On a problem of S. Wainer

(The real ordinal of the 1-section of a continuous functional)

By Dag Normann, Oslo - 76

In [5] S. Wainer introduces a hierarchy for arbitrary type-2-functionals. Given F, he defines a set of ordinal notations 0^F, and for each $a \in 0^F$ a function f_a recursive in F and an ordinal $|a|^F < \omega_1^F$. For any f recursive in F there is an $a \in 0^F$ such that f is primitive recursive in f_a.

Let ρ^F be the least ordinal α such that for any f recursive in F there is an $a \in 0^F$ with $|a|^F < \alpha$ such that f is primitive recursive in f_a. If $\rho^F < \omega_1^F$ the hierarchy breaks down. In Bergstra-Wainer [2] ρ^F is described as "the real ordinal of the 1-section of F".

Using standard methods (originally due to Kleene) one may prove that if F is normal, then $\rho^F = \omega_1^F$.
Feferman has proved that if F is recursive, then $\rho^F = \omega^2$.

Let 1-section (F) = 1-sc(F) = {f; f is recursive in F} where f is a total object of type 1.

Grilliot [4] proved that $F \restriction$ 1-sc(F) is continuous if and only if F is not normal. In Wainer [5] it is stated that if F is not normal, then $\rho^F < \omega_1^F$. We are going to disprove this by proving

<u>Theorem 1</u>

There is a continuous function G of type two such that $\rho^G = \omega_1^G$.

L. Harrington proved the following:

Let F be nonnormal and let h be the canonical associate for F. Then

$$\rho^F < \omega_1^F \Leftrightarrow 1\text{-sc}(F) \in \Delta_1^1(h)$$

The statement in Wainer [5] was proved using this result of Harrington and as a hidden lemma that the right hand side of the equivalence above would always hold. The hidden lemma is false, and we obtain Theorem 1 by combining Harrington's result with:

Theorem 2

There is a continuous function G of type 2 recursive in 0^1 such that $1\text{-sc}(G) \in \Pi_1^1 \setminus \Sigma_1^1$.

Here 0^1 is a complete recursively enumerable set. Theorem 2 is the main result of the paper.

Let Σ_γ^0 consist of those hyperarithmetic sets with notations of order $\leq \gamma$. We define Π_γ^0 and Δ_γ^0 in the obvious way.

Adopting methods from the proof of theorem 2 we may prove

Theorem 3

Let $\gamma < \omega_1^{CK}$. Then there is a continuous functional G of type 2 recursive in 0^1 such that

i $\gamma < \rho^G < \omega_1$

ii $1\text{-sc } G \notin \Sigma_\gamma^0$.

Clearly, for any functional F, $1\text{-sc}(F)$ is closed under recursion, so $1\text{-sc}(F)$ defines an upper semilattice of degrees. We say that $1\text{-sc}(F)$ is topless if $1\text{-sc}(F)$ contains no maximal degree.

Corollary (J. Bergstra [1])

There exists a continuous functional G of type 2 such that $1\text{-sc}(G)$ is topless.

Proof Let G be obtained from theorem 2 or from theorem 3 with $\gamma \geq 5$. If 1-sc(G) is not topless, let $\alpha \in$ 1-sc(G) be of maximal degree. Since α is recursive in 0^1, $\alpha \in \Delta^0_2$. But 1-scG = $\{\beta;\ \beta$ is recursive in $\alpha\} \in \Sigma^0_3(\alpha) \leq \Sigma^0_5$

Many of the ideas in the following construction are due to M. Hyland, J. Bergstra and S. Wainer. The inspiration from Bergstra-Wainer [2] is clear, and several of the technical details are borrowed from Bergstra [1]. We take the liberty to repeat them here.

Lemma 1 (R.O. Gandy [3])

a There is a recursive, linear ordering A on $\mathbb{N}$ such that the maximal wellordered initual segment B is Π^1_1 but not Δ^1_1.

b Let $\gamma < \omega_1$. There is a recursive, linear ordering A on $\mathbb{N}$ such that the maximal well-ordered initial segment B is Δ^1_1 but not Σ^0_γ.

Remark Only a is stated in Gandy [3], but b is proved in the same manner.

We give a quick sketch of the proof:

a Let $\prec$ be the Kleene - Brouwer ordering of the sequence numbers.
Let R be recursive such that

$$(\ast)\ \alpha \in \Delta^1_1 \Leftrightarrow \forall\beta\ \exists n\ \neg R(\overline{\langle\alpha,\beta\rangle}(n))$$

where σ_1 is a subsequence of σ_2 and $R(\sigma_1) \Rightarrow R(\sigma_2)$.
Let A be $\prec$ restricted to R.
A is a recursive linear ordering without hyperarithmetic descending sequences, but A is not well-ordered.
Then the initial wellordered segment must be Π^1_1 but not Δ^1_1.

b A closer analysis of the proof of a gives a k such that when we replace $\ast$ by

$$\alpha \in \Sigma^0_{\gamma+k} \Leftrightarrow \forall\beta\ \exists n \neg R(\overline{\langle\alpha,\beta\rangle}(n))$$

then the maximal initial wellordered segment of A will not be Σ^0_γ, but $\Sigma^0_{\gamma+k_1}$ for some $k_1 \in \omega$.

Lemma 2

Let A be a recursive linear ordering of $\mathbb{N}$. There exists an r.e. set $X \subseteq \mathbb{N}^2$ such that when

$$X_n = \{\langle i,m\rangle \in X;\ m \leq_A n\}$$

and $Y_n = \{\langle i,m\rangle \in X;\ m <_A n\}$

then X_n is not recursive in Y_n.

Proof This is proved by a standard priority argument using the finite injury method.

In lemmas 3-8, let A,B be as in lemma 1.a; X, X_n and Y_n as in lemma 2.

Let $B^\ast = \{\alpha;\ \alpha$ is recursive in X_n for some $n \in B\}$.

Lemma 3

$B^\ast \in \Pi^1_1 \setminus \Sigma^1_1$

The proof is trivial.

We want to construct G so that $1\text{-sc}(G) = B^\ast$.

Conventions

If $n \in \omega$, $\alpha \in tp(1)$, let $n^\frown\alpha(k) = \begin{cases} n & \text{if } k = 0 \\ \alpha(k-1) & \text{if } k > 1 \end{cases}$

Let $\alpha^-(k) = \alpha(k+1)$

If F is a (partial) type two functional, let $F_n(\alpha) = F(n^\frown\alpha)$.

Let T be Kleene's T-predicate with the following properties:

Each r.e.set is on the form $W_a = \{p;\ \exists q T(a,p,q)\}$

For any p,a there is at most one q such that $T(a,p,q)$, and $T(a,p,q) \Rightarrow q \geq 1$

There are recursive functions ϕ and ψ such that

$Y_n = W_{\psi(n)}$ and $X_n = W_{\phi(n)}$.

Field (A) = $\mathbb{N}$.

Definition (Bergstra [1])

a Let σ be a sequence number.

$R_a(\sigma) \Leftrightarrow \exists p,q(1 \le p,q \le lh(\sigma) \wedge T(a,p,q) \wedge \sigma(p) < q)$

b
$$F_a^b(\alpha) = \begin{cases} \mu t[T(b,\alpha(0),t) \wedge \neg R_a(\overline{\alpha(t)})] & \text{if such } t \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

F_a^b is recursive in W_b uniformly in a,b.

Lemma 4 (Bergstra [1])

a $\forall \alpha, n[R_a(\overline{\alpha(n)}) \Rightarrow R_a(\overline{\alpha(n+1)})]$

b If W_a is not recursive in α, then $\exists n R_a(\overline{\alpha(n)})$

c There exists α recursive in W_a such that $\forall n \neg R_a(\overline{\alpha(n)})$

Proof

a Trivial

b Assume $\forall n \neg R_a(\overline{\alpha(n)})$. Then

$p \in W_a \Leftrightarrow \exists q \le \alpha(p) T(a,p,q)$

and W_a is recursive in α

c Let $p > 0$. If there is a q such that $T(a,p,q)$ let $\alpha(p) = q$. Otherwise let $\alpha(p) = 0$. We may let $\alpha(0)$ be anything we want.

Definition

Define the partial recursive function H_a^b by the following instruction for computation:

Find the least t_0 such that $R_a(\overline{\alpha(t_0)})$ (If such t_0 does not exists, $H_a^b(\alpha)$ is undefined.) Then, if there is a $t < t_0$ such that $T(b,\alpha(0),t) \wedge \neg R_a(\overline{\alpha(t)})$, let $H_a^b(\alpha)$ be the one such t. If there is no such $t < t_0$, let $H_a^b(\alpha) = 0$.

Lemma 5

$H^b_a \subseteq F^b_a$, $H^b_a(\alpha)$ is defined if W_a is not recursive in α and H^b_a is recursive uniformly in a,b.

Proof Trivial by lemma 4.

Definition

a Let G be the continuous function defined by

$G_n = F^{\phi(n)}_{\psi(n)}$ for all n.

b Let K^m be the partial functional defined by

$K^m_n = G_n$ if $n <_A m$

$K^m_n = H^{\phi(n)}_{\psi(n)}$ if $m \leq_A n$

c Let L^m be the partial functional defined by

$L^m_n = G_n$ if $n \leq_A m$

$L^m_n = H^{\phi(n)}_{\psi(n)}$ if $m <_A n$

Remark Each F^b_a is uniformly recursive in W_b, a, b, so G is recursive in 0^1.

Lemma 6

There is an index e such that for any $n \in B$ $\lambda m\{e\}(G,n,m)$ is the characteristic function of X_n.

Proof We will show how to compute X_n from Y_n (Bergstra [1]). The lemma then follows by a routine application of the recursion theorem.

For each $m \in \mathbb{N}$, choose α_m such that $\alpha_m(0) = m$ and $\forall k \neg R_{\psi(n)}(\overline{\alpha(k)})$. This can be done uniformly recursive in Y_n, n, m by lemma 4.C. We then have

$$m \in W_{\theta(n)} \Leftrightarrow F^{\phi(n)}_{\psi(n)}(\alpha_m) > 0 \Leftrightarrow G(n\hat{}\alpha_m) > 0.$$

Corollary

$B^* \subseteq 1\text{-sc}(G)$

Lemma 7

a K^m is uniformly recursive in $W_{\psi(n)}, n$

b L^n is uniformly recursive in $W_{\phi(n)}, n$

c If α is recursive in $W_{\psi(n)}$, then $L^n(\alpha)$ is defined.

Proof

a If $\alpha(0) <_A n$, $K^n(\alpha) = F^{\phi(\alpha(0))}_{\psi(\alpha(0))}(\alpha^-)$. This is recursive in $X_{\alpha(0)}$ which again is recursive in Y_n in this situation. If $\alpha(0) {}_A{\geq} n$, then $K^n(\alpha) = H^{\phi(\alpha))}_{\psi(\alpha))}(\alpha^-)$. All H^b_a are recursive uniformly in a,b.

b is proved in the same way.

c For any α such that $\alpha(0) \leq_A n$, $L^n(\alpha)$ is defined.

Let α be recursive in $W_{\psi(n)}$ and assume that $\alpha(0) {}_A{>} n$. Then X_n is recursive in $W_{\psi(\alpha(0))}$ and X_n is not recursive in $Y_n = W_{\psi(n)}$. Then α cannot be recursive in $W_{\psi(\alpha(0))}$ and

$$L^n(\alpha) = H^{\phi(\alpha(0))}_{\psi(\alpha(0))}(\alpha^-)$$ is defined by lemma 5.

Lemma 8

Let $n \in B$, $||n||_B = \gamma < \omega_1^{CK}$. Let $\{e\}(G,\vec{n}) \simeq k$ be a computation of length $\leq \gamma$. Then $\{e\}(L^n,\vec{n}) \simeq k$ by the same computation.

Proof We prove this by induction on γ. The lemma is trivial for all initial cpmputations, and the induction is trivial for all cases except application of G. So assume

$\{e\}(G,\vec{n}) \simeq G(\lambda m\{e_1\}(G,\vec{n},m))$.

By the induction hypothesis there is for each $m \in \omega$ an $n_m <_A n$ such that $\{e_1\}(G,\vec{n},m) \simeq \{e_1\}(L^{n_m},\vec{n},m)$

For each m we have $L^{n_m} \subseteq K^n$, so

$\alpha = \lambda m\ \{e_1\}(K^n,\vec{n},m)$ is total. By lemma 7.a α will be recursive in $W_{\psi(n)}$, and by lemma 7.c $L^n(\alpha)$ is defined and equal to $G(\alpha)$.

Since $K^n \subseteq L^n$, we obtain $\{e\}(G,\vec{n}) = \{e\}(L^n,\vec{n})$, which was what we wanted to prove.

<u>We may now prove theorem 2</u>:

Let G be as constructed above, B^* as defined above. Let $\alpha = \lambda m\{e\}(G,m)$. Let $\gamma = \sup\{|e,G,m|+1;\ m \in \omega\}$, $||n||_B = \gamma$. By lemma 8 then $\alpha = \lambda m\{e\}(L^n,m)$. By lemma 7b, α is recursive in X_n, so $\alpha \in B^*$. This shows, with the corollary of lemma 6, that $B^* = 1\text{-sc}\ G$. Q.E.D.

Now, let A,B be obtained from lemma 1.b with $\gamma \geq \omega$. Define G, B^*, K^n and L^n from A,B as above. We are going to prove the following

<u>Claim</u>

<u>i</u> $B^* = 1\text{-sc}\ G$

<u>ii</u> $||B|| < \rho^G \leq \omega_1$

<u>Proof of theorem 3 from the claim</u>

Let γ_0 be given. Let $\gamma \geq \gamma_0 + \omega$, and let B^*, G, B be as in the claim. If $||B|| \leq \gamma_0$ there is a k such that $B \in \Sigma_{\gamma_{0+k}}$. This contradicts lemma 1.b. By Claim <u>ii</u> $\rho^G > \gamma_0$.

If $B^* \in \Sigma^0_{\gamma_0}$, $B \in \Sigma^0_{\gamma_0+k}$ for some k. But B is not in Σ^0_γ .

Definition

Let C = field $(A) \setminus B$.

Let $C^* = \{\alpha;\ (\forall n \in C)(\alpha \text{ is recursive in } X_n)\}$.

Lemma 6 still gives us that $B^* \subseteq$ 1-sc G.

Lemma 9

Let $\{e\}(G,\vec{n}) \simeq k$ be a computation, $n \in C$. Then $\{e\}(L^n_\alpha,\vec{n}) \simeq k$ by the same computation.

The proof is as in lemma 8 by induction on δ = the length of the computation. In order to prove this for n,δ, we use the induction hypothesis for some $n_0 <_A n$, $n_0 \in C$, and then act as in lemma 8.

Corollary

1-sc(G) $\subseteq C^*$

Now assume that $\alpha \in C^* \setminus B^*$. $\alpha \in \Delta^0_2$ since α is recursive in 0^1. We then have

$$n \in B \Leftrightarrow n \in A \ \& \ \alpha \text{ is not recursive in } X_n.$$

But then $B \in \Delta^0_k$ for some k, contradicting the choice of γ. So $C^* = B^*$ and B^* = 1-sc(G). Claim i is verified.

In order to verify claim ii we prove that if $a \in 0^G$ is a notation in the Wainer-hierarchy such that for some $n \in B$, $|a|^G = ||n||_B$, then f_a is recursive in X_n. We use the same kind of argument as in lemma 8. So, if X_n is primitive recursive in f_a, then $|a|^G \geq ||n||_B$, and we obtain $\rho^G \geq ||B||$. $\rho^G < \omega_1$ since 1-scG $\in \Delta^1_1$.

In this note we have constructed continuous functionals with 1-sections of various degrees of definability. They all have a few properties in common.

1. 1-sc(G) $\in \Pi^1_1$
2. 1-sc(G) $\subseteq \Delta^0_2$
3. 1-sc(G) is generated by its r.e.elements.

It still is an interesting problem to decide the nature of all 1-sections of continuous functionals of type 2, or as partial solutions find criteria that guarantees that a given class of functions is the 1-section of some continuous functional. In this direction, we offer the following problem:

If $A \in \Pi^1_1$, $A \subseteq \Delta^0_2$, A is closed under pairing and recursion and $\alpha \in A$ if and only if there is an r.e.set $\beta \in A$ such that α is recursive in β, is then A the 1-section of some continuous functional?

References

1. J. Bergstra: Computability and continuity in finite types, Disertation, Utrecht 1976.
2. J. Bergstra - S. Wainer, The "real" ordinal of the 1-section of a continuous functional, paper contributed to Logic colloquium'76
3. R.O. Gandy, Proof of Mostowski's conjecture, Bulletin de l'Académie Polonaise des Sciences 9 (1960) 571-575.
4. T. Grilliot, On effectively discontinuous type-2 objects, J.S.L. 36 (1971) 245-248.
5. S.S. Wainer, A hierarchy for the 1-section of any type two object, J.S.L. 39 (1974) 88-94.