THE LOCAL CAUCHY PROBLEM IN $\mathbb{R}^2$ AT A POINT WHERE TWO CHARACTERISTIC CURVES HAVE A COMMON TANGENT

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1. Introduction

We shall consider a linear partial differential operator in $\mathbb{R}^2$ with principal part

$$P(x,D) = x_2^2 D_1^2 - D_2^2$$

where $D_1 = \partial/\partial x_1$ and $D_2 = \partial/\partial x_2$.
The interesting feature of this operator is that its characteristics are simple when $x_2 \neq 0$ but double when $x_2 = 0$. The characteristic curves though the point $(c,0)$ are the parabolae

$$x_1 = \pm\, x_2^2/2 + c \ .$$

They have a common tangent at $(c,0)$. We shall be interested in the Cauchy problem for such operators when data are given on a curve characteristic at the origin.

The very first published results on this problem seem to be the following two theorems by F. Treves [18] concerning the operator

$$(1.1)\quad P_\lambda(x,D) = x_2^2 D_1^2 - D_2^2 - \lambda D_1 \ ,$$

where λ is a real parameter.

Theorem 1.1 (Treves [18, Theorem II]). If λ is an odd positive integer then there exist C^∞-functions u in $\mathbb{R}^2$ such that

$$P_\lambda u = 0, \text{ supp } u = \{x;\ x_1 \geq x_2^2/2\} .$$

Theorem 1.2 (Treves [18, Theorem I]). Suppose that λ in (1.1) is real but not an odd positive integer. Let $\Omega \subset \mathbb{R}^2$ be any open set, and $F \subset \Omega$ a closed subset such that for some real a the set

$$K = \{x;\ x \in \Omega,\ x_1 \leq a\} \cap F$$

is compact. Then there exist an integer $m \geq 2$ depending solely on λ and a neighbourhood U of K such that any function $u \in C^m(\Omega)$ satisfying

$$(1.2) \quad P_\lambda u = 0, \quad \text{supp } u \subset F$$

must vanish in U .

Treves remarks that it seems likely that m can be chosen equal to 2 for all λ [18, footnote p. 230]. We shall prove a stronger result.

Theorem 1.3 Let P_λ, Ω, F and K be as in the hypothesis of of Theorem 1.2. Then there is an open set $U \supset K$ such that any $u \in \mathcal{D}'(\Omega)$ fulfilling (1.2) must vanish in U. The proof is given in Section 2. We use Schwartz' structure theorem for distributions. This idea has been used earlier in proofs of uniqueness theorems See J. Persson [8]. In [9] J. Persson conjectures that uniqueness cones can always be used to decide whether there is uniqueness in the local Cauchy problem when the coefficients are analytic. Theorem 1.3 shows that this conjecture is false.

Theorem 1.1 gives us null solutions with support in $\{x;\ x_1 \geq x_2^2/2\}$ when λ is an odd positive integer. Theorem 1.4 below shows that in a somewhat modified solution space we have "null solutions" for all real λ.

<u>Theorem 1.4</u> There exist continuous functions $u \neq 0$ defined on $\mathbb{R}$ with values in the space $H'(\mathbb{C})$ of analytic functionals over $\mathbb{C}$ such that with $P_\lambda u$ defined in the natural way

$$u(x_1) = 0,\ x_1 \leq 0,\ P_\lambda u = 0,\ (x_1,\ x_2) \in \mathbb{R}\times\mathbb{C}\ .$$

Moreover for each $x_1 \geq 0$ the functional $u(x_1)$ is carried by the set

$$\{x_2\colon\ x_2 \in \mathbb{C},\ |x_2| \leq \sqrt{2x_1}\ \}\ .$$

As to the definition of analytic functionals and elementary facts about them we refer the reader to F. Treves [17 Chap. 9]. From Theorem 1.3 we see that u in Theorem 1.4 cannot be a distribution unless λ is an odd positive integer. We prove Theorem 1.4 in Section 3.

The next problem which we consider is whether there exist non-trivial solutions of $P_\lambda(D)u = 0$ with supports to the right of the leftmost characteristic curve through the origin. It turns out that such solutions always exist at least locally. We shall prove a theorem for a more general situation giving this result as a special case. At first we supplement our notation by letting $|x| = \max(|x_1|,|x_2|)$, $x \in \mathbb{R}^2$ For $\xi = (\xi_1,\xi_2) \in \mathbb{Z}^2$, $\xi_j \geq 0$, and $D = (D_1,D_2) = (\partial/\partial x_1, \partial/\partial x_2)$ we let $D^\xi = D_1^{\xi_1} D_2^{\xi_2}$. For $d = (d_1, d_2) \in \mathbb{R}^2$, $d_1 \geq 1, d_2 \geq 1$ we let $\xi d = \xi_1 d_1 + \xi_2 d_2$. We also let $|\xi| = \xi_1 + \xi_2$ and $\xi d^{\xi d-1} = 1$, $\xi d = 0$, and $(\xi d-2)^{\xi d-3} = 1$, $0 \leq \xi d - 2 \leq 1$.

We notice that our restrictions on ξ and d imply that $\xi \to \xi d^{\xi d-1}$ and $d \to \xi d^{\xi d-1}$ both are non-decreasing.

Now we are ready to state our last theorem.

Theorem 1.5 Let $\varepsilon > 0$ and let $\Omega = \{x;\ x \in \mathbb{R}^2,\ |x| < \varepsilon\}$. Let $a_\alpha \in C^\infty(\Omega)$, $|\alpha| \leq 1$ and $b(x_2) \in C^\infty(\{x_2; |x_2| < \varepsilon\})$. Let $m_\alpha \geq 0$, $|\alpha| \leq 1$, $m > 0$, and r, $0 < r \leq 1$, and $d_1 = d_2$, $1 < d_1 \leq 2$ be constants. We assume that with $d = (d_1,d_2)$

(1.3) $|D^\xi a_\alpha(x)| \leq m_\alpha r^{-|\xi|} \xi d^{\xi d-1}$, $|\alpha| \leq 1$, all ξ, $x \in \Omega$,

and

(1.4) $|D_2^j b(x_2)| \leq mr^{-|\xi|} \xi d^{\xi d-1}$, all $j \geq 0$, $|x_2| < \varepsilon$.

Let

$$\Psi(t) = \int_0^t \min(0,b(s))ds,\ 0 \leq t < \varepsilon,$$

and

$$\Psi(t) = \int_0^t \max(0,b(s))ds,\ 0 < -t < \varepsilon.$$

Let

(1.5) $$P(x,D) = D_2(b(x_2)D_1 + D_2) - \sum_{|\alpha|\leq 1} a_\alpha(x)D^\alpha,\ x \in \Omega.$$

Then there exist a neighbourhood Ω_0 of 0 and a $u \in C^\infty(\Omega_0)$ such that

$$P(x,D)u = 0 \quad \text{in} \quad \Omega_0 \quad \text{and} \quad 0 \in \operatorname{supp} u \subset \{x;\ x_1 \geq \Psi(x_2)\} .$$

Remark 1. If $b(0) \neq 0$, then near the origin we may use new coordinates with the characteristic curves as coordinate axes. We may choose this system such that in this system

$$P(x,D) = D_1D_2 + a_1D_1 + a_2D_2 + a_0.$$

Then we solve the Goursat problem.

$$P(x,D)u = 0,\ u(x_1,0) = \varphi(x_1),\ u(0,x_2) = 0 .$$

Here we choose $\varphi(x_1) = 0,\ x_1 \leq 0,\ \varphi(x_1) > 0,\ x_1 > 0,\ \varphi \in \mathbb{C}^\infty(\mathbb{R})$. The corresponding solution u, see for instance J. Persson [7] and the proof of Theorem 2 there, has $\operatorname{supp} u \subset \{x;\ x_1 \geq 0\}$ and $0 \in \operatorname{supp} u$. This is Goursat's original construction of null solutions with data given on a characteristic line of the wave operator. It is not hard to show that $u \in \mathbb{C}^\infty$ too. Then we transform back and there we have our wanted solution.

Remark 2. The characteristic curves of (1.5) through $(c,0)$ are the line $x_1 = c$ and $x_1 = c + \varphi(x_2)$ where

$$(1.6) \quad \varphi(x_2) = \int_0^{x_2} b(t)\,dt.$$

They have a common tangent at $(c,0)$ precisely when $\varphi'(0) = b(0) = 0$.

Remark 3. After obvious modifications the theorem is also valid for operators which can be transformed into (1.5) by suitable coordinate changes. We mention two such operators:

a) The operator P_λ of (1.1) is transformed into (1.5) by the coordinate shift

$$x_1' = x_1 + x_2^2/2, \; x_2' = x_2.$$

b) If we allow the function b in (1.5) to be of the form $b(x) = b_1(x_1)b_2(x_2)$ with both b_j satisfying estimates of the form (1.4) the resulting operator is transformed back to (1.5) by

$$x_1' = \int_0^{x_1} (b_1(s))^{-1} ds, \; x_2' = x_2.$$

Then it can be shown that the coefficients of the lower order still satisfy estimates of (1.4) type.

The proof of Teorem 1.5 will be given in Section 4. Some soft and some harder auxiliary results are proved in Sections 5 and 6.

How then is Theorem 1.5 related to known results on non-uniqueness. The first construction of a null solution seems to be the one by Goursat already cited. In the constant coefficient case we refer to A. Tihonov [19], S. Täcklind [16], L. Hörmander [3, Theorem 3.2] or [4, Theorem 5.2.2, p. 121], J. Persson [12]. In the case of analytic coefficients we refer to [4, Theorem 5.2.1] when the initial hypersurface is simply characteristic and to J. Persson [10] [11] and [13] when the multiplicity of the initial hypersurface is arbitrary but constant. Later M. D. Bronštejn [2] has extended the results in [10] to non linear problems. H. Komatsu [6] has also constructed null solutions by another method. In all the literature cited above the initial characteristic hypersurface has constant multiplicity. If we let the data of $P_\lambda(D)u = 0$ be given on $x_1 = -x_2^2/2$ with P_λ from (1.1) then the multiplicity of the initial curves is 2 at $x = 0$ and 1 for $x \neq 0$. So this case is not contained in the results cited above. We allow the coefficients to be in non-analytic Gevrey classes in Theorem 1.5. In [10] it is indicated how one may weaken the hypothesis in this direction when the multiplicity of the characteristic initial surface is constant.

If the principal part vanishes on the initial hypersurface L. Hörmander [5, Theorem 2.2] has given some examples of null solutions when the coefficients are analytic. We do not intend to give any complete survey of results on uniqueness and non-uniqueness in the characteristic Cauchy problem but like to cite M.S. Baouendi and C. Goulaouic [1]. They have characterized other types of characteristic Cauchy problems where one cannot construct C^∞-null solutions.

Post Scriptum After this paper was completed we began to think on the construction in the proof of Theorem 1.5. We simply looked at it as the solution of the Cauchy problem when data are given on $x_2 = 0$ in the proper Gevrey class. Then we learned from Zentralblatt about V. Ja. Ivrij [E7] and his striking results on the Cauchy problem for operators with hyperbolic principal part. He treats the case when data are in Gevrey classes and his result covers our result in Theorem 1.5. We still think that our more direct construction and our point of view motivate its publication . Looking at the Cauchy problem in the x_2-direction we also enter into a long range of results. Here we have found no results giving room for one characteristic curve to oscillate around the other one as in Theorem 1.5. However they are more general in other aspects. We have enumerated some of these papers plus the paper by Ivrij in an extra reference list at the end of the references.

It also happened that the author B. B. tried to compute the best constant c in Lemma 5.1. He conjectured that $c = 4$ is the best one. Then Arne Strøm, Oslo, and Robert Fossum and Erik Sparre Andersen, Copenhagen showed us how to prove this fact which goes back to Abel. We thank all this people. Section 5 is rewritten accordingly.

2. Proof of Theorem 1.3

Let $K \subset \mathbb{R}^2$ and $\varepsilon > 0$, $c \in \mathbb{R}$. We define

$$K_\varepsilon = \{x;\ \text{dist}\,(x,K) \leq \varepsilon\}$$

and

$$\Omega_c = \{x;\ x_1 < c\}.$$

Now let the sets Ω, K, F and the number a be as in Theorems 1.2 and 1.3. We look at a distribution solution u of $P_\lambda u = 0$ in Ω with $\text{supp}\, u \subset F$. We want to show that $u = 0$ in some neighbourhood of K.

We start by choosing $\eta > 0$ so small that

$$K_{3\eta} \subset \Omega .$$

It follows that the closure of the set

$$(K_{3\eta} - K_{\eta}) \cap \{x;\ x_1 = a\}$$

is compact and disjoint from F. Thus it has a positive distance to F. Therefore we can choose a real number c such that $a < c < a + \eta$, and such that

$$K_{\eta} \cap F \cap \Omega_c = K_{3\eta} \cap F \cap \Omega_c \tag{2.1}$$

Now we use cut off functions to split u into a sum

$$u = u_1 + u_2,$$

where

(2.2) $\quad \operatorname{supp} u_1 \subset K_{2\eta} \cap F,$

and

$$\operatorname{supp} u_2 \cap K_{\eta} = \emptyset,\ \operatorname{supp} u_2 \subset F.$$

From (2.1) it than follows that

$$\operatorname{supp} u_2 \cap K_{3\eta} \cap \Omega_c = \emptyset. \tag{2.4}$$

Since

$$\operatorname{supp} P_\lambda u_1 \subset (\operatorname{supp} u_1) \cap (\operatorname{supp} u_2),$$

we also have

$$\operatorname{supp} P_\lambda u_1 \cap \Omega_c = \emptyset.$$

We extend u_1 by letting it be zero outside $K_{3\eta}$ to obtain $u_1 \in \mathcal{D}'(\mathbb{R}^2)$ and

(2.5) $P_\lambda u_1 = 0,\ x \in \Omega_c.$

It follows from Schwartz' theorem on the structure of distributions with compact support [14, Théorème 26, p. 91] and (2.2) that there exist a positive integer m and continuous functions f_α, $|\alpha| \leq m$ with $\operatorname{supp} f_\alpha \subset K_{3\eta}$ such that

$$u_1 = \sum_{|\alpha| \leq m} D^\alpha f_\alpha.$$

To simplify notations we choose a real number b such that

(2.6) $K_{4\eta} \subset \{x;\ x_1 > b,\ x_2 > b\}.$

For continuous functions g with $\operatorname{supp} g \subset \{x;\ x_1 > b,\ x_2 > b\}$ we define

$$D_1^{-1} g(x) = \int_b^{x_1} g(s, x_2)\, ds,\quad D_2^{-1} g(x) = \int_b^{x_2} g(x_1, t)\, dt$$

It is clear that $D_1^{-1} g$ and $D_2^{-1} g$ vanish when $x_1 \leq b$ or $x_2 \leq b$.

It is obvious that D_1, D_2, D_1^{-1}, D_1^{-1} all commute and that

$$D_j D_j^{-1} g = D_j^{-1} D_j g = g,\ j = 1,2.$$

Then we let $D_j^{-n} = (D_j^{-1})^n$, $n \geq 0$. We now see that we may write

$$u_1 = D_1^m D_2^m f$$

with

$$f = \sum_{|\alpha| \leq m} D_1^{\alpha_1 - m} D_2^{\alpha_2 - m} f_\alpha .$$

We also notice that (2.4), and supp $f_\alpha \subset K_{3\eta}$ $|\alpha| \leq m$, imply that

(2.7) $\quad$ supp $f \subset \{x\colon x_1 > b + \eta,\ x_2 > b + \eta\}$.

Now we regularize in the x_1-direction.
Let $\varphi \in C_0^\infty(\mathbb{R}^1)$ satisfy $\int\varphi = 1$, $\varphi(t) = 0$ for $|t| > 1/2$. Then we let

$$\varphi_\varepsilon(x_1) = \varepsilon^{-1}\varphi(x_1/\varepsilon)$$

and

$$v = v_\varepsilon = u_1 *' \varphi_\varepsilon$$

where $*'$ denotes convolution in the x_1-variable. The coefficients of P_λ do not depend on x_1 so we have $P_\lambda v = (P_\lambda u_1) *' \varphi_\varepsilon$. Hence (2.5) gives us

(2.8) $\quad P_\lambda v = 0$ in $\Omega_{c-\varepsilon}$.

We choose ε such that

$0 < \varepsilon < c - a < \eta.$

We notice that this and (2.2) gives us

$$\operatorname{supp} v \subset K_{4\eta} \subset \{x;\ x_1 > b,\ x_2 > b\}$$

So we have

$$v(x) = (D_1{}^m D_2{}^m f) *' \varphi_\varepsilon = D_2{}^m (f *' D_1{}^m \varphi_\varepsilon) = D_2{}^m g$$

where

$$g = f *' D_1{}^m \varphi_\varepsilon$$

is a continuous function smooth in x_1 .

Now we rewrite (2.8) as

$$\text{(2.9)} \quad D_2{}^2 D_2{}^m g = x_2{}^2 D_1{}^2 D_2{}^m g - \lambda D_1 D_2{}^m g,\ x \in \Omega_{c-\varepsilon}.$$

We notice that $\operatorname{supp} g \subset \{x;\ x_1 > b,\ x_2 > b\}$.

We also notice that for $m \geq 2$

$$D_2{}^m (x_2{}^2 D_1{}^2 g) = \sum_{j=0}^{2} \binom{m}{j} D_2{}^j (x_2)^2 D_2{}^{m-j} D_1{}^2 g =$$

$$x^2 D_2{}^m D_1{}^2 g + m 2 x_2 D_2{}^{m-1} D_1{}^2 g + m(m-1) D_2{}^{m-2} D_1{}^2 g.$$

We also have

$$D_2^{m-1}(2mx_2D_1^{\,2}g) = m\sum_{j=0}^{1}\binom{m-1}{j}D_2^{\,j}(2x_2)D_2^{\,m-1-j}D_1g =$$

$$m2x_2D_2^{\,m-1}D_1^{\,2}g + m(m-1)2D_2^{\,m-2}D_1^{\,2}g.$$

We put these things together and get

$$D_2^{\,2+m}g = D_2^{\,m}(x_2^{\,2}D_1^{\,2}g) - D_2^{\,m-1}(2mx_2D_1^{\,2}g) +$$

$$D_2^{\,m-2}(m(m-1)D_1^{\,2}g) - D_2^{\,m}(\lambda D_1g) =$$

$$D_2^{\,m}[x_2D_1^{\,2}g - D_2^{-1}(2mx_2D_1^{\,2}g) + D_2^{-2}(m(m-1)D_1^{\,2}g) -$$

$$\lambda D_1^{\,2}g] = D_2^{\,m}h = D_2^{\,m+2}D_2^{-2}h.$$

We see that h is continuous. It is smoth in the x_1-variable and

$$\operatorname{supp} h \subset \{x;\ x_1 > b,\ x_2 > b\}.$$

This shows that

(2.10) $g = D_2^{-2}h,\ x \in \Omega_{c-\varepsilon}.$

For $m = 1$ it is still simpler. For $m = 0$ it is obvious. Now assume that m is the smallest integer positive or not such that for some continuous g with $\operatorname{supp} g \subset \{x;\ x_1 > b,\ x_2 > b\}$

$$v = D_2^{\,m}g \quad \text{in} \quad \Omega_{c-\varepsilon}\ ,$$

g being smooth in x_1 .

The calculation above for $m \geq 0$ and an obvious argument for $m < 0$ shows that (2.10) is true for some h fulfilling the same regularity condition as g. So m was not minimal and v restricted to $\Omega_{c-\varepsilon}$ is in $C^{\infty}(\Omega_{c-\varepsilon})$. Now v_{ε} satisfies all the conditions of Theorem 1.2 in $\Omega_{c-\varepsilon}$ so $v_{\varepsilon} = 0$ in $\Omega_{c-\varepsilon}$. That means that $u_1 = 0$ in Ω_c since $v_{\varepsilon} \to u_1$ there when $\varepsilon \to 0$. Then (2.4) implies that $u_1 = u$ in $\overset{\circ}{K}_{3\eta} \cap \Omega_c$. Let $U = \overset{\circ}{K}_{3\eta} \cap \Omega_c$. The theorem is proved.

3. Proof of Theorem 1.4

In this proof we prefer to abandon the multi-index notation and denote points in $\mathbb{R}^2$ by by (x,y) instead of (x_1,x_2) We also use $D_x = \partial/\partial x$ and $D_y = \partial/\partial y$. Our equation $P_{\lambda}u = 0$ then reads

$$(3.1) \quad (y^2 D_x{}^2 - D_y{}^2 - \lambda D_x) u(x,y) = 0$$

The Fourier-Borel transform with respect to y transforms this into

$$(3.2) \quad (D_z{}^2 D_x{}^2 - z^2 - \lambda D_x) w(x,z) = 0.$$

We are interested in solutions $w(x,z)$ which are continuous functions of $(x,z) \in \mathbb{R}\times\mathbb{C}$, analytic in z for fixed x and vanishing for $x < 0$. We seek solutions of the form

$$(3.3) \quad w(x,z) = \sum_{j=0}^{\infty} \varphi_j(x) z^j/j!$$

where the φ_j are in $C^2(\mathbb{R})$ vanishing for $x \leq 0$. Formal substitution of (3.3) into (3.2) gives us the following differential equations for

φ_j:

$$\varphi_2'' = \lambda\varphi_0' \ , \ \varphi_3'' = \lambda\varphi_1' \ ,$$

and

$$\varphi_{j+2}'' = \lambda\varphi_j' + j(j-1)\varphi_{j-2}, \ j \geq 2.$$

We notice that $\varphi_j \in C^2(\mathbb{R})$, $\varphi_j(x) = 0$, $x \leq 0$ implies that $\varphi_j(0) = \varphi_j'(0) = 0$. We define

$$D^{-1}g(x) = \int_0^x g(t)dt.$$

All this implies that

$$\varphi_2 = \lambda D^{-1}\varphi_0 \ , \ \varphi_3 = \lambda D^{-1}\varphi_1 \ ,$$

and

$$\varphi_{j+2} = \lambda D^{-1}\varphi_j + j(j-1)D^{-2}\varphi_{j-2}, \ j \geq 2.$$

Thus all φ_j can be expressed in terms of the two first ones as follows

$$\varphi_{2j} = \theta(0,\lambda,j)D^{-j}\varphi_0,$$
$$\varphi_{2j+1} = \theta(1,\lambda,j)D^{-j}\varphi_1; \ j \geq 1,$$

where $\theta(i,\lambda,j)$ are complex numbers fulfilling the recursive formulas

$$\theta(0,\lambda,j+1) = \lambda\theta(0,\lambda,j) + 2j(2j-1)\theta(0,\lambda,j-1), \ j \geq 0,$$

and

$$\theta(1,\lambda,j+1) = \lambda\theta(1,\lambda,j) + (2j+1)2j\theta(1,\lambda,j-1) \ j \geq 0,$$

with $\theta(i,\lambda,0) = 1$, $\theta(i,\lambda,j) = 0$, $j < 0$.

It follows by induction that

$$|\theta(0,\lambda,j)| \leq 2^{j+1} j! \prod_{k=1}^{j} (2(k+1) + |\lambda|)(2k+2)^{-1} .$$

Since

$$D^{-j}\varphi_0(x) = ((j-1)!)^{-1} \int_0^x \varphi_0(t)(x-t)^{j-1} dt , \quad j \geq 1 ,$$

a simple computation shows that the series

$$w_0(x,z) = \sum_{j=1}^{\infty} \varphi_{2j} z^{2j}/(2j)!$$

converges in $\mathbb{R}\times\mathbb{C}$, uniformly on compact sets, and that for fixed $x \geq 0$ the function $z \to w_0(x,z)$ is entire and that

$$|w_0(x,z)| \leq C_\varepsilon \exp(|z|(\sqrt{2x}+\varepsilon))$$

for every $\varepsilon > 0$ and

$$w_0(x,z) = 0 , \quad x < 0 .$$

It is also seen that $w_0(x,z)$ solves (3.2). Quite similar statements hold for

$$w_1(x,z) = \sum_{j=0}^{\infty} \varphi_{2j+1}(x) z^{2j+1}/(2j+1)!$$

We see that for any choice of φ_1 and φ_2 vanishing for $x \leq 0$ we obtain a solution $w(x,z) = w_0(x,z) + w_1(x,z)$ of (3.2) in

$C^2(\mathbb{R}\times\mathbb{C})$, entire of exponential growth in z for each fixed x .

When we take the inverse Fourier-Borel transform of $w(x,z)$ in the z-variable, see [17, Theorem 9.1, p. 474], then we obtain a function $u \in C^2(\mathbb{R},H'(\mathbb{C}))$ solving (3.1). We also have that $u \neq 0$ and that for fixed $x > 0$

$$\{y;\ y \in \mathbb{C},\ |y| \le \sqrt{2x}\}$$

is a carrier of $u(x)$. The theorem is proved.

4. Proof of Theorem 1.5

The starting point for the proof is the observation that the differential operator

$$(4.1) \qquad P_2(x,D) = D_2(b(x_2)D_1 + D_2)$$

has a right inverse T which is given explicitly as the integral operator

$$(4.2) \qquad Tg(x) = \int_0^{x_2}\int_0^{t} g(x_1 - \varphi(x_2) + \varphi(t),s)\,ds\,dt$$

where $\varphi' = b$ and $\varphi(0) = 0$. We shall use this fact when we construct null solutions of the full equation $P(x,D)u = 0$ by successive approximations. In order to prove the convergence of these approximations we need some inequalities. They will be proved in Sections 4 and 5. But we shall state them and use them in this section.

Lemma 4.1 Let $\Omega \subset \mathbb{R}^2$ be open and let $f, g \in C^\infty(\Omega)$. Let m, m', r, $d_1 \geq 1$, $d_2 \geq 1$ be positive constants and let $q(x) \geq 0$ in Ω be such that with $d = (d_1, d_2)$

(4.3) $$|D^\xi f| \leq mr^{-|\xi|}\xi d^{\xi d-1}, \ x \in \Omega, \text{ all } \xi ,$$

and

(4.4) $$|D^\xi g| \leq m' r^{-|\xi|}\xi d^{\xi d-1}\exp[(1+\xi d)q(x)], \ x \in \Omega, \text{ all } \xi .$$

Then there exists a constant c , independent of all quantities mentioned above, such that

(4.5) $$|D^\xi (fg)| \leq cmm' r^{-|\xi|}\xi d^{\xi d-1}\exp[(1+\xi d)q(x)] , \ x \in \Omega, \text{ all } \xi .$$

Corollary 4.2 Let f be as above, and let k be a positive integer. Then

(4.6) $$|D^\xi (f^{k+1})| \leq m(mc)^k r^{-|\xi|}\xi d^{\xi d-1} .$$

Proof. Let $q = 0$ in the lemma and use induction in k .

Lemma 4.3 Let $\Omega = \{x; \ x \in \mathbb{R}^2, \ |x_2| < \rho\}$ for some $\rho > 0$. Then (4.2) defines a function $Tg \in C^\infty(\Omega)$ if $g \in C^\infty(\Omega)$ and $\varphi \in C^\infty(\mathbb{R})$.

Let g satisfy (4.4) with r, $0 < r \leq 1$, and Ω as above, and

(4.7) $$q(x) = e^2|x_2|/r, \ \rho \leq re^{-2} ,$$

Let $\varphi \in C^\infty(\{t, |t| < \rho\})$ fulfil

$$(4.8) \qquad |D^{j+1}\varphi(t)| \le mR^{-j}(jd_2)^{jd_2-1}, \; |t| < \rho, \; j = 0, 1, \ldots,$$

where $0 < r/R < 1/4$, and $mc < 1/4$. Here c is taken from Lemma 4.1. Then with T defined in (4.2) we have

$$(4.9) \qquad |D^\xi Tg| \le 4m'r^{2-|\xi|}(\xi d-2)^{\xi d-3}\exp[(\xi d-1)q(x)], \; x \in \Omega, \; |\xi| \ge 2 ,$$

and

$$(4.10) \qquad |D^\xi Tg| \le 4m', \; x \in \Omega, \; |\xi| \le 1 .$$

We like to work with an operator defined in $\Omega = \{x; \, x \in \mathbb{R}^2, |x_2| < \rho\}$ such that (4.8) is fulfilled and such that (1.3) and (1.4) are fulfilled with this Ω and with $\varepsilon = \rho$.

In addition we want to have

$$(4.11) \qquad \Sigma \, m_\alpha \le (8c)^{-1}, \; |\alpha| \le 1 .$$

Here c was introduced in Lemma 4.1. The constants m_α come from (1.3). We also require that

$$(4.12) \qquad |D_2^j b(x_2)| \le mR^{-j}(jd_2)^{jd_2-1}, \; j \ge 0, \; |x_2| < \rho ,$$

where

$$(4.13) \qquad 0 < r/R < 1/4, \; mc < 1/4 .$$

We begin the proof by showing how the general case can be reduced to this one. We may assume that we have adjusted r and R such that the first inequality of (4.13) is satisfied and such that (1.3) is true with the new r and that (1.4) is true with r replaced by R.

We define q by (4.7) and choose ρ such that $0 < \rho < \min(re^{-2},\varepsilon)$. Then we choose a cut-off function $h \in C^{\infty}(\mathbb{R})$ such that

$$0 \le h(x_1) \le 1,\ h(x_1) = 1,\ |x_1| \le \rho/2,\ h(x_1) = 0,\ |x_1| > \rho,$$

also fulfilling

$$|D^j h| \le m' r^{-j} (jd_1)^{jd_1},\ x_1 \in \mathbb{R} .$$

The existence of such a function follows from [4, Lemma 5.7.1, p. 146]. Now we define

$$a'_\alpha(x) = h(x_1)a_\alpha(x),\ |x_1| \le \rho\ ,\ |x_2| < \rho\ ,$$

$$a'_\alpha(x) = 0 \qquad ,\ |x_1| > \rho\ ,\ |x_2| < \rho\ .$$

Then Lemma 4.1 with $q(x) \doteq 0$ shows that the estimates of (1.3) are still true if we replace a_α by a'_α. The only change is that we may have to change the values of the constants m_α. If we replace the coefficients a_α in $P(x,D)$ by a'_α then we get a new operator $P'(x,D)$ defined in $\Omega = \{x;\ |x_2| < \rho,\ x \in \mathbb{R}^2\}$. In a neighbourhood of $x = 0$ the equations $P(x,D)u = 0$ and

$P'(x,D)u = 0$ have the same solutions. From now on we work with $P'(x,D)$ and delete the primes.

If in our original coordinate system (4.11) or the last inequality of (4.13) are not true then we choose

$$x_1' = x_1 \quad \text{and} \quad x_2' = tx_2 \ ,$$

with some constant $t \geq 1$. So we have

$$D_2u(x) = t(D_2'u')(x')$$

where $u(x) = u'(x')$. Now $P(x,D)u = 0$ is transformed into the equivalent equation

$$P'(x',D')u' = D_2'(t^{-1}b(x_2)D_1' + D_2')u' -$$

$$- \sum_{|\alpha|\leq 1} t^{\alpha_2-2} a_\alpha(x)D'^{\alpha}u^1 = D_2'(b'(x_2')D_1' + D_2')u' -$$

$$- \sum_{|\alpha|\leq 1} a_\alpha' D'^{\alpha}u^1 = 0 \ .$$

We have

$$|D'^{\xi}a_\alpha'(x')| \leq m_\alpha t^{\alpha_2-2-\xi_2} r^{-|\xi|} \xi d^{\xi d-1}, \ |x_2'| < t\rho,$$

$$x_1' \in \mathbb{R}, \text{ all } \xi \ ,$$

and

$$|D_2'^j b'(x_2')| \le m t^{-1-j} R^{-j} (jd_2)^{jd_2 - 1}, |x_2'| < t\rho, \ x' \in \mathbb{R}, \ j \ge 0 \ .$$

It is now clear that with a proper choice of $t \ge 1$ and after deleting the primes we may assume that both (4.11) and the last inequality of (4.13) are fulfilled. So we assume that this is true from the beginning.

We notice that with T from (4.2)

$$(4.14) \qquad D_2(bD_1 + D_2)Tg = g, \ g \in C^\infty(\Omega) \ ,$$

and that

$$(4.15) \qquad Tg(x_1,0) = 0, \ x_1 \in \mathbb{R} \ .$$

Let $g \in C^\infty(\Omega)$ be such that $g(x) = 0$ in $M = \{x; \ x \in \Omega, x_1 < \psi(x_2)\}$ with ψ defined in the hypothesis of Theorem 1.5. Then we assert that $Tg(x) = 0$ in M. Let $x \in M, \ x_2 < 0$. In T we have $x_2 \le t \le 0, \ t \le s \le 0$. We notice that $d/dt(\psi - \varphi) = \max(0,b) - b \ge 0, \ t < 0$, and that

$$x_1 - \varphi(x_2) + \varphi(t) - \psi(s) < \psi(x_2) - \varphi(x_2) + \varphi(t) - \psi(t) \le$$

$$\le \psi(x_2) - \psi(t) + \varphi(t) - \varphi(x_2) \le 0 \ .$$

The case $x_2 > 0$ is now also obvious.

Now we construct a solution u of $P(x,D)u = 0$. We start by choosing a function $u^0(x)$ of the form

$$|D_2'^j b'(x_2')| \le mt^{-1-j}R^{-j}(jd_2)^{jd_2-1}, |x_2'| < t\rho,\ x' \in \mathbb{R},\ j \ge 0 .$$

It is now clear that with a proper choice of $t \ge 1$ and after deleting the primes we may assume that both (4.11) and the last inequality of (4.13) are fulfilled. So we assume that this is true from the beginning.

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$$x_1 - \varphi(x_2) + \varphi(t) - \psi(s) < \psi(x_2) - \varphi(x_2) + \varphi(t) - \psi(t) \le$$

$$\le \psi(x_2) - \psi(t) + \varphi(t) - \varphi(x_2) \le 0 .$$

The case $x_2 > 0$ is now also obvious.

Now we construct a solution u of $P(x,D)u = 0$. We start by choosing a function $u^0(x)$ of the form

$$(4.16) \qquad u^0(x) = h(x_1),\ x \in \Omega .$$

We choose $h(x_1)$ such that

$$(4.17) \qquad h(x_1) = 0,\ x_1 \le 0,\ h(x_1) > 0,\ x_1 > 0 ,$$

and such that for some $m'' > 0$

$$(4.18) \qquad |D^j h| \le m'' R^{-j} j^{d_1 - 1},\ x_1 \in \mathbb{R},\ \text{all}\ j \ge 0 .$$

Here d_1 is chosen from the hypothesis. The number R is the constant chosen below formula (4.13). We again refer to [4, Lemma 5.7.1, p. 146]. Then we define

$$(4.19) \qquad f^0(x) = - P(x,D)u^0,\ x \in \Omega ,$$

and recursively for $p \ge 1$

$$(4.20) \qquad u^p(x) = Tf^{p-1}(x),\ x \in \Omega ,$$

and

$$(4.21) \qquad f^p(x) = \Sigma\, a_\alpha D^\alpha u^p,\ x \in \Omega .$$

We are going to prove that for every Ω the series

$$(4.22) \qquad \sum_p D^\xi u^p \quad \text{converges absolutely uniformly on } \Omega \text{ for all } \xi .$$

This means that $u = \Sigma\, u^p$ is a well defined function in

$C^\infty(\Omega)$. Now (4.16) and (4.17) tell us that $u^0(x_1,0) > 0,\ x_1 > 0$, while (4.15) tells us that $u^p(x_1,0) = 0,\ x_1 > 0$, for all $p \geq 1$. That shows us that

$$u(x_1,0) > 0,\ x_1 > 0 .$$

We have

$$\operatorname{supp} u^0 \subset \{x;\ x_1 \geq 0\} \subset \{x;\ x_1 \geq \psi(x_2)\} .$$

The argument after (4.15) then implies that

$$\operatorname{supp} u^p \subset \{x;\ x_1 \geq \psi(x_2)\} .$$

So the same is true for u itself. Finally we have by (4.22), (4.20), (4.14) and (4.21) that

$$P(x,D)u = P(x,D)u^0 + \sum_1^\infty (D_2(bD_1 + D_2)u^p - \sum_{|\alpha|\leq 1} a^\alpha D^\alpha u^p) =$$

$$= - f^0 + \sum_1^\infty f^{p-1} - \sum_1^\infty f^p = 0 .$$

It remains to prove (4.22).

It follows from (4.16) and (4.18) that

$$|D^\xi u^0| \leq m'' r^{-|\xi|} \xi d^{\xi d-1},\ x \in \Omega,\ \text{all } \xi .$$

A short calculation based on (4.18), Lemma 4.1, (1.3) and (4.12) shows that

$$(4.23) \qquad |D^\xi f^0| \le m' r^{-|\xi|} \xi d^{\xi d-1}, \ x \in \Omega, \ \text{all} \ \ \xi \ ,$$

for som constant m' . We want to prove that

$$(4.24) \qquad |D^\xi f^p| \le 2^{-p} m' r^{-|\xi|} \xi d^{\xi d-1} \exp[(1+\xi d) q(x)],$$

$$p \ge 0, \ x \in \Omega, \ \text{all} \ \ \xi$$

where $q(x)$ is still given by (4.7). Now (4.23) shows that (4.24) is true for $p = 0$. So we assume that (4.24) has been established for some $p \ge 0$. Then (4.20), (4.24) and Lemma 4.3 shows that

$$(4.25) \qquad |D^\xi u^{p+1}| \le 2^{2-p} m' r^{2-|\xi|} (\xi d-2)^{\xi d-3} \times$$

$$\times \exp[(\xi d-1) q(x)], \ x \in \Omega, \ |\xi| \ge 2,$$

and

$$(4.26) \qquad |D^\xi u^{p+1}| \le 2^{2-p} m', \ |\xi| \le 1 \ .$$

If $|\xi+\alpha| \ge 2$ and $|\alpha| \le 1$ then (4.25) gives us

$$|D^\xi (D^\alpha u^{p+1}) \le 2^{2-p} m' r^{2-|\alpha|} r^{-|\xi|} (\xi d+\alpha d-2)^{\xi d+\alpha d-3} \times$$

$$\times \exp((\xi d+\alpha d-1) q(x)) \le 2^{2-p} m' r^{-|\xi|} \xi d^{\xi d-1} \exp[(\xi d+1) q(x)],$$

since $r \le 1$, $\alpha d \le 2$.

For $|\xi+\alpha| \le 1$ we use (4.26) and get

$$|D^{\xi}(D^{\alpha}u^{p})| \le 2^{2-p}m'r^{-|\xi|}\xi d^{\xi d-1}\exp[(\xi d+1)q(x)] .$$

Then we use Lemma 4.1 and get

$$|D^{\xi}(a_{\alpha}D^{\alpha}u^{p+1})| \le 2^{2-p}m_{\alpha}m'cr^{-|\xi|}\xi d^{\xi d-1} \times$$

$$\times \exp[(1+\xi d)q(x)] .$$

This and (4.21) tell us that

$$|D^{\xi}f^{p+1}| \le (c\ \Sigma\ m_{\alpha})2^{2-p}m'r^{-|\xi|}\xi d^{\xi d-1} \times$$

$$\times \exp[(1+\xi d)q(x)] .$$

Then a look at (4.11) completes the proof of (4.24). So (4.25) and (4.26) are true for all p too. That implies (4.22) and completes the proof of Theorem 1.5.

5. Estimates of derivatives

As we mentioned in the introduction this section is rewritten. We then also take the opportunity to trace the ideas lying behind Lemma 5.1 below and our use of it. The first example of a problem leading to non-analytical estimates of Gevrey type seems to be a counter-example by S. Kovalevskij [E8] showing that the Cauchy problem for the heat equation $\partial u/\partial t - \partial^2 u/\partial x^2 = 0$ is not always solvable in the class of analytic solutions when the initial datum is given at time $t = 0$. Le Roux [E12] and Holmgren [E6] showed that with data in Gevrey classes 2 on $x = 0$ there exists a solution analytic in the x-variable.

Then M. Gevrey in [E5] introduced the classes nowadays called Gevrey classes. He also solved the Cauchy problem with data on $x = 0$ for the heat equation with added "lower" order terms with coefficients in proper Gevrey classes. There he uses his version of Lemma 5.1. So we may say that he is the one behind it. But he solves the Cauchy problem using an explicit form of the solution of the inhomogeneous heat equation with zero initial data. The first one to use the corresponding idea on successive approximations was C. Pucci [14] when he solved a general linear Cauchy problem for equations with coefficients in proper Gevrey classes. More transparent versions have been used by P. Lax [E11] and A. Friedman [E4] in the proof of different versions of the Cauchy-Kovaleskij theorem. They show that formal power series solutions in the time variable are convergent using estimates of the same type as that in Lemma 5.1. See also J. Persson [E13], [10], [11] and M. Shinbrot and R.R. Welland [E16]. Now we give the "canonical" proof of the lemma.

<u>Lemma 5.1</u> Let $d = (d_1,\dots,d_n) \in \mathbb{R}^n$, $d_j \geq 1$, $1 \leq j \leq n$. Let ν and ξ be multi-indices with non-negative components. Then there exists a constant c independent of n, d, and ξ such that

$$(5.1) \qquad \sum_{\nu \le \xi} \binom{\xi}{\nu} (\xi-\nu)^{(\xi-\nu)d-1} \nu d^{\nu d-1} \xi d^{1-\xi d} \le c \ .$$

Here we have used the natural extension to $\mathbb{R}^n$ of the notation for $\mathbb{R}^2$ in the Introduction. We have let

$$\binom{\xi}{\nu} = \prod_{j=1}^{n} \binom{\xi_j}{\nu_j}, \text{ and } \quad \nu \le \xi \Leftrightarrow \nu_j \le \xi_j,\ 1 \le j \le n \ .$$

<u>Remark</u>. The proof will show that $c = 4$ is the best constant in (5.1).

<u>Proof</u>. Let $x,y \in \mathbb{R}^n$. The "binomial" formula gives

$$(x+y)^{\xi} = \sum_{\nu \le \xi} \binom{\xi}{\nu} x^{\nu} y^{\xi-\nu} \ .$$

Let $x_1 = \ldots = x_n = t$ and let $y_1 = \ldots = y_n = 1$. We get

$$\sum_{j=0}^{|\xi|} \binom{|\xi|}{j} t^j = (1+t)^{|\xi|} = \sum_{\nu \le \xi} \binom{\xi}{\nu} t^{|\nu|} \ .$$

This implies that

$$(5.2) \qquad \binom{|\xi|}{j} = \sum_{\substack{|\nu|=j \\ \nu \le \xi}} \binom{\xi}{\nu}.$$

We also notice that for $\nu \le \xi$, $0 \ne \nu \ne \xi$,

$$(5.3) \qquad (\nu d/\xi d)^{\nu d-1} \le (|\nu|/|\xi|)^{|\nu|-1} \ ,$$

since $d_j \ge 1$, $1 \le j \le n$. Let $|\xi| = k$. Now (5.2) and (5.3) show that the left member of (5.1) is smaller than

$$A = 2 + \sum_{j=1}^{k-1} \binom{k}{j} (k-j)^{k-j-1} j^{j-1} k^{-k+1} .$$

From [E15] p. 20 formula (20) we get the following identity letting $x = y = 1$

$$\sum_{j=0}^{k-2} \binom{k-2}{j} (1+j)^{j-1} (k-j-1)^{k-j-3} = 2k^{k-3}$$

This is equivalent to

$$\sum_{j=1}^{k-1} \binom{k-2}{j-1} j^{j-2} (k-j)^{k-j-2} = 2k^{k-3} ,$$

or

$$(k(k-1))^{-1} \sum_{j=1}^{k-1} \binom{k}{j} j^{j-1} (k-j)^{k-j-1} = 2k^{k-3} .$$

In other words

$$\sum_{j=1}^{k-1} \binom{k}{j} j^{j-1} (k-j)^{k-j-1} k^{-k+1} = 2 - 2/k ,$$

for all $k \geq 2$. The lemma is proved.

Proof of Lemma 4.1. Let f and g satisfy (4.3) and (4.4), respectively. We use Leibniz' formula on $D^{\xi}(fg)$, then the estimates (4.3) and (4.4), and at last Lemma 5 gives

$$|D^{\xi}(fg)| \leq \sum_{\nu \leq \xi} \binom{\xi}{\nu} |D^{\xi-\nu} g| \, |D^{\nu} f| \leq$$

$$\leq \sum_{\nu \leq \xi} \binom{\xi}{\nu} mr^{-|\xi-\nu|} (\xi-\nu)^{(\xi-\nu)d-1} \nu^{\nu d-1} m' r^{-|\nu|} (\nu d)^{\nu d-1}$$

$$\exp[(1+\nu d)q(x)] \leq mm'cr^{-|\xi|} \xi d^{\xi d-1} \exp[(1+\xi d)q(x)] \ .$$

We have also noticed that e^t is increasing. The lemma is proved.

The special case with $q = 0$ shows us that the Gevrey classes with $d \geq (1,\ldots,1)$ are closed under multiplication. An easy argument shows that they are also closed under differentiation. Here we must adjust the r of the estimate, not only choose a new constant m in our estimate. We have used these two facts when we derived (4.24).

6. The hard part

We shall now prove Lemma 4.3. The first statement of the lemma is easy to verify so we concentrate upon the second one. In this section we write $\xi = (i,j)$, instead of $\xi = (\xi_1,\xi_2)$. We also define

(6.1) $\quad A(k) = k^{k-1}, \ k \geq 1, \ A(k) = 1, \ 0 \leq k \leq 1,$

and

(6.2) $\quad E(k,t) = \exp[(1+k)e^2|t|/r], \ k \geq 0 \ .$

We shall let d denote a number here. More specifically we let $d = d_2 \geq 1$. The number k in (6.1) and (6.2) will be of the form $k = jd$, with $j \geq 0$ an integer. In these numbers k both $A(k)$ and $E(k,t)$ are non-decreasing. In this notation Lemma 5.1 takes the form

$$(6.3) \qquad \sum_{j=0}^{k} \binom{k}{j} A((k-j)d)A(jd) \leq c\, A(kd), \text{ all } k .$$

We also notice that

$$(6.4) \qquad A(k) \leq A(k-p)k^p e^p, \ p \leq k ,$$

and if $e^2|t|/r \leq 1$ then

$$(6.5) \qquad E(k,t) \leq E(k-p,t)e^p .$$

We leave the proofs to the reader.

Proof of Lemma 4.3. The first step in the proof is to perform all differentiations in the expression of the left member of (4.9) and (4.10). To facilitate the book-keeping of the arising terms we write the resulting expression in the following form

$$(6.6) \qquad D_1^i D_2^j \int_0^{x_2} \int_0^t g(x_1 - \varphi(x_2) + \varphi(t),s)\,ds\,dt =$$

$$= B^1_{i,j} + B^2_{i,j} + B^3_{i,j} ,$$

where $B^1_{i,j}$ denotes the sum of all terms that contains a double integral, $B^2_{i,j}$ the sum of those containing a single integral, and $B^3_{i,j}$ the sum of those without integral signs.

We define

$$(6.7)\qquad Q_{j,j} = b^j,\ j \geq 0\ ,$$

$$(6.8)\qquad Q_{j,0} = 0,\quad j > 0\ ,$$

and

$$(6.9)\qquad Q_{j+1,k} = bQ_{j,k-1} + D_2 Q_{j,k},\ 0 < k \leq j\ .$$

We remember that $\varphi' = b$. A straightforward calculation shows that

$$(6.10)\qquad B^1_{i,j} = \sum_{k=0}^{j} Q_{j,k} \int_0^{x_2} \int_0^{t} (D_1^{i+k} g)(x_1 - \varphi(x_2) + \varphi(t), s)\, ds\, dt\ ,$$

$$(6.11)\qquad B^2_{i,j} = \sum_{\ell=0}^{j-1} \sum_{k=0}^{\ell} \left(\int_0^{x_2} (D_1^{i+k} g)(x_1, s)\, ds\right) D_2^{j-1-\ell} Q_{\ell,k}$$

and

$$(6.12)\qquad B^3_{i,j} = \sum_{p=1}^{j-1} \sum_{\ell=0}^{p-1} \sum_{k=0}^{\ell} D_2^{j-1-p}[(D_1^{i+k} g)(x_1, x_2) D_2^{p-1-\ell} Q_{\ell,k}]$$

Remark. If $j \geq 1$ the sum in (6.10) actually starts at $k = 1$ since $Q_{j,0} = 0$ then. Similarly if $j \geq 2$ the sums in (6.11) start at $k = 1$, $\ell = 1$ since $D_2^{j-1-\ell} Q_{\ell k} = 0$ if $(\ell,k) = (0,0)$ or $(\ell,k) = (1,0)$. For $j = 1$ there is just one term in (6.11) and none for $j = 0$. The expression of $B^3_{i,j}$ in (6.12) is empty if $j \leq 1$. For $j \geq 2$ it contains just one term with $\ell = k = 0$ namely $D_1^i D_2^{j-2} g$. If that term is taken out the rest of the sum can be taken from $k = 1$ and $\ell = 1$ too and the first summation starts at $p = 2$.

We now assert that

$$(6.13) \qquad |D_2^s Q_{\ell,k}| \leq m(mc)^{k-1}\binom{\ell-1}{k-1}R^{-(s+\ell-k)}A((s+\ell-k)d),$$

$$0 < k \leq \ell,\ s \geq 0,\ |x_2| < \rho\ ,$$

where c is defined in Lemma 4.1, and ρ, m and R in Lemma 4.3.

If $k = \ell$ then we conclude from (6.7), (4.8) and Corollary 4.2 that (6.13) is true for $s \geq 0$. Notice that $d = d_2$ here. Finally assume that (6.13) is true for a certain ℓ, $0 < k \leq \ell$ and all s . This is certainly the case for $\ell = 1$. Now take $0 < k \leq \ell$. We use (6.9), Leibniz' formula, (6.13), and (4.8). For $k > 1$ we get

$$|D_2^s Q_{\ell+1,k}| \leq |D_2^s(bQ_{\ell,k-1})| + |D_2^{s+1}Q_{\ell,k}| \leq$$

$$\leq \sum_{t=0}^{s}\binom{s}{t}|D_2^{s-t}b||D_2^t Q_{\ell,k-1}| + |D_2^{s+1}Q_{\ell,k}| \leq$$

$$\leq \sum_{t=0}^{s}\binom{s}{t}m^2(mc)^{k-2}R^{-(s+\ell-k+1)}\binom{\ell-1}{k-2}A((s-t)d)A(t+\ell-k+1)d) +$$

$$+ m(mc)^{k-1}\binom{\ell-1}{k-1}R^{-(s+1+\ell-k)}A((s+1+\ell-k)d)\ .$$

Now we notice that

$$A((t+\ell-k+1)d)(A((s+\ell-k+1)d))^{-1} \leq A(td)(A(sd))^{-1} .$$

We use this, (6.3), and $\binom{\ell-1}{k-2} + \binom{\ell-1}{k-1} = \binom{\ell}{k-1}$. We get

$$|D_2^s Q_{\ell+1,k}| \leq m(mc)^{k-1} R^{-(s+\ell+1-k)} \binom{\ell}{k-1} A((s+\ell-k+1)d) .$$

By that we have proved that (6.13) is true when ℓ is replaced by $\ell + 1$ and if $1 < k \leq \ell$. We notice that $D_2^s Q_{\ell+1,1} = D_2^{s+1} Q_{\ell,1}$. So (6.13) is also true when $k = 1$ and ℓ is replaced by $\ell + 1$. The case $Q_{\ell+1,\ell+1}$ is treated earlier. So (6.13) is always true.

We can now deduce estimates for the terms $B_{i,j}^k$, $k = 1, 2, 3$, using (6.13) and the assumption (4.4) which in our present notation reads

$$(6.14) \qquad |D_1^i D_2^j g| \leq m' r^{-(i+j)} A((i+j)d) E((i+j)d , x_2), \ x \in \Omega .$$

However, it is a rather complicated task to reduce the resulting estimates into a manageable form so we have to present the computations in detail. We look at $B_{i,j}^1$ first, (6.10). Let $j \geq 1$.

$$|B_{i,j}^1| \leq \sum_{k=1}^{j} m' r^{-i-k} A((i+k)d) m(mc)^{k-1} \binom{j-1}{k-1} \times$$

$$\times R^{-(j-k)} A((j-k)d) \left| \int_0^{x_2} \int_0^t E((i+j)d,s) ds\, dt \right| .$$

We have used $E((i+k)d,s) \leq E((i+j)d,s)$ here. From (4.13) we get $mc < 1/2$, $r < R$. We notice that (6.3) gives

$$\sum_{k=1}^{j}\binom{j-1}{k-1}A((i+k)d)A((j-k)d) \le$$

$$\le \sum_{k=1}^{j}\binom{i+j}{i+k}A((i+k)d)A((j-k)d) \le cA((i+j)d) \ .$$

This together with integration of the double integral gives us

$$(6.15) \qquad |B^1_{i,j}| \le m'r^{-i-j}A((i+j)d)(1+(i+j)d)^{-2}r^2e^{-4} \times$$

$$\times E((i+j)d,x_2) \ .$$

It is clear that (6.15) is also true for $j = 0$. Now we use (6.4) and (6.5) with $p = 2$ for the case $i + j \ge 2$. We get

$$(6.16) \qquad |B^1_{i,j}| \le m'r^{-i-j+2}A((i+j)d-2)E((i+j)d-2),x_2), \ i + j \ge 2 \ .$$

When $i + j \le 1$ a short computation gives

$$(6.17) \qquad |B^1_{i,j}| \le m', \ i + j \le 1 \ ,$$

since $d \le 2$.

The estimate of $B^2_{i,j}$ is obtained in much the same way. We estimate the right member of (6.11) using (6.13) and (6.14). Then we replace $E((i+k)d,s)$ by $E((i+j-1)d,s)$ before we integrate. For $j \ge 2$ we get

$$|B^2_{i,j}| \le \sum_{\ell=1}^{j-1}\sum_{k=1}^{\ell} m'm[1+(i+j-1)d]^{-1}re^{-2}E((i+j-1)d,x_2) \times$$

$$\times r^{-(i+k)}(mc)^{k-1}\binom{\ell-1}{k-1}R^{-(j-1-k)}A((i+k)d)A((j-k-1)d) \ .$$

We interchange the summations. We use $mc < 1/2$, $r < R$ and get

$$|B^2_{i,j}| \le mm'r^{-(i+j)+2}E((i+j-1)d,x_2)e^{-2}[1+(i+j-1)d]^{-1} \times$$

$$\times \sum_{k=1}^{j-1} A((i+k)d)(A((j-k-1)d) \sum_{\ell=k}^{j-1} \binom{\ell-1}{k-1} .$$

We notice that

$$(6.18) \qquad \sum_{\ell=k}^{j-1} \binom{\ell-1}{k-1} = \binom{j-1}{k} .$$

We then argue as in the last step in the estimation of $B^1_{i,j}$. We get

$$|B^2_{i,j}| \le m'r^{-(i+j)+2}E((i+j-1)d,x_2)e^{-2}(1+(i+j-1)d)^{-1} \times$$

$$\times A((i+j-1)d)$$

This estimate also holds for $j = 1$. For $i + j \ge 2$ we use (6.4) and (6.5) with $p = 1$. We notice that $d \ge 1$. We get

$$(6.19) \qquad |B^2_{i,j}| \le m'r^{-(i+j)+2}E((i+j)d-2,x_2)A((i+j)d-2) .$$

For $i + j \le 1$ we only have

$$(6.20) \qquad |B^2_{0,1}| \le m' .$$

The estimation of $B^3_{i,j}$ is more complicated. In our estimate we shall use that $4r \le R$ not just that $r \le R$.

As mentioned before $B^3_{i,j} = 0$ for $j < 2$ and $B^3_{i,2} = D^i_1 g$ which we can estimate by (6.14). We also recall that $D^i_1 D^{j-2}_2 g$ is the only non-vanishing term in (6.12) with $\ell = 0$ or $k = 0$. We use this and Leibniz' formula to rewrite (6.12) as

$$B^3_{i,j} - D^i_1 D^{j-2}_2 g = \sum_{p=2}^{j-1} \sum_{\ell=1}^{p-1} \sum_{k=1}^{\ell} \sum_{q=0}^{j-1-p} \binom{j-2-p}{q} (D^{i+k}_1 D^q_2 g) \times$$

$$\times D^{j-2-\ell-q}_2 Q_{\ell,k}, \ j \geq 3 .$$

We use (6.13) and (6.14). Since $q + k \leq j - 2$ we may replace $E((i+k+q)d, x_2)$ by $E((i+j-2)d, x_2)$ in the estimate of each term. For $j \geq 3$ we get

$$|B^3_{i,j} - D^i_1 D^{j-2}_2 g| \leq mm' r^{-i-j+2} E((i+j-2)d, x_2) \times$$

$$\times \sum_{p=2}^{j-1} \sum_{\ell=1}^{p-1} \sum_{k=1}^{\ell} \sum_{q=0}^{j-1-p} \binom{j-1-p}{q} \binom{\ell-1}{k-1} (mc)^{k-1} (r/R)^{j-2-k-q} \times$$

$$\times A((i+k+q)d) A((j-2-k-q)d) .$$

Just as in the derivation of (6.19) we interchange the ℓ and k summations, and get

$$(6.21) \qquad |B^3_{i,j} - D^i_1 D^{j-2}_2 g| \leq mm' r^{-(i+j-2)} E((i+j-2)d, x_2) \times$$

$$\times \sum_{p=2}^{j-1} \sum_{k=1}^{p-1} \sum_{q=0}^{j-1-p} \binom{j-1-p}{q} \binom{p-1}{k} (mc)^{k-1} (r/R)^{j-2-k-q} \times$$

$$\times A((i+k+q)d) A((j-2-k-q)d) .$$

We shall need the following lemma.

Lemma 6.1 Let $j \geq 3$, p, q, $i \geq 0$, be integers such that $0 \leq q \leq j-1-p$, $2 \leq p \leq j-1$, $1 \leq k \leq p-1$. Let $d \geq 1$. Then we have

$$A((i+k+q)d)A((j-2-k-q)d)(A((i+j-2)d))^{-1} \leq$$

$$\leq A(q)A(j-p-1-q)(A(j-p-1))^{-1} .$$

Proof. At first we notice that

$$A((i+k+q)d)A((j-2-k-q)d)(A(i+j-2)d)^{-1} \leq$$

$$\leq A(k+q)A(j-2-k-q)(A(j-2))^{-1} .$$

Then we notice that

$$k \to A(k+q)A(j-2-q-k)$$

is a convex function. Thus we have

$$A(k+q)A(j-2-k-q) \leq \max(A(q)A(j-2-q), A(q+p-1)A(j-1-p-q))$$

We also notice that

$$A(q)A(j-2-q)(A(j-2))^{-1} \leq A(q)A(j-p-1-q)(A(j-p-1)^{-1} ,$$

and

$$A(q+p-1)A(j-1-p-q)(A(j-2))^{-1} \leq A(q)A(j-p-1-q)(A(j-p-1))^{-1} .$$

The lemma is proved.

Lemma 6.1 applied to (6.21) gives us

$$(6.22) \qquad |B^3_{i,j} - D^i_1 D^{j-2}_2 g| \leq m' r^{-i-j+2} E((i+j-2)d, x_2) \times$$

$$\times \quad A((i+j-2)d) \sum_{p=2}^{j-1} \sum_{q=0}^{j-1-p} \binom{j-1-p}{q} (r/R)^{j-2-q-p-1} \times$$

$$\times \quad A(q)A(j-1-p-q)(A(j-1-p))^{-1} c^{-1} \times$$

$$\times \quad \sum_{k=1}^{p-1} \binom{p-1}{k} (mc)^k (r/R)^{p-1-k} .$$

We notice that $j - 2 - q - p - 1 \geq 0$, $r/R \leq 1$. We also notice that

$$\sum_{k=1}^{p-1} \binom{p-1}{k} (mc)^k (r/R)^{p-1-k} = (mc + r/R)^{p-1} - (r/R)^{p-1} ,$$

and that because of (4.13)

$$\sum_{p=2}^{j-1} (mc + r/R)^{p-1} \leq (mc + r/R)(1 - mc - r/R)^{-1} \leq 1 .$$

These facts together with $d \leq 2$, (6.22) and (6.3) show that for $j \geq 3$

$$|B^3_{i,j} - D^i_1 D^{j-2}_2 g| \leq m' r^{i-j-2} A((i+j)d-2) E((i+j)d-2, x_2) .$$

This and (6.14) shows that

$$(6.23) \qquad |B^3_{i,j}| \le 2m'r^{-i-j-2}A((i+j)d-2)E((i+j)d-2,x_2), \; i \ge 0, \; j \ge 2 \; .$$

Now (6.16), (6.19) and (6.23) give (4.9). At last (6.17) and (6.20) give (4.10). Lemma 4.3 is proved. By this also the full proof of Theorem 1.5 is completed.

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