A jump operator in set recursion

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1. Introduction.

The superjump was introduced by Gandy [3] as a type 3 functional that essentially is a uniform jump operator on the type-two functionals. Harrington [5] gave a description of the sets recursive in the superjump ${}^3S$. He proved

a Let ρ^F be the ordinal for recursion in ${}^3S$, ${}^2F$, i.e. $\rho^F = \omega_1^{{}^3S,F}$. Then ρ^F is the least ordinal recursively Mahlo in F.

b $L^F_{\rho^F} \cap \mathcal{P}(\omega) = 1\text{-sc}({}^3S,{}^2F)$ (= those subsets of ω recursive in ${}^3S$ and F)

To do this, he defined a notion of strong recursion in the superjump. In his Ph.D. Thesis this notion is extended to higher type variants of the superjump, ${}^{k+3}S$. Harrington's strong recursion theory in ${}^{k+3}S$, ${}^{k+2}F$ will have the same total recursive functions, but fewer partial recursive functions. The computation theory will have strong properties such as stage comparison and Grilliot selection.

In Normann [11] we defined a recursion theory on sets called E-recursion. We proved that there are deep connections between E-recursion and Kleene-recursion in normal functionals. In this paper we will add a natural scheme of the jump of a relation to the schemes of E-recursion. We will call the new theory S-recursion. There will be a similarly deep connection between S-recursion and Harrington's strong recursion in the Superjump, as between E-recursion and Kleene-theory in normal functionals. We will leave this connection unproved, but prove Harrington's results for S-recursion. Some of the arguments are adjustments of ideas from [6], particularly in Theorems 1 and 2 and lemma 3.

We will use S-recursion to give some characterizations of the envelopes and sections connected to strong recursion in the superjump.

In the sequel we will assume familiarity with set-recursion and the companion theory (theory of codes for sets) for E-recursion. We will concentrate on the special arguments needed for lifting results from E-recursion to S-recursion.

2. E-recursion and S-recursion

E-recursion as defined in Normann [11] is obtained by adding indices to the schemes for rudimentary functions, and then a scheme of reflection (diagonalization). For a relation R and a set $\mathbf{x}$, we defined

$$\mathrm{Spec}(R;\mathbf{x}) = \langle M_y(R;\mathbf{x})\rangle_{y\subseteq x,\ y \text{ finite}}$$

where $M_y(R;x) = \{\{e\}^{E(R)}(x,y_1,\ldots,y_n);\ e\in\omega,\ \{y_1,\ldots,y_n\}\subseteq y\}$
($\{e\}^{E(R)}$ is the partial function in E(R)-recursion theory with index e. We give $\{e\}^{S(R)}$ the same meaning for S(R)-recursion, $\{e\}^K$ will mean the Kleene-recursive function with index e).

If we let $I = \mathrm{type}(k)$, $a \in I$ and F a functional of type k+2 we prove that

<u>a</u> For $A \subseteq I$, A is Kleene-recursive in ${}^{k+2}F, a, {}^{k+2}E$ if and only if $A \in M_a(F;I)$

b For $A \subseteq I$, A is Kleene-semirecursive in ${}^{k+2}F, a, {}^{k+2}E$ if and only if A is Σ^*_a-definable over Spec(F) (if and only if for some Δ_o-formula φ with parameters from $M_a(F;I)$

$$b \in A \iff \exists x \in M_{\{a,b\}}(F;I)\varphi(x,b)).$$

The superjump ${}^{k+3}S$ is defined as the functional

$${}^{k+3}S(e,F) = \begin{cases} 0 & \text{if } \{e\}(F) \text{ has a value } (\{e\}(F)\downarrow) \\ 1 & \text{if } \{e\}(F) \text{ does not have a value } (\{e\}(F)\uparrow) \end{cases}$$

${}^{k+3}S$ is not a normal functional. Recursion in ${}^{k+3}S$ does not satisfy stage comparison and that a subset of I is recursive in ${}^{k+3}S$ if and only if both it and it's complement are semirecursive. The reason for this misbehaviour seems to be that ${}^{k+3}S(e,F)$ is defined only for total F, while we need information only from a part of F to compute ${}^{k+3}S(e,F)$. In E-recursion there are two natural candidates for the jump, either a complete Σ^*-definable set, or the spectrum itself. In defining S-recursion we choose the latter.

Definition.

Define S-recursion from E-recursion by adding the following scheme:

$\{e\}(x,\vec{y}) = \mathrm{Spec}(\lambda z\{e_1\}(x,\vec{y},z);x)$ if $\lambda z\{e_1\}(x,\vec{y},z)$ is total on it's spectrum over x $\qquad e = \langle 8,e_1,n\rangle$

As usual, we identify a function with its graph. S-recursion is, like E-recursion, relativezed to arbitrary relations.

Remark. It is essential that we require that $\lambda z\{e_1\}(x,\vec{y},z)$ is total on its spectrum. If we remove that requirement, we may let e_1 be the index for diagonalization $\{e_1\}(e_2,a) = \{e_2\}(a)$. Then $\mathrm{Spec}(\lambda\langle e_2,a\rangle\{e_1\}(e_2,a);I)$ would have as an element $\{\langle e_2,a\rangle:\{e_2\}(a)\simeq 0\}$. But that set cannot be recursive.

On the other hand, there is no justification for requiring that $\lambda z\{e_1\}(x,\vec{y},z)$ is defined outside its spectrum.

Inspecting the inductive definition of the $S(R)$-computations we see that

$$\{\langle 8,e_1,n\rangle\}^{S(R)}(x,\vec{y}) \simeq \langle M_y\rangle_{y\in x^f} \quad \text{iff}$$

$$\forall z \in M\,(= \bigcup_{y\in x^f} M_y)\{e_1\}^{S(R)}(x,\vec{y},z)\downarrow$$

and

$$\forall y \in x^f(z\in M_y \Rightarrow \exists e\in\omega(z=\{e\}^{\Sigma(F_{e_1})}(\vec{y},x)$$

and

each M_y is rudimentary closed relative to F_{e_1}

and

$$\langle M_y\rangle_{y\in x^f} \models \Sigma^*(F_{e_1})\text{-collection}$$

where x^f means the set of finite subsets of x, $F_{e_1} = \lambda z\{e_1\}^{S(R)}(x,\vec{y},z)$.

The length of this computation will then naturally be

$$\operatorname{Sup}\{\alpha, \|\langle e_1,x,\vec{y},z\rangle\|^{S(R)}\,;\, \alpha\in\operatorname{Spec}(F_{e_1};x) \text{ and } z\in\operatorname{Spec}(F_{e_1};x)\}.$$

<u>Definition</u>. Let R be a relation, x a set, $y\in x^f$

$$\underline{SM_y(R;x)} = \{\{e\}^{S(R)}(y_1,\ldots,y_n,x);\ \{y_1,\ldots,y_n\}\subseteq y,\ e\in\omega\}$$

$$\text{S-Spec}(R;x) = \langle SM_y(R;x)\rangle_{y\in x^f}$$

3. <u>S-recursion and the Superjump</u>.

In this section we will let $I = tp(k)$ for some fixed $k \geq 0$. We also let F be a functional of type $k+2$. Before we can prove our main reduction theorem for $S(F)$-recursion, we need some machinery for companion-theory.

Definition. Let $A \subseteq I \times I$. Assume that A is a transitive relation. Define $\simeq$ by $a \simeq b$ if $A(a,b)$ and $A(b,a)$. We say that A is a code for a set x if $A/\simeq$ is isomorphic to $\langle TC(x), \in \rangle$. Suitable references to the theory of codes will be Sacks [12] and [13], and Normann [10] and [11].

Lemma 1. The relation 'A is a code' is recursive in ${}^{k+3}S$.

Proof. The relation 'A is a code' may be defined by some quantifiers over I + 'A is a well-founded relation'. Since ${}^{k+2}E$ is recursive in ${}^{k+3}S$, the lemma is trivial for $k > 0$.
For $k = 0$: The relation 'A is well founded' is semirecursive in ${}^{2}E$, and since ${}^{3}S$ is a jump-operator it will be recursive in ${}^{3}S$.

Lemma 2. There is a function f partially recursive in ${}^{k+3}S, F$ such that if $A_1, \ldots, A_n$ are codes for sets $x_1, \ldots, x_n$ and $\{e\}^{S(F)}(x_1, \ldots, x_n)\downarrow$, then $\lambda a,b\, f(e, \langle A_1, \ldots, A_n \rangle, a, b)$ is a total characteristic function for a code for $\{e\}^{S(F)}(x_1, \ldots, x_n)$.
If $A_1, \ldots, A_n$ are codes for $x_1, \ldots, x_n$ and $\{e\}^{S(F)}(x_1, \ldots, x_n)\uparrow$, then

$$f(e, \langle A_1, \ldots, A_n \rangle, a, b) \text{ will not be defined for any } a, b .$$

Proof. We use the recursion-theorem, and define f by induction on the length of the computation $\{e\}^{S(F)}(x_1, \ldots, x_n)$. There will be 8 cases, according to the type of the index e. The proof is by standard manipulations on codes (see Sacks [13] or Normann [10]) in all cases except scheme 8:

$$\{e\}(x, \vec{y}) = \mathrm{Spec}(\lambda z \{e_1\}(x, \vec{y}, z); x) .$$

As an induction hypothesis, assume that f is defined and recursive and acts as it shall for all shorter computations.

Let codes for $x,\vec{y}$ be given. What we will do will be uniformly recursive in these codes.

Let $F_{e_1} = \lambda z\{e_1\}(x,\vec{y},z)$.

The idea is to define another function G which is total, recursive in ${}^{k+3}S$, F and the codes and 'equivalent' to F_{e_1}. Then we can apply ${}^{k+3}S$ on G to define $\mathrm{Spec}(G) = \mathrm{Spec}(F_{e_1})$.

Claim 1. Let A be a code for a computation tree T for some $\{e\}^{F(R)}(\vec{y},x)$-computation for $\vec{y} \in x^f$. We may then recursively decide if we may replace R by F_{e_1} in the computation or not.

Proof. We here assume that F_{e_1} is total on $\mathrm{Spec}(F_{e_1})$. If F_{e_1} diverges on some critical argument in T, our procedure will diverge.

We will use the recursion theorem to define the following recursive function ρ on A.

Let $a \in$ field A. We will let $\rho(a) = 1$ if we in the computation coded by a have used a part of R different from F_{e_1}. Otherwise we will let $\rho(a) = 0$.

ρ is precisely defnined this way:

If a codes a computation σ and for some code b for a subcomputation τ of σ, $\rho(b) = 1$, then $\rho(a) = 1$.

If for all codes b for subcomputations τ of σ, $\rho(b) = 0$, and σ is not an application of R, let $\rho(a) = 1$. (This takes care of the initial computation.)

If for all codes b for subcomputations τ of σ, $\rho(b) = 0$ and σ is an application of R, we must check if this application actually is an application of F_{e_1}. We may assume as an induction-hypothesis that all applications of R in subcomputations of σ actually are applications of F_{e_1}. Let the application be $Z_1 \cap R$. We get a

code for z_1 and $z_1 \in \mathrm{Spec}(F_{e_1}:x)$. So F_{e_1} is total on $\mathrm{dom}(z_1) = \{y_1; \exists y_2 \langle y_1, y_2\rangle \in z_1\}$. Using f, the code for z_1, and standard manipulations on codes, we may compute a code for $z_1 \cap F_{e_1}$. From T we have a code for $z_1 \cap R$.
If these two codes code the same set, let $\rho(a) = 0$, otherwise, let $\rho(a) = 1$.

Now A codes a computation in F_{e_1} if ρ is constant 0 on A.

□ Claim 1.

Now, define

$$G(A,B) = \begin{cases} 0 & \text{if } A \text{ is a code for a computation-tree in } E(F_{e_1})\text{-recursion leading from } x \text{ and some } y \in x^f \text{ to a set } z, \text{ and } B \text{ is a code for } F_{e_1}(z). \\ 1 & \text{otherwise} \end{cases}$$

<u>Claim 2</u>. G is recursive in ${}^{k+3}S, F$ and the codes.

<u>Proof</u>. We use the same assumptions as in claim 1. If they do not hold, our procedure for computing G will give a partial functional.

We will describe an algorithm for computing G.
Let A, B be given. First decide of both A and B are codes. If they are not, let $G(A,B) = 1$. Assume they are codes.

We have already noticed that well-foundedness is recursive in ${}^{k+3}S$, so we may recursively decide if A is the code of a relativized computation-tree or not. If not, let $G(A,B) = 1$. If it is, we may by claim 1 decide if A is coding a tree for a computation relative to F_{e_1}. If not, let $G(A,B) = 1$. If it is, we get a computation tree T computing a set z in $\mathrm{Spec}(F_{e_1};x)$, and we may from A effectively compute a code C for $F_{e_1}(z)$. If B and C code the same set, we let $G(A,B) = 0$. Otherwise we let $G(A,B) = 1$.

This ends the proof of claim 2.

Now, let $A,\vec{B}$ be the codes for $x,\vec{y}$ resp. Let C be a complete r.e. - $G,A,\vec{B}$ subset of I. Using ${}^{k+3}S$ we see that C will be recursive in ${}^{k+3}S,F$ uniformly in $A,\vec{B}$. From C, G we may effectively construct a code for $\mathrm{Spec}(G,A,\vec{B};I)$.

$x \in \mathrm{Spec}(G,A,\vec{B};I)$ since A is a code for x.

Let $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(G,A,\vec{B};I)$. For $z \in \mathrm{Spec}(F_{e_1};x)$ the following definitions of the relation '$F_{e_1}(z) = u$' are valid

'$F_{e_1}(z) = u$' $\iff \forall b(z,u \in M_b \Rightarrow \exists C,B \in M_b$(B is a code for u, C is a code for a computation from x and some $y \in x^f$ leading to z, and $G(C,B) = 0$)

$\iff \forall b(z,u \in M_b \Rightarrow \forall C,B \in M_b$(B is a code for u, C is a code for a computation from x and some $y \in x^f$ leading to $z \Rightarrow G(C,B) = 0$)

This shows that $\mathrm{Spec}(F_{e_1};x)$ will be $w - \Sigma^*(G)$-definable over $\mathrm{Spec}(G,A,\vec{B};I)$. But then we may extract a code for $\mathrm{Spec}(F_e;x)$ from A and $\mathrm{Spec}(G,A,\vec{B};I)$.

By the effectiveness of these arguments we may use the recursion theorem to prove lemma 2.

<u>Theorem 1</u>. Let $I = tp(k)$, F a functional of type $k+2$.

<u>a</u> The relation

$$\{(e,a) : \{e\}^{S(F)}(a,I)\downarrow\}$$

is semirecursive in ${}^{k+3}S$, F.

<u>b</u> If a subset A of I is $S(F)$-recursive in $a \in I$, then A is recursive in ${}^{k+3}S$, F, a.

<u>c</u> If a subset A of I is $S(F)$-semirecursive in $a \in I$, then A is semirecursive in ${}^{k+3}S$, F, a.

These are all immediate consequences of lemma 2.

Our next result will show that the recursive sets will be the same in the two theories. This will not hold for semirecursion.

Theorem 2. There is a primitive recursive function ρ such that if

$$\{e\}^K(F,{}^{k+3}S,\vec{f},\vec{a}) \simeq k$$

then

$$\{\rho(e)\}^{S(F)}(\vec{f},\vec{a},I) \simeq k$$

Proof. We will use the recursion theorem for primitive recursion. In all cases except when we apply ${}^{k+3}S$ we will just imitate what happens in Kleene-recursion. When the Kleene computation seems to apply ${}^{k+3}S$, we will in the $S(F)$-theory forget the requirements of totality and thereby introduce some more computations.

So we regard the case

$$\{e\}(F,{}^{k+3}S,\vec{f},\vec{a}) \simeq {}^{k+3}S(\lambda f\{e_1\}(F,{}^{k+3}S,f,\vec{f},a))$$

Let $\rho(e)$ be an index for the following $S(F)$-computation:

Find $\mathrm{Spec}(\lambda f\,\{\rho(e_1)\}^{S(F)}\,(f,\vec{f},I))$

and by inspection compute ${}^{k+3}S(G)$ for any total extension G of

$$\lambda f\{\rho(e_1)\}^{S(F)}(f,\vec{f},I)$$

(All computation-trees for G will be in

$$\mathrm{Spec}(G;I) = \mathrm{Spec}(\lambda f\{\rho(e_1)\}^{S(F)}(f,\vec{f},I)))$$

This ends the proof of theorem 2.

We have now verified that the concepts of total recursion are the same for Kleene-recursion in ${}^{k+3}S$ and S-recursion over type(k). Our next task is to show that semi-recursion in S-recursion behaves better than semirecursion in ${}^{k+3}S$. This is shown by proving that S-recursion satisfies stage comparison.

Lemma 3. Let R be a relation. Uniformly in R there is an index e such that for any pair σ, τ of computation-tuples

$$\{e\}^{S(R)}(\sigma,\tau) \simeq \begin{cases} 0 & \text{if } \|\sigma\|^{S(R)} < \|\tau\|^{S(R)} \text{ (where } \|\tau\|^{S(R)} = \infty \text{ if } \tau\uparrow) \\ 1 & \text{if } \|\tau\|^{S(R)} \leq \|\sigma\|^{S(R)} \text{ and } \|\tau\|^{S(R)} < \infty . \end{cases}$$

Proof. We will drop the superscript S(R). We define e by use of the recursion theorem. The definition is by 64 cases according to the schemes used in σ and τ. The 49 cases where there is no use of scheme 8 are treated as in E-recursion. (Normann [11]) Moreover, all cases where one of the computations is an initial one, are trivial. We give case 8.8, which is the most complicated.
Let

$$\sigma : \{e_1\}(x,\vec{y}) \simeq \mathrm{Spec}(\lambda y\{e_2\}(y,x,\vec{y});x)$$

$$\tau : \{d_1\}(u,\vec{w}) \simeq \mathrm{Spec}(\lambda w\{d_2\}(w,u,\vec{v});u)$$

We will assume that either $\sigma\downarrow$ or $\tau\downarrow$, and as an induction hypothesis that the lemma is established for any subcomputation of σ or τ. It will be clear from the definition that if both σ and τ diverge, then the described computation on σ and τ will diverge.

We will use the assumptions to define another function G which will be total, and such that $\mathrm{Spec}(G;x \cup u \cup \{e\} \cup \{u\})$ will contain sufficient information to decide if $\|\sigma\| < \|\tau\|$ or $\|\tau\| \leq \|\sigma\|$. G will be defined just on the ordinals, which is no real restriction. We will let $G(\alpha)$ describe what we, with the help of stage comparison so far, can say about the part of the two spectra that is constructed at level α.

For the purpose of this definition, let

$$\langle X,Y,U,Z\rangle = \{0\}\times X \cup \{1\}\times Y \cup \{2\}\times U \cup \{3\}\times Z$$

Definition of G. Each $G(\alpha)$ will be a tuple $\langle X_1^\alpha, H_1^\alpha, X_2^\alpha, H_2^\alpha \rangle$ where H_i^α is a partial function on X_i^α, X_1^α a family of spaces indexed over x^f and X_2^α a family of spaces indexed over u^f.

If λ is a limit ordinal, we let $X_i^\lambda = \bigcup_{\gamma<\lambda} X_i^\gamma$, $H_i^\gamma = \bigcup_{\gamma<\lambda} H_i^\gamma$.
(It will follow from the construction that this makes sense.)

To compute $G(\alpha+1)$ we regard two cases

Case 1. If $X_1^\alpha = \mathrm{Spec}(H_1^\alpha;x)$ or $X_2^\alpha = \mathrm{Spec}(H_2^\alpha;u)$, let $G(\alpha+1) = G(\alpha)$.

Case 2. Otherwise. We define $X_1^{\alpha+1}$ as follows.

For $x_1 \in x^f$, let

$$(X_1^{\alpha+1})_{x_1} = \{\{s\}^{E(H_1^\alpha)}(\vec{x}_1,x);\ s\in\omega \wedge \|\langle s,\vec{x}_1,x\rangle\|^{E(H_1^\alpha)} \leq \alpha\}$$

For $u_1 \in u^f$, let

$$(X_2^{\alpha+1})_{u_1} = \{\{s\}^{E(H_2^\alpha)}(\vec{u}_1,u);\ s\in\omega \wedge \|\langle s,\vec{u}_1,u\rangle\|^{E(H_2^\alpha)} \leq \alpha\}$$

For $y \in X_1^{\alpha+1}$, let

$$H_1^{\alpha+1}(y) = z \quad \text{if} \quad \exists w \in X_2^{\alpha+1}(\{e\}(\langle e_2,y,x,\vec{y}\rangle,\langle d_2,w,u,\vec{v}\rangle) = 0 \wedge \{e_2\}(y,x,\vec{y}) = z)$$

For $w \in X_2^{\alpha+1}$ let

$$H_2^{\alpha+1}(w) = q \quad \text{if} \quad \exists y \in X_1^{\alpha+1}(\{e\}(\langle d_2,w,u,\vec{v}\rangle,\langle e_2,y,x,\vec{y}\rangle) = 0 \wedge \{d_2\}(w,u,\vec{v}) = q)$$

It is E-recursive to decide between case 1 and 2. Letting $G(0) = \langle \emptyset,\emptyset,\emptyset,\emptyset \rangle$ we use the induction hypothesis and the assumption to prove the following by induction on α:

a X_1^α is an initial segment of $\mathrm{Spec}(\lambda y\{e_2\}(y,x,\vec{y});x)$ if the latter exists b X_2^α is an initial segment of $\mathrm{Spec}(\lambda w\{d_2\}(w,u,\vec{v});u)$ if the latter exists

c For $y \in X_1^\alpha$,

$\lambda w \in X_2^\alpha(\{e\}(\langle e_2,y,x,\vec{y}\rangle,\langle d_2,w,u,\vec{v}\rangle)$ is total, and if the value of the computation is 0 for some w then $\{e_2\}(y,x,\vec{y})\downarrow$

d For $w \in X_2^\alpha$

$\lambda y \in X_1^\alpha(\{e\}(\langle d_2,w,u,\vec{v}\rangle,\langle e_2,y,x,\vec{y}\rangle)$ is total, and if the value of the computation is 0 for some y then $\{d_2\}(w,u,\vec{v})\downarrow$

e For at least one $i \in \{1.2\}$ is H_i^α total on X_i^α.

The proof is straightforward.

So G will be recursive in $\tau,\sigma,x,u,\vec{y},\vec{v}$ and G will be total.

Let $M = \mathrm{Spec}(G;x \cup u \cup \{x\} \cup \{u\})$.

<u>Claim</u>. At least one spectrum obtained by σ or τ will be included in M.

<u>Proof</u>. Let $\alpha = \mathrm{On} \cap M$. We regard two cases.

<u>Case 1</u>. In defining $G(\alpha+1)$ we are in case 1.

By symmetry we may assume that $X_1^\alpha = \mathrm{Spec}(H_1^\alpha)$.

Since $G \cap M$ is definable from G and M we see that $H_1^\alpha \cap M$ is definable from G and M. But then $\mathrm{Spec}(H_1^\alpha) \subseteq M$, so $X_1^\alpha \subseteq M$.

<u>Case 2</u>. In defining $G(\alpha+1)$ we are in case 2.

By the argument of case1, this actually means that both H_1^α and H_2^α are partial. But this is impossible by e above.

By inspection of M and $M \cap G$ one may decide the proper value of $\{e\}(\sigma,\tau)$.

The other cases where scheme 8 is involved, are treated in the same manner, by a recursive function G one tries to imitate the construction of the actual spectrum until either the spectrum is completed or the other computation terminates.

This ends the proof of lemma 3.

As usual we now obtain Gandy's selection theorem for numbers, and that a set is recursive in some parameters if and only if both it and its complement are semirecursive in the parameters. We also have established sufficient properties to claim that $S^{k+2}(F)$-recursion theory over $tp(k)$ is equivalent to Harrington's strong recursion in ${}^{k+3}S$, ${}^{k+2}F$.

4. Functions and relations

One of the properties of set-recursion is that for any relation R, there is a function F of type $k+2$ such that $E(R)$-recursion over type k is the same as $E(F)$-recursion over type k.
This is established for $k=0$ in Harrington-Kechris-Simpson [7] and for $k>0$ independently in Kechris [8] and Normann [10]. For a general proof, see Normann [11].

We will prove that this result also is true for S-recursion. We prove the result for $I = tp(k)$, but it may be proved with the same generality as the parallell result in Normann [11].

Definition. Let R be a relation. Define F_R by

$$F_R(f) = \begin{cases} 0 & \text{if } f \text{ is a code for a set } x \text{ and } R(x) \\ 1 & \text{otherwise.} \end{cases}$$

where we say that $f: I \to \omega$ is a code if f is the characteristic function of a code.

Lemma 4. F_R is S-recursive in I relative to R.

Proof. By lemma 1, the relation ' f is a code' is ${}^{k+3}$S-recursive, and thus S-recursive by theorem 2. In E-recursion there is an index e such that if f is a code for x, then $\{e\}(f,I) = x$. The lemma then follows.

We cannot prove the other direction of the lemma, since R may contain information about sets not codable. But for our purpose it will be sufficient to do so for $R \cap$ S-Spec(R).

Lemma 5. In S(R)-recursion there is an index e such that if for some $a \in I$ and some x, $\{e_1\}^{S(R)}(a,I) = x$, then $\{e\}^{S(R)}(e_1,a,I)$ is a code for x.

Proof. We define e by the recursion-theorem. We may use the same argument as in lemma 2, except in case 6, relativization to R. But there we may use lemma 4.

Theorem 3.

a S-Spec(F_R) = S-Spec(R) for any relation R

b For $a \in I$, $A \subseteq I$

A is S(R)-recursive in $a,I \iff A$ is S(F_R)-recursive in a,I

($\iff A$ is Kleene-recursive in $a, F_R, {}^{k+3}S$)

c For $a \in I$, $A \subseteq I$

A is S(R)-semirecursive in $a,I \iff A$ is S(F_R)-semirecursive in a, I

($\iff A$ is strongly semirecursive in $a, F_R, {}^{k+3}S$)

<u>Proof</u>.

<u>a</u> Let $x = \{e_1\}^{S(F_R)}(a,I)$. By lemma 4, there is an index e_2 such that $x = \{e_2\}^{S(R)}(a,I)$.
Now, let $y = \{e_2\}^{S(R)}(a,I)$. From the proof of lemma 5 we see that the construction of a code for y is actually a computation in F_R. But then y is $S(F_R)$-recursive in a,I.

<u>b</u> This is just a special case of <u>a</u>.

<u>c</u> To obtain <u>c</u> we need the following claim:
A subset A of I is $S(R)$-semirecursive in a,I if and only if there is a set Q recursive in a,I such that

$$b \in A \iff \exists x \in SM_{\langle a,b\rangle}(R)\ Q(b,x)$$

<u>Proof</u>. If $A = \{b\,;\,\{e\}^{S(R)}(a,b,I)\downarrow\}$, let

$$Q(b,x) \iff x \text{ is a computation-tree for } \{e\}^{S(R)}(b,a,I).$$

On the other hand, let Q be given and let A be defined from Q as above. By the Gandy selection operator obtained from lemma 3, we describe a partial function recursive in a,I and defined just on A.

5. <u>Equivalences to the S(R)-theories</u>

The following considerations are valid for most notions of computation-theories, see e.g. Fenstad [1] or [2] or Moldestad [9]. So, let Θ be a computation-theory on a computation domain I. We say that Θ is p-normal if we Θ-recursively may compare lengths of computations in Θ, i.e. Θ satisfies lemma 3 of this paper. If Θ is p-normal, Θ will allow a selection operator for numbers (Grilliot [4], see also Moldestad [9]).

Definition. Let Θ be a computation-theory on the computation-domain I. We call Θ weakly normal if $=$ on I is Θ-recursive, Θ is p-normal and there is an index e such that

$$\{e\}_\Theta(e_1,\vec{a}) = \begin{cases} 0 & \text{if } \forall b \in I \ \{e_1\}_\Theta(\vec{a},b) = 0 \\ 1 & \text{if } \forall b \in I \ \{e_1\}_\Theta(\vec{a},b)\downarrow \text{ and } \exists b \in I \ \{e_1\}_\Theta(\vec{a},b) \neq 0 . \end{cases}$$

We define the notion of a code over I as in section 3, and we define <u>$\mathrm{Spec}(\Theta)$</u> $= \langle M_a(\Theta)\rangle_{a \in I}$ by

$$x \in M_a(\Theta) \iff \text{there is a code for } x \text{ that is } \Theta[a]\text{-recursive.}$$

Let $R_\Theta = \{\langle\sigma,\alpha\rangle;\ \sigma \in \Theta \wedge \|\sigma\|_\Theta = \alpha\}$.
We call $\langle \mathrm{Spec}(\Theta), R_\Theta\rangle$ the <u>companion of Θ</u>.

<u>Remark</u>. $\mathrm{Spec}(\Theta)$ will be $E(R_\Theta)$-recursively closed and satisfies $\Sigma^*(R_\Theta)$-collection.

<u>Lemma 6</u>. A subset A of I is $\Theta[a]$-semi-computable if and only if it is $\Sigma^*(R_\Theta,a)$ definable over $\mathrm{Spec}(\Theta)$.

<u>Proof</u>. Assume A is $\Theta[a]$-semicomputable,

$$b \in A \iff \{e\}_\Theta(a,b)\downarrow \text{ for some } a .$$

Then

$$b \in A \iff \exists n \in \mathbb{N} \ \exists \alpha \in M_{a,b}(\Theta)(\langle\langle e,a,b,n\rangle,\alpha\rangle \in R_\Theta)$$

On the other hand, let A be $\Sigma^*(R_\Theta,a)$-definable. Let φ be a Δ_o-formula such that

$$\begin{aligned} b \in A &\iff \exists x \in M_{a,b}(\Theta)\, \varphi(x,a,b,R_\Theta) \\ &\iff \exists e \in \mathbb{N}\ (e \text{ is an index for a code for a set } x \\ &\qquad\qquad \text{such that } \varphi(x,a,b,R_\Theta) . \end{aligned}$$

Given a code for x, we may decide $\varphi(x,a,b,R_\Theta)$ $E(R_\Theta)$-recursively in that code. The relation 'A is a code' is semirecursive in E-recursion, and we may compute x uniformly in a code for x. By Gandy-selection we see that A is semirecursive.

Definition.

a Let α be an ordinal. Let $f: \alpha \to \alpha$. We call f normal if f is strictly monotone and continuous.

b Let $\langle M_a \rangle_{a \in I}$ be a family of structures indexed over I, R a relation. $\langle M_a \rangle_{a \in I}$ is R-admissible over I if each M_a is rudimentary closed in R, and $\langle M_a \rangle_{a \in I}$ satisfies Σ^*-collection over I. A function $f: M \to M$ is closed in $\langle M_a \rangle_{a \in I}$ if for each $a \in I$, if $x \in M_a$ then $f(x) \in M_a$.
f is $w - \Delta^*$ if the graph of f is weakly Δ^*-definable.

c Let $\langle M_a \rangle_{a \in I}$ be a family R-admissible over I.
$\langle M_a \rangle_{a \in I}$ is weakly R-Mahlo if the following is satisfied:

Let $\alpha = \sup(On \cap M_a ; a \in I)$

Let $f: \alpha \to \alpha$ be normal, closed in $\langle M_a \rangle_{a \in I}$ and weakly $\Delta^*(R)$.

Then there is a family $\langle N_a \rangle_{a \in I}$ R-admissible over I such that f is closed in $\langle N_a \rangle_{a \in I}$, for each a $N_a \subseteq M_a$, and for at least one $a \in I$, the inclusion is proper.

(We write this $\langle N_a \rangle_{a \in I} \subsetneq \langle M_a \rangle_{a \in I}$)

The following theorem is proved with various degrees of generality in Harrington-Kechris-Simpson [7], Normann [10], Kechris [8] and Moldestad [9].

Theorem 4. Let Θ be a weakly normal computation-theory on I. Then Θ is equivalent to E(R)-recursion in I for some R if and only if Spec(Θ) is not weakly R_Θ-Mahlo.

We will prove a similar result for S-recursion.

<u>Definition</u>. Let $\langle M_a \rangle_{a \in I}$ be R-admissible over I.

<u>a</u> $\langle M_a \rangle_{a \in I}$ is <u>strongly R-Mahlo</u> if the following is satisfied: Let $f : \alpha \to \alpha$ be normal, closed in $\langle M_{\langle a,b \rangle} \rangle_{b \in I}$, f is $w - \Delta_a^*(R)$. Then there is a family $\langle N_b \rangle_{b \in I}$ that is R,a-admissible over I such that f is closed in $\langle N_b \rangle_{b \in I}$ and $\langle N_b \rangle_{b \in I} \in M_a$.

<u>b</u> $\langle M_a \rangle_{a \in I}$ is <u>weakly hyper-R-Mahlo</u> if the following is satisfied: Let $f : \alpha \to \alpha$ be normal, closed in $\langle M_a \rangle_{a \in I}$ and $w - \Delta^*(R)$. Then there is a strongly R-Mahlo family $\langle N_a \rangle_{a \in I} \subsetneq \langle M_a \rangle_{a \in I}$ such that f is closed in $\langle N_a \rangle_{a \in I}$.

<u>Remark</u>. If $k = 0$, then $I = \omega$ so $M_a = M_b$ for all $a,b \in I$. Then these notions coincide with admissible, recursively Mahlo and recursively hyper Mahlo. There will be no distinction between the weak and strong Mahlo-property.

<u>Lemma 7</u>. Let R be an arbitrary relation. S-Spec(R;I) is the least strongly R-Mahlo family over I.

<u>Proof</u>.

<u>i</u> S-Spec(R;I) is strongly R-Mahlo. Let S-Spec(R;I) = $\langle M_a \rangle_{a \in I}$.

<u>Proof</u>. Let f be $\Delta_a^*(R)$, normal and closed in $\langle M_{\langle a,b \rangle} \rangle_{b \in I}$. By the Gandy selection operator for numbers, f is S(R)-recursive and there is an S(R)-recursive function $f_1 : On \to$ Codes such that $f_1(\alpha)$ is a code for $f(\alpha)$.

$$\text{Let } G(\alpha,b) = \begin{cases} 1 & \text{if } b \in f_1(\alpha) \\ 0 & \text{if } b \notin f_1(\alpha) \end{cases}$$

Then Spec(R,G,a ; I) $\in M_a$. Spec(R,G,a ; I) will have the wanted properties.

<u>ii</u> S(R)-recursion is closed within any strongly R-Mahlo family locally of type k+1 (or type I) (i.e. $x \in M_a \iff x$ has a code in M_a).

<u>Proof</u>. Let $\langle M_a \rangle_{a \in I}$ be strongly R-Mahlo. By induction on the length of computations we will prove that the partial function $\lambda e, \vec{x}\ \{e\}^{S(R)}(\vec{x})$ is closed in $\langle M_a \rangle_{a \in I}$ and that the relation $\{e\}^{S(R)}(\vec{x}) \simeq y$ is w-Σ^*-definable in $\langle M_a \rangle_{a \in I}$, by proving that the computation-tree also will be in M_a when $\vec{x} \in M_a$.
For all schemes except scheme 8 this is known from E-recursion.
So assume $\{e\}^{S(R)}(\vec{x}) = \mathrm{Spec}(\lambda y \{e_1\}(y,\vec{x}); x_1)$.
By the induction hypothesis, the function $\lambda y \{e_1\}(y,\vec{x})$ will be w-Σ^*-definable. in the parameters. Define f by

$$f(0) = 1 \quad f(\lambda) = \bigcup_{\gamma < \lambda} f(\gamma) \text{ when } \lambda \text{ is a limit.}$$

Let $f(\alpha+1)$ be the least ordinal $\gamma \geq f(\alpha)+1$ such that we in order to define $\mathrm{Spec}_{f(\alpha)}(\lambda y\{e_1\}(y,\vec{x});x_1)$ only need computations $\{e_1\}(y,\vec{x})$ of length $< \gamma$.

We use the fact that $\langle M_a \rangle_{a \in I}$ is locally of type k+1 to prove that whenever the parameters are in M_a, then f is closed in $\langle M_{\langle a,b \rangle} \rangle_{b \in I}$ and f is w-Δ_a^*-definable. Let $\langle N_b \rangle_{b \in I} \in M_a$ be R-admissible such that f is closed in $\langle N_b \rangle_{b \in I}$. Then $\lambda y \{e_1\}(\vec{x},y)$ is w-Δ^*-definable over $\langle N_b \rangle_{b \in I}$, so $\mathrm{Spec}(\lambda y\{e_1\}(\vec{x},y);x_1)$ is a definable subfamily of $\langle N_b \rangle_{b \in I}$ and thus an element of M_a. By a similar argument we see that the computation-tree will be in M_a.

In theorem 4 we used the notion of equivalent theories. We must make this notion precise.

Definition. Let Θ_1 and Θ_2 be two weakly normal theories on I (E and S-recursion restricted to I may be regarded as such theories)

Θ_1 and Θ_2 are equivalent if $\text{Spec}(\Theta_1) = \text{Spec}(\Theta_2)$

and R_{Θ_1} and R_{Θ_2} are Δ^*-definable in each other.

The two theories will be equivalent iff the semirecursive sets are the same.

Theorem 5. Let Θ be a weakly normal theory on I. Then the following two statements are equivalent:

i There is a relation R such that Θ is equivalent to S(R)-recursion over I.

ii $\text{Spec}(\Theta)$ is strongly R_Θ-Mahlo but not weakly R_Θ-hyper-Mahlo.

Corollary 5.1. Let $I = tp(k)$.

Let Θ be a weakly normal theory on I.

Then the two statements are equivalent:

i There is a type k+2-functional F such that Θ is equivalent to strong recursion in ${}^{k+3}S, F$.

ii $\text{Spec}(\Theta)$ is strongly R_Θ-Mahlo but not weakly hyper-R_Θ-Mahlo.

This is a consequence of theorems 3 and 5.

When $k = 0$, we are regarding normal theories over $\mathbb{N}$, and we then observe:

Corollary 5.2. Let Θ be a normal theory on ω. Then Θ is equivalent to strong recursion in ${}^2F, {}^3S$ for some F if and only if the companion of Θ is recursively Mahlo, but not recursively hyper-Mahlo.

Proof of theorem 5.

i => ii Let Θ and R be given, $\mathrm{Spec}(\Theta) = \text{S-Spec}(R;I) = \langle M_a \rangle_{a\in I}$, R_Θ and R are Δ^* in each other.

Θ and R will induce two different hierarchies on $\langle M_a \rangle_{a\in I}$, call them $\langle M_a^\alpha \rangle_{a\in I}$ from Θ and $\langle N_a^\alpha \rangle_{a\in I}$ from S(R). Let R be defined from R_Θ by 'the Δ^*-formula Φ' and R_Θ from R by 'the Δ^*-formula Ψ'.

Define $g_1(\alpha) = \mu\beta \geq \alpha$ such that $R \cap \langle M_a^\alpha \rangle_{a\in I}$ is definable over $\langle M_a^\beta \rangle_{a\in I}$ by Φ.

Define $g_2(\alpha) = \mu\beta \geq \alpha$ such that $R_\Theta \cap \langle N_a^\alpha \rangle_{a\in I}$ is definable over $\langle N_a^\beta \rangle_{a\in I}$ by Ψ.

g_1 and g_2 will be Δ^*-definable and closed in $\langle M_a \rangle_{a\in I}$. Moreover, at fix-points λ for g_1 resp. g_2, $R \cap \langle M_a^\lambda \rangle$ will be $\Delta^*(R_\Theta)$-definable over $\langle M_a^\lambda \rangle_{a\in I}$ (resp. statement for R_Θ, $\langle N_a^\lambda \rangle_{a\in I}$ and R).

We may now show that $\langle M_a \rangle_{a\in I}$ have the same Mahlo-properties w.r.t. R and R_Θ.

Let f be a normal Δ^*-function. Let $f_1 = f\circ g_1 \circ g_2$.
f_1 is normal, Δ^*-definable and if f is closed in $\langle M_a \rangle_{a\in I}$, f_1 is closed in $\langle M_a \rangle_{a\in I}$.
Moreover, a fix-point for f_1 will be a fix-point for f, g_1 and g_2.

Claim 1. $\mathrm{Spec}(\Theta)$ is strongly R_Θ-Mahlo.

Proof. Let f be a normal Δ_a^*-function closed in $\langle M_{a,b} \rangle_{b\in I}$. Since $\langle M_a \rangle_{a\in I}$ is strongly R-Mahlo (lemma 7), f_1 is closed in an R-admissible subfamily $\langle N^\alpha_{\langle a,b\rangle} \rangle_{b\in I} \in M_a$. But α is a fix-point of f_1, so R_Θ is Δ^* over $\langle N^\alpha_{\langle a,b\rangle} \rangle_{b\in I}$, so $\langle N^\alpha_{\langle a,b\rangle} \rangle$ is R_Θ-admissible.

Claim 2. Spec(Θ) is not weakly hyper-R_Θ-Mahlo.

Proof. Assume it is. By the argument of claim 1 it follows that S-Spec(R,I) is weakly hyper-R-Mahlo, which is false by lemma 7.

ii => i Assume that Spec(Θ) is strongly R_Θ-Mahlo but not weakly R_Θ-hyper-Mahlo. Let f be a counterexample to hyper-Mahloness. Define

$$R_\gamma = \{\langle a,b,\gamma\rangle ; \ a \text{ and } b \text{ are } \Theta\text{-computations and } \|a\|_\Theta < \|b\|_\Theta < f(\gamma)\}. \text{ Let } R = \bigcup_{\gamma \in On} R_\gamma . \ R \text{ is clearly } \Delta^*(R_\Theta) \text{ over Spec}(\Theta).$$

Claim 3. Spec(Θ) is the least strongly R-Mahlo family over I.

Proof. Since Spec(Θ) is strongly R_Θ-Mahlo and R is $\Delta^*(R_\Theta)$ we may use the arguments from i => ii to see that Spec(Θ) is strongly R-Mahlo. On the other hand, if there is a subfamily $\langle N_a\rangle_{a\in I}$ that is strongly R-Mahlo, it is sufficient to prove that f is closed in $\langle N_a\rangle_{a\in I}$ and that R_Θ is $\Delta^*(R)$-definable over $\langle N_a\rangle_{a\in I}$.

a f is closed in $\langle N_a\rangle_{a\in I}$.
Let $\alpha \in N_a$. Then $\{\langle a,b\rangle : a \text{ and } b \text{ are } \Theta\text{-computations and } \|a\|_\Theta < \|b\|_\Theta < f(\alpha)\} \in N_a$ since N_a is rudimentary closed in R. But this is a prewellordering of length $f(\alpha)$, so $f(\alpha) \in N_a$.

b R_Θ is $\Delta^*(R)$-definable over $\langle N_a\rangle_{a\in I}$:
$\langle a,\alpha\rangle \in R_\Theta$ if for some $b \in I$, $\langle a,b,\alpha+1\rangle \in R$ and a has rank α in the prewellordering $\{\langle c,d\rangle ; \langle c,d,\alpha+1\rangle \in R\}$.

This proves claim 3.

By lemma 7, ii => i will hold, and theorem 5 is established.

6. The sections of S(R)-theories

In Sacks [12] and [13] the notion of abstract k+1-sections is defined and it is proved that they are exactly the k+1-sections of normal functionals ^{k+2}F.

In this section we will give a similar characterization of the k+1-section of ^{k+3}S, ^{k+2}F, i.e. the subsets of tp(k) recursive in ^{k+3}S, ^{k+2}F. Both characterizations and proofs are suitable adjustments of the arguments of Sacks.

In this section we will restrict ourselves to recursion in $I = tp(k)$. The b-part may however always be generalized to normal recursion on two domains (Moldestad [9]).

Definition of Abstract ^{k+3}S-section

a k = 0. A is an abstract ^{3}S-section if

i A is an abstract 1-section (Sacks [12]) i.e.

Each element in A has a code in A

A is admissible and satisfies Δ_0 - DC

ii If φ is a Δ_0-formula and $\vec{y} \in A^n$ and

$A \models \forall x \exists y\, \varphi(x,y,\vec{y})$

then there is an abstract 1-section B such that

$\vec{y} \in B \in A$ and $B \models \forall x \exists y\, \varphi(x,y,\vec{y})$

b k > 0. A is an abstract ^{k+3}S-section if there exists a set B such that $A \in B$, A is countable in B, $A \prec_{\Sigma_1} B$ and both A and B have the following properties:

i They are rudimentary closed and satisfies Σ_1-collection (admissible with gaps) and are locally of type k+1.

ii They are closed under S-recursion.

<u>Remark</u>. <u>i</u> and <u>ii</u> play the same rôle in both definitions. <u>ii</u> gives the appropriate variant of the Mahlo-property.

<u>Theorem 6</u>. Let F be of type $k+2$. Let $x \in A \iff x$ has a code recursive in $F, {}^{k+3}S$. Then A is an abstract ${}^{k+1}S$-section.

<u>Proof</u>. Let $\langle M_a \rangle_{a \in I} = S\text{-}Spec(F)$. $A = M_a$ for a recursive.

<u>a</u> <u>$k = 0$</u>. Here $A = L^F_{\rho^F}$, where ρ^F is the least ordinal recursively Mahlo in F (Harrington [5]). Assume

$$A \models \forall x \, \exists y \, \varphi(x,y,\vec{y}) .$$

Define

$$g(\gamma) = \mu\beta \, \forall x \in L^F_\gamma \, \exists y \in L^F_\beta \, \varphi(x,y,\vec{x}) .$$

g is closed in some F-admissible ordinal α_0, and $L^F_{\alpha_0}$ will be an abstract 1-section.

<u>b</u> <u>$k > 0$</u>. Let c be a complete $S(R)$-semirecursive subset of ω. Let $B = M_c$. By Gandy's selection operator for numbers we may use a proof due to Harrington [6] (see also Moldestad [9]) to see that $A <_{\Sigma_1} B$. Since c in a way acts as an enumeration of A, $A \in B$ and A is countable in B. Clearly both A and B are closed under S-recursion.

This proves theorem 6.

<u>Theorem 7</u>. If A is an abstract ${}^{k+3}S$-section, then there is some functional F of type $k+2$ such that A is the $k+1$-section of $F, {}^{k+3}S$.

<u>Proof</u>. By theorem 3 it is sufficient to find a relation P such that A is the sets $S(P)$-recursive in $tp(k)$.

<u>a</u> <u>$k = 0$</u>. We want to find P such that A is the least P-recursively Mahlo structure.

Define a set of conditions $\mathbb{P}$ by

$p \in \mathbb{P}$ if $p \subseteq On$, $p \in A$ and no ordinal $\leq$ rank(p) is p-recursively Mahlo. $p \leq q$ if $p = q \cap rn(p)$.

If φ is a Δ_0-formula with parameters $\vec{x}$, and $p \in \mathbb{P}$, we say $p \Vdash \varphi(\vec{x},P)$ if $L^p_{rn(p)} \models \varphi(\vec{x},p)$.

For other formulas

$p \Vdash \exists x \varphi$ if for some $x \in L^p_{rn(p)}$ $p \Vdash \varphi(x)$

$p \Vdash \varphi \vee \psi$ if $p \Vdash \varphi$ or $p \Vdash \psi$

$p \Vdash \neg\varphi$ if $\forall q \geq p$ $q \not\Vdash \varphi$ (q does not force φ)

Let P be $\mathbb{P}$-generic over A (or actually the union of a $\mathbb{P}$-generic set from $\mathbb{P}$).

i $L^P_{rn(P)} = A$.

Proof. $L^P_{rn(P)} \subseteq A$ since each $p \in A$.
Let $x \in A$, and let $y \subseteq \omega$ be a code for x.
Let p be any condition. Let $q = p \cup \{rn(p)+n ; n \in y\}$
Then $y \in L^q_{rn(q)+1}$. Since P is generic, y will be in $L^p_{rn(p)}$ for some initial $p \subseteq P$, and since rn(P) is admissible, $y \in L^P_{rn(P)}$.

ii rn(P) is P-recursively Mahlo.

Proof. Assume $\langle A,P\rangle \models \forall x \exists y \varphi(x,y,P)$.
Since P is generic, there is an initial $p \subseteq P$ such that

$p \Vdash \forall x \exists y \varphi(x,y,P)$,

so $\forall q \geq p \forall x \exists r \geq q$ $r \Vdash \exists y \varphi(x,y,P)$.
Let $B \in A$ be an abstract 1-section such that $p \in B$ and $B \models \forall q \geq p \forall x \exists r \geq q$ $r \Vdash \exists y \varphi(x,y,P)$.

Let p' be an extention of p that is generic over B.

Then $B = L^{p'}_{rn(p')}$. Let $\beta \geq rn(p)$ be the least admissible ordinal such that

$$L^{p'}_{\beta} \models \forall x \exists y\, \varphi(x,y,p').$$

Let $q = p' \cap \beta$. Then $q \in \mathbb{P}$, $q \geq p$ and for any proper extention $r > q$,

$$r \Vdash \exists B \;(B \text{ is admissible in } P \text{ and } \forall x \in B\; \exists y \in B\; \varphi(x,y,P)).$$

Since P is generic, this shows a variant of recursive Mahloness.

b <u>$k > 0$</u>. Define a set of conditions $\mathbb{P}_A$ ($\mathbb{P}_B$) by

$p \in \mathbb{P}_A$ if $p \in A$, $p \subseteq On \times I$ and for $\alpha = rn(p)$,

α is $S(p)$-recursive in I (without parameters outside ω)

$q \geq p$ if q is an end-extention of p (i.e. $\langle \beta, a \rangle \in q \setminus p \Rightarrow \beta \geq rn(p)$).

Let P be $\mathbb{P}_A$-generic over A. (It will be clear from the argument what sort of generity we need.) We may assume that $P \in B$ since A is countable in B.

Let $\langle N_a \rangle_{a \in I} = S\text{-Spec}(P,I)$, o recursive. We will prove that $N_o = A$.

i $A \subseteq N_o$.

<u>Proof</u>. Let $x \in A$, p a condition. Let y be a code for x.

Let $q = p \cup \{(rn(p),a) ; a \in y\}$.

Clearly q is a condition, and if P extends q, $rn(p) \in N_o$ (since p is a condition) and $y \in N_o$. Then $x \in N_o$ as well. Since P is generic, $x \in N_o$.

ii $N_o \subseteq A$.

Let $x \in N_o$. Then $x = \{e\}^{S(P)}(I)$ for some $e \in \omega$.

If this computation is

$$x = \{e\}^{S(p)}(I)$$

for some initial condition $p \subseteq P$, then $x \in A$ since A is closed under S-recursion.

If we need cofinally much information about P to compute x, the computation still takes place in B since $P \in B$ and B is closed under S-recursion. Let α be the supremum of the ordinals occuring in the computation, and let $P_0 = P \cup \langle \alpha, 0 \rangle$. Then $P_0 \in \mathbb{P}_B$.

Let $p \subseteq P$ be initial. Then

$$B \models \exists q \geq p \ \{e\}^{S(q)}(I)\downarrow .$$

Since $A <_{\Sigma_1} B$,

$$A \models \exists q \geq p \ \{e\}^{S(q)}(I)\downarrow .$$

This contradicts the fact that P is generic. So $N_0 \subseteq A$.

This ends the proof of theorem 7.

Corollary 7.1. Let Θ be a weakly normal computation-theory on I such that $\mathrm{Spec}(\Theta)$ is strongly R_Θ-Mahlo. Then for some functional F of type $k+2$, $k+1-\mathrm{sc}(\Theta) = k+1-\mathrm{sc}({}^{k+3}S, F)$.

Corollary 7.2. For each k, there is a normal functional F of type $k+2$ such that

$$k+1-\mathrm{sc}({}^{k+3}S, F) = k+1-\mathrm{sc}({}^{k+3}E).$$

References

1. J.E. Fenstad, Computation theories: An axiomatic approach to recursion on general structures, in: Lecture Notes in Mathematics no 499 (Springer-Verlag 1975) 143-169.

2. J.E. Fenstad, Monograph on generalized recursion theory, to appear.

3. R.O. Gandy, General recursive functionals of finite type and hierarchies of functions, Ann. Fac. Sci. Univ. Clermond-Ferrand No 35 (1967) 5-24.

4. T.J. Grilliot, Selection functions for recursive functionals, Notre Dame Jour. Formal Log. X (1969) 225-234.

5. L.A. Harrington, The Superjump and the first recursively Mahlo ordinal, in Fenstad-Hinman (eds) Generalized Recursion Theory (North Holland, Amsterdam 1974) 43-52.

6. L.A. Harrington, Contributions to recursion theory on higher types. Ph.D. Thesis, M.I.T. (1973).

7. L.A. Harrington, A.S. Kechris, S.G. Simpson, 1-envelopes of type 2 objects, American Math. Soc. Notices 20 (1973) A-587.

8. A.S. Kechris, The structure of envelopes, A survey of recursion in higher types, M.I.T. Logic Seminar Notes, December 1973.

9. J. Moldestad, Computations in higher types, Lecture Notes in Mathematics 574, Springer-Verlag 1977.

10. D. Normann, Imbedding of higher type theories, Preprint series in mathematics, No 16 - 1974, Oslo.

11. D. Normann, Set recursion, Proceedings from conference in generalized recursion theory, Oslo - 77.

12. G.E. Sacks, The 1-section of a type n object, in: Fenstad-Hinman (eds) generalized Recursion Theory (North-Holland, Amsterdam 1974) 81-93.

13. G.E. Sacks, The k-section of a type-n object, to appear.