\mathfrak{A}$ is separately continuous in each variable with respect to the ultraweak topology (see e.g. [Sa; 1.8.5]). From this it follows that $a \to b^*ab$ is a weak* continuous map of A into itself for each $b \in \mathfrak{A}$. Recall also that an ascending net from A will converge pointwise (on K) to an element of A iff it converges in the weak* topology. Hence the requirement (3.1) of § 3 of Part I is satisfied.

In the sequel we shall sometimes think of the elements of V as functions on the von Neumann algebra $\mathfrak{A}$ (or on its self-adjoint part A), and sometimes we shall think of the elements of A as functions on the pre-dual $\mathfrak{A}_*$ (or on the state space $K \subset \mathfrak{A}_*$) . Therefore we find it convenient to use the symmetric notation $\langle a,x \rangle$ from the first section of Part I.

For later references we recall that by application of the Cauchy-Schwartz inequality to the semi-inner product defined on $\mathfrak{A}$ by a state x (i.e. $(a,b) \to \langle b^*a,x \rangle$) , we obtain the following well known formula valid for all $a \in \mathfrak{A}^+$ and $b \in \mathfrak{A}$:

$$(3.1) \qquad |\langle ba,x \rangle|^2 = |\langle (a^{\frac{1}{2}}b^*)^*a^{\frac{1}{2}},x \rangle|^2 \leq \langle a,x \rangle \langle bab^*,x \rangle$$

Proposition 3.1. A map $P: A \to A$ is a P-projection iff it is of the form

$$(3.2) \qquad Pa = pap$$

for a (self-adjoint) projection $p \in \mathfrak{A}$, and in this case $P'a = (e-p)a(e-p)$.

Proof. 1.) Assume first that P is of the form (3.2) for some projection $p \in \mathfrak{A}$ and let the map $Q: A \to A$ be defined by $Qa = (e-p)a(e-p)$. Clearly P is positive, weak* (=ultraweakly) continuous, $P = P^2$, and $\|P\| \leq 1$. Similarly for Q .

If $a \in \ker^+P$ then $a \in A^+$ and $pap = 0$. By (3.1) $\langle pa,x \rangle = 0$ for all $x \in K$, and so $pa = 0$ and $ap = (pa)^* = 0$. Hence

$$Qa = a - pa - ap + pap = a ,$$

and so $a \in \operatorname{im}^+Q$.

On the other hand, $a \in \operatorname{im}^+P$ means $pap = a$, which implies

$$Qa = (e-p)a(e-p) = (e-p)pap(e-p) = 0 ,$$

and so $a \in \ker^+ Q$.

Hence $\ker^+ P \subset \mathrm{im}^+ Q$ and $\mathrm{im}^+ P \subset \ker^+ Q$. The opposite relations follow by interchanging P and Q . Hence P and Q are quasi-complementary.

By Proposition I.2.4 it remains to prove that P^* and Q^* are neutral. We shall do this for Q^* . The verification for P^* is similar.

Let $x \in K$ and assume that $\|Q^* x\| = 1$. Then $1 = \langle e, Q^* x\rangle = \langle Qe, x\rangle = \langle e-p, x\rangle$. Hence $\langle p, x\rangle = 0$, and by application of (3.1) we find for all $a \in A^+$:

$$\begin{aligned}\langle a,x\rangle &= \langle pap,x\rangle + \langle pa(e-p),x\rangle + \langle (e-p)ap,x\rangle + \langle (e-p)a(e-p),x\rangle \\ &= \langle (e-p)a(e-p),x\rangle = \langle Qa,x\rangle = \langle a,Q^* x\rangle .\end{aligned}$$

Hence $x \in \mathrm{im}^+ Q^*$, and the first part of the proof is complete.

2.) Assume next that $P: A \to A$ is a P-projection, and let $p = Pe$. By Corollary I.2.12, p is an extreme point of $[0,e]$. But the extreme points of the positive part of the unit ball of a von Neumann algebra are known to be projections, and the P-projection defined on A by the formula $a \to pap$ is seen to have the same positive image as the given P-projection P (Cor. I.2.11). Therefore the two P-projections must coincide (Cor. I.2.9). This completes the proof. □

<u>Corollary 3.2</u>. <u>The projective units in A are precisely the projections in $\mathfrak{A}$</u> .

<u>Proof</u>. By definition the projective units are the elements $Pe = pep = p$ where p is a projection in $\mathfrak{A}$. □

<u>Corollary 3.3. An element $a \in A$ is compatible with a P-projection $P: a \to pap$ iff it commutes with the projection $p \in \mathfrak{A}$.</u>

<u>Proof</u>. By definition a is compatible with P iff $a = Pa + P'a$, or equivalently

$$(3.3) \qquad a = pap + (e-p)a(e-p) .$$

Generally

$$a = pap + (e-p)ap + pa(e-p) + (e-p)a(e-p) ;$$

hence $pa = ap$ will imply (3.3).

Conversely, from (3.3) we obtain $pa = pap$ and $ap = pap$ by multiplication by p from the left and the right; hence $pa = ap$. □

From Corollary 3.3 it follows that the $\mathcal{P}$-bicommutant of an element $a \in A$ consists of all P-projections $P: a \to pap$ such that p is a projection in the (relative) bicommutant of a in $\mathfrak{A}$.

<u>Proposition 3.4</u>. <u>If (A,e) and (V,K) are defined as above, then they are in spectral duality, and the functional calculus in A coincides with the customary functional calculus for self-adjoint elements of $\mathfrak{A}$</u> .

<u>Proof</u>. Let $a \in A$, $\lambda_o \in \mathbb{R}$, and let e_{λ_o} be the spectral projection of the self-adjoint element a of $\mathfrak{A}$ for the value λ_o. By elementary theory of von Neumann algebras, e_{λ_o} is in the bicommutant of a in $\mathfrak{A}$, and by Corollary 3.2 and Corollary 3.3 above e_{λ_o} is a projective unit in A bicompatible with the given element a. We denote by F the projective face of K which corresponds to the projective unit e_{λ_o}.

If $x \in F$, then $\langle e_{\lambda_0}, x\rangle = 1$, and so

$$\langle a,x\rangle = \int \lambda \, de_\lambda(x) \leq \lambda_0 .$$

If $x \in F^\#$, then $\langle e_{\lambda_0}, x\rangle = 0$. This implies that the measure on $\mathbb{R}$ with distribution $\lambda \to \langle e_\lambda, x\rangle$ has no mass in the interval $\langle -\infty, \lambda_0]$, and therefore

$$\langle a,x\rangle = \int \lambda \, de_\lambda(x) > \lambda_0 .$$

Thus by Proposition I.5.2 every exposed face of K is projective, so the requirement (3.2) of Part I is satisfied. Now by Corollary I.6.7 (A,e) and (V,K) will be in spectral duality.

Clearly the functional calculus defined by this spectral duality will be the customary functional calculus, since the spectral units are precisely the ordinary spectral projections. □

We are now going to look more closely into the geometry of state spaces, and we shall need some slightly less elementary results on operator algebras.

If F is a norm closed face of the normal state space of a von Neumann algebra $\mathfrak{A}$, then there exists a (unique) projection $p_F \in \mathfrak{A}$, called the <u>carrier projection</u> of F, such that

(3.4) $$F = \{x \in K \mid \langle p_F, x\rangle = 1\} .$$

The idea of the proof of this result is to show that the ultra-weakly closed left ideal

(3.5) $$J_F = \{a \in \mathfrak{A} \mid \langle a^*a, x\rangle = 0 \quad \text{all} \quad x \in F\}$$

contains a maximal projection q, which will be a right unit for J_F (the proof of this fact is essentially due to Kaplansky [Kap]), and then to verify that $p_F = e - q$ has the desired property (3.4). (For details, see Prosser's memoir [P]) .

The theorem below will describe the facial structure of K as far as the norm closed faces are concerned. In this theorem, as in the rest of this section, the terms "projective unit", "projective face" etc. are used with reference to the duality of $(A^b(K),e) \cong (\mathfrak{A}_{sa},e)$ and $(V,K) \cong ((\mathfrak{A}_*)_{sa},K)$.

Theorem 3.5. Let K be the normal state space of a von Neumann algebra $\mathfrak{A}$ and let F be a norm closed face of K. Then F is a projective face of K; the associated projective unit is the carrier projection p_F (considered as an element of $A^b(K)$) ; the associated P-projection P is given by $Pa = p_F a p_F$ for $a \in \mathfrak{A}_{sa}$ $(\cong A^b(K))$; the quasicomplement $F^\#$ consists of all $x \in K$ such that $\langle p_F, x\rangle = 0$; and the unique affine retraction ρ of K onto $co(F \cup F^\#)$ is given by

$$(3.6) \qquad \langle a,\rho x\rangle = \langle pap + (e-p)a(e-p), x\rangle \qquad \text{all } a \in \mathfrak{A}_{sa}$$

Moreover, $F \to p_F$ is a bijection of the set of norm closed faces of K onto the set of projections in $\mathfrak{A}$, which maps split faces onto central projections.

Proof. By the existence of carrier projections for norm closed faces and by use of Proposition 3.1 and Corollary 3.2 we conclude that every norm closed face F of K is projective and that the associated projective unit, P-projection and quasicomplement are of the form described in the present theorem. Generally the affine retraction $\rho : K \to co(F \cup F^\#)$ is given by $\rho = P^* + P'^*$;

hence (3.6) follows by use of the formula $Pa = p_F a p_F$ and the corresponding formula for P' .

The map $F \to p_F$ is injective (cf. (3.4)), and it is also surjective since every projection $p \in \mathfrak{A}$ determines a norm closed face $F = \{x \in K | \langle p,x \rangle = 1\}$, and we see that $p_F = p$.

As we remarked after Proposition 2.2, split faces of K correspond to central projections, and thus to projective units compatible with all elements of $A^b(K)$. By Corollary 3.3 these are precisely the central projections in $\mathfrak{A}$, and the proof is complete. □

<u>Remark.</u> If $\mathfrak{A}$ is acting on a Hilbert space H , and if H_1 and H_2 are range and null space of a projection $p \in \mathfrak{A}$, then every $a \in \mathfrak{A}$ can be represented by a 2×2-matrix (a_{ij}) where a_{ij} is a bounded linear operator from H_i into H_j . Now the element $pap + (e-p)a(e-p)$ is represented by the matrix obtained from the former by cancellation of the "cross terms" a_{12} and a_{21} . By Theorem 3.5 the dual of this "cancellation map" is the only possible affine retraction of the state space K onto $co(F \cup F^{\#})$ where $F = \{x \in K | \langle p,x \rangle = 1\}$ and $F^{\#} = \{x \in K | \langle p,x \rangle = 0\}$. This result (or rather its dual version) can also be obtained as a special case of a uniqueness theorem for conditional expectations on $\mathfrak{A}$ (see [Ar; Th. 6.22] for a specific reference). In fact the theory of conditional expectations on von Neumann algebras grew out of "diagonal processes" like the maps $a \to \sum_{i=1}^{n} p_i e p_i$ where $\{p_1,\dots,p_n\}$ is a set of orthogonal projections with $\sum_{i=1}^{n} p_i = e$. (Cf. e.g. [Neu_2] and [KS]) .

After this brief account of von Neumann algebras we pass to the C^*-algebra case. From now on we assume that (A,e) is the self-adjoint part of a C^*-algebra $\mathfrak{A}$ with identity e organized to an order-unit space in the usual way, and that (V,K) is the self-adjoint part of the dual space $\mathfrak{A}^*$ with state space K organized to a base-norm space in the usual way. We recall that the enveloping von Neumann algebra is the bidual space $\mathfrak{A}^{**}$ endowed with the multiplication and involution obtained by identifying $\mathfrak{A}^{**}$ with the weak (= ultraweak) closure of $\pi(\mathfrak{A})$ in $\mathcal{B}(H_\pi)$ where π is the universal representation of $\mathfrak{A}$, (see [D_2; § 12]). The space $\mathfrak{A}^*$ will be the predual of $\mathfrak{A}^{**}$ (consisting of ultraweakly continuous linear functionals on $\mathfrak{A}^{**}$), and K will be the set of normal states on $\mathfrak{A}^{**}$. Therefore we may apply the previous results to the duality of $\mathfrak{A}^*$ and $\mathfrak{A}^{**}$.

It is appropriate here to add a few words on the connections with the theory of affine functions on convex sets. By a known result (due to Kadison) the space (A,e) is isometrically (linear- and order-) isomorphic to the linear space $A(K)$ of all weak* continuous affine functions on the state space K provided with the uniform norm. (This holds for every complete order unit space; see e.g. [A_1; Th. II. 1.8]). The self-adjoint part $\mathfrak{A}^{**}{}_{sa}$ of $\mathfrak{A}^{**}$ will be the dual of the space $\mathfrak{A}^*_{sa} = V$; hence it can be identified with the space $A^b(K)$ of all bounded affine functions on the base K. Thus we obtain an isometric (linear- and order-) isomorphism of $\mathfrak{A}^{**}_{sa}$ onto $A^b(K)$, and this representation takes ultraweak convergence into pointwise convergence. (Note that the elements of K act as ultraweakly continuous linear functionals on $\mathfrak{A}^{**}$). Moreover, this representation of $\mathfrak{A}^{**}_{sa}$ onto $A^b(K)$ will reduce to the original representation for elements of $A = \mathfrak{A}_{sa} \subset \mathfrak{A}^{**}_{sa}$ (In fact, the original representation of $\mathfrak{A}_{sa}$ is nothing but an embedding into $\mathfrak{A}^{**}$ followed by restriction of the functionals from the domain $\mathfrak{A}^*$ to the smaller domain K).

Our main result on state spaces will now follow, essentially by summing up our previous discussion of operator algebras. As usual the topology of the state space of a C^*-algebra will be understood to be the weak* topology unless otherwise is stated.

<u>Theorem 3.6.</u> <u>The state space K of a C^*-algebra $\mathfrak{A}$ is a strongly spectral convex compact set.</u>

<u>Proof</u>. Identifying $\mathfrak{A}_{sa}^{**}$ with $A^b(K)$ and using Proposition 3.4, we conclude that $(A^b(K),e)$ and (V,K) are in spectral duality. Hence K is a spectral convex set.

By Proposition 3.4 the functional calculus in $A^b(K)$ coincides with the customary functional calculus for self-adjoint elements of $\mathfrak{A}^{**}$. By elementary spectral theory for operators $\mathfrak{A}_{sa}$ is closed under (real) continuous functions. Therefore $A(K)$ is closed under continuous functions, and by Theorem 2.6 K is strongly spectral. □

We turn now to weak*-closed faces, and we shall need the following fundamental result: Every weak*-closed face F of K is semi-exposed in the duality of (A,e) and (V,K), i.e. F contains all $x \in K$ such that $\langle a,x\rangle = 0$ for all $a \in A^+$ vanishing on F.

This theorem is due to Effros [E; Th. 4.9], and it is also implicit in Prosser's memoir [P; § 4). We shall briefly sketch the idea of the proof, and we first recall that for given $x \in K$ and $a \in \mathfrak{A}^{**}$ the functional $a \circ x \in \mathfrak{A}^*$ is defined on $\mathfrak{A}$ by $b \to \langle ab,x\rangle$ (here the product ab is formed in $\mathfrak{A}^{**}$ and x acts as a normal state on $\mathfrak{A}^{**}$). Now it can be proved that for every weak*-closed face F of K the set

$$L = \{a \circ x \mid a \in \mathfrak{A}^{**},\ x \in F\} \qquad (3.7)$$

is a weak*-closed subspace of $\mathfrak{A}^*$ and that the equality $L \cap K = F$ holds. The proof of this fact is based on a polar decomposition for linear functionals in $\mathfrak{A}^*$ due to Tomita [To], Effros [E; Th. 3.2, Lem 3.3] and Prosser [P; Lem 3.4]. (See also [D_2; Th. 12.2.4, Prop. 12.2.9]). With the weak* closed subspace L at hand one may proceed by a standard Hahn-Banach argument. Specifically, let $x \in K$ satisfy $\langle a,x\rangle = 0$ for all $a \in A^+$ vanishing on F, and assume for contradiction that $x \notin F$. By the above result $x \notin L$; hence there exists $b \in \mathfrak{A}$ vanishing on L such that $\langle b,x\rangle \neq 0$. Then the element $b^*b \in A^+$ will satisfy $\langle b^*b,y\rangle = \langle b,b^*\circ y\rangle = 0$ for all $y \in F$. But Schwartz' inequality gives $0 < |\langle b,x\rangle|^2 \leq \langle b^*b,x\rangle$, which is the desired contradiction.

The fact that the weak* closed faces of K are semi-exposed, is used in a crucial way in the proof of the next theorem.

<u>Theorem 3.7</u>. <u>Let F be a norm closed, hence projective, face of the state space K of a C^*-algebra $\mathfrak{A}$, and let p_F be the corresponding projective unit (= carrier projection). Then $p_F \in A^{\downarrow}(K)$ iff F is weak*-closed, and $p_F \in A(K)$ iff F and $F^{\#}$ are both weak* closed</u>.

<u>Proof</u>. 1.) Assume first that F is weak* closed. It is well known (and easily verified) that the set

$$(3.8) \qquad I_F = \{a \in \mathfrak{A} \mid \langle a^*a,x\rangle = 0 \quad \text{all} \quad x \in F\}$$

is a closed left ideal in $\mathfrak{A}$; hence it contains an ascending right approximate unit $\{u_\lambda\}$ (see [D_2; Prop. 1.7.3]). Since $\{u_\lambda\}$ is a bounded ascending net, it has an ultraweak limit $q \in \mathfrak{A}^{**}$. For every $a \in \mathfrak{A}$ the net $\{au_\lambda\}$ will converge in norm to a, but it will also converge ultraweakly to aq since multiplication is ultraweakly continuous in each variable. From this it follows that

$$(3.9) \qquad aq = a \qquad \text{all } a \in I_F$$

In particular $u_\lambda q = u_\lambda$ for all λ. Passing to ultraweak limits, we obtain $q^2 = q$. Hence q is a projection in $\mathfrak{A}^{**}$.

We claim that $q = e - p_F$.

To prove this claim we shall verify that (3.4) holds with $e - q$ in place of p_F, or equivalently that the following equivalence is valid for $x \in K$:

$$(3.10) \qquad x \in F \iff \langle q,x\rangle = 0 .$$

Assume first $x \in F$. Then $\langle u_\lambda^* u_\lambda, x\rangle = 0$ since $u_\lambda \in I$, and by Schwartz' inequality

$$|\langle u_\lambda,x\rangle|^2 = |\langle e^* u_\lambda,x\rangle|^2 \leq \langle e,x\rangle\langle u_\lambda^* u_\lambda,x\rangle = 0 .$$

Hence $\langle u_\lambda,x\rangle = 0$ for all λ. Since x acts as a normal state on $\mathfrak{A}^{**}$ we can pass to the ultraweak limit q, and we get $\langle q,x\rangle = 0$.

Assume next that $\langle q,x\rangle = 0$. Then for every $a \in A^+$ vanishing on F we have $a^{\frac{1}{2}} \in I$. Since I_F is a left ideal, $a \in I$. Using (3.9), the equality $q^2 = q$ and Schwartz' inequality, we find

$$|\langle a,x\rangle|^2 = |\langle aq,x\rangle|^2 \leq \langle q,x\rangle\langle aa^*,x\rangle = 0 .$$

Since F is semi-exposed in the duality of (A,e) and (V,K), this gives $x \in F$.

Now we have proved that $q = e - p_F$ as claimed. It follows that $p_F = e - q$, and so $e - u_\lambda \searrow p_F$ (pointwise convergence on K). Hence $p_F \in A^{\downarrow}(K)$.

2.) If $p_F \in A^{\downarrow}(K)$ then p_F is a weak* upper semicontinuous function on K with values in $[0,1]$ and the set $F = \{x \in K | \langle p_F, x\rangle = 1\}$ must be weak* closed.

3.) If F and $F^{\#}$ are both weak* closed, then $p_F \in A^{\downarrow}(K)$ and $p_F' = e - p_F \in A^{\downarrow}(K)$. Hence p_F is both upper and lower semicontinuous in the weak* topology, and so $p_F \in A(K)$.

4.) If $p_F \in A(K)$ then p_F is weak*-continuous and $F = \{x \in K | \langle p_F, x\rangle = 1\}$ and $F^{\#} = \{x \in K | \langle p_F, x\rangle = 0\}$ are both closed.∎

Remark. The discussion above differs from the standard treatment in that we have emphasized the facial structure of K rather than the ideal structure of $\mathfrak{A}$ and $\mathfrak{A}^{**}$. For the sake of completeness we recall a few basic facts on ideals, which are closely related to the results above. By (3.5) every norm closed face F of K determines an ultraweakly closed left ideal J_F of $\mathfrak{A}^{**}$, and by (3.8) every weak*-closed face F of K determines a closed left ideal I_F of $\mathfrak{A}$. It can be proved (essentially by the correspondence of norm closed faces of K and projections in $\mathfrak{A}^{**}$, cf. [P; Th. 3.16]), that the map $F \to J_F$ is bijective and that the inverse map is given by

$$(3.11) \qquad F = \{x \in K | \langle a^*a, x\rangle = 0 \text{ all } a \in J_F\} .$$

Similarly it can be proved (cf. [P; Th. 5.11]) that the map $F \to I_F$ is bijective and that the inverse map is given by the similar formula

$$(3.12) \qquad F = \{x \in K | \langle a^*a, x\rangle = 0 \text{ all } a \in I_F\} .$$

Also it can be verified that J_F and I_F are two-sided ideals iff F is a split face. (This was first observed for I_F in [AA_1; Prop. 7.1]). Finally we note that a result related to the first assertion of Theorem 3.7 was given in [GR; Appendix].

The next result is also well known, but since it is essential for understanding the geometry of state spaces, we have included the simple proof.

Proposition 3.8. If K is the state space of a C^*-algebra $\mathfrak{A}$, then the following statements are equivalent:

(i) K is a simplex

(ii) $\mathfrak{A}$ is commutative

(iii) K is a Bauer simplex (i.e. K is a simplex and $\mathcal{E}(K)$ is weak*-closed).

Proof. (i) => (ii). If K is a simplex then every norm closed face of K is a split face [As - E1]. Hence every projection in $\mathfrak{A}^{**}$ is central (Theorem 3.5). From this it follows that $\mathfrak{A}^{**}$, and then also $\mathfrak{A}$, is commutative.

(ii) => (iii) If $\mathfrak{A}$ is commutative, then $\mathfrak{A} \cong C(X)$ for a compact Hausdorff space X, and hence K must be affinely isomorphic and homeomorphic to the Bauer simplex $M_1^+(X)$ of all probability measures on X provided with the vague (or weak*) topology.

(iii) => (i) Trivial. □

We shall close this section by a discussion of the geometry of the state spaces of finite dimensional algebras. Here all ideals and faces are closed in all relevant topologies, so we do not have to worry about topology at all. Note also that we can restrict our attention to simple algebras, for if $\mathfrak{A}$ is a C^*-algebra with a proper two-sided ideal, then K can be decomposed into a direct convex sum of two complementary split faces. Next we recall that a simple finite-dimensional C^*-algebra is a full matrix algebra M_n for some $n \in \mathbb{N}$ (Wedderburn's Structure

Theorem). Hence it suffices to study the state spaces of M_n for $n \in \mathbb{N}$.

It is well known (and easily verified) that the states of M_n are of the form

$$(3.13) \qquad\qquad p(a) = \mathrm{Tr}(ta)$$

where t is a positive $n \times n$-matrix of trace 1, and that (3.13) defines an affine isomorphism of the state space K of M_n onto the set of all such matrices. Counting variables we conclude that $\dim K = n^2-1$ (i.e. K can be realized as a convex body in $\mathbb{R}^{n^2-1}$).

By the general theory of this section the faces of K will all be projective. They will occur in quasi-complementary pairs (F,G), and for such a pair there will exist one and only one affine retraction $\rho: K \to \mathrm{co}(F \cup G)$. To every face F of K is associated a number $k = 0,1,\dots,n$ viz. $k = \dim p(H)$ where $H = \mathbb{C}^n$ and $p \in M_n$ is the carrier projection of F. The face F will then be affinely isomorphic to the state space of M_k. Hence there will exist proper faces of K of dimension k^2-1 for $k = 1,\dots,n-1$, and no other. For example the state space of M_3 will be of dimension 8, and it will have proper faces of dimension 0 (extreme points) and 3 (state space of M_2).

The only state space which is easily visualized, is that of M_2. By (3.13) a state p of M_2 is given by a positive 2×2-matrix t of trace 1, i.e. a matrix of the form

$$t = \begin{pmatrix} \frac{1}{2}+\xi, & \eta+i\zeta \\ \eta-i\zeta, & \frac{1}{2}-\xi \end{pmatrix}, \quad \det(t) \geq 0 .$$

Evaluating the determinant we find that the condition $\det(t) \geq 0$

is equivalent to

$$\xi^2 + \eta^2 + \zeta^2 \leq \frac{1}{4} .$$

Therefore the state space of M_2 will be a solid sphere in $\mathbb{R}^3$, and the center of the sphere corresponds to the normalized trace functional.

From the discussion above it also follows that of the six convex sets pictured at the end of § 1 and § 2, only those of Fig. 1 and Fig. 3 are state spaces, whereas those of Fig. 4, Fig. 5 and Fig. 6 are spectral without being state spaces, and that of Fig. 2 is not even spectral.

§ 4. Some further results on spaces in spectral duality.

In this section (A,e) and (V,K) will denote order-unit and base-norm spaces in spectral duality. In addition we will assume $A = V^*$, so that $A \cong A^b(K)$. Our purpose here will be to prove various results which are valid in this setting and which generalize known results from operator theory. When the proofs are specialized to the context of operator algebras, they will usually be more geometric than previous ones.

Lemma 4.1. If $\{u_\alpha\}$ is a monotone net of projective units of A, then the pointwise limit of this net exists and is a projective unit.

Proof. It suffices to consider a decreasing net $\{u_\alpha\}$. For such a net $a = \lim_\alpha u_\alpha = \inf_\alpha u_\alpha$ exists by the pointwise monotone completeness of $A \cong A^b(K)$. Recall that for $a \in A^+$ we have defined $rp(a)$ to be the smallest projective unit u such that $a \in face(u)$. Recall also that if $0 \leq a \leq e$ then $a \leq rp(a)$ (cf. (3.23) of Part I). Thus for any projective unit v, $0 \leq a \leq v$ implies $a \leq rp(a) \leq v$. Now $0 \leq a \leq u_\alpha$ implies $a \leq rp(a) \leq u_\alpha$, so

$$a \leq rp(a) \leq \inf_\alpha u_\alpha = a .$$

Thus $a = rp(a)$ is a projective unit. ∎

Keeping the terminology of Part I, we shall say that a net in A or in V converges weakly if it converges in the weak topology determined by the given duality of A and V. (For nets in $A = V^*$, the term "weak*" would be equally appropriate).

Lemma 4.2. *If $a \in A^+$, then $rp(a)$ is the pointwise, hence weak, limit of the ascending sequence $\{\varphi_n(a)\}$ where $\varphi_n(\lambda) = 1 \wedge n\lambda$ for $\lambda \in [0, \infty\rangle$.*

Proof. We have $rp(a) = e - e_o^a$ by formula (6.7) of Part I and the remarks on the following lines. (It is also easy to prove this directly from the spectral integral for a). Hence $rp(a) = \chi_{\langle o, \infty\rangle}(a)$, and since $\varphi_n \nearrow \chi_{\langle o, \infty\rangle}$ on $[0, \infty\rangle$, we find that $\varphi_n(a) \nearrow rp(a)$. Since $V = \text{lin}\, K$ then $\varphi_n(a) \to rp(a)$ pointwise on V, i.e. weakly. $\square$

Theorem 4.3. *Every weakly closed face H of A^+ is of the form $H = \text{im}^+P$ where $P \in \mathcal{P}$.*

Proof. Let $\mathcal{X}$ be a maximal collection of non-zero pairwise orthogonal projective units in H and let $\mathcal{F}(\mathcal{X})$ be the collection of all finite sums $u = u_1 + \ldots + u_n \in H$ where $u_i \in \mathcal{X}$ for $i = 1, \ldots, n$. By Proposition I.3.4 we also have $u = u_1 \vee \ldots \vee u_n \in \mathcal{U}$, and it follows that $\mathcal{F}(\mathcal{X})$ is directed upwards. By Lemma 4.1 $u_o = \sup\{u \mid u \in \mathcal{F}(\mathcal{X})\}$ is a projective unit which will be in the weakly closed face H.

We will now show $H = \text{im}^+P$, where $P \in \mathcal{P}$ satisfies $Pe = u_o$. For fixed $a \in H$ we define $v = rp(a + u_o)$. By Lemma 4.2, $v \in H$. Note that $a \in \text{face}(v)$; we are going to show $v = u_o$ so that $a \in \text{face}(u_o) \subset H$. Since a was an arbitrary element of H, $\text{face}(u_o) = H$ will follow. By Corollary I.2.11 $\text{face}(u_o) = \text{im}^+P$, so this will complete the proof that $H = \text{im}^+P$.

To prove $v = u_o$, we note that $u_o \in \text{face}(v)$, so $u_o = rp(u_o) \leq v$. Thus $v - u_o = v \wedge u_o' \in \mathcal{U}$ (see the proof of Theorem I.3.5). For arbitrary $u \in \mathcal{X}$ we have $u \leq u_o$ so

$$u + (v - u_o) \leq u_o + (v - u_o) = v \leq e ,$$

We shall now apply Theorem 4.3 in the context of operator algebras. Suppose that A is the self-adjoint part of a von Neumann algebra $\mathfrak{A}$ and that V is the self-adjoint part of the predual $\mathfrak{A}_*$ with K the normal state space. Now the weak topology of A determined by the duality with V, will be the ultraweak topology. (If $\mathfrak{A}$ is the enveloping von Neumann algebra of a C^*-algebra, then the ultraweak topology on $\mathfrak{A}$ is the same as the weak operator topology when $\mathfrak{A}$ is acting on the Hilbert space of the universal representation of the given C^*-algebra, cf. [D; § 12]).

It follows from Theorem 4.3 that for every ultraweakly closed face H of A^+ there exists a P-projection P on A such that $H = \mathrm{im}^+ P$. By Proposition 3.1 P is of the form $Pa = pap$ where p is a (self-adjoint) projection in $\mathfrak{A}$. Hence H will be of the form

$$H = pA^+p .$$

It is clear from the argument above that $p \to pA^+p$ is a $1-1$ correspondence between projections in $\mathfrak{A}$ and ultraweakly closed faces of A^+.

For brevity we write $q = e - p$ and we recall that P' is of the form $P'a = qaq$ (Proposition 3.1). We also define

$$J_H = \{a \in \mathfrak{A} \mid a^*a \in H\} .$$

Now it is easily seen that J_H is an ultraweakly closed left ideal and that $H \subset A^+ \cap J_H$. On the other hand for every $a \in J_H$ we have $a^*a \in H = \ker^+ P'$, and so $0 = \|qa^*aq\| = \|aq\|^2$, which implies $aq = 0$ and then also $qaq = 0$. Therefore $A^+ \cap J_H \subset \ker^+ P' = H$, and so we have proved

$$H = A^+ \cap J_H . \tag{4.1}$$

For any ultraweakly closed left ideal J of $\mathfrak{A}$ one has

$$(4.2) \qquad J = \{a \in \mathfrak{A} \mid a^*a \in A^+ \cap J\} .$$

(This follows from the polar decomposition of $\mathfrak{A}$, cf. e.g. [Sa; Th. 1.12.1]). Thus by (4.1) and (4.2) $J \to A^+ \cap J$ is a 1-1 correspondence of ultraweakly closed left ideals of $\mathfrak{A}$ and ultraweakly closed faces of A^+ .

The connections between ultraweakly closed faces of A^+ , projections in $\mathfrak{A}$, and ultraweakly closed left (or right) ideals in $\mathfrak{A}$, were first established by Effros [E] and Prosser [P] .

We turn now to a result which will be of use later and which is also of interest in itself. Recall that every element x of a base-norm space admits a decomposition $x = y - z$ where $y,z \geq 0$ and $\|x\| = \|y\| + \|z\|$ (cf. [A_1; Prop. II. 1.4]). Under certain conditions this decomposition is unique (e.g. see [El_1]). In particular, the existence and uniqueness of such a decomposition in the predual of a von Neumann algebra (and thus also in the dual of a C^*-algebra) was proved by Grothendieck [Gr]. We will now generalize this result to the context of this section (i.e. A and V in spectral duality and $A = V^*$).

Proposition 4.6. Every $x \in V$ admits a unique decomposition of the form $x = y - z$ where $y,z \geq 0$ and $\|x\| = \|y\| + \|z\|$. Moreover, there exists $P \in \mathcal{P}$ such that $y = P^*x$ and $z = -P'^*x$.

Proof. Assume $\|x\| = 1$ and let $x = y - z$ be a decomposition of the type described in the proposition.

Note that the unit ball A_1 of $A = V^*$ is weakly (i.e.weak*-) compact, so the set $E = \{a \in A_1 \mid \langle a,x \rangle = 1\}$ is a non-empty compact face of A_1 . By the Krein-Milman Theorem E will contain an extreme point s , and s will also be an extreme point of A_1 since E is a face of A_1 .

The map $a \to 2a - e$ is an affine isomorphism of the convex set $[0,e]$ (order interval) onto $[-e,e] = A_1$. Since the extreme points of $[0,e]$ will be mapped onto the extreme points of $[-e,e]$, and since the extreme points of $[0,e]$ are precisely the projective units (Proposition I.7.7), there exists a projective unit u such that $s = 2u - e$. Setting $u' = e - u$, we can write $s = u - u'$.

Let P be the P-projection on A such that $u = Pe$. We shall prove that $y = P^*x$ and $z = -P'^*x$.

Since $s \in E$ then

$$(4.3) \qquad 1 = \langle s,x\rangle = \langle u-u',x\rangle = \langle u-u',y-z\rangle =$$

$$= \langle u,y\rangle + \langle u',z\rangle - \langle u,z\rangle - \langle u',y\rangle .$$

Since by assumption $1 = \|x\| = \|y\| + \|z\|$, we also have

$$(4.4) \qquad 1 = \langle e,y\rangle + \langle e,z\rangle \geq \langle u,y\rangle + \langle u',z\rangle .$$

Subtracting (4.3) from (4.4) we find

$$0 \geq \langle u,z\rangle + \langle u',y\rangle ,$$

from which it follows that each term on the right must be zero. Otherwise stated:

$$(4.5) \qquad \langle Pe,z\rangle = \langle P'e,y\rangle = 0 .$$

From (4.5) we conclude that $\|P^*z\| = \langle e,P^*z\rangle = 0$ and that $\|P'^*y\| = \langle e,P'^*y\rangle = 0$. Hence $P^*z = 0$ and $P'^*y = 0$, and so

$$z \in \ker^+P^* = \operatorname{im}^+P'^* , \quad y \in \ker^+P'^* = \operatorname{im}^+P^* .$$

Applying P^* and P'^* to both sides of the equation $x = y - z$, we now obtain the desired conclusion $P^*x = y$, $P'^*x = -z$. ∎

Our final goal will be to generalize the result that the traces of a C^*-algebra form a simplex. We begin by giving the general definitions and results in the context of the present section (A and V in spectral duality and $A = V^*$); then we specialize to operator theory.

<u>Definition</u>. An element $z \in V$ is said to be <u>central</u> if $(P+P')^*z = z$ for all $P \in \mathcal{P}$, and the central elements of K are called <u>traces</u>.

Note that if one considers a von Neumann algebra and its predual in the context of spectral duality, then there are two notions available of "trace" and "central". However, this ambiguity is only apparent: we shall show later that the condition $(P+P')^*z = z$ for all $P \in \mathcal{P}$ characterizes central elements and traces of the operator algebra variety.

<u>Theorem 4.7</u>. <u>The central elements of V form a vector lattice and the traces form a linearly compact simplex</u>.

<u>Proof</u>. We first prove that if $z \in V$ is central and if $z = z^+ - z^-$ is the (unique) decomposition given in Proposition 4.6, then z^+ and z^- are also central.

For every $P \in \mathcal{P}$ we have

$$(4.6) \qquad z = (P+P')^*z = (P+P')^*z^+ - (P+P')^*z^- .$$

By assumption $\|z\| = \|z^+\| + \|z^-\|$, and since $(P+P')^*$ preserves norms in V^+ (see Prop. I. 2.1), we have

$$(4.7) \qquad \|z\| = \|(P+P')^*z^+\| + \|(P+P')^*z^-\| .$$

But by the uniqueness statement of Proposition 4.6, (4.6) and (4.7) imply that

$$z^+ = (P+P')^*z^+ \ ; \quad z^- = (P+P')^*z^- \ .$$

Since $P \in \mathcal{P}$ was arbitrary, z^+ and z^- are central.

It follows in particular that the central elements form a positively generated linear subspace $Z(V)$ of V. We will prove that $Z(V)$ is a vector lattice, and to this end it suffices to prove that for every $z \in Z(V)$ the element z^+ is the least upper bound of z and 0 within $Z(V)$. Clearly $z^+ \geq z$ and $z^+ \geq 0$. Suppose $y \in Z(V)$ and $y \geq z$, $y \geq 0$. Let $P \in \mathcal{P}$ be chosen such that $z^+ = P^*z$ and $z^- = -P'^*z$ (Proposition 4.6); then

$$y = (P+P')^*y \geq P^*y \geq P^*z = z^+ \ ,$$

as desired.

Finally we note that the traces form a base for the positive cone $Z(V)^+$ of the vector lattice $Z(V)$ (since K is a base for V^+). This implies that the set of traces is a linearly compact simplex (see [Ken]). $\square$

We will next prove that the concepts of a "central element" of V and of a "trace" really generalize the corresponding concepts used in operator theory. We will need some auxiliary results. For the statement of the first of these we agree to say that two elements a,b of a C^*-algebra are <u>exchanged</u> by a unitary u if $u^*au = b$.

<u>Lemma 4.8.</u> <u>If projections p and q in a von Neumann algebra $\mathfrak{A}$ can be exchanged by a unitary $u \in \mathfrak{A}$, then there exist self-adjoint unitaries $s,t \in \mathfrak{A}$ such that p and q are exchanged by st.</u>

Proof. If p and q can be exchenged by a unitary, then by a theorem of Fillmore [Fil] p and q are perspective (i.e. have a common complement in the lattice of projections). Now by a result of Topping two perspective projections can be exchanged by the product of two self-adjoint unitaries. (This result is in fact valid in the more general context of JW-algebras, see [T; Th. 8]). The proof is now complete. □

We recall that a bounded linear functional z on a C^*-algebra $\mathfrak{A}$ with identity e is said to be central if it is "invariant under conjugation by unitaries", i.e. if

$$\langle u^*au, z\rangle = \langle a, z\rangle \tag{4.8}$$

for all $a \in \mathfrak{A}$ and all unitaries $u \in \mathfrak{A}$.

It is easily verified that this is equivalent to the alternative condition

$$\langle ab, z\rangle = \langle ba, z\rangle \tag{4.9}$$

for all $a,b \in \mathfrak{A}$.

We will use the word trace to denote a central state on $\mathfrak{A}$. (Note that with this definition we require that the traces shall be "normalized", i.e. that $\langle e,z\rangle = \|z\| = 1$).

We shall need an auxiliary result by which the traces of a C^*-algebra $\mathfrak{A}$ with identity can be identified with the normal traces of the enveloping von Neumann algebra $\mathfrak{A}^{**}$. This result is elementary, but since we have been unable to find a satisfactory reference, we include a proof.

Lemma 4.9 If t is a trace on a C^*-algebra $\mathfrak{A}$, with identity, then t (in its natural identification) is a normal trace on the enveloping von Neumann algebra, and conversely.

Proof. It is evident from the definition that a normal trace on $\mathfrak{A}^{**}$ restricts to a trace on $\mathfrak{A}$.

Let t be a trace on $\mathfrak{A}$. Then (4.9) holds for all $a,b \in \mathfrak{A}$; we will show that (4.9) is also valid for all $a,b \in \mathfrak{A}^{**}$. Clearly it suffices to prove this for $a,b \in \mathfrak{A}^{**}$ with $\|a\| = \|b\| = 1$.

By the Kaplansky density theorem there exists nets $\{a_\alpha\}$ and $\{b_\alpha\}$ in the unit ball of $\mathfrak{A}$ which converge to a and b in the ultrastrong topology (i.e. the strong operator topology when $\mathfrak{A}^{**}$ is acting on the Hilbert space H of the universal representation of $\mathfrak{A}$). Since multiplication is jointly continuous with respect to the ultrastrong topology on the unit ball of $\mathfrak{A}^{**}$, then $a_\alpha b_\alpha \to ab$ and $b_\alpha a_\alpha \to ba$ ultrastrongly. Since ultraweakly continuous linear functionals are ulstrastrongly continuous on the unit ball, then

$$\langle ab, t\rangle = \lim_\alpha \langle a_\alpha b_\alpha, t\rangle = \lim_\alpha \langle b_\alpha a_\alpha, t\rangle = \langle ba, t\rangle .$$

This completes the proof. $\square$

Theorem 4.10. Let A be the self-adjoint part of a von Neumann algebra $\mathfrak{A}$ and let V be the self-adjoint part of the predual $\mathfrak{A}_*$ with K the normal state space. An element $z \in V$ determines a central linear functional on $\mathfrak{A}$ iff z is central in the general sense defined in this section, i.e. iff

$$(4.10) \qquad (P + P')^* z = z \qquad \text{for all } P \in \mathcal{P} .$$

Proof. Suppose first that $z \in V$ determines a central linear functional on $\mathfrak{A}$, and consider an arbitrary $P \in \mathcal{P}$ expressed by $Pa = pap$ where p is a projection in $\mathfrak{A}$ (Proposition 3.1). For brevity we write $q = e - p$, and we recall that $P'a = qaq$ for $a \in A$.

Now we observe that $p-q$ is unitary, so by (4.8)

(4.11) $$\langle (p-q)a(p-q),z\rangle = \langle a,z\rangle .$$

But since $p+q = e$, we also have

(4.12) $$\langle (p+q)a(p+q),z\rangle = \langle a,z\rangle .$$

Adding (4.11) and (4.12) and dividing by 2, we obtain

(4.13) $$\langle pap+qaq,z\rangle = \langle a,z\rangle ,$$

which is equivalent to (4.10).

Conversely, suppose that (4.10) holds for a given $z \in V$. Subtracting (4.12) from twice (4.13) we are again back to (4.11). Since $p-q$ (with $q = e-p$) is the general form of a self-adjoint unitary, this proves that z is invariant under conjugation by self-adjoint unitaries.

Now suppose that $u \in \mathfrak{A}$ is any unitary and $r \in \mathfrak{A}$ is any projection. Then u^*ru and r are projections exchanged by a unitary; by Lemma 4.8 there exist self-adjoint unitaries s,t such that r and u^*ru are exchanged by st , i.e.

$$u^*ru = (ts)r(st) .$$

Hence

$$\langle u^*ru,z\rangle = \langle t(srs)t,z\rangle = \langle r,z\rangle .$$

Since the linear span of projections is norm dense in $\mathfrak{A}$, and since the map $a \to u^*au$ and the linear functional determined by z are norm continuous, we have

$$\langle u^*au,z\rangle = \langle a,z\rangle \quad \text{for all } a \in \mathfrak{A} .$$

Thus, z determines a central linear functional on $\mathfrak{A}$, and this completes the proof. $\square$

Corollary 4.11. Let $\mathfrak{A}$ be a C^*-algebra with identity, let A be the self-adjoint part of the enveloping von Neumann algebra $\mathfrak{A}^{**}$, and let V be the self-adjoint part of $\mathfrak{A}^*(\cong(\mathfrak{A}^{**})_*)$. Then a state t on $\mathfrak{A}$ will be a trace iff t (in its natural identification) is a trace on A in the general sense defined in this section.

Proof. The corollary follows at once from Lemma 4.9 and Theorem 4.10. $\square$

Corollary 4.12. (Thoma [Th], Effros-Hahn [EH; Cor. 2.14]) The traces on a C^*-algebra with identity form a weak* compact simplex.

Proof. By Theorem 4.7 and Corollary 4.11 the traces form a linearly compact simplex. But it follows immediately from the definition (4.9) that the traces form a weak* closed, therefore compact, subset of the state space. $\square$

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