Contents

DIVISORS OF FINITE CHARACTER

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"Therefore, when one widens the realm of elements to that of ideals in a given ring, one sometimes gains and sometimes loses. One gets the impression that, generally speaking, the truth lies halfway: if the domain of integers in many cases is too narrow, the domain of ideals is in most cases too wide."

Hermann Weyl (in [45] p. 38)

1. Introduction [1)] In its most general and purest form, the study of the notion of divisibility appears as a strictly multiplicative theory. In spite of this, the majority of the abstract investigations concerning the notion of divisibility have been carried out within the setting of integral domains. The tradition of studying divisibility properties in rings or fields rather than in monoids or groups [2)] goes back to the early days of algebraic number theory. Dedekinds ideal concept is a ring-theoretic concept and not a purely multiplicative one (although it turned out later that in the classical case of algebraic integers his ideals may be given a purely multiplicative interpretation as 'divisorial ideals'). Thus, a somewhat blurring and irrelevant additive ingredient was brought into the general theory of divisibility right from the start.

On the other hand, ideals reappeared much later in a more truly additive context, namely as kernels of ring homomorphisms. Viewed from the standpoint of present day mathematics it is really this latter fact which is at the root of the widespread use of ideals, going far beyond

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2) The groups, rings and monoids considered in this paper are all commutative.

their historical and arithmetical origin. The Dedekind notion of an ideal acquires its full significance in connection with its additive and linear aspects, which are also tied up in an essential way with general module theory. It seems, however, that the prestige which this ideal concept has acquired from additive sources also has tended to give the Dedekind notion an unjustified position in purely multiplicative contexts. There are many signs of this and the development toward a 'multiplicative liberation' has been slow.

Only around 1930, more than fifty years after the pioneering work of Kummer, Dedekind and Kronecker, did there appear several investigations by Arnold, van der Waerden, Artin, Prüfer and Krull dealing with a purely multiplicative ideal concept - the so-called v-ideals or divisorial ideals (simply called 'divisors' by Bourbaki). But characteristically enough, these ideals were (apart from Arnolds work) still considered in the setting of rings and were only viewed as a more restricted brand of Dedekind ideals (the latter being called <u>d-ideals</u> in the sequel).

The true multiplicative liberation came with Lorenzen's thesis [33] in 1939. It is the purpose of the present paper to try to revive and continue some of the work of Lorenzen. It seems to us that although his 1939 paper is widely cited it is rather poorly understood. Papers (and also several books such as [12],[18] ,[19] and [32]) which deal with divisors and multiplicative ideal theory are still being published without taking account of Lorenzen's most basic ideas. Their treatment of several topics is decidedly inferior to what can be extracted (admittedly, sometimes with pain) from Lorenzen's work. Only Jaffard's monograph [21] seems to us to do full justice to Lorenzen's ideas. This is really a very fine book, but it is written in a style and uses a terminology which may have prevented many from reading it who otherwise could have been attracted by its rich content.

We shall let the present paper revolve around the concept of a divisorial ideal of finite character - called t-ideals for short. Our main objective will be to present some of the evidence which points in favour of t-ideals as the building blocks of a general arithmetic. They seem to form the true arithmetical divisors with nice properties, shared neither by the d-ideals nor by the v-ideals. In view of this evidence it may be hard to understand how the v-ideals and even the d-ideals have survived in many multiplicative contexts where the t-ideals turn out ot be superior. Bourbaki's treatment of divisors, in Chapter VII of his Commutative Algebra, is for instance based on v-ideals instead of t-ideals, therby missing (both in the main text and in the exercises) a smoother treatment and a better understanding of such matters as Krull rings, factorial domains, localization, Kronecker function rings etc.

In particular, we should like to point at the very basic, but much neglected concept of a Lorenzen group, which advantagously replaces that of a Kronecker function ring. The concept of a Lorenzen group leads to a functor - here called the GCD-functor - which gives the ultimate solution to the classical problem of providing greatest common divisors, and at the same time as it ties up with valuation theory in a very satisfactory way. This functor also exhibits the distinguished and universal role played by the t-ideals. In fact the GCD-functor appears as the left adjoint of a forgetful inclusion functor which is defined in terms of t-ideals.

It should also be mentioned in passing that the concept of a Lorenzen class group represents a natural generalization of the ideal class group of a Dedekind domain and the divisor class group of a Krull domain, putting these two concepts on an equal footing. Another feature of the Lorenzen groups is that they make us fully understand the intimate ties that exist between the two basic arithmetical notions of greatest common divisor and integral closure. More specifically they give us the precise relationship between various notions of integral closure and the ways in which a directed group can be embedded into a GCD-group [3] (= lattice ordered group, Theorem 3 and its corollaries). The further embedding of such a GCD-group into a direct product of totally ordered groups is also best achieved by using the t-system - namely by localization with respect to prime t-ideals. It is really a tour de force to use rings and d-ideals in order to get this embedding via the so-called Krull-Kaplansky-Jaffard-Ohm theorem (as is for instance done in [34]).

The two instances which we have just described are typical of the philosophy which emerges from Lorenzen's work: The use of the Kronecker function ring (as defined by Prüfer and Krull) conceals the fact that it is really the property of being t-Bezout (every finitely generated t-ideal is principal) which matters and not the fact that this ring is d-Bezout. Similarly the essential property of a GCD-group is that it is t-Bezout. The property that such a group can be represented as the divisibility group of an integral domain which is d-Bezout (the Krull-Kaplansky-Jaffard-Ohm theorem) is interesting in itself, but introduces an unneccessary complication which is alien to the purely multiplicative problems at hand.

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3) In our arithmetical context we prefer the more suggestive term of a GCD-group to that of a lattice ordered group or l-group. This also achieves a uniform terminology which is in harmony with the term 'GCD-functor' and the already established notion of a GCD-domain.

Another topic which is illuminiated by the introduction of Lorenzen groups is the axiomatic approach to the theory of divisors as treated for instance by Krull in [29] and by Borevic-Shafarevic in [11]. We indicate how the present point of view leads to a generalization and a sharpening of the exposition of Borevic-Shafarevic.

We should also like to emphasize two other general features of considerable importance in connection with t-ideals. Firstly, in contrast to the v-ideals, the t-ideals are defined by means of a closure property which is of finite (algebraic) character, meaning that a t-ideal generated by a set A is the set-theoretic union of the t-ideals generated by finite subsets of A. This is an essential property when it comes to such matters as the use of Zorn's lemma, the creation of a reasonable theory of localization, the proof that invertible t-ideals are finitely generated etc.

Secondly, there is a useful kind of 'duality' between the prime t-ideals and the prime l-ideals in a GCD-group (an l-ideal being an absolutely convex (isolated) subgroup of such a group). This duality may be quite helpful in the study of t-ideals because it may reduce this study to the case of the simpler and more manageable l-ideals. The simplicity of this latter ideal system has at least two sources: In the first place, it is defined relative to the multiplication $|a| \wedge |b|$ which is essentially an idempotent operation. Secondly, the l-ideals have certain pleasant 'additive' aspects, being just the kernels of morphisms of GCD-groups.

In the two last paragraphs of the present paper we shall show that the notion of a prime t-ideal and that of a t-valuation seem to provide the best foundation for a coherent theory of both sectional and functional representation of ordered groups. Again, the GCD-functor plays an organizing and clarifying role and brings about ameliorations and precisions of earlier work of Keimel [25] and Fleischer [17] on sectional, respectively functional representation of ordered groups.

The present paper is to a certain extent expository and it does not really presuppose much specific knowledge from the theory of ideal systems (x-systems), although our own inspiration comés from this more general theory. However, it is only when viewed against this more general background that the special virtues and the distinguished role of the t-ideals become appearant. In particular, this is the case in connection with the notion of integral closure where the theory of ideal systems offers a more refined and satisfactory treatment than the classical set-up of ring theory. If the reader feels that some preparatory reading is needed in connection with the Lorenzen groups, he should in particular consult Lorenzen's own paper [33] and Jaffards book [21]. Other points where we come into closer contact with the general theory of x-systems are in connection with localization and especially with a counter-example of Dieudonné. (Here [3] may serve as a supplementary reference). We have included some remarks on this latter example because it concerns t-ideals and seems to be best understood in the light of the general notion of additivity for x-systems. Dieudonné's example together with a general theorem on additive ideal systems disclose that the t-system is not in general additive. This seemingly negative property opens up some new problems. The lack of additivity of the t-system makes it doubtful whether a Krull domain may be characterized as an integral domain where every integral t-ideal can be written as a t-product of prime t-ideals without imposing unicity. We also show that the above reasoning may be used to prove that a score of other ideal systems are non-additive. For instance, it follows that the s_a-system in a GCD-group G is additive if and only if G is totally ordered.

2. The basic problems of divisibility theory. Our topic will be a part of algebra, sometimes referred to as divisibility theory, sometimes as general arithmetic. It is concerned with the most general and basic questions surrounding the notion of divisibility in a set D where there is given a commutative and associative multiplication. We put $b|a$ (or $b \leq a$) for two elements $a,b \in D$ if there exists a third element $c \in D$ such that $a = bc$, and we then say that a is divisible by b. The notion of divisibility is generally studied within the setting of rings, especially in integral domains or fields, within ordered (abelian) groups - or simply within monoids. In order to emphasize the purely multiplicative nature of divisibility theory we shall work within a monoid D (i.e. a commutative semigroup with an identity element e). For simplicity we shall also assume that D satisfies the cancellation law and hence can be embedded in a group G. If U denotes the group of units (invertible elements) of D we define the devisibility group of D as the factor group G/U equipped with order which is induced by the divisibility relation in G, taking D as the monoid of integral elements. Equivalently, we may regard the divisibility group as the group of (fractionary) principal ideals, putting $(b) \leq (a)$ whenever $(a) \subset (b)$. An alternative approach is to start out with a directed abelian group G and recover $D = G^+$ as the monoid of all integral (positive) elements of G. (Note that in the case of an integral domain R where $D = R - \{0\}$, the latter approach is more general

than the first: There are directed groups which cannot be represented as the divisibility group of an integral domain.)

Among the most basic topics in divisibility theory are the following three:

1. The problems surrounding the notion of a greatest common divisor (g.c.d). Especially to find constructive methods for adjoining g.c.d's in case they are missing, and to determine the exact conditions under which such an extension process is possible (and can be achieved in a 'minimal' and unique way).
2. The similar problems concerning unique factorization into a product of prime (irreducible) elements: On the one hand to find necessary and sufficient conditions assuring such a unique decomposition. On the other hand to determine the exact conditions under which such a unique factorization can be restored by an extension process and how this extension can be achieved in a 'minimal', unique and constructive way.
3. Decomposition of a divisibility relation into a conjunction of total (linear) divisibility relations.

The main bulk of the present paper will consist in showing how the notion of a divisorial ideal of finite character (t-ideal) plays a crucial role in connection with giving optimal solutions to these three problems.

3. <u>Ideal systems</u>. Although our main concern in the sequel will be the t-ideals, their universal and distinguished role will only appear clearly when viewed against the more general background of ideal systems. For the convenience of the reader we shall therefore collect some of the pertinent material from this theory.

(i) x-systems and r-systems. An ideal system or x-system (D,x) is a monoid D together with a closure operation $A \to A_x$ defined on the subsets of D such that this closure operation is algebraic (of finite character) and is related to the multiplication in D by the following axiom

$$(3.1) \qquad AB_x \subset B_x \cap (AB)_x$$

whenever A and B are subsets of D. If $A = A_x$ we say that A is an x-ideal. The operations of sum (x-union) and product (x-product), denoted by $+$ and $\circ$, are defined by $A + B = (A \cup B)_x$ and $A \circ B = (A \cdot B)_x$.

Let (D_1,x_1) and (D_2,x_2) be two ideal systems. A mapping φ from D_1 into D_2 is said to be an (x_1,x_2)-morphism (or simply a morphism) if the following three conditions are satisfied: (i) $\varphi(e_1) = e_2$ where e_i is the identity element in D_i. (ii) $\varphi(ab) = \varphi(a)\varphi(b)$ and (iii) $\varphi(A_{x_1}) \subset (\varphi(A))_{x_2}$.

The notion of an x-system is more general than the notion of an r-system as originally defined by Lorenzen in [33]. Lorenzen's theory is directed exclusively towards arithmetical goals, using integral domains and their groups of divisibility as the model. Accordingly, D is in his theory supposed to be the integral (positive) part of a directed group G with the inclusion $aB_x \subset (aB)_x$ strengthened to an equality $aB_x = (aB)_x$ and such that $(a)_x = Da$ (the x-system is principal). With these extra hypothesis we speak of a Lorenzen system (r-system in Lorenzen's terminology). The main motivation for restricting the attention to Lorenzen systems in an arithmetical context lies in their principality and the fact that they are exactly the ideal systems which allow for a reasonable theory of fractionary ideals. See [2] p. 29 for the definition of a fractionary ideal system.

(ii) Additive ideal systems. The notion of 'additivity' is absent from Lorenzen's theory, but seems to be crucial for an abstract commutative algebra based on the notion of an x-system. Our considerations in connection with Dieudonné's counterexample in paragraph 11 will show, however, that the concept of additivity is also of relevance in contexts with a more distinctly arithmetical flavour.

To any x-ideal A_x in an x-system (D,x) we can associate a canonical congruence relation by putting $b \equiv c(A_x)$ whenever $A_x + \{b\} = A_x + \{c\}$. This is the unique coarsest congruence relation in D such that any x-ideal containing A_x is a union of congruence classes. This leads to a factor monoid $\bar{D} = D/A_x$ and a canonical map $\varphi : D \to \bar{D}$. There exists a unique finest ideal system $\bar{x}$ such that $\varphi : (D,x) \to (\bar{D},\bar{x})$ is a morphism. This means in particular that $\varphi(A_x) \subset (\varphi(A))_{\bar{x}}$ for any $A \subset D$. If this inclusion is an equality we say that the given ideal system is additive. Equivalently, an x-system is additive iff any canonical map φ of the above kind is closed, in the sense that x-ideals are mapped onto $\bar{x}$-ideals. For more information on additive ideal systems see [3].

(iii) Integral closure. One of the arithmetical assets of ideal systems is that this concept allows for a more satisfactory and refined treatment of the notion of integral closure than is possible when restricting ourselves to the classical ring-theoretic situation.

Let (G,x) denote a directed (abelian) group (written multiplicatively and with an identity element e) equipped with a (fractionary) Lorenzen system x. We then say that G - or its integral part $D = G^+ = \{a | a \geq e\}$ - is integrally x-closed, or shortly x-closed if $A_x : A_x \subset D$ for any finite set $A \subset G$. This notion reduces to the ordinary notion of integral closure if G is the group of divisibility of an integral domain R equipped with the Lorenzen system which comes from the ordinary d-ideals in R.

Two other cases are of particular importance. To any directed group there is canonically attached a unique finest Lorenzen system (the s-system) as well as a unique coarsest Lorenzen system (the t-system). An s-closed (semi-closed) group G is characterized by the implication $a^n \in G^+ \Rightarrow a \in G^+$ whereas the property of being t-closed is a generalization of what Bourbaki calls 'regularly integrally closed' in the case of integral domains.

(iv) Localization. The method of localization may be generalized to ideal systems as follows. Let (D,x) denote an ideal system and let S be a submonoid of D. There then exists a unique ideal system $(S^{-1}D,x_S)$ which solves the universal problem of factorizing uniquely those morphisms $g:(D,x) \to (D',x')$ such that $g(s)$ is invertible in D' whenever $s \in S$. The x_S-ideals of $S^{-1}D$ are exactly the sets of the form $S^{-1}A_x$ where A_x is an x-ideal in D. The property that the family of x_S-ideals is closed under arbitrary intersections relies heavily on the fact that an x-system is supposed to be of finite character. If the given ideal system is a Lorenzen system, then its localizations are also Lorenzen systems. In this case $S^{-1}G^+$ may be identified with a submonoid of G containing G^+. This induces a new divisibility relation on G with $S^{-1}G^+$ as its integral part and one proves the globalization formula

$$A_x = \bigcap_S A_{x_S}$$

where S runs over all complements of maximal x-ideals of G^+. For more details consult [4].

(v) Shadow functors. When dealing with the application of the general theory of ideal systems to particular cases one encounters a kind of forgetful functors which we have termed shadow functors (see [5]). These are functors from the category of commutative rings, commutative differential rings, distributive lattices, lattice ordered groups etc., into the category of ideal systems. We have for instance a shadow functor I_d from the category of commutative rings into the category of ideal systems which takes a ring R into the usual ideal system (R,d) and a ring homomorphism $\varphi : R_1 \to R_2$ into the induced morphism of ideal systems $I_d(\varphi) : (R_1,d) \to (R_2,d)$. We say that (R,d) is the d-shadow (or just the shadow) of R and similarly that $I_d(\varphi)$ is the shadow of φ.

A series of problems arises as to the behaviour of the various shadow functors. In particular whether they give rise to full embeddings or not, and to what extent they commute with various operations such as product formation, factor (quotient) formation, localization etc. In our situation it is of particular interest that the natural shadow functor I_t relating GCD-groups to the t-system produces a full embedding and that this functor commutes with localization.

4. Divisors and t-ideals. A directed group G is said to be factorial if it is isomorphic to an ordered direct sum of copies of $\mathbb{Z}$ (a free abelian group with pointwise order). Such a factorial group is written $\mathbb{Z}^{(I)}$ for some set I and is interpreted as the set of all functions from I to $\mathbb{Z}$, zero outside of a finite set - with pointwise addition and ordering. If G is order-isomorphic to a subgroup of a factorial group we shall say that G

is a <u>préfactorial group</u>. A unique factorization domain (respectively a Krull domain) is an integral domain whose divisibility group is factorial (respectively préfactorial). (We note, however, that in the case of an arbitrary directed group one ought to make a distinction between a préfactorial group and a Krull group. We define a <u>Krull group</u> as a préfactorial group which admits an embedding φ into a group $\mathfrak{D} = \mathbb{Z}^{(I)}$ with the following approximation property: To any element $\delta \in \mathfrak{D}$ and any finite subset $J \subset I$ there exists an element $g \in G$ such that $\varphi(a)$ agrees with δ on J and $\varphi(g) \geq \delta$ elsewhere. In case of the divisibility group of an integral domain, the notions of a préfactorial group and a Krull group coincide (see [21] Theorem 6 p. 84). Since there are examples of préfactorial groups which are not Krull groups, this shows that there are directed groups which are not divisibility groups of integral domains.)

The situation of a préfactorial group exhibits the original arithmetical content of the concept of a 'divisor' and a 'prime divisor'. The divisors which are adjoined in order to achieve unique factorzation are conceived of as finite products (or sums) of the canonical generators (the prime divisors) of the free abelian group $\mathbb{Z}^{(I)}$. It is reasonable, however, to restrict the use of the term 'divisor' somewhat further. For we are not really interested in 'unnecessarily big' extensions with no definite ties between G and $\mathbb{Z}^{(I)}$. It turns out that for a préfactorial group G we can always choose $\mathfrak{D} = \mathbb{Z}^{(I)}$ in a unique minimal way (i.e. such that $\mathfrak{D}$ is contained in all factorial groups containing G as an ordered subgroup) - namely as the group of fractionary t-ideals of G . Thus the t-ideals - which we are now finally going to introduce in some more detail - appear as the true arithmetical divisors.

Let A denote a bounded subset of the directed group G (i.e. there exists an element $g \in G$ such that $gA \subset G^{+} \pm D$). The set

$$A_v = \bigcap_{A \subset (a)} (a)$$

or equivalently $A_v = D : (D:A)$ is then the divisorial ideal or the v-ideal generated by A. We define the t-ideal generated by A as the set-theoretic union of all the v-ideals generated by finite subsets of A:

$$A_t = \bigcup_{\substack{N \subset A \\ N \text{ finite}}} N_v$$

An important technical difference between v-ideals and t-ideals is given by the fact that the t-generation is of finite character whereas the v-generation is not. The t-system forms the unique coarsest Lorenzen system in G.

If G is a GCD-group with the g.c.d.-operation denoted by $\wedge$, the definition of a t-ideal assumes a more appealing form as the conjunction of the two properties

1. $DA_t \subset A_t$
2. $a,b \in A_t \Rightarrow a \wedge b \in A_t$

As opposed to ordinary d-ideals, the presence of a g.c.d. for two (or a finite number of) elements is measured faithfully in terms of t-ideals: Two elements a and b have a g.c.d. if and only if the t-ideal generated by a and b is principal. Otherwise expressed: The divisibility group of the monoid D is a GCD-group if and only if D is t-Bezout (every finitely generated t-ideal is principal). Already at this elementary level the advantage of t-ideals over d-ideals is hence clear (also apart from the fact that d-ideals only make sense in the case of of integral domains, divisibility groups[√]). For a d-ideal (a,b) may fail to be principal also in case a and b have a g.c.d. For a d-ideal (a,b) to be principal it is not only required that a and b have a g.c.d., but that this g.c.d. be a linear combination of a and b. Thus d-ideals bring in an extraneous additive condition which is alien to the purely multiplicative situation at hand.

The problem of providing g.c.d.'s by a suitable extension process will be taken up in connection with the notion of a Lorenzen group and the associated GCD-functor.

Here we shall content ourselves by summing up the result which essentially takes care of the second problem formulated in paragraph 2.

Theorem 1 (i) A directed group G is préfactorial if and only if the (fractionary) t-ideals of G form a group under t-multiplication. This group $\mathfrak{D}$ of t-ideals is automatically factorial and any factorial extension of G contains $\mathfrak{D}$ as an ordered subgroup. In other words, if unique factorization can at all be restored by extension, it can also be achieved by means of t-ideals and this in a canonical and minimal way.

(ii) G is préfactorial if and only if every integral t-ideal can be written uniquely as a t-product of prime t-ideals. This, in turn, is equivalent to the condition that G is t-Noetherian and integrally t-closed ("regularly integrally closed").

(iii) G is factorial if and only if every t-ideal in G is principal.

Although t-ideals are conspicuously absent from the most well-known books treating divisibility theory - we can nevertheless refer the reader to various sources for the proof of the above results. Parts of it go back to Arnold [1] and Clifford [13]. Essential ingredients of the theorem may also be found in Lorenzen [33] (especially on pages 542, 543 (footnote) and p. 552), although Lorenzen is strangely casual about this central issue. For (i) and (ii) see especially his Satz 7 and commenting lines, whereas (iii) is relegated to a footnote on p. 543. A full proof of the theorem can also be put together by consulting Jaffards book [21] (Theorem 5 and its Corollary 2 on p. 82, Proposition 4 on p. 83 and Corollary 1 on p. 32). We shall also have occasion to return to the above theorem in paragraph 9.

When the above theorem is applied to the divisibility group of an integral domain it shows in particular that Krull domains basically exhibits the same behaviour with respect to t-ideals as Dedekind domains with respect to d-ideals. One should not expect, however, that this analogy between Krull domains and Dedekind domains goes all the way, in the sense that any characterization of Dedekind domains in terms of d-ideals may be translated into a similar characterization of Krull domains in terms of t-ideals, simply substituting t for d. Considered from the viewpoint of the general theory of ideal systems there are some notable differences between the d-system and the t-system, stemming from the fact that the former is additive whereas the latter is not. This fact may very well disturb the above mentioned analogy between Dedekind domains and Krull domains at certain points. (See paragraph 11.)

Another comment on the above theorem is perhaps in place. Namely, that the difference between t-ideals and v-ideals may seems to be only slight in this connection since the t-ideals of a préfactorial group are finitely generated and hence are v-ideals anyhow. But this in only a consequence of the theorem and not a fact which allows us to substitute v for t in the characterizations themselves. The factorial groups are far from being those directed groups where every v-ideal (or every finitely generated v-ideal) is principal - and the préfactorial groups are certainly not characterized by the v-ideals (or the finitely generated v-ideals) forming a group. On this background it rather appears as a surprising fact that we may substitute v for t in (ii) and still have a characterization of préfactorial groups. (See Satz 7 in [33] and p.119 in [27].)

5. Lorenzen groups. We shall now enter a subject which, in spite of being almost entirely neglected, seems to us to form the deepest and most interesting part of the general theory of divisibility.

Exploiting the original ideas of Kronecker, Prüfer and especially Krull defined and used the so-called Kronecker function rings in order to study the arithmetic of integral domains. The main virtue of the extension process which leads from an integrally closed domain R to its Kronecker function ring is the fact that the latter is a Bezout domain (finitely generated d-ideals are principal) and hence provide g.c.d.'s. This enables us to get a better grasp of the valuation overrings of R, establishing in particular that these are in one-to-one correspondence with the prime ideals of the corresponding function ring.

The subject of the Kronecker function rings was generalized, clarified and simplified by Lorenzen when he defined the purely multiplicative object of a 'Lorenzen group', freeing the initial construction of a Kronecker function ring from any intervention of an additive operation as well as from the Kroneckerian scheme of adjunction of indeterminates. In spite of this face lift, however, the Kronecker function rings have also in their new disguise as Lorenzen groups remained a neglected and poorly understood area. The following presentation of the rudiments of this subject is offered in the hope of contributing to a better understanding of Lorenzen's ideas. We shall do this by stressing functorial properties as well as the universal role which is played by the t-system in this connection. This will also bring out some facts which are not made sufficiently explicit in [33] and [21], the only sources we know of, treating the subject of Lorenzen groups.

The main way of motivating the introduction of Lorenzen groups is via the old problem of providing g.c.d.'s by a suitable extension process. On an entirely general level, this extension problem is related to ideal systems as follows: To inject a directed group G isomorphically into an ordered monoid M possessing g.c.d.'s amounts to the same thing as to define a Lorenzen system in G (see [21]p.22). This almost trivial observation may be considered as the arithmetical raison d'être of ideal systems. It provides a systematic method of adjoining 'divisors' (in the form of x-ideals) so as to obtain a more well-behaved theory of divisibility. In this generality, however, this extension process is of little use. What one wants is a condition which assures that the monoid M of finitely generated x-ideals satisfies the cancellation law such that it can be further embedded into a GCD-group. From this point on one may in turn have an embedding into a factorial group, this being the ultimate goal of any arithmetical extension process of this kind.

We now proceed to fill in some of the most essential technical details. Let G be a directed group equipped with a (fractionary) Lorenzen system x. We suppose that G is x-closed in the sense of paragraph 3. To the given x-system we can associate another fractionary ideal system in G which is denoted by x_a and which is determined by

$$A_{x_a} = \{c \mid cN_x \subset A_x \circ N_x \text{ for some finite } N \subset G\}$$

whenever A is a finite subset of G. The x_a-ideal generated by a (general) bounded subset B of G is then equal to the set-theoretic union of all the x_a-ideals generated by finite subsets of B.

The crucial property of the x_a-system is that the monoid of finitely generated x_a-ideals (under x_a-multiplication) satisfy the cancellation law and hence possesses a group of quotients $\Lambda_x(G) = \Lambda_x$ (see [21] p. 41-42 for a proof). This group is made into an ordered group by putting $\frac{A_{x_a}}{B_{x_a}} \in \Lambda_x^+$ whenever $A_{x_a} \subset B_{x_a}$ and is as such called the Lorenzen x-group associated to G. The main property of the Lorenzen x-group of G is that it is a GCD-group which contains G as an ordered subgroup. It prvides the g.c.d.'s which may be missing in G and when the x-system is suitably chosen it does this in the most economical way. In fact, whenever G is an ordered subgroup of a GCD-group H, it is also an ordered subgroup of a suitable Lorenzen group $\Lambda_x(G)$ sitting in H. It is sufficient to choose the x-closed system in G which is the trace of the t-system in H (see [21] p. 44-45 and for the notion of a trace-system on p. 52 in the same book). A similar minimality property holds for an embedding of G in a factorial group H. In this case the Lorenzen t-group is factorial and sits between G and H (see Proposition 4 p. 83 in [21] and see also paragraph 9).

The construction of $\Lambda_x(G)$ depends heavily on the condition that G is x-closed. The weakest assumption is here obtained by considering the finest possible x-system - i.e. the s-system in G, defined by $A_s = G^+A$. The resulting condition of s-closure (see (iii) paragraph 3) represents the necessary and sufficient condition for a directed group to be embeddable with all its structure in a GCD-group (see Corollary 3 of theorem 3 in paragraph 7). We shall later return to the more precise ties which exist between a directed group G and its various Lorenzen groups.

In passing, let us just mention that the concept of a Lorenzen group gives a most natural generalization of the notion of a class group - comprising the ideal class group of a Dedekind domain and the divisor class group of a Krull domain as special instances.

By the Lorenzen x-class group of an x-closed group G, we understand the factor group

$$\Lambda_x(G)/G$$

In case of an x-Prüfer group G (see paragraph 8), $\Lambda_x(G)$ may be identified with the group of finitely generated x-ideals and when G is x-Dedekind, $\Lambda_x(G)$ becomes the group of all (fractionary) x-ideals under x-multiplication. By specializing this latter case to the d-system and the t-system of the divisibility group of an integral domain, we obtain the two classical instances of class groups mentioned above.

6. <u>The GCD-functor</u>. Let $\mathfrak{J}$ denote the category of integrally closed directed groups. An object in this category is a directed

(abelian) group G equipped with a Lorenzen system x such that G is x-closed. A morphism in $\mathfrak{J}$ is a morphism of ideal systems $\varphi : (G,x) \to (H,y)$ where (G,x) and $(H,y) \in \mathfrak{J}$.

The category $\mathfrak{J}$ contains in particular two distinguished full subcategories, corresponding to the cases $x = s$ and $x = t$ respectively: The category $\mathcal{S}$ of all semi-closed directed groups with orderpreserving group homomorphisms as morphisms and the category GCD of all GCD-groups with homomorphisms of GCD-groups as morphisms. The proof of these two facts is simple and we shall content ourselves by treating the case which interests us most:

Lemma. The t-shadow functor I_t provides a full embedding of the category of GCD-groups into the category of integrally closed directed groups.

Proof. Obviously, any GCD-group is t-closed. It hence suffices to show that the natural map

$$\mathrm{Hom}_{GCD}(G_1, G_2) \to \mathrm{Hom}_{\mathfrak{J}}((G_1,t),(G_2,t))$$

is a surjection, i.e. any (t,t)-morphism of GCD-groups is really a homomorphism of GCD-groups. First of all, any morphism $\varphi : (G,x) \to (H,y)$ between two Lorenzen systems (and hence in particular any (t,t)-morphism) is order preserving. For $a \geq b$ is equivalent to $a \in (b)_x$ which implies $\varphi(a) \in \varphi((b)_x) \subset (\varphi(b))_y$ which in turn is equivalent to $\varphi(a) \geq \varphi(b)$. On the other hand $\varphi((a,b)_t) \subset (\varphi(a),\varphi(b))_t$ reduces to $\varphi(a \wedge b) \geq \varphi(a) \wedge \varphi(b)$. Since $\varphi(a \wedge b) \leq \varphi(a) \wedge \varphi(b)$ is a consequence of φ being order preserving, it follows that $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ and φ is a homomorphism of GCD-groups.

The following theorem could appropriately be termed 'Main theorem

<u>of divisibility theory</u>'. It shows how the Lorenzen groups act as universal objects with respect to the basic arithmetical completion process of providing g.c.d.'s.

<u>Theorem 2.</u> <u>The passage from an x-closed group</u> (G,x) <u>to its Lorenzen group</u> $\Lambda_x(G)$ <u>defines a faithful functor from the category</u> $\mathcal{J}$ <u>onto the category</u> GCD <u>such that</u> GCD <u>appears as a full reflective subcategory of</u> $\mathcal{J}$ - <u>i.e. the indicated functor is the left adjoint of the t-shadow functor</u>.

We shall call the functor alluded to here for the GCD-<u>functor</u> and denote it by Λ .

<u>Proof.</u> So far we have only defined how the functor Λ acts on the objects of $\mathcal{J}$. If $\varphi : (G,x) \to (H,y)$ is a morphism in $\mathcal{J}$ we define $\Lambda(\varphi) = \Phi$ by putting

$$\Phi\left(\frac{A_{x_a}}{B_{x_a}}\right) = \frac{(\varphi(A))_{y_a}}{(\varphi(B))_{y_a}} \tag{6.1}$$

When we identify G with its group of principal ideals it is clear that the restriction of Φ to G is just φ , showing that Λ is faithful. To verify that Φ is a homomorphism of GCD-groups is routine and we content ourselves by showing that Φ is a lattice homomorphism - the proof that Φ is a group homomorphism being similar. We can assume that the two given quotients have the same denominator and then we get

$$\Phi\left(\frac{A_{x_a}}{C_{x_a}} \wedge \frac{B_{x_a}}{C_{x_a}}\right) = \Phi\left(\frac{A_{x_a} + B_{x_a}}{C_{x_a}}\right) = \frac{(\varphi(A \cup B))_{y_a}}{(\varphi(C))_{y_a}}$$

$$= \frac{(\varphi(A) \cup \varphi(B))_{y_a}}{(\varphi(C))_{y_a}} = \frac{(\varphi(A))_{y_a}}{(\varphi(C))_{y_a}} \wedge \frac{(\varphi(B))_{y_a}}{(\varphi(C))_{y_a}} = \Phi\left(\frac{A_{x_a}}{C_{x_a}}\right) \wedge \Phi\left(\frac{B_{x_a}}{C_{x_a}}\right)$$

where A and B are finite subsets of G. That Λ is compatible with composition is obvious. We have a commutative diagram

(6.2)
$$\begin{array}{ccc} \Lambda_x(G) & \xrightarrow{\Phi} & \Lambda_y(H) \\ \uparrow{\tau_x} & & \uparrow{\tau_y} \\ (G,x) & \xrightarrow{\varphi} & (H,y) \end{array}$$

where the natural inclusion maps τ_x and τ_y are an (x_a,t)-morphism and a (y_a,t)-morphism, respectively. Since every finitely generated t-ideal in $\Lambda_x(G)$ is principal it suffices to show that $\tau_x^{-1}((c)_t)$ is an x_a-ideal in G whenever $c \in \Lambda_x(G)$. If $b_1,\ldots,b_n \in \tau_x^{-1}((c)_t)$ and $b \in (b_1,\ldots,b_n)_{x_a}$, $\tau_x(b)$ may be identified with the principal ideal it generates in G and hence

$$\tau_x(b) \geq (b_1,\ldots,b_n)_{x_a} = \tau_x(b_1) \wedge \ldots \wedge \tau_x(b_n) \geq c$$

with respect to the order relation which is defined in $\Lambda_x(G)$. This entails $b \in \tau_x^{-1}((c)_t)$ as required. (Since the x_a-system is coarser than the x-system this shows in particular that τ_x is an (x,t)-morphism).

By letting H be a GCD-group and putting $y = t$, the diagram 6.2 gives rise to the following one

(6.3)
$$\begin{array}{ccc} & & (\Lambda_x(G),t) \\ & \nearrow^{\tau_x} & \downarrow{\Phi} \\ (G,x) & \xrightarrow{\varphi} & (H,t) \end{array}$$

Here φ and τ_x are (x,t)-morphisms, whereas Φ is a homomorphism of GCD-groups, or equivalently a (t,t)-morphism. The diagram (6.3) exhibits the universal role of the Lorenzen group with respect to (x,t)-morphisms into GCD-groups. For Φ is in fact uniquely determined by the formula

$$(6.4) \qquad \Phi\left(\frac{(a_1,\dots,a_m)_{x_a}}{(b_1,\dots,b_n)_{x_a}}\right) = (\varphi(a_1)\wedge\dots\wedge\varphi(a_m))(\varphi(b_1)\wedge\dots\wedge\varphi(b_n))^{-1}$$

which is just a particular case of (6.1). We know already that Λ is faithful, such that the above remarks establish an injection

$$(6.5) \qquad \Lambda : \mathrm{Hom}_{\mathfrak{J}}((G,x),I_t(H)) \to \mathrm{Hom}_{GCD}(\Lambda_x(G),H)\ .$$

It remains to be shown that this map is also a surjection, thereby proving that Λ is the left-adjoint of the shadow functor I_t. Let $\theta \in \mathrm{Hom}_{GCD}(\Lambda_x(G),H)$ and put $\varphi = \theta\circ\tau_x$. Since τ_x is an (x,t)-morphism, the same is true of φ. Furthermore $\Lambda(\varphi) = \theta$, because there is just <u>one</u> extension of φ to a (t,t)-morphism of $\Lambda_x(G)$ (given by the formula (6.4)). ■

We want to specialize Theorem 2 in such a way as to obtain Lorenzen's main result on the groups $\Lambda_x(G)$ and to establish contact with Krull's researches on the Kronecker function rings. Both of these applications will stress the links with valuation theory.

The natural generalization of the classical notion of a valuation to the setting of ideal systems is the following one: By an x-<u>valuation</u> of a directed group G equipped with a Lorenzen system x we understand an (x,t)-morphism of G onto a totally ordered group Γ. (Note that a totally ordered group is characterized by the fact that s = t, i.e. it carries only one Lorenzen system (of finite character). We could hence equally well speak of an x-valuation as an (x,s)-morphism onto Γ.)

In the case of the divisibility group of an integral domain, equipped with the d-system, the notion of a d-valuation is nothing but an ordinary Krull valuation. The condition that inverse images of t-ideals are d-ideals is in fact equivalent to the classical inequality $v(a \pm b) \geq \mathrm{Min}(v(a), v(b))$.

Corollary 1 (Lorenzen). There is a bijection between the x-valuations of an x-closed group G and the t-valuations of the corresponding Lorenzen group $\Lambda_x(G)$. Furthermore these t-valuations are in one-to-one correspondence with the prime t-ideals of $\Lambda_x(G)^+$. (See [33] Satz 13 and [21] Theorem 4 p. 49.)

The first and main part of this corollary is nothing but a specialization of the bijection (6.5) to the case where H is a totally ordered group. The correspondence between t-valuations and prime t-ideals is not contained in Theorem 2, but is a rather simple matter to which we shall return later in connection with t-localization. It is also a special case of Theorem 9.

Among the consequences of Corollary 1 is the fact that a group G is x-closed if and only if G^+ is an intersection of x-valuation monoids. We shall have occasion to return to this fact in the next section (Corollary 2 of Theorem 3). Here we specialize Corollary 1 one step further:

Corollary 2. There is a bijection between the Krull valuations of an integrally closed domain R and the Krull valuations of its corresponding Kronecker function ring.

The Kronecker function ring $\mathcal{K}(R)$ alluded to here is the canonical one corresponding to the d_a-system. In order to derive this corollary from the preceding one we first notice that the monoid $\Lambda_d(G)^+$, where G is the divisibility group of R , is isomorphic to the monoid of the principal and integral d-ideals of $\mathcal{K}(R)$. This allows us, in a multiplicative context, to consider a Kronecker function ring as a special case of a Lorenzen group. Having established this identification it remains only to see that any d-valuation of $\mathcal{K}(R)$ is in fact a t-valuation. This follows from the fact that $\mathcal{K}(R)$ is a Bezout domain, since this implies that finitely generated d-ideals are t-ideals.

The following corollary extends one of Krull's other results on the Kronecker function ring. ([28] , Satz 19. See also [21], Proposition 2 and subsequent remarks on p. 45.)

Corollary 3. All the Lorenzen groups of an s-closed group G appear as the localizations (or factors) of the Lorenzen s-group $\Lambda_s(G)$.

Proof. If G is x-closed, it is automatically s-closed and the identity map $(G,s) \to (G,x)$ is an (s,x)-morphism giving, by the above theorem, rise to a surjective homomorphism of GCD-groups $\Phi : \Lambda_s(G) \to \Lambda_x(G)$. We thus have an isomorphism of GCD-groups

$$\Lambda_x(G) \simeq \Lambda_s(G)/\mathrm{Ker}\,\Phi$$

where $\mathrm{Ker}\,\Phi$ is an l-ideal (absolutely convex subgroup) of $\Lambda_s(G)$. Alternatively, this factor GCD-group may be considered as a localization arising from $\Lambda_s(G)$ by choosing $S^{-1}\Lambda_s(G)^+$ as a monoid of integral elements in $\Lambda_s(G)$, S designating the multiplicatively closed subset $\Lambda_s(G)^+ \setminus (\Lambda_s(G)^+ \cap \mathrm{Ker}\,\Phi)$.

7. Greatest common divisors and integral closure. The construction of the GCD-functor Λ relies heavily on the condition of integral closure (x-closure). We shall now give a result which clarifies the exact relationship between integral closure and the embeddability in

a GCD-group. For this purpose we shall give a few preparatory remarks.

To any morphism of Lorenzen systems $\varphi : (G_1, x_1) \to (G_2, x_2)$ we can associate a map Φ between their respective monoids of ideals:

$$(7.1) \qquad \Phi(A_{x_1}) = (\varphi(A))_{x_2} (= (\varphi(A_{x_1}))_{x_2}) .$$

Just as for the functor Λ it is a routine matter to verify that Φ is a morphism of monoids: $\Phi(A_{x_1} \circ B_{x_1}) = \Phi(A_{x_1}) \circ \Phi(B_{x_1})$. A directed group equipped with a Lorenzen system x is said to be regularly x-closed if the implication

$$A_x \circ C_x = B_x \circ C_x \Rightarrow A_x = B_x$$

holds true for any finitely generated x-ideal C_x.

With the above notation and terminology we have the following obvious

Lemma. *If G_2 is regularly x_2-closed and Φ is injective, then G_1 is regularly x_1-closed.*

With this in mind we can now prove the following

Theorem 3. *A directed group G is x-closed if and only if it can be considered as an ordered subgroup of a GCD-group in such a way that the resulting injection is an (x,t)-morphism.*

Proof. That an x-closed group can be isomorphically (x,t)-injected into a GCD-group is part of the proof of Theorem 2 where it was established that the canonical injection $G \to \Lambda_x(G)$ is an (x,t)-morphism. That this map identifies G with an ordered subgroup of $\Lambda_x(G)$ is clear.

totally ordered group and v_i is an x-valuation of G into Γ_i leads to an (x,t)-injection

$$G \rightarrow \prod \Gamma_i$$

and vice versa.

As another consequence of Theorem 3 we note the following well-known result

Corollary 3. G is semi-closed (s-closed) if and only if it is an ordered subgroup of some GCD-group.

This is a consequence of Theorem 3, simply because the notion of an order-preserving group homomorphism is the same thing as an (s,t)-morphism.

The two following corollaries give specializations to the cases $x = t$ and $x = d$ respectively.

Corollary 4. G is regularly integrally closed (t-closed) if and only if it can be considered as an ordered subgroup of a GCD-group in such a way that the resulting injection is a (t,t)-morphism.

Note that the notion of a (t,t)-morphism is the same as what is called a V-homomorphism in [34] p. 5. When Corollary 4 is applied to the divisibility group of an integral domain it gives the Corollary 3.3 of [34] p. 8.

Corollary 5. An integral domain is integrally closed if and only if its divisibility group can be isomorphically (d,t)-injected into a GCD-group.

This latter corollary is not surprising since the reader will have no difficulty in showing that the morphism condition $\varphi(A_d) \subset (\varphi(A))_t$ for an arbitrary bounded set A is equivalent to the familiar inequality $\varphi(a \pm b) \geq \mathrm{Min}(\varphi(a), \varphi(b))$ of a Krull valuation (taking the purely multiplicative condition for granted). Combining this observation with Corollary 1 or 2, we get the usual characterization of an integrally closed domain as an intersection of valuation rings.

8. Regularly x-closed groups and Prüfer groups. In his fundamental paper [39] , Prüfer considered in particular two conditions on the divisibility group of a domain, each of which are stronger than integral closure. One of these is Prüfers condition Γ , which by Krull was given the name 'arithmetisch brauchbar' or rather 'endlich arithmetisch brauchbar'. Bourbaki ([12] p. 554) introduces this notion only in the case of v-ideals (divisors in his terminology) and then speaks of an integral domain as being 'regularly integrally closed'. The general notion is the one introduced above as a regularly x-closed group.

A slightly stronger condition is offered by the following definition: G is said to be an x-Prüfer group if the finitely generated x-ideals in G form a group under x-multiplication. For many Lorenzen systems (G,x) there is no difference between the concepts of a regularly x-closed group and an x-Prüfer group. It is for instance well known that in the case $x = d$, a Prüfer domain may be characterized by either of these two properties. A more comprehensive result of this kind will be given in paragraph 11. Here we shall characterize the concepts of a regularly x-closed group and an x-Prüfer group in terms of the map Φ introduced in the preceding paragraphs.

In the following theorem, G is an x-closed group, φ denotes the canonical (x,t)-injection $(G,x) \to (\Lambda_x(G),t)$ and Φ is defined by $\Phi(A_x) = (\varphi(A_x))_t = A_t$ where A is any bounded set in G. If there exists a family $\mathcal{V}$ of <u>valuations</u> (= s-valuations) of the group G such that for any bounded $A \subset G$,

$$A_x = \bigcap_{v \in \mathcal{V}} v^{-1}((v(A))_t)$$

we say that the given x-system is <u>defined by a family of valuations</u>. (See [21] p. 47 and [19] p. 398.)

<u>Theorem 4. The following conditions are equivalent for an x-closed group G:</u>

1. <u>G is regularly x-closed.</u>
2. <u>The map Φ is injective.</u>
3. <u>The x-system in G is the trace of the t-system in some GCD-group which contains G as an ordered subgroup.</u>
4. <u>The x-system coincides with the x_a-system in G.</u>
5. <u>The x-system is defined by a family of valuations.</u>

<u>Furthermore the following two conditions are also equivalent</u>

6. <u>G is an x-Prüfer group.</u>
7. <u>The map Φ is bijective.</u>

<u>In case the given x-system is additive, all the above seven conditions are equivalent.</u>

We shall not go into any details with respect to the proof of this theorem since such a proof can be more or less extracted from [21] (especially from Proposition 7, p. 49, Theorem 5, p. 50 and Theorem 3, p. 55). The only statement in the theorem which really needs a proof, is the last one concerning additivity. This will follow, however, from Theorem 6 below. For a further elaboration on the properties 5. and 6. in the case $x = d$, the reader should consult [19] p. 303 and Theorem 32.12 p. 402.

9. Divisors revisited. The axiomatic approach of Borevic-Shafarevic.

We shall now indicate how the material developed so far may be used in order to put the axiomatic introduction of divisors of Borevic-Shafarevic into a slightly different perspective. This will lead to both a generalization and a sharpening of their treatment.

Few introductory books on algebraic number theory take the trouble to explain the notion of a divisor properly. Hasse in his classical 'Zahlentheorie' puts considerable emphasis on the concept of a divisor, but without clarifying the most fundamental issues. A step towards such a clarification is taken by Borevic-Shafarevic in Chapter 3 ('The theory of divisibility') of their book 'Number Theory'. Here the notion of 'a theory of divisors' is introduced axiomatically as a map φ from the group of divisibility G of an integral domain into a factorial group $\mathfrak{D}$ verifying the following three conditions:

(1) φ is an isomorphism which identifies G with an ordered subgroup of $\mathfrak{D}$

(2) If $\varphi(a) \geq \mathfrak{d}$ and $\varphi(b) \geq \mathfrak{d}$ then also $\varphi(a \pm b) \geq \mathfrak{d}$

(3) If $\mathfrak{a}$ and $\mathfrak{b}$ are elements in $\mathfrak{D}$ such that $\{g \in G \mid \varphi(g) \geq \mathfrak{a}\} = \{g \in G \mid \varphi(g) \geq \mathfrak{b}\}$ then $\mathfrak{a} = \mathfrak{b}$

The elements of $\mathfrak{D}$ are called divisors and the divisors of the form $\varphi(a)$ are said to be principal divisors.

An equivalent formulation of (3) is to say that any $\mathfrak{d} \in \mathfrak{D}$ is the infimum of a finite number of principal divisors. Both these formulations of (3) express our wish to leave out "unnecessary" divisors - i.e. to consider only minimal factorial extensions of G .

The main questions surrounding this notion of a theory of divisors are the questions of existence and unicity and methods for an actual construction of $\mathfrak{D}$ from objects definable in terms of G . By unicity we mean that if $(\mathfrak{D}_1, \varphi_1)$ and $(\mathfrak{D}_2, \varphi_2)$ are two theories of divisors for G , then there exists an isomorphism between $\mathfrak{D}_1$ and $\mathfrak{D}_2$ which extends the canonical isomorphism between $\varphi_1(G)$ and $\varphi_2(G)$.

The exposition of Borevic-Shafarevic is in spite of its virtues still blurred by the presence of the additive operation. The additive operation is irrelevant for the general treatment of divisors and should be discarded. But also in case one insists on a ring-theoretic treatment, the axiom (2) of Borevic-Shafarevic is redundant, as was also noticed by L. Skula in [43]. (An earlier axiomatic treatment of divisors due to Krull ([29], p. 123), which is essentially equivalent to the one by Borevic-Shafarevic, suffers from the same redundancy.) A significant forerunner of Skula's purely multiplicative treatment is Clifford's paper [13] .

Our aim here is to look at the axiomatic introduction of divisors in the light of the Lorenzen groups. It is then natural to start out with a more general situation where the above axiom (2) is discarded and the factorial group $\mathfrak{D}$ is replaced by a GCD-group $\mathfrak{G}$ in the axioms (1) and (3). In that case we shall speak of a theory of quasi-divisors for G . We shall not enter into a discussion of the exact conditions which assures the unicity of a theory of quasi-divisors for G. We shall content ourselves with the following result.

Theorem 5. *A t-closed group G has a unique theory of quasi-divisors determined by the canonical injection of G into its Lorenzen t-group $\Lambda_t(G)$. In case G is the group of divisibility of an integral domain this injection will automatically be a (d,t)-morphism and hence also verify the axiom (2) of Borevic-Shafarevic (and Krull [29]).*

Proof: Let $\varphi : G \to \mathcal{G}$ be a theory of quasi-divisors for G and let y denote the trace-system which the t-system in $\mathcal{G}$ induces on G. We shall show that this trace-system actually is the t-system in G, thus making φ into a (t,t)-morphism. Since every t-ideal is a d-ideal (in case G stems from an integral domain) this will in particular prove the latter part of the theorem. By Theorem 4 (property 3), G is regularly y-closed with corresponding Lorenzen group $\Lambda_y(G)$ and we have a commutative diagram

$$\begin{array}{ccc} & \Lambda_y(G) & \\ {\scriptstyle\tau}\nearrow & & \searrow{\scriptstyle\Phi} \\ G & \xrightarrow{\ \varphi\ } & \mathcal{G} \end{array}$$

where all maps are injections. (That Φ is an injection is again a consequence of Theorem 4 (property 2).) By the minimality property expressed in axiom 3 we infer the isomorphism $\Lambda_y(G) \simeq \mathcal{G}$ (see also [21] p. 44-45).

On the other hand G is t-closed with a canonical (t,t)-injection $\psi : G \to \Lambda_t(G)$. Regarding this latter map as a (y,t)-morphism we get another commutative diagram of injective maps

$$\begin{array}{ccc} & \Lambda_y(G) & \\ {}^{\tau}\nearrow & & \searrow^{\Phi'} \\ G & \xrightarrow{\ \psi\ } & \Lambda_t(G) \end{array}$$

establishing in the same manner as above an isomorphism $\Lambda_y(G) \simeq \Lambda_t(G)$. Combining these two isomorphisms we get $\mathcal{G} \simeq \Lambda_t(G)$ as a GCD-isomorphism which extends the canonical isomorphism between $\varphi(G)$ and $\psi(G)$.

By imposing stronger conditions we obtain a variety of more precise results. We first note the following

Corollary 1. *A t-Prüfer group* G *has a unique theory of quasi-divisors* $G \to \Lambda_t(G)$ *where* $\Lambda_t(G)$ *is isomorphic to the group of finitely generated t-ideals in* G.

Proof: The elements of $\Lambda_t(G)$ are of the form

$$\frac{(a_1 \ldots a_m)_{t_a}}{(b_1 \ldots b_n)_{t_a}}$$

Since $t_a = t$ and G is t-Prüfer we can identify this fraction with the finitely generated (fractionary) t-ideal C_t which solves the equation $(b_1, \ldots, b_n)_t \circ C_t = (a_1, \ldots, a_m)_t$.

Another way to look at Corollary 1 is to use the equivalence between the properties 6 and 7 in Theorem 4 which gives an isomorphism from the group of finitely generated t-ideals of G onto the group of principal t-ideals of $\Lambda_t(G)$, (taking into account that the latter group is a GCD-group).

We may also take the trouble to spell out Corollary 1 in a more classical situation:

Corollary 2. The divisibility group G of a Prüfer domain R has a unique theory of quasi-divisors $G \to \Lambda_t(G)$ where $\Lambda_t(G)$ is isomorphic to the group of all (non-zero) finitely generated fractionary t-ideals of R.

This follows from Corllary 1 in view of the implication d-Prüfer $\Rightarrow$ t-Prüfer.

Moving from quasi-divisors to divisors we first have the following

Corollary 3. If a theory of divisors (verifying (1) and (3)) exists for a directed group G it is uniquely determined as the canonical injection $G \to \Lambda_t(G)$.

Proof: By Theorem 5 it suffices to verify that G is t-closed. Since a factorial group is obviously completely integrally closed, the same is true of G considered as an ordered subgroup of such a group. Corollary 3 then follows from the fact that complete integral closure implies t-closure.

Finally, the following corollary gives a general existence and unicity result which in particular comprises the classical case of a Dedekind domain – and more generally that of a Krull domain.

Corollary 4. A directed group G which is t-Prüfer and satisfies the ascending chain condition for finitely generated (integral) t-ideals has a unique theory of divisors given by the factorial group of all the fractionary t-ideals of G.

Proof: The ascending chain condition for finitely generated t-ideals will in view of Corollary 1 correspond to the descending chain condition for integral elements in $\Lambda_t(G)$. Since $\Lambda_t(G)$ is a GCD-group, this means that $\Lambda_t(G)$ is factorial. Every t-ideal of $\Lambda_t(G)$ is then principal (the descending chain condition for integral elements assures that any t-ideal in $\Lambda_t(G)$ is finitely generated and the GCD-property entails that they are in fact principal). The equivalence of properties 6. and 7. of Theorem 4 gives the desired isomorphism between $\Lambda_t(G)$ and the group of fractionary t-ideals of G .

Corollary 4 has a converse in the sense that if a directed group G has a theory of divisors then it must be t-Prüfer and satisfy the ascending chain condition for finitely generated t-ideals. We know already from the proof of Corollary 3 that G must be completely integrally closed, i.e. that the v-ideals form a group under v-multiplication. It will hence suffice to show that any t-ideal of G is finitely generated and thus a v-ideal. For as a result of this the two hypothesis of Corollary 4 will immediately follow. (An easy proof of the fact that every t-ideal in a préfactorial group is finitely generated can for instance be found on p. 82 in [21].)

Many different characterizations of préfactorial groups have been presented. In the present context we shall content ourselves with giving the following

Theorem 6. The following properties are equivalent for a directed group G .

1. G has a theory of divisors.
2. G has a unique theory of divisors.
3. G is préfactorial.
4. The t-ideals of G form a group under t-multiplication.

5. G is t-Prüfer and satisfies the ascending chain condition for integral t-ideals.
6. G is t-closed and satisfies the ascending chain condition for integral t-ideals.
7. There exists a Lorenzen system x such that G is x-closed and satisfies the ascending chain condition for integral x_a-ideals.

Proof: All the implications in the sequence $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ can easily be extracted from what has been said above, with the possible exception of $7 \Rightarrow 3$ which is proved thus: If G is x-closed it is also x_a-closed and if every x_a-ideal is finitely generated this means that G is completely integrally closed, i.e. the v-ideals form a group $\mathfrak{D}$ under v-multiplication. Since the a.c.c. for x_a-ideals implies the a.c.c. for v-ideals it follows that the GCD-group $\mathfrak{D}$ is indeed factorial.

The contents of this section also gives, essentially, a proof of Theorem 1.

10. t-Localization versus the Krull-Kaplansky-Jaffard-Ohm theorem.

In the preceding sections we have dealt with the relevance of t-ideals in connection with the problem of restoring basic arithmetical properties by a suitable extension process. This comprises in particular the problems 1 and 2 mentioned in paragraph 2.

The third one of the main problems of the theory of divisibility is concerned with the decomposition of a divisibility relation into a conjunction of linear (total) ones. This issue has also been touched upon above in connection with the topic of Lorenzen groups (Corollaries 1 and 2 of Theorem 2 and Corollaries 1 and 2 of Theorem 3). In ring theory this problem takes the form of writing an integrally closed (d-closed) domain as an intersection of valuation rings. The purely multiplicative problem consists in embedding a GCD-group into a direct product of totally ordered groups - taking for granted that the embedding of a directed group into a GCD-group has already been clarified by Theorem 3 and its corollaries.

In connection with this question some authors have advocated a point of view which may be said to be strictly opposite to the one which underlies the present paper. These authors have tried to solve problems concerning GCD-groups by reducing them to ring theory via the so-called Krull-Kaplansky-Jaffard-Ohm theorem (see in particular [34]). This theorem tells us that any GCD-group is order isomorphic to the divisibility group of a suitably chosen Bezout domain. In this way the general theory of GCD-groups can profit from what is known about Bezout domains. This method can in particular be used in order to realize the embedding of a GCD-group into a direct product of totally ordered groups (a result which was first obtained by Lorenzen). For if G is a GCD-group which is the divisibility group of a Bezout domain R we can argue as follows: Being a Bezout domain, R is in particular integrally closed (d-closed) and as such equal to an intersection of valuation rings V_i sitting in the quotient field of R. If Γ_i denotes the totally ordered divisibility group of V_i then

(10.1) $$G \to \prod_i \Gamma_i$$

gives an embedding of the desired type.

This is simple enough, once the K-K-J-O-theorem has been proved. Still, it is fair to say thet this proof procedure succeeds - not because of its relavance for the problem at hand, but rather in spite of its irrelevance. It seems far fetched to use d-ideals, d-closure and d-valuations in connection with this purely multiplicative problem, just because the sufficient amount of commutative algebra happens to be readily available in the d-case. The recipe should rather be to use the concept of a t-ideal which matches the multiplicative situation perfectly - and develop the relevant piece of commutative algebra in the t-case. In fact, only the bare rudiments of a theory of t-localization is all that is needed. This was already recognized by Lorenzen although he did not develop any systematic theory of localization for ideal systems. The globalization formula of paragraph 3 (iv) gives in the case $x = t$ and $A = G^+$:

(10.2) $$G^+ = \bigcap_{S_i} S_i^{-1} G^+$$

with $S_i = G^+ - P_t^{(i)}$ running over all the complements of maximal (prime) t-ideals $P_t^{(i)}$ in G^+.

Let us now elucidate a little bit the remarks given at the very end of paragraph 3. concerning the relationship between localization in GCD-groups and the t-shadow functor.

Let G be a GCD-group with $D = G^+$ as its monoid of integral elements and let S be a submonoid of D. According to the general procedure described in paragraph 3 (iv) we can, on the basis of the ideal system (D,t), form the localized ideal system $(S^{-1}D,t_S)$. This integral ideal system is a Lorenzen system and will hence define a fractionary ideal system in G where the new order relation in G is having $S^{-1}D$ as its monoid of integral elements. It is easy to see that the corresponding ordered group is isomorphic to the factor group

$$G/_{S^{-1}D \cap SD^{-1}}$$

and is hence again a GCD-group since $S^{-1}D \cap SD^{-1}$ is an l-ideal (absolutely convex subgroup) of G. This fact can also be seen by explicitely computing the g.c.d.'s relative to the new 'localized' ordering, according to the formula

$$\frac{d_1}{s_1} \wedge \frac{d_2}{s_2} = \frac{d_1 s_2 \wedge d_2 s_1}{s_1 s_2} \tag{10.3}$$

Using (10.3) we also see that the t_S-system defined in $S^{-1}D$ is the same as the t-system in $S^{-1}D$ defined intrinsically in terms of the order relation given by (10.3). By (10.3) the t_S-ideal $S^{-1}A_t$ $(A \subset D)$ is a t-ideal in $S^{-1}D$ and for any t-ideal B_t in $S^{-1}D$ we have $B_t = S^{-1}(B_t \cap D)$ where $B_t \cap D$ is a t-ideal in D.

The contents of these remarks may be summarized as follows: We have a localization procedure going on at two levels - one for GCD-groups and one for ideal systems (the t-system). These localization procedures are linked by the t-shadow functor in such a way that we obtain an obvious commutative diagram.

Let Γ_i denote the ordered group which is associated to the préordering of G, given by specifying $S_i^{-1}D$ as the monoid of integral elements. The injectivity of (10.1) then follows from (10.2) and the fact that (10.1) is a morphism of GCD-groups follows from the map $D \to S^{-1}D$ being a (t,t_S)-morphism by construction (see [4]), together with the fullness of the t-shadow functor (see the Lemma of paragraph 6). Finally each Γ_i is totally ordered, due to the fact that $S_i^{-1}D$ is a t-local (préordered) monoid in the sense that it contains a unique maximal t-ideal $M_t = S_i^{-1}P_t^{(i)}$ which in the associated ordered group simply consists of all elements $> e$. Since M_t is closed under intersection this means that we have the implication $a > e \ \& \ b > e \Rightarrow a \wedge b > e$ and this is characteristic of a GCD-group which is totally ordered.

One of the features of the duality between prime t-ideals and prime l-ideals in GCD-groups is that the localization with respect to a prime t-ideal is order isomorphic to the factor group with respect to the dual prime l-ideal. Alternatively one may therefore obtain the embedding (10.1) by replacing (10.2) by the fact that the intersection of all prime l-ideals in a GCD-group reduces to the identity element and that any factor group modulo a prime l-ideal is totally ordered. It seems to us, however, that the method of localization may have an advantage because of its broader perspective. This will come up again in connection with sheaf representation.

11. Additive ideal systems and a counterexample of Dieudonné.

The relative strength between the notions of an x-closed group, a regularly x-closed group and an x-Prüfer group has been touched upon in paragraph 8. For the t-system we have already noticed that a t-closed group and a regularly t-closed group is one and the same thing, simply due to the fact that the t-system is the coarsest Lorenzen system (of finite character) which exists in a directed group - and hence $t = t_a$. Theorem 6 shows that 'x-closed' is even equivalent to 'x-Prüfer' in the presence of the ascending chain condition for integral t-ideals. It was shown by Lorenzen (in [33]p. 551) that there exist directed groups which are t-closed, but which are not t-Prüfer groups.

Dieudonné, (in [16]), sharpened this result by showing that there is a distinction between these two notions also within the more restricted realm of divisibility groups of integral domains.

Our interest in this somewhat marginal question comes from the general theory of additive ideal systems. As we see it, it is in the light of the below Theorem 7 that the counterexamples of Lorenzen and Dieudonné acquire some additional interest by exhibiting the reason for the existence of these examples - namely the lack of additivity.

Theorem 7 will generalize a result of Prüfer to the effect that a regularly d-closed domain is a Prüfer domain. Our proof will closely follow the proof of this result as given in [21], p. 26-28. In this generality the theorem was first proved by H. Bie Lorentzen in [9].

Lemma 1. G is an x-Prüfer group if and only if every x-ideal with two generators is invertible.

Proof: Assume that we have shown that any x-ideal with less than $n+1$ generators is invertible and let $A_x = (a_1, \ldots, a_{n+1})_x$ with $n \geq 2$. We then have finitely generated x-ideals B_x, C_x and D_x such that

(11.1) $$(a_1, \ldots, a_n)_x \circ B_x = (e)$$

(11.2) $$(a_2, \ldots, a_{n+1})_x \circ C_x = (e)$$

(11.3) $$(a_1, a_{n+1})_x \circ D_x = (e)$$

By putting $E_x = a_1 B_x \circ D_x + a_{n+1} C_x \circ D_x$, a computation, using an easy consequence of the continuity axiom (3.1) as well as the equations (11.1-3), shows that $A_x \circ E_x = (e)$ as desired. (See [21] p. 27 for details in the case $x = d$.)

Lemma 2. Let (G,x) be an additive Lorenzen system and assume that G is x-closed. G will then be an x-Prüfer group if and only if $a \in (e,a^2)_x$ for all $a \in G$.

Proof: We are here mainly interested in proving the 'if'-part. (The proof of the 'only if'-part is contained in the proof of Theorem 7.) By Lemma 1 it suffices to show that any x-ideal of the form $(b,c)_x$ is invertible. Since $(b,c)_x = (b)_x \circ (e,\frac{c}{b})_x$ it is in turn sufficient to show that $(e,a)_x$ is invertible for any $a \in G$. From the assumption $a \in (e,a^2)_x$ if follows by additivity that

(11.4) $$(g,a^2)_x = (a,a^2)_x$$

for a suitable $g \in G^+$. In particular $a \in (g,a^2)_x$ which by additivity and principality gives

(11.5) $$(g,a)_x = (g,ha^2)_x$$

for some $h \in G^+$. Putting $A_x = (ga^{-1},h)_x \circ (e,a)_x = (ga^{-1},g,h,ah)_x$ it will be sufficient to show that $A_x = G^+$. From (11.5) we infer that

(11.6) $$(ga^{-1},e)_x = (ga^{-1},ha)_x$$

which entails $e \in (ga^{-1},ha)_x \subset A_x$ showing that $G^+ \subset A_x$. It remains to be shown that ga^{-1} and ha belong to G^+ since this will give $A_x \subset G^+$. We get $g(e,a)_x = (g,ga)_x \subset (g,a)_x = (g,ha^2)_x \subset (g,a^2)_x = (a,a^2)_x = a(e,a)_x$ using (11.4) and (11.5) as well as the gact that g and h are integral elements of G.

From $g(e,a)_x \subset a(e,a)_x$ we get $ga^{-1} \in G^+$ since G is x-closed. Together with (11.6) this also yields $ha \in G^+$.

Theorem 7. Any regularly x-closed group is an x-Prüfer group provided that the given fractionary x-system is additive.

Proof: By Lemma 2 it is sufficient to show that the property of regular x-closure implies that $a \in (e,a^2)_x$ for all $a \in G$. We have

$$(a)_x \circ (e,a)_x = (a,a^2)_x \subset (e,a^2)_x \circ (e,a)_x$$

and $(a)_x \subset (e,a^2)_x$ results by cancellation (noting that cancellation with respect to equalities is equivalant to cancellation with respect to inclusions).

In [16] Dieudonné gives an example of an integral domain which is regularly t-closed but not t-Prüfer (regularly integrally closed but not pseudo-Prüfer in Bourbaki's terminology [12] p. 554 and 561). When this is combined with the above Theorem 7 we get the following

Corollary 1. There exists an integral domain where the divisorial ideals of finite character do not form an additive ideal system.

A sharpening of this result is the following

Corollary 2. There exists a t-closed divisibility group where no x_a- system is additive.

Proof: If the directed group G is t-closed it is x-closed for any Lorenzen system x in G. If an x_a-system in G were additive for some x it would follow from Theorem 7 that G is x_a-Prüfer, hence also t-Prüfer (according to [21] Theorem 1,2° p. 25) contradicting Theorem 7.

A more explicit result in the same direction is the following corollary which exhibits an abundance of non-additive ideal systems.

Corollary 3. The s_a-system in a GCD-group G is additive if and only if G is totally ordered.

Proof: If G is totally ordered, all ideal systems in G coincides with the s-system which is additive. Assume conversely that G is a GCD-group which is not totally ordered. There then exist strictly positive elements $a, b \in G^+$ such that $a \wedge b = e$. This entails $(a,b)_t = (e)$ and $(a,b)_{s_a} \neq (e)$. The latter fact follows from a result of Lorenzen ([33]p. 538) and shows that $(a,b)_{s_a}$ cannot be invertible since it as such would be a t-ideal, contradicting $(a,b)_{s_a} \neq (a,b)_t$.

This latter corollary shows that the property of additivity is not generally transmitted from an x-system to the corresponding x_a-system.

12. Sheaf representation over the t-spectrum. Among the most important types of ordered groups are on the one hand the multiplicative groups arising from the theory of divisibility (divisibility groups, groups of ideals, groups of divisors, Lorenzen groups, etc.), and on the other hand additive groups of real-valued functions. Although these two types of ordered abelian groups arise in different contexts, the preceding paragraphs have shown that there is a common meeting ground for them within the theory of divisibility. In fact, the most satisfactory arithmetical situations arise exactly when either the divisibility group itself or a suitable group of ideals form a nice function-group like an additive group of integer-valued functions vanishing outside of finite sets.

Viewing factorial and prefactorial groups from the point of view of a functional representation of these groups over the family of prime t-ideals, this suggests a more general representation theory for ordered groups which closely parallels the well-known sectional representation of commutative rings.

We shall here content ourselves by giving the full sectional representation of the integral part of a GCD-group. This also accomplishes a sectional representation of a semi-closed group via the embedding into its Lorenzen s-group.

Let $D = G^+$ denote the monoid of integral (positive) elements of a GCD-group G. By the <u>t-spectrum</u> of D, denoted by $X = \mathrm{Spec}_t D$ (or $\mathrm{Spec}_t G$), we understand the family of all prime t-ideals of D, equipped with the usual spectral topology where the basic open sets are given by the sets of the form $D(a) = \{P_t \mid a \notin P_t\}$. Whenever S is a submonoid of D we can form the usual monoid of quotients $S^{-1}D$ with $D \subset S^{-1}D \subset G$. As explained on pages 40-41 the monoid $S^{-1}D$ induces a preorder in G, and it is the restriction of this preorder to $S^{-1}D$ which will be considered in the sequel. This makes $S^{-1}D$ into a preordered GCD-monoid according to (10.3) p. 41. The particular case where S is of the form $S_a = \{e, a, a^2, \ldots\}$ gives rise to a presheaf of preordered GCD-monoids over $\mathrm{Spec}_t D$. For $D(b) \subset D(a)$ is by the Krull-Stone theorem for x-ideals ([2] Theorem 12) equivalent to $b \in \sqrt{(a)}$. By putting $b^n = ga$ this gives rise to a well-defined homomorphism of GCD-monoids

$$\varphi_b^a : S_a^{-1}D \to S_b^{-1}D$$

where $\varphi_b^a(\frac{d}{a^m}) = \frac{dg^m}{b^{m\cdot n}}$. Obviously $\varphi_c^b \circ \varphi_b^a = \varphi_c^a$ whenever $D(c) \subset D(b) \subset D(a)$. In this way the assignment $D(a) \to S_a^{-1}D$ defines a presheaf of GCD-monoids on the basis $\{D(a), a \in D\}$ and hence determines a presheaf $\mathcal{T}_X$ on $X = \mathrm{Spec}_t D$. In much the same way as for commutative rings we can prove the following

Theorem 8. The presheaf $\mathcal{T}_X$ is a sheaf. In particular there is an isomorphism of GCD-monoids $D \simeq \Gamma(X, \mathcal{T}_X)$. Furthermore the stalk of $\mathcal{T}_X$ at P_t is isomorphic to the totally preordered monoid $S^{-1}D$ where $S = D \setminus P_t$.

Proof: As usual one must verify that the presheaf $\mathcal{T}_X$ satisfies the two defining properties of a sheaf. These two properties correspond, respectively, to the injectivity and the surjectivity of the natural map $D \to \Gamma(X, \mathcal{T}_X)$. The injectivity is obvious in this case, since we operate within a group where cancellation is available. Let us show the surjectivity, i.e. that any global section of the given presheaf comes from an element in D. By the (quasi)compactness of X ([2] p. 35) the problem reduces to the following one: Given a finite covering of X by basic open sets $X = D(a_1) \cup D(a_2) \cup \ldots \cup D(a_k)$ and given a corresponding family of elements $s_i \in S_a^{-1}D$ such that s_i and s_j have the same 'restriction' to $D(a_i) \cap D(a_j) = D(a_i a_j)$ - we want to exhibit an element $d \in D$ whose 'restriction' to $D(a_i)$ is s_i.

Since we are dealing with a finite covering we can adjust the representation of s_i as a quotient in such a way that the exponent in the denominator is independent of i, i.e. $s_i = \frac{d_i}{a_i^n}$ for all i. The fact that s_i and s_j by the presheaf restriction maps are mapped onto the same element in $S_{a_i a_j}^{-1}D$ gives rise to the equations

(12.1) $$a_j^n d_i = a_i^n d_j$$

Using the equality $D(a_i^n) = D(a_i)$ and the fact that the sets $D(a_i)$ form a covering of X. we deduce the identity

$$(a_1^n,\dots,a_k^n)_t = (a_1^n \wedge \dots \wedge a_k^n) = D$$

which simply means that

(12.2) $$a_1^n \wedge \dots \wedge a_k^n = e .$$

Putting $d = d_1 \wedge \dots \wedge d_k$ and using (12.1) and (12.2) we get

$$a_i^n d = a_i^n(d_1 \wedge \dots \wedge d_k) = a_i^n d_1 \wedge \dots \wedge a_i^n d_k =$$
$$= a_1^n d_i \wedge \dots \wedge a_k^n d_i = (a_1^n \wedge \dots \wedge a_k^n) d_i = d_i .$$

This shows that $d = \frac{d}{e} = \frac{d_i}{a_i^n} = s_i$ when compared in $S_{a_i}^{-1}D$ and thus proves that $d \in D$ gives rise to the given section $s \in \Gamma(X, \mathcal{T}_X)$.

The verification of the isomorphism $\varinjlim S_a^{-1}D \simeq S^{-1}D$ is routine and may be left to the reader. (Here $S = D \setminus P_t$ and the inductive limit is taken with respect to all $a \notin P_t$.)

By replacing each stalk $S^{-1}D$ in the sheaf $\mathcal{T}_X$ by the group G equipped with the preordering which is induced by choosing $S^{-1}D$ as the monoid of integral elements - we can easily extend the above sheaf representation from D to G. In fact, any element $g \in G$ may be written uniquely in the form $g = g^+(g^-)^{-1}$ where $g^+ = g \vee e$ and $g^- = g^{-1} \vee e$ both belong to $D = G^+$. The section s_g corresponding to g is then defined by

$$s_g(P_t) = s_{g^+}(P_t)(s_{g^-}(P_t))^{-1} .$$

This will indeed be a section if we extend the definition of the topology on the disjoint union of the stalks by declaring all sets which may be written as a union of sets of the form

$$\{s_g(P_t) \mid P_t \in D(a)\}$$

as open.

We have thus obtained a sheaf representation of a GCD-group in terms of a sheaf which is built up of totally preordered groups as stalks. From there on we can easily go one step further by passing from the preorder to the associated order in each stalk, i.e. to pass from G to the (totally) ordered factorgroup $G_S = {}^{G}/_{SS^{-1}}$ and redefine the sections accordingly. We may formulate this as a

Corollary 1. *Every (ordered) GCD-group G may be represented as the GCD-group of all sections in a sheaf of totally ordered groups over the quasi-compact space* $\mathrm{Spec}_t\, G$.

Let us also give a more special corollary concerning representations by "real-valued" sections. By a *real group* we shall understand an ordered subgroup of the ordered additive group of real numbers. We shall further say that a GCD-group G is *regular* if every prime t-ideal in G^+ is maximal. (This terminology is chosen because von Neumann regular rings and regular GCD-groups both yield examples of the notion of a regular ideal system, introduced in the next paragraph.)

Corollary 2. *Any regular GCD-group G is isomorphic to the GCD-group of all sections in a sheaf of real groups over the quasi-compact space* $\mathrm{Spec}_t\, G$.

According to Theorem 8 the stalk at P_t is isomorphic to the factor group ${}^{G}/_{H_P} = {}^{G}/_{SS^{-1}}$ where H_P is the prime l-ideal corresponding to P_t . If every prime t-ideal of G is maximal, it will also be minimal. Hence, each H_P will be maximal and the corresponding factor group will be totally ordered and archimedian, thus a real group.

Corollary 1 gives a sharpening of the purely algebraic embedding (10.1) of a GCD-group into a direct product of totally ordered groups. Using a language which corresponds to the one which we used in connection with divisors we may say that the 'principal sections' corresponding to the image of G in the general and 'discontinuous' representation

$$G \hookrightarrow \prod \Gamma_i$$

of paragraph 10 are here characterized (selected) as the <u>continuous</u> ones with respect to the topological restrictions imposed by the given sheaf.

The above approach seems to give the simplest and most general access to a full sectional representation of GCD-groups by means of totally ordered groups. It is based on a Grothendieck approach in terms of localization rather than on a Gelfand-like approach in terms of factor formation. The sheaf-representation of various classes of lattice ordered groups and rings has been extensively studied by Klaus Keimel ([10], [24] and [25]) who has preferred to use a Gelfand-type of approach. As far as we can see this seems to have some slight disadvantages in the case of GCD-groups: (1) It is less simple than the approach in terms of localizations. (2) It is less general in the sense that it requires extra conditions on the given GCD-group in order to obtain a full representation over a quasi-compact space. (3) The stalks are not in general totally ordered and hence less simple and appealing. This latter disadvantage may be compensated for in Keimels approach by passing to the subspace of minimal prime l-ideals which is in addition Hausdorff and zero-dimensional (but generally not compact). We shall return to a somewhat closer comparison with Keimels approach in the next paragraph.

In a sense localization and factor formation are dual procedures. In ring theory the 'self-dual' case (where $R_{\mathfrak{p}} \simeq R/\mathfrak{p}$ for all prime ideals $\mathfrak{p}$) is represented by the class of von Neumann regular rings. In this case the two representation procedures coincide as far as the stalks are concerned. The classical representation theory of Boolean rings may thus be considered from either point of view, although it is the Grothendiek approach which allows us to extend Stone's theory to general commutative rings. A similar advantage of the approach in terms of localization also prevails in the case of GCD-groups. These groups bear in fact a considerable resemblance to regular rings in that they exhibit a similar duality, although this duality for GCD-groups involves two different ideal systems rather than one. We have already alluded to the bijection between the prime t-ideals and the prime l-ideals of a GCD-group and the correspondence which it induces between localization with respect to a prime t-ideal and the factor formation with respect to the corresponding prime l-ideal. One aspect of this duality which is of particular relevance to functional and sectional representation of GCD-groups is the fact that the 'semi-simplicity' for the l-system (the Krull-Stone theorem applied to the zero-ideal) corresponds to the globalization formula (10.2) for the t-system. (In terms of our notation the bijection between prime t-ideals and prime l-ideals is given by $P_t \to H_P = SS^{-1}$ where $S = G^+ - P_t$. See remarks at the end of paragraph 10.) See also Theorem 11 and its consequences for a further substantiation of the analogy between regular rings and GCD-groups.

We shall now further clarify the relative virtues of the different candidates for a notion of a 'spectrum' for a partially ordered group. As we have indicated, the prime t-ideals are superior to the prime l-ideals even in the case of GCD-groups although this is more visible in connection with sectional representation than in the functional case. We shall next show that the applicability of the prime t-spectrum for a sectional representation of partially ordered groups, which are not necessarily GCD-groups, is in a certain precise sense limited to the Prüfer groups. For integrally closed groups which are not Prüfer groups one preferably passes to a spectrum consisting of x-valuations. Again it is the GCD-functor and Lorenzens theorem (Corollary 1 of Theorem 2) which gives the clue to this insight. Thus it is the concept of an x-valuation which turns out to have the widest scope when it comes to the problem of picking the points of the representation space.

Definition. The topological space $\mathrm{Specval}_x G$ (called the x-valuation spectrum of G) consists of all (equivalence classes of) x-valuations of an x-closed group G with the sets $D(a) = \{v \mid v(a) = e, a \in G^+\}$ as basic open sets. (The notion of equivalence of x-valuations extends in an obvious way the usual notion of equivalence between Krull-valuations.)

For every x-closed group G we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec\,val}_t(\Lambda_x(G)) & \xrightarrow{\beta} & \mathrm{Spec}_t(\Lambda_x(G)) \\ \alpha\downarrow & & \downarrow\delta \\ \mathrm{Spec\,val}_x G & \xrightarrow{\gamma} & \mathrm{Spec}_x G \end{array} \tag{12.3}$$

where α is the restriction map related to Lorenzens theorem (Corollary 1 of Theorem 2), δ is the map $P_t \to P_t \cap G$ and γ is the map $v \to v^{-1}((\mathrm{Im}\, v)^+ \setminus \{e\})$. Finally β is just the specialization of γ to the case $x = t$.

By Lorenzens theorem, α is a bijection. This bijection is obviously continuous, but seemingly not in general a homeomorphism. It follows from the following theorem, however, that α is surely a homeomorphism when G is an x-Prüfer group. This theorem also shows that β is a homeomorphism for any x-closed group G. The maps γ and δ are both continuous but in general not bijective. If they are bijective they are also homeomorphisms. More precisely:

Theorem 9. The following conditions are equivalent for an x-closed group G.

1. G is an x-Prüfer group
2. Every localization at a prime x-ideal of G^+ yields an x-valuation monoid in G.
3. The map $\gamma : \text{Spec val}_x G \to \text{Spec}_x G$ is a (surjective) homeomorphism.
4. The map $\delta : \text{Spec}_t(\Lambda_x(G)) \to \text{Spec}_x G$ is a (surjective) homeomorphism.

Proof: We first show that 1. and 2. are equivalent. If G is x-Prüfer it is clear that G is also x_S-Prüfer where S is the complement of a prime x-ideal P_x in $D = G^+$. It is sufficient to observe that the equality $A_x \circ B_x = D$ entails the equality $A_{x_S} \circ B_{x_S} = S^{-1}D$. (We have quite generally that $S^{-1}(A_x \circ B_x) = S^{-1}A_x \circ S^{-1}B_x$ where the latter $\circ$ denotes the x_S-multiplication.) In order to establish the implication $1 \Rightarrow 2$ it is hence sufficient to show that an x-local and x-Prüfer monoid is an x-valuation monoid (observing that $S^{-1}D$ is an x_S-local monoid in the sense that the set $S^{-1}P_x$ of all non-units of $S^{-1}D$ forms a maximal x_S-ideal of $S^{-1}D$). The fact that $S^{-1}D$ produces a total order in G is proved in the case $x = d$ in Proposition 4 p. 67 in [21] and this proof carries over to the general case without change. By an x-valuation monoid in G we understand a set of the form $v^{-1}(\Gamma^+)$ where $v : G \to \Gamma$ is an x-valuation of G, (see Corollary 2 of Theorem 3). In the present situation the canonical map $v : G \to G/_{SS^{-1}} = \Gamma$ will in fact be an x-valuation with $S^{-1}D$ as corresponding valuation monoid. For if $\{a_1, \ldots, a_n\} \subset S^{-1}D$, there exists an element $s \in S$ such that $s\{a_1 \ldots a_n\} \subset D$ and hence also $s\{a_1 \ldots a_n\}_x \subset D$ since D is (by definition) an x-ideal in G. Thus $\{a_1, \ldots, a_n\}_x \subset S^{-1}D$ and $v^{-1}(\Gamma^+)$ is an x-ideal in G. By 'translation' it follows that inverse images of principal ideals in Γ are x-ideals in G. Since the given x-system is supposed to be of finite character we conclude that $v^{-1}(A_t)$ is an x-ideal in G for any bounded set $A \subset \Gamma$.

In order to show that $2 \Rightarrow 1$ it is (according to Lemma 1 in paragraph 11) enough to prove that every x-ideal of the form $(a,b)_x$ is invertible. By the fact that every localization at a prime x-ideal gives rise to a total order, we must have

$$(a)_{x_S} \subset (b)_{x_S} \quad \text{or} \quad (b)_{x_S} \subset (a)_{x_S} .$$

This entails easily that

$$S^{-1}((a)_x \circ (b)_x) = S^{-1}((a,b)_x \circ ((a)_x \cap (b)_x))$$

which by the globalization formula of paragraph 3, (iv) gives

$$(ab)_x = (a)_x \circ (b)_x = (a,b)_x \circ ((a)_x \cap (b)_x) .$$

Since a principal x-ideal is invertible, it follows that $(a,b)_x$ is invertible.

By assuming 2. we see that the map γ has an inverse, as constructed in the first part of the proof. In fact, γ is then a homeomorphism because the basic open sets in the two topologies correspond to each other as follows:

$$\{v \mid v(a) = e\} \leftrightarrow \{\gamma(v) \mid a \notin \gamma(v)\} .$$

That 3. implies 2. is obvious. From the implication $1 \Rightarrow 3$ and the fact that a GCD-group is always a t-Prüfer group it follows that there is a bijection between the t-valuations and the prime t-ideals in such a group. This establishes of course that β is a homeomorphism for any x-closed group G. It follows that γ is bijective if and only if δ is bijective. This shows in particular that $4 \Rightarrow 1$ (since the bijectivity of γ implies 1.). On the other hand if G is an x-Prüfer group (i.e. γ is bijective) then δ will be bijective. More precisely, it follows in conjunction with the equivalence of 6. and 7. in Theorem 4 that δ and the map Φ of that theorem are inverses of each other when Φ is restricted to $\mathrm{Spec}_x G$. From this we can infer that a basic open set $D(a) = \{P_t | a \notin P_t\} \subset \mathrm{Spec}_t \Lambda(G)$ by δ corresponds to an open set in $\mathrm{Spec}_x G$. For $a \in G^+$ this is obvious since then $\delta(D(a)) = \{P_x | a \notin P_x\}$. In case $a \in \Lambda(G)^+ \setminus G^+$ we can prove that

$$(12.4) \qquad \delta(D(a)) = \{P_x | (a) \cap G^+ \not\subset P_x\}$$

or equivalently

$$(12.5) \qquad a \notin P_t \Leftrightarrow (a) \cap G^+ \not\subset \delta(P_t) .$$

Since $a \in P_t \Rightarrow (a) \cap G^+ \subset P_t \cap G^+$ the implication $\Leftarrow$ in (12.5) is clear. Conversely, since Φ is the inverse of δ it follows that the t-ideal in $\Lambda_t(G)$ which is generated from $(a) \cap G^+$ is (a). If $(a) \cap G^+ \subset P_t \cap G^+$ we therefore obtain $a \in P_t$ as desired.

Since the right-hand side of (12.4) is evidently a union of basic open sets in $\mathrm{Spec}_x G$ it follows that δ is an open map and this completes the proof of the theorem. ∎

It is clear from the above proof that the mere bijectivity of either of the maps γ or δ is sufficient to assure that G is an x-Prüfer group. In case of γ, the bicontinuity follows immediately from the bijectivity whereas our proof of the openness of δ relies on Theorem 4.

We spell out two special cases

Corollary 1. An integrally closed domain R is a Prüfer domain if and only if the map δ induces a homeomorphism between the prime spectra of R and its Kronecker function ring $\mathcal{K}(R)$.

(See Corollary 2 of Theorem 2 and succeeding remarks.)

Corollary 2. A t-closed group G is a t-Prüfer group if and only if the map δ gives a homeomorphism between the prime t-spectra of G and its Lorenzen t-group.

We shall say that a subgroup G of a GCD-group $\mathcal{D}$ is dense if the axiom (3) of 'a theory of quasi-divisors' is satisfied (see p. 32). As a joint corollary of Theorems 8 and 9 we get

Corollary 3. Every x-Prüfer group G may be represented as a dense subgroup of the GCD-group of all sections in a sheaf of totally ordered groups over the quasi-compact space $\mathrm{Spec}_x G$.

In fact, the axiom (3) of paragraph 9 amounts to the condition that any element in the GCD-group is an infimum of a finite number of elements of the given dense subgroup. In the case of a pair $G \hookrightarrow \Lambda_x(G)$ the latter denseness property is equivalent to G being an x-Prüfer group. (See Theorem 3 in [21] p. 55.)

In all the cases where the map α (in the commutative diagram (12.3)) is a homeomorphism we obtain a sheaf representation of the group G over $\mathrm{Spec\,val}_x G$, simply by restricting the full sectional representation of $\Lambda_x(G)$ to G. In case of an arbitrary x-closed group we can obtain the same type of representation by transferring the topology of $\operatorname{Spec val}_t(\Lambda_x(G))$ to $\operatorname{Spec val}_x G$ via the bijection α. It seems reasonable to conjecture that α is a homeomorphism if and only if G is an x-Prüfer group. When trying to prove that α is an open map one encounters a problem which is analogous to the one in connection with the openness of δ. By the very definition of the GCD-functor (see (6.4)) we get

$$\text{(12.6)} \quad \alpha(D(\frac{A_{x_a}}{B_{x_a}})) = \alpha\{v \in \operatorname{Spec val}_t(\Lambda_x(G)) \mid v(a_1) \wedge \ldots \wedge v(a_m)[v(b_1) \wedge \ldots \wedge v(b_n)]^{-1} = e\}$$

where $A_{x_a} = (a_1 \ldots a_m)_{x_a} \subset (b_1 \ldots b_n)_{x_a} = B_x$.

Without any further hypothesis it is not clear how the set (12.6) can be written as a union of basic open sets $D(a) \subset \operatorname{Spec val}_x G$ with $a \in G^+$. If G is an x-Prüfer group, however, we know that an element in $\Lambda_x(G)^+$ may be identified with an integral and finitely generated x-ideal $C_x = (c_1 \ldots c_k)_x$ (i.e. with all $c_i \in G^+$). In this case

$$\alpha(D(C_x)) = \{v \in \operatorname{Spec val}_x G \mid v(c_1) \wedge \ldots \wedge v(c_k) = e\}$$

$$= D(c_1) \cup \ldots \cup D(c_k)$$

and α is hence an open map.

Although this seems to reconfirm that the openness of α depends on the x-Prüfer condition we have not been able to prove the converse: α is open $\Rightarrow$ G is an x-Prüfer group.

13. Germinal ideals and real representations. We shall now relate the material of the preceding paragraph to Keimel's sectional representation theory for GCD-groups. His approach is based on the notion of a germinal l-ideal which in a purely algebraic form imitates the analytical notion of an ideal of vanishing germs at a

given point. Without using Keimels general machinery this notion will quickly lead us to a quite satisfactory sectional representation theorem for regular GCD-groups with a formal unit (bearing in fact a considerable resemblance to Stone's representation theorem for Boolean algebras).

The l-ideals of a GCD-group form an ideal system with respect to the 'multiplication' $a \cdot b = |a| \wedge |b|$. Let $\mathrm{Spec}_l\, G$ denote the family of prime l-ideals P equipped with the spectral topology where the basic open sets are given by $E(a) = \{P \in \mathrm{Spec}_l G \mid a \notin P\}$. (For simplicity we are dropping the subscript l in the prime l-ideals, thereby also avoiding any confusion with t-ideals.) For any subset $A \subset G$, $E(A)$ denotes the open set $\{P \mid A \not\subset P\} = \bigcup_{a \in A} E(a)$.

We now fix $P \in \mathrm{Spec}_l\, G$ and let U denote an open neighbourhood of P . We put

$$0_U = \bigcap_{Q \in U} Q \quad \text{and} \quad 0_P = \bigcup 0_U$$

(where the latter union is taken over all open neighbourhoods U of P).

The set 0_P is an l-ideal contained in P which is called the <u>germinal l-ideal</u> associated with P . A sheaf of GCD-groups may now be defined over $\mathrm{Spec}_l\, G$ by choosing $G/0_P$ as the stalk corresponding to P . Every element $g \in G$ will give rise to a 'section' $\hat{g}$ in the disjoint union F of these stalks by putting

$$\hat{g}(P) = g_P$$

where g_P denotes the residue class in $G/0_P$ to which g belongs. This induces a projection map $\pi : F \to \mathrm{Spec}_l\, G$ by putting $\pi(\hat{g}(P)) = P$. In order to make $(\mathrm{Spec}_l\, G, F, \pi)$ into a sheaf of GCD-groups we equip F with the finest topology making all the maps $\hat{g}$ continuous.

An alternative approach, leading to the same sheaf is to start out with the presheaf $U \to G/0_U = G(U)$ where every inclusion $V \subset U$ gives rise to a canonical homomorphism of GCD-groups $G/0_U \to G/0_V$.

In case G has a formal unit (i.e. an element u such that $\{u\}_1 = G$) Keimel proves that the map $g \to \hat{g}$ gives an isomorphism of G onto the GCD-group $\Gamma(\mathrm{Spec}_1 G, F)$ consisting of all global sections of F. As already indicated, this sectional representation has the disadvantage that the stalks need not be totally ordered. A natural condition which assures this is the condition that every prime 1-ideal is identical with its associated germinal 1-ideal: $P = 0_P$. This condition is in turn equivalent to the condition that every prime 1-ideal is maximal. This equivalence results from the fact that 0_P equals the intersection of all (minimal) prime 1-ideals contained in P (see Proposition 6.6. in [25]).

The notion of a germinal ideal offers another opportunity to spell out the analogy between (von Neumann) regular rings and regular GCD-groups in a more precise way than we have done so far. In fact we have here two situations which give rise to a regular ideal system in the sense of the following

Definition. An x-system (D,x) (see paragraph 3 (i)) is said to be (von Neumann) regular if it satisfies the following conditions

(i) D has an x-zero element 0 satisfying $\{0\}_x = \{0\}$.

(ii) D is reduced in the sense that it has no non-zero nilpotent elements.

(iii) Every prime x-ideal in D is a maximal x-ideal.

(One can easily verify that under the assumption of (i) and (ii),

(iii) is equivalent to $\mathrm{Spec}_x D$ being Hausforff and also to $\mathrm{Spec}_x D$ being a regular (Hausdorff) space. There is thus a happy and unexpected coincidence between the algebraic and the topological terminology.)

On the basis of this definition and the fact that the notion of a germinal ideal carries over to arbitrary x-systems we can give a more general and satisfactory answer to the problem of characterizing those situations where a prime ideal always coincides with its associated germinal ideal:

Theorem 10. An additive x-system (D,x) with an x-zero element is regular if and only if every prime x-ideal in D coincides with its associated germinal x-ideal.

Since this theorem is somewhat of a digression with respect to the main content of the present paper, we shall omit its proof and rather refer the reader to a forthcoming paper [6]. For the proof of Theorem 10 one should notice that it is not necessary to assume that D has a multiplicative identity (as has been the tacit assumption throughout this paper). What we have in mind is really that D has an x-identity in the sense of [2] p. 34. This is an element u such that $u \in D^2$ and $\{u_x\} = D$. The identity element of a ring and a formal unit of a GCD-group (with respect to the 'multiplication' $|a|\wedge|b|$) are both examples of x-identities. With this in mind we note the following consequences of Theorem 10.

A. A commutative ring R (with an identity element) is a von Neumann regular ring if and only if any of its prime ideals coincides with the associated germinal ideal.

B. A GCD-group with a formal unit is regular if and only if every prime l-ideal coincides with its associated germinal l-ideal.

C. A distributive lattice with a greatest and a least element is a Boolean lattice if and only if any of its prime lattice ideals is equal to the associated germinal lattice ideal.

(In the case C. the distributivity assures that the lattice ideals form an ideal system with respect to $\wedge$ as a 'multiplication'.)

Superficially, the case B. seems to give a link between Keimels sheaf representation and ours (see Corollary 2 of Theorem 8). There are, however, considerable differences between the two situations. Whereas our approach yields quasi-compactness of the base space and total order of the stalks for general GCD-groups, the corresponding properties are obtained in Keimels approach only when G has a formal unit and the germinal l-ideal which is associated to a prime l-ideal is itself prime. (See Theorem 10.6.2 in [10] and its corollaries.) For regular GCD-groups the two approaches give sectional representations which bear a certain resemblance to each other in that they both have real groups as stalks. But apart from this there are marked differences, stemming above all from the different topological properties of $\mathrm{Spec}_t\, G$ and $\mathrm{Spec}_l\, G$.

It should be noted, however, that Keimel is able to dispense with the condition that O_P is a prime l-ideal and still obtain a sheaf representation with totally ordered stalks. This is done by restricting the given sheaf to $\mathrm{Spec\,min}_l\, G$ consisting of the minimal prime l-ideals with the subspace topology induced from $\mathrm{Spec}_l\, G$. For a minimal prime l-ideal is always identical with its associated germinal l-ideal and the stalk is hence totally ordered. It seems, however, that the restriction to $\mathrm{Spec\,min}_l\, G$ further damages the fullness of the representation. Without a formal unit Keimel can only claim that sections with quasi-compact support on $Spec_l\,G$ come from elements in G. When restricting the sheaf to $Spec\,min_l\,G$ even this is no longer true.

Although this is a digression from the main theme of the present paper we shall close these considerations on sheaf representation of GCD-groups by proving the following rather specialized representation theorem (which in spirit comes close to Stones topological representation of Boolean algebras).

Theorem 11. Every regular GCD-group with a formal unit is isomorphic to the GCD-group of all sections in a sheaf of real groups over a totally disconnected, compact Hausdorff space.

Proof: We shall give a direct proof of this theorem which is based on the notion of a germinal l-ideal but which avoids any use of the material in Chapter 10 of [10]. In particular we shall avoid the use of Keimel's 'standard construction' (10.4.7 p. 212 in [10]) and the succeeding main theorem 10.6.2. Instead we shall base the proof on the consideration of the presheaf $\mathcal{L}_Y$ defined over the space $Y = Spec_l\,G$ by the assignment $U \to G/O_U = G(U)$ and combine this with the use of Nakano's chinese remainder theorem for l-ideals [36].

Let us first verify the topological properties of $Spec_l\,G$ announced in the theorem. A formal unit is an (integral) element u in G such that $\{u\}_l = G$. It is easily seen that the existence of a formal unit in G is equivalent to the quasi-compactness of $Spec_l\,G$ (see p. 16 in [25]). The Hausdorff property is likewise an immediate consequence of the fact that there exists no inclusion relation between two different prime l-ideals in G. That $Spec_l\,G$ is totally disconnected results from the fact that the basic open sets $U_a = E(a)$ are also closed. In fact, for any a and

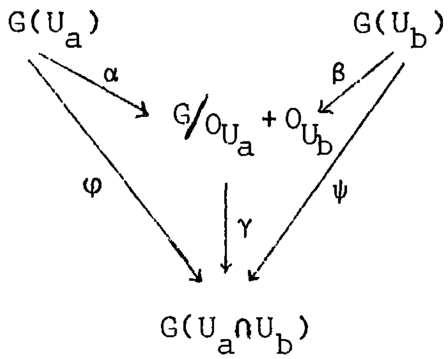
$a^{\perp} = e : a = \{b \mid |b| \wedge |a| = e\}$ we have the relations

$$E(a) \cup E(a^{\perp}) = Y \quad \text{and} \quad E(a) \cap E(a^{\perp}) = \emptyset .$$

This follows from the fact that exactly one of the two relations $a \in P$ or $a^{\perp} \subset P$ holds for each $a \in G$.

We shall next verify that the presheaf $\mathcal{L}_Y$ is a sheaf. Hence, let $\{U_a\}$ with $a \in A \subset G$ be a covering of Y by basic open sets and let the family $\{g_a \in G(U_a) \mid a \in A\}$ be selected in such a way that for each pair of elements $a, b \in A$ the presheaf images of g_a and g_b in $G(U_a \cap U_b)$ are equal. We must show that there exists a unique $g \in G = G(Y)$ whose image in $G(U_a)$ is g_a for all $a \in A$.

Since the unicity is obvious let us pass to the existence. Consider the diagram



The two 'exterior' maps φ and ψ are ordinary presheaf maps whereas the 'inner' maps α, β, γ are canonical maps induced on the factor groups by the inclusions $0_{U_a}, 0_{U_b} \subset 0_{U_a} + 0_{U_b} \subset 0_{U_a \cap U_b}$. The crucial point is that the regularity condition in the theorem (every prime l-ideal is maximal) assures that also $0_{U_a \cap U_b} \subset 0_{U_a} + 0_{U_b}$ such that γ becomes the identity map. In fact, when this latter inclusion is interpreted in the spectral topology of Y it simply amounts to the inclusion $\bar{U}_a \cap \bar{U}_b \subset \overline{U_a \cap U_b}$ which is trivially true since U_a and U_b are closed sets.

By the compactness of Y we can select a subcovering $\{U_b\}$ of $\{U_a\}$ with $b \in B$ for some finite subset B of A. We now apply Nakano's chinese remainder theorem for l-ideals [36] to the finite families $\{0_{U_b}\}$ and $\{g_b\}$. Actually, by the initial compatibility condition on the g_a's we have $g_b \equiv g_c (\mathrm{mod}\ 0_{U_b \cap U_c})$ which by the identity $0_{U_b \cap U_c} = 0_{U_b} + 0_{U_c}$ amounts to

$$g_b \equiv g_c \ (\mathrm{mod}(0_{U_b} + 0_{U_c}))$$

for all $b, c \in B$. By Nakano's theorem there exists a $g \in G$ such that

(13.1) $$g \equiv g_b \ (\mathrm{mod}\ 0_{U_b}) .$$

This means that g is mapped onto g_b for all $b \in B$ by the given presheaf maps $G(Y) \to G(U_b)$. We now claim that

(13.2) $$g \equiv g_a \ (\mathrm{mod}\ 0_{U_a})$$

for all $a \in A$. Since $0_{U_b} \subset 0_{U_a \cap U_b}$ (13.1) gives

(13.3) $$g \equiv g_b \ (\mathrm{mod}\ 0_{U_a \cap U_b}) .$$

Combining (13.3) with the initial condition $g_a \equiv g_b (\mathrm{mod}\ 0_{U_a \cap U_b})$ we obtain

(13.4) $$g \equiv g_a \ (\mathrm{mod}\ 0_{U_a \cap U_b})$$

for all $b \in B$.

Using (13.4) together with $0_{U_a} = \bigcap_{b \in B} 0_{U_a \cap U_b}$ we get (13.2) as desired. This finishes the proof that $\mathcal{L}_Y$ is a sheaf and that we hence have an isomorphism of GCD-groups $G \simeq \Gamma(Y, \mathcal{L}_Y)$.

For the remaining part of the theorem we observe that the very definition of a direct limit gives

$$\varinjlim G/0_U = G/0_P$$

where O_P is the germinal l-ideal belonging to P and the limit is taken over all spectral (basic) open neighbourhoods of P. Since P is a minimal prime l-ideal it follows that $O_P = \bigcap_{Q \subset P} Q = P$ and the stalk at P of the sheaf $\mathcal{L}_Y$ will hence be isomorphic to the totally ordered group G/P. Since P is a maximal l-ideal of G this stalk will be order isomorphic to a subgroup of the group of real numbers and this completes the proof of the theorem∎

This paper deals with basic arithmetical questions linked to the notion of a t-ideal. With respect to this perspective, one may say that our considerations on germinal l-ideals and the associated sheaf representation are somewhat marginal. Prime l-ideals are, however, intimately linked to the prime t-ideals and it is essential to be able to play on both of these types of objects and the duality between them. It should also be noted that the crux of the preceding proof (i.e. the chinese remainder theorem of Nakano) has a distinctly arithmetical origin. Nakano's theorem arose directly out of considerations by Krull [30] and Ribenboim [42] concerning approximation theorems in valuation theory. (For a more general treatment of the relationship between sheaf representations and chinese remainder theorems see Cornish [14].)

Theorem 11 deals with real sectional representation of GCD groups. Let us now turn to real functional representation of (partially) ordered groups. The literature on this topic is somewhat confusing and difficult to penetrate. There seems to be a need for a comprehensive exposition which surveys the whole field and which clarifies the interrelations between the different approaches and the different underlying assumptions. A comparison is made difficult by the fact that different authors have different candidates as to the objects which are chosen as the points of the representation space (i.e. the points making up the domain of definition for the representing functions). We shall now show that a neat exposition of the topic of real functional representation of ordered groups is achieved by the use of the Lorenzen t-group and the GCD-functor. This is really nothing more than applying the language of the present paper in order to give a more clear exposition of the main content of an interesting but rather cryptic paper by I. Fleischer [17].

Theorem 12. A completely integrally closed group G $(\neq \{e\})$ with an archimedean element (strong unit) is order isomorphic to a separating group of continuous real-valued functions on a compact (Hausdorff) space.

Proof: We recall that an archimedean element of G is an element $u > e$ such that for every $g \in G$ there exists $n \geq 1$ with $u^n \geq g$. Since G is completely integrally closed, it can be embedded (order-isomorphically) in its group of v-ideals G^*. In particular, G is t-closed (Proposition 4 in [21] p. 26) and we have a commutative diagram of (not necessarily surjective) isomorphisms of ordered groups

(13.5)
$$\begin{array}{ccc} & \Lambda_t(G) & \\ {\scriptstyle\tau}\nearrow & & \searrow{\scriptstyle\Phi} \\ G & \xrightarrow{\ \varphi\ } & G^* \end{array}$$

All the maps in (13.5) are (t,t)-morphisms and the diagram (13.5) is just a particular case of (6.3) with Φ defined as in (6.4) (from which its injectivity results). The GCD-group $\Lambda_t(G)$ is again completely integrally closed with the same archmedian element as G. It is known that for a GCD-group with an archimedian element the condition of complete integral closure amounts to the property that the intersection of the maximal l-ideals of G reduces to the identity element (or equivalently to the fact that its monoid of integral elements is equal to the intersection of all the t-valuation monoids arising from localization at minimal prime t-ideals). A short proof of the part of this result which interests us here runs as follows: For any GCD-group H with an archimedian element $u > e$ we can to each prime l-ideal P_i select a maximal l-ideal M_i containing P_i (which is itself prime due to the presence of u). This gives rise to obvious homomorphisms of GCD-groups

$$H \xrightarrow{\varphi} \prod_{i \in I} H/P_i \xrightarrow{\psi} \prod_{i \in I} H/M_i$$

where $\varphi = \{\varphi_i\}_{i \in I}$ is known to be injective. Assume now that H is completely integrally closed and that $\mathrm{Ker}(\psi \circ \varphi) \neq \{e\}$. Since $\mathrm{Ker}(\psi \circ \varphi)$ is an l-ideal we can assume $a \in \mathrm{Ker}(\psi \circ \varphi)$ with $a > e$ and we must then have $a^n \nleq u$ for a certain $n \geq 1$, because of the complete integral closure. In view of the fact that φ is an isomorphism and H/P_i is totally ordered, this entails $\varphi_i(a)^n > \varphi_i(u)$ for some i. Since $\varphi_i(u)$ is an archimedian element in H/P_i it follows that $\varphi_i(u) \notin M_i$ and hence that $(\psi_i \circ \varphi_i(a))^n$ is strictly positive in H/M_i. From this we infer that $\psi_i \circ \varphi_i(a)$ is different from the identity element in H/M_i, contradicting that $a \in \mathrm{Ker}(\psi \circ \varphi)$.

Once the "strong l-semisimplicity" has been proved, the functional representation of $\Lambda_t(G)$ over the <u>set</u> $\mathrm{Spec\,min}_t(\Lambda_t(G))$ (or equivalently over the <u>set</u> $\mathrm{Spec\,max}_l(\Lambda_t(G))$, results immediately since G/P_l is a real group for any maximal l-ideal P_l. Endowing the set $\operatorname{Spec}\min_t(\Lambda_t(G))$ with the coarsest topology making all the representing functions continuous, we clearly obtain a representation of $\Lambda_t(G)$ which has the properties announced in the theorem.

It remains to be seen how the representation of $\Lambda_t(G)$ induces the desired representation of G via the inclusion map τ in (13.5) and how the representation space may be described in terms of entities in G. It is convenient to do the latter part first: We know already that the maps α and β in the diagram (12.3) are bijections (α is a bijection because of Lorenzens theorem and β is a bijection since a GCD-group is a t-Prüfer group). These two bijections induces the bijections

(13.6) $\operatorname{Spec}\max \operatorname{val}_t G \xleftrightarrow[-]{} \operatorname{Spec}\max \operatorname{val}_t \Lambda_t(G) \longleftrightarrow \operatorname{Spec}\min_t \Lambda_t(G)$

where the left hand side denotes the set of all maximal t-valuation monoids of G - or equivalently the set of all real-valued t-valuations of G. We thus only transport the above-mentioned weak topology of the right-hand side of (13.6) to the left-hand side, which indeed consists of a family of objects directly attached to G.

We must finally show that the restriction of the representation from $\Lambda_t(G)$ to G retains the property of point-separation. Assume hence that $\hat{g}(v_1) = \hat{g}(v_2)$ for $v_1, v_2 \in \operatorname{Spec}\max \operatorname{val}_t G$ and all $g \in G$. This means that $v_1(g) = v_2(g)$ for all $g \in G$. By the Lorenzen theorem, v_1 and v_2 are uniquely extendible to $v_1' = \alpha^{-1}(v_1)$ and $v_2' = \alpha^{-1}(v_2) \in \operatorname{Spec}\max \operatorname{val}_t \Lambda_t(G)$ (using the notation of (12.3)). Thus $v_1'(h) = v_2'(h)$ or $\hat{h}(v_1') = \hat{h}(v_2')$ for all $h \in \Lambda_t(G)$. This means that $v_1' = v_2'$ and hence $v_1 = v_2$ as desired. ∎

At first sight, the reader will probably have some difficulty in recognizing the above proof as a precision of Fleischers proof, which hardly contains more than hints. But if one observes that the group $\bar{G}$ occurring at the bottom of page 261 of [17] is nothing but the Lorenzen t-group of G and that the 'maximal closed semigroups' in the second paragraph of page 262 coincide with our maximal t-valuation monoids, one sees that the spirit of our proof is in fact quite close to Fleischers proof-suggestions - although we make a much more explicit use of our heritage from Lorenzen. Another exposition of Fleischers work has been given by P. Ribenboim in [41]. As to the origin of Theorem 12, it goes back to more analytical work of Yosida and Stone and a later paper by Ky Fan [31] The present neat formulation seems to be due to Fleischer. Ribenboim [41] (Theorem 11 p 75) gives reference to Jaffard [22] for a similar result, but this reference does not seem to be quite accurate. Theorem 12 occurs also, essentially, as a corollary of a more complicated and more general representation theory given in [38].

14. Historical remarks. There does not seem to exist any comprehensive and satisfactory account of the history of the theory of divisibility. Here, we shall content ourselves by stressing a few points of this history which are of particular relevance to the present paper.

When considering Dedekinds work in algebraic number theory on the one hand and his introduction of real (irrational) numbers on the other, one is struck by the close analogy between the two situations. In the former case we have the multiplicative group of non-zero elements of an algebraic number field giving rise to an ordered group with respect to the divisibility relation which is induced by the algebraic integers of the given field. In the latter case we have the rational numbers considered as an ordered group with respect to addition and the usual ordering of the rationals. Dedekind was here faced with two classical completion problems which he solved in essentially the same manner - although he did not himself realize how far the analogy goes. In retrospect we can now see that the objects he adjoined in order to achieve the completion are the same in both cases. In fact, the upper half of a Dedekind cut is nothing but a v-ideal ('divisor') in the ordered additive group of rational numbers. On the other hand Dedekind was fully aware of the fact that the fractional ideals of an algebraic number field form a group under ideal multiplication (see his Supplement XI to Dirichlet's 'Vorlesungen über Zahlentheorie' p 553 in the fourth edition from 1894). From this one infers immediately that any ideal $\mathfrak{a} = (\mathfrak{a}^{-1})^{-1}$ is a divisorial ideal and that $\mathfrak{a}$ is equal to the intersection of all the fractional principal ideals containing it. With respect to divisibility as ordering relation, this amounts to saying that $\mathfrak{a}$ is the set of all upper bounds of the set of all lower bounds of $\mathfrak{a}$ - i.e. the upper half of a Dedekind cut. It does not seem that Dedekind was aware of the fact that his 'cuts' and his 'ideals' are formally identical objects.

Viewed against this background it would not have been very surprising if Dedekind had discovered divisorial ideals more than half a century before they finally appeared on the scene around 1930 in the works of Arnold, Artin and van der Waerden. But such an early discovery by Dedekind would not have been of any great importance for the development of the theory of algebraic number fields. This is so simply because rings of algebraic integers are not only Krull domains, but are Dedekind domains where the destinction between ordinary ideals and divisorial ideals is obliterated. One is rather tempted to say that a purely multiplicative concept of ideals, in the form of divisorial ideals, could have been harmful to progress at this early stage of the game. In algebraic number theory one is also highly interested in the additive services which are rendered by Dedekind ideals in connection with the generalization to rings of algebraic integers of the familiar notion of a congruence in elementary number theory.

Several books on algebraic number theory show a surprisingly great preoccupation with the "rivalry" between Dedekind and Kronecker with respect to their alternative foundations of algebraic number theory. (In addition to the difference between the approaches of Dedekind and Kronecker one can also distinguish a third line of development which emphasizes the 'local' and 'valuation-theoretic' point of view and which is associated with such names as Kummer, Weierstrass, Hensel and Hasse.)

Among the classical textbooks on algebraic number theory which make a point of expressing a spiritual alliance with Kronecker rather than with Dedekind one can mention the following three: H. Weyl: Algebraic Theory of Numbers (Princeton 1940), H. Hasse: Zahlentheorie (Berlin 1963) and M. Eichler: Einführung in die Theorie der algebraischen Zahlen und Funktionen (Basel 1963). We do not contest the interest and the relevance of many of the remarks made by these authors concerning their inclination towards a Kroneckerian point of view. But one gets the impression that these authors are not always equally well informed about the developments in divisibility theory which have taken place since 1930.

Weyl gives in [45] an interesting account of the Kroneckerian theory of divisors, but apparently without knowing of previous work of Prüfer and Krull on this subject. In particular, Weyl shows no awareness of the basic connection between the Kronecker function ring and valuation theory - a connection which first showed up in Krull's work [28]. Krull discovered that there is a bijection between the valuation rings which are canonically associated with an integrally closed domain and the prime ideals of the corresponding Kronecker function ring.

One of the paragraphs in Weyl's book has the headnig 'Our disbelief in ideals'. Weyl's disbelief seems here to be rooted in the misconception that there exists only one brand of ideals - namely Dedekind ideals. Only in an 'amendment' at the end of his book does Weyl hastily mention v-ideals.

It is of some historical interest to note that Weyl seems to have been the first to present an axiomatic introduction of divisors (p 38 in [45]), preceding the works of Krull [29], Borevic-Shafarevic [11] and Skula [43] which we have referred to in paragraph 9. Weyl enumerates altogether 17 axioms, but a major bulk of them just expresses that the group of divisors is supposed to contain the divisibility group of a field as an ordered subgroup (relative to a given notion of divisibility).

Bourbaki [12, p 584] gives an account of the history related to the 'rivalry' between Dedekind and Kronecker, but plays down the differences when he says: "Kroneckers adjunction of indeterminates, when the Theory of Numbers is concerned, is scarcely in our eyes more than a variant of Dedekinds". But neither Bourbaki gives any account of the developments in divisibility theory after 1930, developments which open up for new clues as to the relationships between the approaches of Dedekind and Kronecker. Prüfers paper [39] is a milestone in this connection, representing the main inspiration for succeeding papers by Krull and Lorenzen. Lorenzen's multiplicative version of a Kronecker function ring has brought about a very harmonious fusion of the approaches of Dedekind and Kronecker and at the same time incorporating the valuation-theoretic point of view. This fusion is in the present paper summarized in one basic theorem which we have taken the liberty of calling 'the main theorem of divisibility theory' (Theorem 2) and which is the pivotal result around which our paper turns. The completion processes of Dedekind and Kronecker are here united in the explicit solution of a universal problem - formulated in a convenient categorical language.

It is an intrigueing question to try to explain the total absence of the notion of a 'divisor of finite character' (t-ideal) from virtually all the main texts on divisibility theory - be it books on the multiplicative ideal theory of rings or be it books on ordered groups. It is certainly quite understandable that Weyl as a non-specialist (in 1940) did not know of the small paper by Arnold on t-ideals, published in 1929 in a Russian journal. And if he had known about it, one can hardly expect that he should have perceived that the t-ideals really furnish the answer to the quotation from his 'Algebraic Theory of Numbers' which we have chosen as the epigram of the present paper (see p. 1).

In spite of his life-long enthusiasm for algebraic number theory Hermann Weyl must still be said to have been somewhat of an outsider when it came to divisibility theory. The same thing can hardly be said about Krull who made outstanding contributions to the arithmetic theory of integral domains over a period of more than 30 years. In view of this it is surprising that Krull never seemed to grasp the relevance of t-ideals for divisibility theory, at least if we are to judge from his published works. We have not been able to find any trace of t-ideals in his 'Idealbericht' nor in his long series of papers entitled "Beiträge zur Arithmetik kommutativer Integritätsbereiche", nor in his papers on Krull rings ('Endliche diskrete Hauptordnungen'). In his paper [30] from 1957 there are some rather casual remarks about prime t-ideals, but only in the case of GCD-groups where the full importance of this notion is not yet apparent. The absence of t-ideals from Krull's works must seem rather astonishing to anybody who is familiar with Krull's life-long and remarkable occupation with the arithmetic of integral domains. The notion of a t-ideal lies at the cross-road of several of Krull's pet topics such as "Endliche diskrete Hauptordnungen" and "Gruppensätze". In his last paper (Beitrag VII from 1943) in the aforementioned series, Krull makes extensive reference to Lorenzen's paper [33] and to the crucial 'finite character property of ideal systems' which is shared by the t-ideals but not by the v-ideals. With this in mind it becomes an outright puzzle when one reads his short bibliographical note on v-ideals on p. 121 in his 'Idealbericht' where he refers among other things to Arnolds paper [1]. In order to describe the content of that paper, Krull writes in a parenthesis: ("v-Ideale in Halbgruppen"). But Arnolds paper is concerned with t-ideals and not with v-ideals! The only conclusion one can draw from all this seems to be that Krull had not really studied the works of Arnold and Lorenzen very carefully.

There is not much to be added to the history of t-ideals after Lorenzen. The only other author who has let them play a prominent role in his works is Jaffard who used them in sereral of his papers. For instance, he introduces in [23] the spectral topology for prime t-ideals although he only makes rather superficial use of this in connection with a characterization of irreducible representations of the kind (10.1). The most important reference to t-ideals besides Lorenzen [33] is certainly Jaffards monograph [21], where the notion of a t-ideal plays a significant role throughout the book.

15. Remarks on terminology. The theory of divisibility is crowded by a varying and rather confusing terminology. It seems that the language of ideal systems may here offer a more systematic viewpoint. General concepts such as 'x-closed', 'regularly x-closed', 'x-Prüferian', 'x-Bezoutian', etc. give by putting $x = s,d,t,v \ldots$ a host of special notions which have been considered in the literature and which have been given varying names which do not sufficiently reflect the underlying systematic ties between these notions. We shall here mention a few instances which are particularly pertinent to the present paper.

(i) 't-ideal'. This terminology is employed by Lorenzen [33] and Jaffard [21] whereas for instance Arnold [1] and Brandal [46] simply talk about ideals instead of t-ideals. In the case of a GCD-group the term 'prime filter' is frequently used instead of prime t-ideal ([7], [19]). Krull [30] speaks in this case of a 'Primhalbgruppe'.

(ii) 'regularly t-closed'. Here Bourbaki [12] p. 554 speaks of 'regularly integrally closed' and so do I. Beck [8] p. 88 and N. Railland [40]. The German term "endlich arithmetisch brauchbar" was employed especially by Krull, but it is also in use in non-German literature (as for instance in [19] p. 394). Jaffard uses the term 't-γ fini' inspired by Prüfers original terminology used in [39] . In case of rings the term 'v-domain' is in use ([19] p. 418).

(iii) 't-Prüfer(ian)' In the case of rings, this is the notion 'pseudo-Prüferian' in Bourbaki [12] p. 561. Jaffard uses the designation 'v-β fini' for this notion (again a remnant from Prüfers original terminology). Griffin [20] speaks of a 'v-multiplication ring' and Gilmer [19] p. 427 of a 'Prüfer v-multiplication rings'.

(iv) 't-Bezout(ian)'. In the case of rings the most frequently used term seems to be the term of a 'GCD-domain' (also called a HCF-domain). Bourbaki's term is here 'pseudo-Bezoutian' [12] p. 551. Prüfers original terminology was 'vollständig' whereas Lorenzen speaks of 't-vollständig' (or 'v-vollständig') as a particular case of 'r-vollständig' for ordered groups in general.

In the present arithmetical context we have found it more natural and suggestive to use the term 'GCD-group' instead of the familiar term of a lattice (ordered) group or l-group. This is also in better harmony with the term 'GCD-functor' and the already existing notion of a GCD-domain. Actually, the term 'GCD-group' would have been a very natural choice of terminology right from the start since this notion was first formulated and studied abstractly by Dedekind in the context of divisibility theory (see [15] § 6).

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